Fast Exponential Algorithms for Maximum r-Regular Induced Subgraph Problems

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Abstract. Given a graph $G = (V, E)$ on n vertices, the MAXIMUM r-REGULAR INDUCED SUBGRAPH (M-r-RIS) problems ask for a maximum sized subset of vertices $R \subseteq V$ such that the induced subgraph on R, $G[R]$, is r-regular. We give an $\mathcal{O}(c^n)$ time algorithm for these problems for any fixed constant r, where c is a positive constant strictly less than 2, solving a well known open problem. These algorithms are then generalized to solve *counting* and *enumeration* version of these problems in the same time. An interesting consequence of the enumeration algorithm is, that it shows that the number of maximal r-regular induced subgraphs for a fixed constant r on any graph on n vertices is upper bounded by $o(2^n)$.

We then give combinatorial lower bounds on the number of *maximal* r-regular induced subgraphs possible on a graph on n vertices and also give matching *algorithmic* upper bounds.

We use the techniques and results obtained in the paper to obtain an improved exact algorithm for a special case of INDUCED SUBGRAPH ISOMORPHISM that is INDUCED r-REGULAR SUBGRAPH ISOMORPHISM, where r is a constant.

All the algorithms in the paper are simple but their analyses are not. Some of the upper bound proofs or algorithms require a *new and different measure* than the usual number of vertices or edges to measure the progress of the algorithm, and require solving an interesting system of polynomials.

1 Introduction

The problem of finding a MAXIMUM/MINIMUM INDUCED SUBGRAPH having properties like acyclicity [6,14], bipartiteness [3,13], regularity [4,5,7,15,16] and regularity with dominance [2] is among the fundamental problems in graph algorithms. Here we study one such problem, namely the MAXIMUM r-REGULAR INDUCED SUBGRAPH problem. The problem is defined as follows:

> MAXIMUM r-REGULAR INDUCED SUBGRAPH (M-r-RIS): Given an undirected graph $G = (V, E)$, find a maximum subset of vertices $R \subseteq V$ such that the induced subgraph on R, $G[R]$, is r-regular.

When r is 0 or 1, it corresponds to the well studied MAXIMUM INDEPENDENT SET and MAXIMUM INDUCED MATCHING problems respectively. While MAXIMUM INDEPENDENT SET problem is among the six classical NP-complete problems [9], MAXIMUM INDUCED MATCHING problem was introduced by Stockmeyer and Vazirani in

S. Arun-Kumar and N. Garg (Eds.): FSTTCS 2006, LNCS 4337, pp. 139–151, 2006.

[17] who showed it to be NP-complete [17]. But only recently, has it been shown [5] that the problem of finding a maximum sized r-regular induced subgraph is NP-complete for any value of r.

In this paper we look at the M-r-RIS problems (a) from *exact exponential time algorithm* paradigm and (b) from the view point of *combinatorial bounds* on the number of maximal r-regular induced subgraphs possible on a graph on n vertices.

An exact algorithm to find a MAXIMUM INDEPENDENT SET or M-0-RIS problem has attracted a lot of attention in the area of exact exponential time algorithms [7,15] and the current fastest known exact algorithm runs in time $\mathcal{O}(1.2108^n)$ [1][15]. There is no algorithm better than $\Theta(2^n)$ is known for larger values of r.

Here, we give a *simple-generic* algorithm for MAXIMUM r-REGULAR INDUCED SUBGRAPH problems taking $\mathcal{O}(c^n)$ time, $c < 2$, a constant, depending on r alone. As a corollary, we obtain $\mathcal{O}(1.6957^n)$, $\mathcal{O}(1.7069^n)$ and $\mathcal{O}(1.7362^n)$ time algorithms for MAXIMUM INDUCED MATCHING, MAXIMUM 2-REGULAR INDUCED SUBGRAPH and MAXIMUM INDUCED CUBIC SUBGRAPH problems respectively. We then generalize the algorithm to solve the counting and enumeration version of M-r-RIS problems in the same time.

Another interesting consequence of the algorithm is that it gives an algorithmic upper bound of $o(2^n)$ on the number of maximal r-regular induced subgraphs on n vertices, if r is some constant. We then investigate the lower bounds on the number of maximal r-regular induced subgraphs of a graph and observe that for larger values of r, the lower bounds and the upper bounds (mentioned above) on the number of maximal r-regular induced subgraphs on n vertices are "almost identical". For small values of r, we improve the upper bounds using a different technique and give a matching *lower* and *upper* bounds on the number of maximal r-regular induced subgraphs. This generalizes the result of Moon and Moser [12] who showed an upper and lower bound of $3^{n/3}$ on the number of maximal independent sets on a graph on n vertices.

Applications of the algorithms developed in this paper include non trivial exact algorithms for a special case of INDUCED SUBGRAPH ISOMORPHISM problem, that is INDUCED r-REGULAR SUBGRAPH ISOMORPHISM problem, where r is a constant, δ-SEPARATING MAXIMUM MATCHING problem [17] and EFFICIENT EDGE DOMINATING SET problem [10].

All our algorithms are simple but their analyses are non trivial. These algorithms are based on one of the most important and widely used tool of exact algorithms, namely the *Branch & Reduce* paradigm. In this paradigm we obtain an optimal solution to a problem by combining solutions to many subproblems of smaller size. We also use a *new measure* not just the number of vertices or edges to measure the progress of the algorithms and use it extensively in many of our upper bound proofs. Measure other than the number of vertices has been source of many recently developed non trivial exact algorithms [6,7,14]. See recent surveys by Woeginger [18] and Fomin et al. [8] for an overview and recent developments in designing exponential time exact algorithms.

Organization of the Rest of the Paper: In Section 2, we give a *generic* algorithm for MAXIMUM r-REGULAR INDUCED SUBGRAPH problems and then generalize it

[1] We round the base of the exponent in all our algorithms which allows us to ignore polynomial terms and write $O(c^n n^{O(1)})$ as $\mathcal{O}(c^n)$.

to solve the counting and enumeration version of the problems. In Section 3 we give matching *lower* and *upper* bounds on the number of maximal r-regular induced subgraphs for various values of r. In Section 4 we conjure all that we develop that far to give faster exact algorithms for M-r-RIS problems for $r = 1$ and 2 than that is possible from the general theorem. We also obtain a non trivial exact algorithm for INDUCED r-REGULAR SUBGRAPH ISOMORPHISM problem in this section. We conclude with some remarks and open problems in Section 5.

In the rest of the paper, we assume that all our graphs are simple and undirected. Given a graph $G = (V, E)$, n represents the number of vertices, and m represents the number of edges. For a subset $V' \subseteq V$, by $G[V']$ we mean the subgraph of G induced on V'. By $N(u)$ we represent all vertices (excluding u) that are adjacent to u, and by $N[u]$, we refer to $N(u) \cup \{u\}$. Similarly, for a subset $D \subseteq V$, we define $N[D] = \cup_{v \in D} N[v]$.

2 Maximum r-Regular Induced Subgraph

Our algorithm is based on the Branch & Reduce paradigm. It selects a vertex v and on one branch of recursion grows a maximum r-regular induced subgraph without v and on the other a maximum r-regular induced subgraph containing v and then outputs the one with the maximum size. At any point of time in our algorithm we maintain a set R (of possible vertices of a M-r-RIS) and construct one *connected component* of this R. Once we finish one connected component, say R_i, we remove all the neighbors of vertices of R_i which are not in R_i, that is $N[R_i] - R_i$, from the graph and then proceed. Based on the structure of $G[R]$, we divide our algorithm into two phases:

1. ACTIVE PHASE : $G[R]$ is $\emptyset$ or a r regular induced subgraph.
2. GROWTH PHASE : There exists a unique component R_i of $G[R]$ such that $G[R_i]$ is not a r regular subgraph.

In ACTIVE PHASE we initiate constructing a new connected component for the possible M-r-RIS. We select a vertex v and at one branch construct a solution not including v and at other branches we construct a solution containing v and a r-subset of $N(v)$. This leads to $\binom{|N(v)|}{r} + 1$ way branching. In the GROWTH PHASE, we choose a vertex v of an unique component R_i of $G[R]$ ($G[R_i]$ is not a r regular subgraph) such that degree of v in $G[R_i]$ is $r_v < r$ and branch on all possible subsets of size $r - r_v$ of $N(v) - R$, which leads to $\binom{|N(v)-R|}{r-r_v}$ way branching.

At any point of time, our algorithm has a 4 tuple $(G' = (V', E'), G, r, R)$. Here, G' contains the unexplored vertices (vertices which are neither in R nor those which have been removed from the consideration). G is the initial *input* graph. This graph never changes during recursion and is only used for checking whether or not $G[R]$ is induced r-regular. R is a set of vertices already chosen for a possible maximum r-regular induced subgraph. We return $-\infty$ if we detect that the corresponding branch can not lead to a r-regular induced subgraph; for an example if in GROWTH PHASE, we find a vertex $v \in R$ having degree r_v in $G[R]$ but strictly less than $r - r_v$ neighbors in V'. In our algorithm until we state otherwise $N(v)$ and $N[v]$ mean $N_G(v)$ and $N_G[v]$ respectively. The details of our algorithm are presented in Figure 1.

Algorithm Max-r-RIS ($G' = (V', E'), G, r, R$)

Step 1: [active phase] If $G[R]$ is not r regular and not empty then go to Step 2.

Step 1a: Obtain a new G' by removing $N[R]$ from G'.

Step 1b: Remove all vertices of degree $< r$ recursively from G'.

Step 1c: If G' is non empty then select a vertex v of maximum degree $d \geq r$ and branch in following ways: (1) $v \notin R$, and (2) $v \in R$ and then some r neighbors of v are in R.

1. $R_1 \leftarrow$ `Max-r-RIS`$(G' - v, G, r, R)$
2. for ($S \subseteq N_{G'}(v)$ & $|S| = r$),
   $R_S \leftarrow$ `Max-r-RIS`$(G' - N_{G'}[v], G, r, R \cup S \cup \{v\})$.

return *the set (or number) of maximum size between* $\{R_1\}$ *and* $\{R_S \mid S' \subseteq N_{G'}(v)\ |S'| = r\}$.

Step 2: `[growth phase]` Let R' be *the unique* component of $G[R]$ such that $G[R']$ is not a r regular induced subgraph. $R_1 \leftarrow -\infty$. Choose a vertex v with degree say r_i in $G[R']$ such that $1 \leq r_i \leq r-1$ and $|N(v) \cap V'| \geq r - r_i$.

1. for ($S \subseteq (N(v) \cap V')$ & $|S| = r - r_i$ & maximum degree of $G[R' \cup S]$ is $\leq r$)
   $R_S \leftarrow$`Max-r-RIS`$(G' - (N(v) \cap V'), G, r, R \cup S)$

return *the set (or number) of maximum size between* $\{R_1\}$ *and* $\{R_{S'} \mid S' \subseteq (N(v) \cap V')\ \&\ |S'| = r - r_i\}$.

Fig. 1. A Generic Algorithm to find a Maximum r-Regular Induced Subgraph

Theorem 1. *Let $G = (V, E)$ be a graph on n vertices and r be a fixed constant. Then there exists a constant c, $c < 2$ such that the* MAXIMUM r-REGULAR INDUCED SUBGRAPH *problem can be solved in $\mathcal{O}(c^n)$ time.*

Proof. The correctness of the algorithm is clear. The analysis of time complexity is involved and we present the details here.

From now onwards let r be a fixed positive constant. Observe that the above algorithm is guided by the following recurrences:

$$T(n) \leq T(n-1) + \binom{d}{r} T(n-d-1) \quad d \geq r \qquad \text{[Active Phase]}. \quad (1)$$

$$T(n) \leq \binom{d}{t} T(n-d) \qquad d \geq t,\ \ 1 \leq t \leq r-1 \ \text{[Growth Phase]}. \quad (2)$$

The smallest positive roots of the following inequalities,

$$h_d(x,r) = x^{d+1} - x^d - \binom{d}{r} \geq 0\ , d \geq r \ \textbf{and}\ g_d(x,t) = x^d - \binom{d}{t} \geq 0\ \ , d \geq t, 1 \leq t \leq r-1,$$

are solutions to the above recurrences. It is clear that $x = 2$ satisfies these inequalities. Now we show that if r is a constant then we can find a c, a function of r alone, and $c < 2$ satisfying these set of inequalities. We need the following easy lemma for our proof.

Lemma 1. *For any* $r \geq 5$, $\binom{2r}{r} \leq \frac{2^{2r}}{4}$.

We concentrate on the polynomials coming from the ACTIVE PHASE as they represent the dominating recurrences. Observe that

$$x^d - \binom{d}{r} \geq x^d(x-1) - \binom{d}{r} \geq x^{d+1} - x^d - \binom{d}{r}.$$

The inequality holds as $x \leq 2$. This shows that if there exists $c = f(r)$ such that $h_d(f(r), r) \geq 0$ then $g_d(f(r), r) \geq 0$.

Now we show that if there exists a $c = f(r)$ such that $h_{2r}(c, r) \geq 0$ then we can choose a c' such that $h_d(c', r) \geq 0$ for any d. We take $c' = \max\left\{c, \frac{2r+1}{r+1}\right\}$. We prove this using forward induction for $d \geq 2r$ and backward induction for $d \leq 2r$. For the base case observe that $h_{2r}(c', r) \geq h_{2r}(c, r) \geq 0$. Now assume that $h_d(c, r) \geq 0$ for some $d \geq 2r$. Then

$$h_{d+1}(c', r) = c'^{d+2} - c'^{d+1} - \binom{d+1}{r} = c'(c'^{d+1} - c'^d) - \binom{d+1}{r} \geq c'\binom{d}{r} - \binom{d+1}{r} \geq 0.$$

The second last inequality follows from induction hypothesis while the last inequality follows as: $c' \geq \frac{\binom{d+1}{r}}{\binom{d}{r}} = \frac{d+1}{d+1-r} \geq \frac{2r+1}{r+1}$, for $d \geq 2r$. Similarly using backward induction we can show that $h_d(c', r) \geq 0$ for $d \leq 2r$. Observe that for $r \geq 0$, $1 \leq \frac{2r+1}{r+1} < 2$, is a constant depending on r alone. So now we are left with showing a $c = f(r)$ for $h_{2r}(x, r)$. For $r \geq 5$, we know that $\binom{2r}{r} \leq \frac{2^{2r}}{4}$. We choose a c such that $c^{2r+1} - c^{2r} \geq \frac{2^{2r}}{4}$ which will prove the desired result. We take $c = 2^{1-\frac{1}{2r}}$ for $r \geq 5$ and $c = 1.761$ for $r \leq 4$. For small values of r we get the desired number by directly solving the corresponding equations.

Hence for any $r \geq 0$, we choose $c = \max\left\{1.761,\ 2^{1-\frac{1}{2r}},\ \frac{2r+1}{r+1}\right\}$. This proves that our generic algorithm **Max-r-RIS** takes $\mathcal{O}(c^n)$ time, $c < 2$, for any positive constant r. □

We gave a conservative bound on the value of c in the Theorem 1, as our main aim there was to obtain a $c < 2$ for any fixed constant r. For smaller values of r, we obtain improved bounds on c by directly finding the roots of the polynomials coming from the recurrences of MAX-r-RIS algorithm. Without going into the details, we list c for various values of r in the table below where $\mathcal{O}(c^n)$ is the runtime of our **Max-r-RIS** algorithm.

Table 1. Improved Upper Bounds on c for Various r

$r=$	1	2	3	4	5	6	7	8	9
$c=$	1.69562	1.70688	1.73615	1.76357	1.78554	1.80351	1.81846	1.83111	1.84195
$r=$	10	15	20	30	50	75	100	125	150
$c=$	1.85136	1.88452	1.90486	1.92868	1.95138	1.96458	1.97186	1.97652	1.97979

We observe that the **Max-r-RIS** algorithm can be generalized to solve the counting versions of M-r-RIS problems. The counting version of M-r-RIS problems (#M-r-RIS) asks for the number of maximum r regular induced subgraphs of the given graph

G. We can also consider counting the number of maximal r-regular induced subgraphs of the given graph G which we call #MAXIMAL-r-RIS problems. To solve these problems we allow our algorithm **Max-r-RIS** to enumerate all the R's it finds during the recursion for G and check whether they are maximal if we want to count maximal r-regular induced subgraphs alone. If we want to count maximum r-regular induced subgraphs then we also need to check the size of R. Thus we give the following theorem.

Theorem 2. *Let $G = (V, E)$ be a graph on n vertices and r be a fixed constant. Then (a) #M-r-RIS problems and (b) #MAXIMAL-r-RIS problems can be solved in $\mathcal{O}(c^n)$ time, where c is max of $\{1.761, 2^{1-\frac{1}{2r}}, (2r+1)/(r+1)\}$.*

We observed above that our algorithm enumerates all maximal r-regular induced subgraphs. Hence Theorem 2 also implies that the number of maximal r-regular induced subgraphs of a graph on n vertices is upper bounded by the time complexity of the algorithm. Let $\mathscr{M}_r(n)$ denote the number of maximal r-regular induced subgraph of graphs on n vertices, then we get following theorem.

Theorem 3. *Let $G = (V, E)$ be a graph on n vertices and r be a fixed constant. Then $\mathscr{M}_r(n)$ is upper bounded by c^n, where c is max of $\{1.761, 2^{1-\frac{1}{2r}}, (2r+1)/(r+1)\}$, i.e. $\mathscr{M}_r(n)$ is upper bounded by $o(2^n)$, if r is a fixed constant.*

In the next section we consider the lower bounds on the number of maximal r-regular induced subgraphs on graphs on n vertices and improve the upper bounds coming from Theorem 3 to match the lower bounds for various r.

3 Bounds on Number of Maximal r-Regular Induced Subgraphs

Moon and Moser [12] gave a matching lower and upper bound of $3^{n/3}$ on the number of maximal independent sets on a graph on n vertices. We generalize this result and give matching *algorithmic* lower and upper bounds on $\mathscr{M}_r(n)$ for larger values of r.

3.1 Bounds on $\mathscr{M}_1(n)$ or Number of Maximal Induced Matching

For lower bound assume that $n \equiv (0 \mod 5)$. Consider the graph $G = \bigcup_{i=1}^{\frac{n}{5}} K_5^i$ that is $n/5$ disjoint copies of K_5 (K_n^i represents the complete graph on n vertices). Observe that we need to include one edge from each copy of the K_5 (we can include exactly one edge from each copy) to obtain a maximal induced matching for G. Since a K_5 has 10 edges and for any K_5 we can select any edge, we get $10^{n/5}$ distinct maximal induced matching for G, giving a lower bound of $10^{n/5}$ on $\mathscr{M}_1(n)$. This shows the following theorem.

Theorem 4. *$\mathscr{M}_1(n)$ is at least $10^{n/5} \approx 1.58489^n$.*

For an upper bound proof, we obtain recurrences for $\mathscr{M}_1(n)$ by considering various cases based on the maximum degree of the graph. The proof is long and is similar to the upper bound proof in Theorem 6 which we prove in detail below.

Theorem 5. *$\mathscr{M}_1(n)$ is at most $10^{n/5} \approx 1.58489^n$ and all the maximal induced matching of a graph G can be enumerated with polynomial delay.*

3.2 Bounds on $\mathscr{M}_r(n)$ for $r \geq 2$

Now we extend the matching upper and lower bounds for larger values of $r(\geq 2)$. To give the upper bound on $\mathscr{M}_r(n)$, we define the following generalized problem.

> GEN-r-RIS (G-r-RIS): Given a graph $G = (V, E)$ and $R \subseteq V$, such that $G[R]$ is connected induced subgraph of degree at most r. The objective is to find a maximum $R' \subseteq V - R$ such that $G[R \cup R']$ is a r regular subgraph extending R.

Observe that given any instance (G, R), where R satisfies the constraints in the definition of G-r-RIS problem, if we can give a bound on the number of R' such that $G[R' \cup R]$ is a maximal r-regular subgraph then by setting $R = \emptyset$ we have an upper bound on $\mathscr{M}_r(n)$. Given an instance (G, R) where R satisfies the constraints in G-r-RIS problem, we define μ as follows:

$$\mu = \alpha|N^R| + \beta|U|$$

Here $N^R = N[R] - R$ and $U = V - N[R]$. In other words, we assign a weight of α to the vertices of N^R and β to the vertices of U. The value of α and β depend on the problem. The weight of a vertex changes in following situation:

1. If a vertex goes to N^R from U then the weight changes from β to α and the μ changes by $\delta = \beta - \alpha$.
2. If a vertex has current weight either α or β and the vertex is either included in R or removed from the graph then the weight changes to 0. In this case μ changes either by α or β.

We use μ as a measure rather than the number of vertices and give an upper bound on $\mathscr{M}_r(n)$ as a function f of μ. We exemplify the approach by giving the matching lower and upper bound on the number of *maximal* 2*-regular induced subgraphs*.

Theorem 6. *$\mathscr{M}_2(n)$ is at most $35^{n/7} \approx 1.66181^n$ and there exists a graph on n vertices such that $\mathscr{M}_2(n)$ is at least $35^{n/7} \approx 1.66181^n$. Moreover, all the maximal* 2*-regular induced subgraphs of a graph G can be enumerated with polynomial delay.*

Proof. For the lower bound on $\mathscr{M}_2(n)$, assume that $n \equiv (0 \mod 7)$ and consider the graph $G = \bigcup_{i=1}^{\frac{n}{7}} K_7^i$, $n/7$ disjoint copies of K_7. Any maximal 2-regular induced subgraph of G contains a 2 regular induced subgraph (a triangle) from each copy of K_7. Every K_7 has 35 distinct triangles and hence G has $35^{n/7}$ distinct maximal 2-regular induced subgraphs. This shows the desired lower bound on $\mathscr{M}_2(n)$.

For upper bound, we consider the generalized problem where we have been given $(G = (V, E), R)$ and R satisfies the constraints in the definition of the G-2-RIS problem. We give a bound on the number of R''s, i.e. is the size of the set $\{R' \mid G[R' \cup R]$ is a maximal 2-regular $\}$ as a function f of μ. Depending on various cases we give recurrence relation for f.

Case 1: **($G[R] \neq \emptyset$)** Here we have two cases based on the degree of a vertex in $G[R]$. For a subset $X \subseteq V$, by $deg_X(v)$ we mean the number of neighbors of v in $G[X]$.

Suppose we have a vertex $v \in R$ such that $deg_R(v) = 2$ and have l neighbors in $V - R$ then

$$f(\mu) \leq f(\mu - \alpha l);$$

as none of the l neighbors of v in $V - R$ can be selected in any R' extending R and hence can be removed from the graph, leading to decrease in μ by at least αl. Now suppose we have a vertex v such that degree of v is d in G and $deg_R(v) = 1$.

Now any maximal 2-regular induced subgraph extending R must contain one of the neighbors of v in $V - R$. Hence when we include a neighbor u of v in R we remove all other neighbors of v from G as they can not be part of any R' extending R. This reduces μ by $\alpha(d-1)$. Since there are $d-1$ neighbors of v in $V - R$, we get the following recurrence:

$$f(\mu) \leq (d-1) f(\mu - \alpha(d-1)).$$

We can assume that $(d-1) \geq 1$, otherwise in this case R can not be extended to any maximal 2 regular induced subgraphs resulting in $f(\mu) = 0$.

Case 2: ($R = \emptyset$) We assume that the minimum degree of G is at least 2, as the vertices of degree at most 1 can never be part of any maximal 2 regular induced subgraphs. Also note that every vertex has weight β now. Let v be a vertex of maximum degree d. A maximal 2-regular induced subgraph of G either does not contain v or contains v and its two neighbors. In the first case μ reduces by β and in the other cases where v and its two neighbors are selected in R and other neighbors of v are removed from the graph, μ decreases by $(d+1)\beta$. This gives the following worst case recurrence on $f(\mu)$:

$$f(\mu) \leq f(\mu - \beta) + \binom{d}{2} f(\mu - (d+1)\beta).$$

When $d \geq 7$ this recurrence itself gives us the desired bound on $\mathscr{M}_2(n)$. So from now on we assume that the maximum degree of G is at most 6. To obtain the desired bound in this case we refine the recurrences on $f(\mu)$ based on following three cases. These cases are applied in order of their appearance.

(a) CON-COM CASE: There exists a vertex v such that $G[N[v]]$ is one of the connected component of G. Call the connected component containing v $\mathcal{C}_v$. Now the number of maximal 2-regular induced subgraphs of G is maximized when we have $\mathcal{C}_v$ such that $\mathcal{C}_v$ has maximum number of maximal 2-regular induced subgraphs. This happens precisely when $\mathcal{C}_v = K_t$ where $t = deg_V(v)$. So for this case we get:

$$f(\mu) \leq \binom{d+1}{3} f(\mu - \beta(d+1)), \quad 2 \leq d \leq 6.$$

(b) CUT-EDGE CASE: We have a vertex v such that it has an unique neighbor u having an unique neighbor x such that $x \notin N[v]$. Since the edge (u, x) is a cut edge it is not part of any maximal 2-regular induced subgraph. So the number of maximal 2 regular subgraphs of G is upper bounded by the number of maximal 2 regular subgraphs of G' obtained from G by removing the edge (u, x). This reduces it to CON-COM CASE.

(c)AT-LEAST-2-IN-$N_2[v]$ CASE: In this case every vertex $v \in V$ either has a neighbor u such that u has at least 2 neighbors not in $N[v]$ or there are at least two neighbors of v which don't have neighbors in $N[v]$. For this case we give a generic recurrence. Partition the neighbor set $N(v)$ of v into W_1, W_2 and W_3 such that every vertex $u \in W_1$ has $N(u) \subseteq N[v]$, each vertex in W_2 has an unique neighbor x such that $x \notin N[v]$ while every vertex $u \in W_3$ has at least 2 neighbors not in $N[v]$. By S_y^v we mean the set $N(y)$-$N[v]$. Let $2 \leq \sum_{i=1}^{3} |W_i| = d \leq 6$. We consider the recurrence on $f(\mu)$ based on whether or not v is a part of maximal 2-regular induced subgraph. When $v \notin R$ μ changes by $\mu - \beta$. Now we consider the case when v and its two neighbors u_1, u_2 and $u_1 \neq u_2$ are in R and see the change in μ based on which W_i's, $1 \leq i \leq 3$, u_1 and u_2 belong.

(A) $[(\mathbf{u_1}, \mathbf{u_2}) \in \mathbf{W_1} \times \mathbf{W_1}]$ μ changes to $\mu - \beta(d+1)$.

(B) $[(\mathbf{u_1}, \mathbf{u_2}) \in \mathbf{W_1} \times \mathbf{W_2}]$ The only way we can have a 2-regular induced subgraph is when (u_1, u_2) is an edge and v, u_1, u_2 is a triangle. This implies that x, the unique neighbor of u_2 not in $N[v]$ will be removed from the graph. This reduces μ to $\mu - \beta(d+1) - \beta$.

(C) $[(\mathbf{u_1}, \mathbf{u_2}) \in \mathbf{W_1} \times \mathbf{W_3}]$ Similar to the previous case we can argue that μ at least reduces to $\mu - \beta(d+1) - 2\beta$.

(D) $[(\mathbf{u_1}, \mathbf{u_2}) \in \mathbf{W_2} \times \mathbf{W_2}]$ The worst case is when u_1 and u_2 have a common neighbor x which is not in $N[v]$. In this case μ changes to $\mu - \beta(d+1) - \beta$.

(E) $[(\mathbf{u_1}, \mathbf{u_2}) \in \mathbf{W_2} \times \mathbf{W_3}]$ If (u_1, u_2) is an edge or u_1 and u_2 have a common neighbor x then either $\{v, u_1, u_2\}$ or $\{v, u_1, u_2, x\}$ forms a 2 regular induced subgraph leading to a reduction of $\beta(d+1) - 2\beta$ in μ. When none of these cases arise then since x is an unique neighbor of u_1, x gets included in R and two neighbor of u_2 become elements of N^R, leading to change in μ by $\beta(d+1) - \beta - 2\delta$.

(F) $[(\mathbf{u_1}, \mathbf{u_2}) \in \mathbf{W_3} \times \mathbf{W_3}]$ Here the worst case is when u_1 and u_2 have exactly two neighbors not in $N[v]$ and $S_{u_1}^v = S_{u_2}^v$, that is u_1 and u_2 have common neighbors not in $N[v]$. This reduces μ by $\beta(d+1) - 2\delta$ as both neighbors of u_1 and u_2 which are not in $N[v]$ become element of N^R.

Above discussion gives us following recurrence on $f(\mu)$.

$$
\begin{aligned}
f(\mu) \leq f(\mu - \beta) &+ \binom{|W_1|}{2} f(\mu - \beta(d+1)) + |W_1||W_2| f(\mu - \beta(d+1) - \beta) \\
&+ |W_1||W_3| f(\mu - \beta(d+1) - 2\beta) + \binom{|W_2|}{2} f(\mu - \beta(d+1) - \beta) \\
&+ |W_2||W_3| f(\mu - \beta(d+1) - \beta - 2\delta) + \binom{|W_3|}{2} f(\mu - \beta(d+1) - 2\delta).
\end{aligned}
$$

We assume that $\binom{l_1}{l_2} = 0$ if $l_1 < l_2$. Note that, $|W_1| \leq d - 1$ and if there is an unique neighbor u of v having a neighbor x such that $x \notin N[v]$ then $W_2 = \emptyset$ because of the CUT-EDGE CASE.

We numerically obtain $\alpha = 1.45$, $\beta = 2$ and $\delta = \beta - \alpha = 0.55$, as values which minimizes the above set of recurrences on f.

We used a program to generate the above set of recurrences based on different partitions of $N(u)$ and found that the worst case recurrence among the above set after setting $\alpha = 1.45$ and $\beta = 2$ corresponds to the following scenario:

$$d = 5,\ W_1 = W_2 = \emptyset \text{ and } \forall (y, z) \in W_3 \times W_3,\ |S_y^v \cup S_z^v| = 2.$$

The recurrence corresponding to this scenario is: $f(\mu) \leq f(\mu - \beta) + 10 f(\mu - 6\beta - 2\delta)$.

r	3	4	5	6	7	8	10	15
lb_r	1.71149	1.7468	1.7734	1.7943	1.8113	1.8253	1.8474	1.8828
ub_r	1.73615	1.76357	1.78554	1.80351	1.81846	1.83111	1.85136	1.88452
$ub_r - lb_r$	0.02466	0.016782	0.012131	0.0091762	0.0071727	0.0057618	0.0039415	0.0017377

Fig. 2. Bounds on the Number of Maximal-r-Regular Induced Subgraphs for Small Values of r

All the recurrences occurring in all the above cases (Cases 1 & 2) are dominated by

$$f(\mu) \leq \binom{7}{3} f(\mu - 7\beta)$$

which solves to $(35)^{\frac{\mu}{7\beta}}$. Now given a graph G, $\mu(G) \leq n\beta$, and hence

$$\mathscr{M}_2(n) \leq f(\beta n) \leq 35^{\beta n/7\beta} = 35^{n/7}.$$

This proves the required upper bound. These cases can be changed in branching steps leading to an enumeration algorithm running in $\mathcal{O}(35^{n/7}) = \mathcal{O}(1.66181^n)$ time. □

To obtain a lower bound on $\mathscr{M}_r(n)$ for larger values of r we need to find a function $g(r)$ such that when we take G as $\frac{n}{g(r)}$ disjoint copies of $K_{g(r)}$ then $\binom{g(r)}{r+1}^{1/g(r)}$ is maximized. We obtain the following description for $g(r)$.

Lemma 2. *Given r, $g(r)$ defined below*

$$g(r) = \begin{cases} 2r+3 & 0 \leq r \leq 11 \\ 2r+4 & 12 \leq r \leq 100 \\ 2r+2+\left\lfloor \frac{1}{2}\ln\left(\frac{(2r+1)\pi}{2}\right) + O\left(\frac{(\ln r)^2}{r}\right)\right\rfloor & r > 100 \end{cases}$$

maximizes $\binom{g(r)}{r+1}^{1/g(r)}$. Hence $\mathscr{M}_r(n)$ is at least $\binom{g(r)}{r+1}^{n/g(r)}$.

The proof of Lemma 2 is based on estimates on binomial coefficients and will appear in the longer version of the paper.

For a fixed r, let lb_r and ub_r denote a base of exponent in lower bound and upper bound on $\mathscr{M}_r(n)$, i.e., $lb_r^n \leq \mathscr{M}_r(n) \leq ub_r^n$. When $r \geq 3$, we obtain tighter upper bounds on $\mathscr{M}_r$ by directly finding the roots of the polynomials coming from the recurrences in MAX-r-RIS algorithm. We can see that the upper bound obtained this way and the lower bound coming from Lemma 2 are already very close, as Figure 2 shows. For small values of r, these upper bounds could be made equal to lower bound by choosing α and β appropriately in the definition of μ and by doing the analysis similar to the one in Theorem 6. For an example, when $r = 3$ we can take $\alpha = 1.73$ and $\beta = 2$ and show that $lb_3 = ub_3$. We do not go into the details due to lack of space and the details will appear in the full version of the paper.

4 Improved Algorithms for $r = 1$ and 2 and Applications

Our generic algorithm **Max-r-RIS** finds a maximum r-regular induced subgraph in time $\mathcal{O}(1.6957^n)$ and $\mathcal{O}(1.7069^n)$ for $r = 1$ and 2 respectively. Our algorithmic upper bound proofs (on $\mathscr{M}_r(n)$) of Section 3 enumerates all maximal r-regular subgraphs in time $\mathcal{O}(1.58469^n)$ and $\mathcal{O}(1.66181^n)$ for $r = 1$ and 2 respectively, already improving the bounds given in Section 2. Here we further improve these algorithms for $r = 1$ and 2 and give an application of algorithms developed so far in the paper.

4.1 Maximum Induced Matching (MIM) and M-2-RIS Problems

Let $G = (V, E)$ be a graph and v be a vertex having a neighbor u such that $N(u) \subseteq N[v]$. Consider the set $\mathcal{M}_v$ of maximum sized induced matching having v (these may not be the maximum sized induced matching of G). Then the following is easy to see.

Lemma 3. *Let G be a graph and v be a vertex and $u \in N(v)$ such that $N(u) \subseteq N[v]$. Then there exists a $M' \in \mathcal{M}_v$ such that it contains the edge (v, u).*

The other observation relates MIM of G to MAXIMUM INDEPENDENT SET (MIS) of square of the *line graph* of G. The *line graph*, $L(G)$ of $G = (V, E)$ is the graph whose vertices are edges of G, and two edges e_1, e_2 are adjacent if and only if they are adjacent edges in G. G^i (ith power of G) is a graph on V and there are edges between two vertices v_1 and v_2 if and only if there is a path of length at most i between v_1 and v_2.

Lemma 4 ([4]). *Let G be a graph then $MIM(G) = MIS(L(G)^2)$.*

So our algorithm uses branching on a vertex v when the maximum degree of the graph is at least 5 and distinguishes cases based on Lemma 3. When the maximum degree of the graph is at most 4, we use the well known algorithms to find a maximum independent set [7,15] in $L(G)^2$. Without going into further details we state the following theorem.

Theorem 7. *Let $G = (V, E)$ be a graph on n vertices, then a* MIM *can be found in (a) $\mathcal{O}(1.4904^n)$ time and space polynomial in n or in (b) $\mathcal{O}(1.4786^n)$ time and space exponential in n.*

We obtain an improved algorithm for M-2-RIS by refining the measure defined in the Section 3 and by using new branching rules. We omit the details and simply state the following theorem.

Theorem 8. *Let $G = (V, E)$ be a graph on n vertices, then the* MAXIMUM 2-REGULAR INDUCED SUBGRAPH *problem can be solved in $\mathcal{O}(1.62355^n)$ time.*

4.2 Induced r-Regular Subgraph Isomorphism

Here we consider a special case of INDUCED SUBGRAPH ISOMORPHISM (IND-SI) problem.

> IND-SI: Given a graph $G = (V, E)$ and H, the question is to determine whether there exists a $H' \subseteq V$ such that $G[H'] \cong H$.

A brute force algorithm for this is to enumerate all subsets H' of size $|H|$ of G and check whether $G[H'] \cong H$, using the $O(n^{o(n)})$ time graph isomorphism algorithm [1]. The question is: *can we do this in time $\mathcal{O}(c^n)$ time where n is the number of vertices* *in G and $c < 2$, a constant?* Here, we answer this question for a special class of H, that is when *H is a r-regular graph with r a constant*. Even with such restrictions this problem is NP-hard as it contains problems like INDEPENDENT SET. We show the following theorem.

Theorem 9. *Given a graph $G = (V, E)$ on n vertices and a graph H, where H is r-regular, for a constant r, we can determine whether there exists a $H' \subseteq V$ such that $G[H'] \cong H$ in $\mathcal{O}(c^n)$ time, where $c < 2$ a constant depending on r alone.*

Proof. Let $H = \{H_1, H_2, \cdots, H_r\}$ where each H_i is a connected component of H. If there exists a $H' \subseteq V$ such that $G[H'] \cong H$ then H' can also be written as $\{H'_1, H'_2, \cdots H'_r\}$, H'_i connected component of $G[H']$, such that $G[H'_i] \cong H_i$ for $1 \leq i \leq r$.

The crucial observation is that if there exists a H' such that $G[H'] \cong H$ then there exists a maximal r-regular induced subgraph R extending H' such that each of the connected component of H' appears as a connected component of $G[R]$. By applying Theorem 3, we enumerate all maximal r-regular induced subgraphs of a graph on n vertices in $\mathcal{O}(c^n)$ time, $c < 2$ a constant depending on r alone. Now given a R, a maximal r-regular induced subgraph of G, we check the isomorphism of each connected component of $G[R]$ with each of H_i using the polynomial time bounded degree graph isomorphism algorithm of Luks [11]. If we obtain a H' such that $G[H'] \cong H$ then we return H' else we return no. The correctness and the time complexity of the algorithm follow easily. □

5 Conclusion

In this paper we developed an $\mathcal{O}(c^n)$ time exact algorithms for MAXIMUM r-REGULAR INDUCED SUBGRAPH problems for any fixed constant r, where $c < 2$ is a constant depending on r alone. We also showed that if r is a constant then the number of maximal r-regular induced subgraphs on a graph on n vertices is bounded by $o(2^n)$. Then we gave very tight lower and upper bounds on the number of maximal r-regular induced subgraphs on n vertices. All our algorithms were simple to describe but their analyses were non-trivial and involved a different measure than the usual number of vertices to measure the progress of the algorithms. We analyzed recurrences having binomial coefficients and believe that these may trigger some new results in the area of exact algorithms. Finally, we used the results obtained on the enumeration version of MAXIMUM r-REGULAR INDUCED SUBGRAPH problems to give a non trivial exact algorithm for INDUCED r-REGULAR SUBGRAPH ISOMORPHISM when r is a constant. The other problems for which we can give non trivial exact algorithms based on the algorithms and the techniques developed in this paper include EFFICIENT EDGE DOMINATING SET [10], δ-SEPARATING MAXIMUM MATCHING [17] and MAXIMUM BOUNDED DEGREE INDUCED SUBGRAPH problems.

It will be interesting to find other applications of the algorithms developed in this paper. Finding a non trivial exact algorithm for INDUCED SUBGRAPH ISOMORPHISM problem, even for special classes of H, remains open. Here we obtained an efficient algorithm for INDUCED SUBGRAPH ISOMORPHISM when H is a r-regular graph for a constant r.

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