On factorization by similarity of fuzzy concept lattices with hedges*

Radim Bělohlávek, Jan Outrata and Vilém Vychodil

Department of Computer Science, Palacky University, Olomouc
Tomkova 40, CZ-779 00 Olomouc, Czech Republic
{radim.belohlavek, jan.outrata, vilem.vychodil}@upol.cz

Abstract. The paper presents results on factorization by similarity of fuzzy concept lattices with hedges. Factorization of fuzzy concept lattices including a fast way to compute the factor lattice was presented in our earlier papers. The basic idea is to have, instead of a whole fuzzy concept lattice, its factor lattice. The factor lattice results by factorizing the original fuzzy concept lattice by a similarity relation which is specified by a user by a single parameter (similarity threshold). The main purpose is to have a smaller lattice which can be seen as a reasonable approximation of the original, possibly large, fuzzy concept lattice. In this paper, we extend the existing results to the case of fuzzy concept lattices with hedges, i.e. with parameters controlling the size of a fuzzy concept lattice.

1 Introduction and motivation

The present paper is a continuation of our previous papers on formal concept analysis (FCA) of data with fuzzy attributes. In particular, it is a continuation of two ways to reduce the size of fuzzy concept lattices.

The first way, see [2, 8, 9], consists in considering, instead of a possibly large fuzzy concept lattice $\mathcal{B}(X, Y, I)$ associated to the input data $\langle X, Y, I\rangle$, a factor lattice $\mathcal{B}(X, Y, I)/{}^a{\approx}$. Note that here, $\langle X, Y, I\rangle$ (sometimes called a formal fuzzy context) consists of a finite set X of objects, a finite set Y of attributes, and a fuzzy relation I between X and Y indicating for each $x \in X$ and $y \in Y$ a degree to which object x has attribute y. In addition to that, $\mathcal{B}(X, Y, I)$ is a fuzzy concept lattice in the sense of [3, 24]. Finally, $\mathcal{B}(X, Y, I)/{}^a{\approx}$ is a factor lattice of $\mathcal{B}(X, Y, I)$ by a compatible tolerance relation ${}^a{\approx}$ on $\mathcal{B}(X, Y, I)$ (see e.g. [18] for the notion of a factor lattice by a tolerance). The relation ${}^a{\approx}$ results as an a-cut of $\approx$ where a is a user-specified threshold (a particular truth degree, e.g. $a = 0.5$) and $\approx$ is a naturally defined fuzzy equivalence relation on $\mathcal{B}(X, Y, I)$ (see later). In [8, 9], two methods to compute the factor lattice $\mathcal{B}(X, Y, I)/{}^a{\approx}$ directly from data, i.e. without the need to compute the whole $\mathcal{B}(X, Y, I)$ first, have been described.

* Supported by grant No. 1ET101370417 of GA AV ČR, by grant No. 201/05/0079 of the Czech Science Foundation, and by institutional support, research plan MSM 6198959214.

The second way, see e.g. [10, 11, 13], consists in introducing two additional parameters into FCA of data with fuzzy attributes. These parameters, called hedges, are particular unary functions ^{*_X} and ^{*_Y} on the scale of truth degrees. The hedges are used to modify the basic operators associated to $\langle X, Y, I\rangle$, i.e., the extent and intent forming operators $^{\Uparrow}$ and $^{\Downarrow}$. Then, instead of $\mathcal{B}(X, Y, I)$, one considers $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ which is defined to be the set of fixed points of the modified operators. The basic idea is that stronger hedges lead to smaller $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$. An interesting point here is that the approach via hedges subsumes some of the earlier approaches to FCA of data with fuzzy attributes. First, if both ^{*_X} and ^{*_Y} are identities, $\mathcal{B}(X, Y, I)$ coincides with $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$. Second, if one of the hedges is identity and the other one is globalization (see later), the resulting $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is in fact the fuzzy concept lattice considered independently in [12, 15, 23]. Note also that, as shown in [11], the approach using hedges subsumes the approach using thresholds as presented in [17] and also [16].

The main aim of this paper is to look to what extent the idea of factorization by similarity given by a user-specified threshold can be applied to fuzzy concept lattices with hedges. We present some preliminary results and illustrative examples. Due to lack of space, we present only sketches of proofs and postpone full proofs to a full version of this paper. Section 2 presents preliminaries. Section 3 presents the results. An illustrative example is contained in Section 4. Section 5 presents a summary and outline of a future research.

2 Preliminaries

2.1 Fuzzy sets and fuzzy logic

In this section, we recall necessary notions from fuzzy sets and fuzzy logic. We refer to [3, 21] for further details. The concept of a fuzzy set generalizes that of an ordinary set in that an element may belong to a fuzzy set in an intermediate truth degree not necessarily being 0 or 1. As a structure of truth degrees, equipped with operations for logical connectives, we use complete residuated lattices, i.e. structures $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$, where $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); and $\otimes$ and $\rightarrow$ satisfy so-called adjointness property, i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ for each $a, b, c \in L$. Elements a of L are called truth degrees, $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication".

The most applied set L of truth degrees is the real interval $[0, 1]$; with $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$, and with one of the three important pairs of fuzzy conjunction and fuzzy implication: Łukasiewicz ($a \otimes b = \max(a + b - 1, 0)$, $a \rightarrow b = \min(1 - a + b, 1)$), minimum ($a \otimes b = \min(a, b)$, $a \rightarrow b = 1$ if $a \leq b$ and $= b$ else), and product ($a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$ and $= b/a$ else). In applications, we usually need a finite chain $\{a_0 = 0, a_1, \ldots, a_n = 1\}$ ($a_0 < \cdots < a_n$); with corresponding Łukasiewicz ($a_k \otimes a_l = a_{\max(k+l-n,0)}$, $a_k \rightarrow a_l = a_{\min(n-k+l,n)}$) or minimum ($a_k \otimes a_l = a_{\min(k,l)}$, $a_k \rightarrow a_l = a_n$ for

$a_k \leq a_l$ and $a_k \to a_l = a_l$ otherwise) connectives. Note that complete residuated lattices are basic structures of truth degrees used in fuzzy logic, see [19, 21]. Residuated lattices cover many structures used in applications.

For a complete residuated lattice $\mathbf{L}$, a (truth-stressing) hedge is a unary function * satisfying (i) $1^* = 1$, (ii) $a^* \leq a$, (iii) $(a \to b)^* \leq a^* \to b^*$, (iv) $a^{**} = a^*$, for all $a, b \in L$. A hedge * is a (truth function of) logical connective "very true" [22]. The largest hedge (by pointwise ordering) is identity (i.e. $a^* = a$), the least hedge is globalization which is defined by $a^* = 1$ for $a = 1$ and $a^* = 0$ for $a < 1$. Note that for $L = \{0, 1\}$, there exists exactly one complete residuated lattice $\mathbf{L}$ (the two-element Boolean algebra) and exactly one hedge (the identity on $\{0, 1\}$).

By $\mathbf{L}^U$ or L^U we denote the set of all fuzzy sets ($\mathbf{L}$-sets) in universe U, i.e. $L^U = \{A \mid A \text{ is a mapping of } U \text{ to } L\}$, $A(u)$ being interpreted as a degree to which u belongs to A. If $U = \{u_1, \ldots, u_n\}$ then A is denoted by $A = \{ {}^{a_1}/u_1, \ldots, {}^{a_n}/u_n\}$ meaning that $A(u_i)$ equals a_i. For brevity, we omit elements of U whose membership degree is zero. A binary fuzzy relation R between sets X and Y is a fuzzy set in universe $U = X \times Y$. For $A \in L^U$ and $a \in L$, a set ${}^aA = \{u \in U \mid A(u) \geq a\}$ is called an a-cut of A (the ordinary set of elements from U which belong to A to degree at least a); a fuzzy set $a \to A$ in U defined by $(a \to A)(u) = a \to A(u)$ is called an a-shift of A; $a \otimes A$ is defined similarily. Given $A, B \in \mathbf{L}^U$, we define a subsethood degree $S(A, B) = \bigwedge_{u \in U} \big(A(u) \to B(u)\big)$, which generalizes the classical subsethood relation $\subseteq$. $S(A, B)$ represents a degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A, B) = 1$ (A is fully contained in B). As a consequence, $A \subseteq B$ iff $A(u) \leq B(u)$ for each $u \in U$.

2.2 Fuzzy concept lattices (with hedges and thresholds)

A data table with fuzzy attributes (formal fuzzy context in terms of FCA) can be identified with a triplet $\langle X, Y, I \rangle$ where X is a non-empty set of objects (table rows), Y is a non-empty set of attributes (table columns), and I is a (binary) fuzzy relation between X and Y, i.e. $I : X \times Y \to L$. For $x \in X$ and $y \in Y$, a degree $I(x, y) \in L$ is interpreted as a degree to which object x has attribute y (table entry corresponding to row x and column y). For $L = \{0, 1\}$, formal fuzzy contexts can be identified in an obvious way with ordinary formal contexts.

Let ^{*_X} and ^{*_Y} be hedges. For fuzzy sets $A \in L^X$ and $B \in L^Y$, we define fuzzy sets $A^\uparrow \in L^Y$ and $B^\downarrow \in L^X$ (denoted also $A^{\uparrow_I}$ and $B^{\downarrow_I}$ to make I explicit) by

$$A^\uparrow(y) = \textstyle\bigwedge_{x \in X} (A^{*_X}(x) \to I(x, y)), \qquad (1)$$

$$B^\downarrow(x) = \textstyle\bigwedge_{y \in Y} (B^{*_Y}(y) \to I(x, y)). \qquad (2)$$

Using basic rules of predicate fuzzy logic, $A^\uparrow$ is a fuzzy set of all attributes common to all objects (for which it is very true that they are) from A, and $B^\downarrow$ is a fuzzy set of all objects sharing all attributes (for which it is very true that they are) from B. The set

$$\mathcal{B}(X^{*X}, Y^{*Y}, I) = \{\langle A, B\rangle \mid A^{\uparrow} = B,\ B^{\downarrow} = A\}$$

of all fixed points of $\langle^{\uparrow}, ^{\downarrow}\rangle$ is called a fuzzy concept lattice (with hedges) of $\langle X, Y, I\rangle$; elements $\langle A, B\rangle \in \mathcal{B}(X^{*X}, Y^{*Y}, I)$ will be called formal concepts of $\langle X, Y, I\rangle$; A and B are called the extent and intent of $\langle A, B\rangle$, respectively. For the sake of brevity, we will sometimes write also $\mathcal{B}(X^{*}, Y^{*}, I)$ instead of $\mathcal{B}(X^{*X}, Y^{*Y}, I)$. Under a partial order $\leq$ defined on $\mathcal{B}(X^{*X}, Y^{*Y}, I)$ by

$$\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle \quad \text{iff} \quad A_1 \subseteq A_2,$$

$\mathcal{B}(X^{*X}, Y^{*Y}, I)$ happens to be a complete lattice and we refer to [13] for results describing the structure of $\mathcal{B}(X^{*X}, Y^{*Y}, I)$. Note that $\mathcal{B}(X^{*X}, Y^{*Y}, I)$ is the basic structure used for formal concept analysis of the data table represented by $\langle X, Y, I\rangle$.

Remark 1. Operators $^{\uparrow}$ and $^{\downarrow}$ were introduced in [10, 13] as a parameterization of operators $A^{\Uparrow}(y) = \bigwedge_{x\in X}(A(x) \to I(x, y))$ and $B^{\Downarrow}(x) = \bigwedge_{y\in Y}(B(y) \to I(x, y))$ which were studied before, see [1, 4, 24]. Clearly, if both ^{*X} are ^{*Y} are identities on L, $^{\uparrow}$ and $^{\downarrow}$ coincide with $^{\Uparrow}$ and $^{\Downarrow}$, respectively. If ^{*X} or ^{*Y} is the identity on L, we omit ^{*X} or ^{*Y} in $\mathcal{B}(X^{*X}, Y^{*Y}, I)$, e.g. we write just $\mathcal{B}(X^{*X}, Y, I)$ if $^{*Y} = \mathrm{id}_L$.

Inspired by the "thresholded approach" of [17] (see also [16]), another parameterization of operators $^{\uparrow}$ and $^{\downarrow}$ was introduced in [11]: For $\delta, \varepsilon \in L$, fuzzy sets $A \in L^X$ and $B \in L^Y$, consider fuzzy sets $A^{\uparrow_{I,\delta}} \in L^Y$ and $B^{\downarrow_{I,\varepsilon}} \in L^X$ defined by

$$A^{\uparrow_{I,\delta}}(y) = \delta \to \bigwedge_{x\in X}(A^{*X}(x) \to I(x, y)), \tag{3}$$

$$B^{\downarrow_{I,\varepsilon}}(x) = \varepsilon \to \bigwedge_{y\in Y}(B^{*Y}(y) \to I(x, y)). \tag{4}$$

$A^{\uparrow_{I,\delta}}(y)$ can be thought of as a truth degree of *the degree to which y is shared by all objects from A is at least δ*, and similarily for $B^{\downarrow_{I,\delta}}(x)$. We will often write just $A^{\uparrow}$ and $B^{\downarrow}$ if I, δ, and ε are obvious, particularly if $\delta = \varepsilon$. The set

$$\mathcal{B}(X_{\delta}^{*X}, Y_{\varepsilon}^{*Y}, I) = \{\langle A, B\rangle \mid A^{\uparrow} = B,\ B^{\downarrow} = A\}$$

of all fixed points of $\langle^{\uparrow}, ^{\downarrow}\rangle$ is called a (thresholded) fuzzy concept lattice (with hedges) of $\langle X, Y, I\rangle$. Describing the structure of $\mathcal{B}(X_{\delta}^{*X}, Y_{\varepsilon}^{*Y}, I)$ (under a partial order $\leq$) is an open problem which remains to be studied.

Remark 2. Since $1 \to a = a$ for each $a \in L$, we have $A^{\uparrow_{I,1}} = A^{\uparrow_I}$ and $B^{\downarrow_{I,1}} = B^{\downarrow_I}$ and, therefore, $\mathcal{B}(X_1^{*X}, Y_1^{*Y}, I) = \mathcal{B}(X^{*X}, Y^{*Y}, I)$.

For existing results on some basic relationships to earlier approaches we refer to [11]. In this article we will focus on the case $\delta = \varepsilon$ only.

3 Factorization of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ by similarity

3.1 The case of $\mathcal{B}(X, Y, I)$

We need to recall the parametrized method of factorization introduced in [2] to which we refer for details. Given $\langle X, Y, I\rangle$, introduce a binary fuzzy relation $\approx_{\rm Ext}$ on $\mathcal{B}(X, Y, I)$ (we will use it for $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ later on) by

$$(\langle A_1, B_1\rangle \approx_{\rm Ext} \langle A_2, B_2\rangle) \;=\; \bigwedge_{x\in X}(A_1(x) \leftrightarrow A_2(x)) \tag{5}$$

for $\langle A_i, B_i\rangle \in \mathcal{B}(X, Y, I)$, $i = 1, 2$, where $\bigwedge$ denotes infimum and $\leftrightarrow$ is a so-called biresiduum (truth function of equivalence connective) defined by $a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$. $(\langle A_1, B_1\rangle \approx_{\rm Ext} \langle A_2, B_2\rangle)$, called the degree of similarity of $\langle A_1, B_1\rangle$ and $\langle A_2, B_2\rangle$, is just the truth degree of "for each object x: x is covered by A_1 iff x is covered by A_2". One can also consider a fuzzy relation $\approx_{\rm Int}$ defined by $(\langle A_1, B_1\rangle \approx_{\rm Int} \langle A_2, B_2\rangle) \;=\; \bigwedge_{y\in Y}(B_1(y) \leftrightarrow B_2(y))$. It can be shown [3] that measuring similarity of formal concepts via intents B_i coincides with measuring similarity via extents A_i, corresponding naturally to the duality of extent/intent view. As a result, we write also just $\approx$ instead of $\approx_{\rm Ext}$ and $\approx_{\rm Int}$. Note also that $\approx$ is a fuzzy equivalence relation on $\mathcal{B}(X, Y, I)$.

Given a truth degree $a \in L$ (a threshold specified by a user), consider the thresholded relation ${}^a\!\approx$ on $\mathcal{B}(X, Y, I)$ defined by $(\langle A_1, B_1\rangle, \langle A_2, B_2\rangle) \in {}^a\!\approx$ iff $(\langle A_1, B_1\rangle \approx \langle A_2, B_2\rangle) \geq a$. That is, ${}^a\!\approx$ is the relation "being similar to degree at least a" and we thereby call it simply similarity (relation). ${}^a\!\approx$ is reflexive and symmetric (i.e., a tolerance relation), but need not be transitive (it is transitive if, e.g., $a \otimes b = a \wedge b$ holds true in $\mathbf{L}$). A similarity ${}^a\!\approx$ on $\mathcal{B}(X, Y, I)$ is said to be compatible if it is preserved under arbitrary suprema and infima, i.e. if $c_j\,{}^a\!\!\approx c'_j$, implies both $(\bigwedge_{j\in J} c_j)\,{}^a\!\!\approx(\bigwedge_{j\in J} c'_j)$ and $(\bigvee_{j\in J} c_j)\,{}^a\!\!\approx(\bigvee_{j\in J} c'_j)$ for any $c_j, c'_j \in \mathcal{B}(X, Y, I)$, $j \in J$. We call $\approx$ compatible if ${}^a\!\approx$ is compatible for each $a \in L$.

Call a subset B of $\mathcal{B}(X, Y, I)$ an ${}^a\!\approx$-block if it is a maximal subset of $\mathcal{B}(X, Y, I)$ such that each two formal concepts from B are similar to degree at least a (the notion of a ${}^a\!\approx$-block generalizes that of an equivalence class: if ${}^a\!\approx$ is an equivalence relation, ${}^a\!\approx$-blocks are exactly the equivalence classes). Denote by $\mathcal{B}(X, Y, I)/{}^a\!\approx$ the collection of all ${}^a\!\approx$-blocks. It can be shown that, if ${}^a\!\approx$ is compatible, then ${}^a\!\approx$-blocks are special intervals in the concept lattice $\mathcal{B}(X, Y, I)$. For a formal concept $\langle A, B\rangle \in \mathcal{B}(X, Y, I)$, denote $\langle A, B\rangle_a$ and $\langle A, B\rangle^a$ the infimum and the supremum of the set of all formal concepts which are similar to $\langle A, B\rangle$ to degree at least a. Operators $\ldots_a$ and $\ldots^a$ are important in description of ${}^a\!\approx$-blocks [18]:

Lemma 1. *${}^a\!\approx$-blocks are exactly intervals of $\mathcal{B}(X, Y, I)$ of the form $[\langle A, B\rangle_a, (\langle A, B\rangle_a)^a]$, i.e.*

$$\mathcal{B}(X, Y, I)/{}^a\!\approx \;=\; \{[\langle A, B\rangle_a, (\langle A, B\rangle_a)^a] \mid \langle A, B\rangle \in \mathcal{B}(X, Y, I)\}.$$

Now, define a partial order $\preceq$ on blocks of $\mathcal{B}(X, Y, I)/{}^a\!\approx$ by $[c_1, c_2] \preceq [d_1, d_2]$ iff $c_1 \leq d_1$(iff $c_2 \leq d_2$) where $[c_1, c_2], [d_1, d_2] \in \mathcal{B}(X, Y, I)/{}^a\!\approx$. Then we have [2]:

Theorem 1. *$\mathcal{B}(X,Y,I)/^a\!\approx$ equipped with $\preceq$ is a partially ordered set which is a complete lattice, the so-called factor lattice of $\mathcal{B}(X,Y,I)$ by similarity $\approx$ and threshold a.*

Elements of $\mathcal{B}(X,Y,I)/^a\!\approx$ can be seen as similarity-based granules of formal concepts/clusters from $\mathcal{B}(X,Y,I)$. $\mathcal{B}(X,Y,I)/^a\!\approx$ thus provides a granular view on (the possibly large) $\mathcal{B}(X,Y,I)$. For further details and properties of $\mathcal{B}(X,Y,I)/^a\!\approx$ we refer to [2].

3.2 The case of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$

We now turn our attention to factorization by similarity of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$.

Note first that one cannot directly apply the approach which works for $\mathcal{B}(X,Y,I)$. Namely, due to employing hedges, some important properties are no longer available (for instance, the composite mappings ${}^{\uparrow\downarrow}$ and ${}^{\downarrow\uparrow}$ are not fuzzy closure operators in general). Nevertheless, we propose a feasible approach to factorization of concept lattices with hedges. In some cases, however, we restrict ourselves to the case when one of the hedges is identity and leave the fully general case to future investigation. Note that in $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ corresponding to both "one-sided" fuzzy concept lattices, see [15] and [23], one of the hedges is globalization.

Remark 3. If one would define $\approx_{\mathrm{Ext}}$ (or $\approx_{\mathrm{Int}}$) by (5), compatibility would be lost. This is still true even if one of the hedges is identity. Consider e.g. ${}^{*_X} = \mathrm{id}_L$. Then, $\approx_{\mathrm{Ext}}$ is compatible with $\bigwedge$. Namely, $\bigwedge_{j\in J} A_j = \bigcap_{j\in J} A_j$ for $A_j = A_j^{\Uparrow\downarrow}$ [13]. However, $\approx_{\mathrm{Ext}}$ need not be compatible with $\bigvee$ as shown by the following example. The dual situation applies to $\approx_{\mathrm{Int}}$.

Example 1. Take a Łukasiewicz structure on $[0,1]$, let ^{*_X} be identity and ^{*_Y} be globalization, and consider the following data table

I	y_1	y_2	y_3
x_1	1	0.5	0
x_2	0	0	1
x_3	0.5	1	0

One can check that for $A_1 = \{\,{}^{0.5}/x_1,\ {}^{0.5}/x_3\}$, $B_1 = \{\,{}^{1}/y_1,\ {}^{1}/y_2,\ {}^{0.5}/y_3\}$, $A_2 = \{\,{}^{0.5}/x_1,\ {}^{1}/x_3\}$, $B_2 = \{\,{}^{0.5}/y_1,\ {}^{1}/y_2\}$, $A_3 = \{\,{}^{1}/x_1,\ {}^{0.5}/x_3\}$ and $B_3 = \{\,{}^{1}/y_1,\ {}^{0.5}/y_2\}$, $\langle A_i, B_i\rangle \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I)$, $i = 1,2,3$ and $\langle A_1, B_1\rangle {}^a\!\approx \langle A_2, B_2\rangle$, $\langle A_1, B_1\rangle {}^a\!\approx \langle A_3, B_3\rangle$, $(\langle A_1, B_1\rangle \wedge \langle A_1, B_1\rangle) = \langle A_1, B_1\rangle {}^a\!\approx \langle A_1, B_1\rangle = (\langle A_2, B_2\rangle \wedge \langle A_3, B_3\rangle)$, but $a \not\leq (\langle A_1, B_1\rangle \vee \langle A_1, B_1\rangle) \approx (\langle A_2, B_2\rangle \vee \langle A_3, B_3\rangle)$.

In order to propose our way to factorize $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$, we need the following notion. Let $\approx$ be a fuzzy relation in $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$, $a \in L$ be a truth degree, and * be a hedge (particularly, * will be ^{*_X} or ^{*_Y}). We say that $\approx$ is compatible with * and a if for each $c_1, c_2 \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ we have that

$$\text{if } a \leq (c_1 \approx c_2), \text{ then } a \leq (c_1 \approx c_2)^*. \tag{6}$$

Consider the following fuzzy relations on $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$: By $\approx_{\text{Ext}}$ we denote the fuzzy relation defined by (5); similarly for $\approx_{\text{Int}}$; by $\approx^{*_X}_{\text{Ext}}$ we denote a fuzzy relation defined by

$$(\langle A_1, B_1\rangle \approx^{*_X}_{\text{Ext}} \langle A_2, B_2\rangle) \;=\; \left(\bigwedge_{x\in X}(A_1(x) \leftrightarrow A_2(x))\right)^{*_X}; \tag{7}$$

similarly for $\approx^{*_Y}_{\text{Int}}$. Occasionally, we write also $(A_1 \approx^{*_X}_{\text{Ext}} A_2)$ instead of $(\langle A_1, B_1\rangle \approx^{*_X}_{\text{Ext}} \langle A_2, B_2\rangle)$, etc.

The following assertion is easy to see.

Lemma 2. *(1) If $a \in L$ is a fixed point of ^{*_X}, i.e. $a^{*_X} = a$, then $\approx_{\text{Ext}}$ is compatible with ^{*_X} and a; similarly for ^{*_Y} and $\approx_{\text{Int}}$.*
(2) For any $a \in L$, $\approx^{*_X}_{\text{Ext}}$ is compatible with ^{*_X} and a; similarly for ^{*_Y} and $\approx^{*_Y}_{\text{Int}}$.

We need the following two assertions (here, $\approx$ is defined by $(A_1 \approx A_2) = \bigwedge_{x\in X}(A_1(x) \leftrightarrow A_2(x))$).

Lemma 3. *Let $A_1, A_2 \in L^X$. Then $(A_1 \approx A_2)^{*_X} \leq (A_1^{*_X} \approx A_2^{*_X})$.*

Proof. Denote ^{*_X} by *. We have $(A_1 \approx A_2)^* \leq (A_1^* \approx A_2^*) = \bigwedge_{x\in X}(A_1(x)^* \leftrightarrow A_2(x)^*)$ iff $(A_1 \approx A_2)^* \leq (A_1(x)^* \leftrightarrow A_2(x)^*)$ for all $x \in X$. Since $(A_1 \approx A_2)^* \leq (A_1(x) \leftrightarrow A_2(x))^*$ for all $x \in X$ it suffices to show $(A_1(x) \leftrightarrow A_2(x))^* \leq (A_1(x)^* \leftrightarrow A_2(x)^*)$, which is true. Indeed, $(A_1(x) \leftrightarrow A_2(x))^* \leq (A_1(x) \to A_2(x))^* \wedge (A_2(x) \to A_1(x))^* \leq (A_1(x)^* \to A_2(x)^*) \wedge (A_2(x)^* \to A_1(x)^*) = (A_1(x)^* \leftrightarrow A_2(x)^*)$.

Lemma 4. *For $A_1, A_2 \in L^X$ we have $(A_1 \approx A_2)^{*_X} \leq (A_1^{\uparrow} \approx A_2^{\uparrow})$.*

Proof. Follows directly from Lemmma 3 and $(A_1 \approx A_2) \leq (A_1^{\Uparrow} \approx A_2^{\Uparrow})$ [2].

Suppose we have two fuzzy equivalence relations on $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$, $\approx_X$ and $\approx_Y$ such that $\approx_X$ is compatible with ^{*_X} and a, and $\approx_Y$ is compatible with ^{*_Y} and a. Although, in general, $\approx_X$ may be different from $\approx_Y$, the following theorem shows that their a-cuts coincide.

Theorem 2. *Let $\approx_X$ and $\approx_Y$ be fuzzy equivalence relations on $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ compatible with ^{*_X} and a, and with ^{*_Y} and a, respectively. Then $^a{\approx_X} = {^a{\approx_Y}}$.*

Proof. Using Lemma 4, the proof is similar to the proof of $\approx_{\text{Ext}} = \approx_{\text{Int}}$ in [2].

We can therefore write $^a{\approx}$ instead of $^a{\approx_X}$ and $^a{\approx_Y}$. Note that Theorem 2 applies in particular to the fuzzy relations from Lemma 2. With the above notation, the following theorem shows a way to factorize $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$.

Theorem 3. *$^a{\approx}$ is a compatible tolerance on $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$.*

Proof. Theorem can be proved by applying (6) and Lemma 4 twice at the end of the proof of compatibility of $\approx$ on $\mathcal{B}(X, Y, I)$ in [2].

Therefore, we can consider the factor lattice $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)/{}^a\!\approx$ of lattice $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ by tolerance ${}^a\!\approx$. In what follows, we present a way to obtain $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)/{}^a\!\approx$ directly, without the need to compute $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ first and then to compute the blocks of ${}^a\!\approx$. Basically, we follow and appropriately modify the method from [8]. The method from [8] makes use of the fact that for each fuzzy set A (extent/intent) we have

$$\langle A, a \otimes A\rangle \in {}^a\!\approx \text{ and } \langle A, a \to A\rangle \in {}^a\!\approx. \tag{8}$$

If ${}^a\!\approx$ has this feature, we can proceed also for fast factorization of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ by ${}^a\!\approx$. Note that (8) is satisfied, for instance, for ${}^a\!\approx_{\mathrm{Ext}}$ if a is a fixed point of ^{*_X} and ^{*_Y}, cf. Lemma 2. In the remainder of the paper we will suppose that ${}^a\!\approx$ always satisfies (8). The following assertion shows that $\langle A, B\rangle_a$ (the least formal concept ${}^a\!\approx$-similar to $\langle A, B\rangle$) and $\langle A, B\rangle^a$ (the greatest formal concept ${}^a\!\approx$-similar to $\langle A, B\rangle$) can be computed from $\langle A, B\rangle$ directly.

Lemma 5. *For $\langle A, B\rangle \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I)$, we have (a) $\langle A, B\rangle_a = \big\langle (a \otimes A)^{\uparrow\downarrow}, (a \to B)^{\downarrow\uparrow}\big\rangle$ and (b) $\langle A, B\rangle^a = \big\langle (a \to A)^{\uparrow\downarrow}, (a \otimes B)^{\downarrow\uparrow}\big\rangle$.*

Proof. Due to duality we sketch only the proof of (a). We need to prove, that $(a \otimes A)^{\uparrow\downarrow}$ is an extent of the least formal concept similar to $\langle A, B\rangle$ to degree at least a and $(a \to B)^{\downarrow\uparrow}$ is the corresponding intent. That is (1) $(a \otimes A)^{\uparrow\downarrow}$ is an extent of a formal concept $\langle (a \otimes A)^{\uparrow\downarrow}, D\rangle$ which is similar to $\langle A, B\rangle$ to degree at least a; (2) if $\langle C, F\rangle$ is a formal concept similar to $\langle A, B\rangle$ to degree at least a then $\langle (a \otimes A)^{\uparrow\downarrow}, D\rangle \leq \langle C, F\rangle$; and similarily for intent $(a \to B)^{\downarrow\uparrow}$. Both (1) and (2) can be easily proved using (8) and (6), Lemma 4 and adjointness property.

Remark 4. Thus we have $(\langle A, B\rangle_a)^a = \big\langle (a \to (a \otimes A)^{\uparrow\downarrow})^{\uparrow\downarrow}, (a \otimes (a \to B)^{\downarrow\uparrow})^{\downarrow\uparrow}\big\rangle$.

Another property, analogous to the case of $\mathcal{B}(X, Y, I)$, is the following.

Lemma 6. *If ^{*_X} is identity on L and A is an extent then we have $a \to A = (a \to A)^{\uparrow\downarrow}$; similarly for ^{*_Y} and an intent B.*

Proof. We sketch the proof for extents. The inequality $\subseteq$ follows directly from $A = A^{*_X} \subseteq A^{\uparrow\downarrow}$ and the converse inequality $\supseteq$ can be proved the same way as the corresponding inequality in analogous lemma in [8], with application of (8), (6) and Lemma 4 at appropriate places.

One way to obtain the factor lattice directly is based on the following theorem. Recall that an $\mathbf{L}_{\{1\}}$-closure operator in U is a mapping $C : \mathbf{L}^U \to \mathbf{L}^U$ satisfying $A \subseteq C(A)$; $A_1 \subseteq A_2$ implies $C(A_1) \subseteq C(A_2)$; $C(A) = C(C(A))$. A fixed point of C is any fuzzy set A in U such that $A = C(A)$.

Theorem 4. *Let ^{*_X} be identity on L. Then the mapping $C_a : A \mapsto (a \to (a \otimes A)^{\uparrow\downarrow})^{\uparrow\downarrow}$ is an $\mathbf{L}_{\{1\}}$-closure operator in $\mathbf{L}^X$ such that the fixed points of C_a are just the extents of suprema of ${}^a\!\approx$-blocks of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)/{}^a\!\approx$.*

Proof. The idea of the proof remains the same as in the proof of analogous theorem for $\mathcal{B}(X, Y, I)/^a\!\approx$ in [8]. Briefly, we need to (a) verify that C_a is an $\mathbf{L}_{\{1\}}$-closure operator and (2) prove the equality of the set of fixed points of C_a and the set of extents of suprema of $^a\!\approx$-blocks. First is a nice exercise on checking the tree conditions from the definition of $\mathbf{L}_{\{1\}}$-closure operator and second is also easy, see full version of the paper.

Now, fixed points of $\mathbf{L}_{\{1\}}$-closure operators can be efficiently computed by an extension of Ganter's NextClosure algorithm, see [6].

Remark 5. $C_a : A \mapsto a \to (a \otimes A)^{\Uparrow\downarrow}$ is an $\mathbf{L}_{\{1\}}$-closure operator, but not an $\mathbf{L}$-closure operator in general, since we do not have $\mathrm{S}(A_1, A_2) \leq \mathrm{S}(C_a(A_1), C_a(A_2))$ for all $A_1, A_2 \in L^X$ as the following example shows.

Example 2. Consider the setting and data table from Example 1. Take $A_1 = \{\, {}^{0.5}/x_1,\ {}^{1}/x_2,\ {}^{0.5}/x_3\}$ and $A_2 = \{\, {}^{1}/x_2\}$. One can check that given $a = 1$, $C_a(A_1) = A_1^{\Uparrow\downarrow} = \{\, {}^{1}/x_1,\ {}^{1}/x_2,\ {}^{1}/x_3\}$ and $C_a(A_2) = A_2^{\Uparrow\downarrow} = \{\, {}^{1}/x_2\}$, hence $0.5 = \mathrm{S}(A_1, A_2) \nleq \mathrm{S}(C_a(A_1), C_a(A_2)) = 0$.

Another way to obtain the factor lattice directly is based on the following. From [11] we know that $A^{\uparrow_{I,a}} = a \to A^{\uparrow_I}$ equals $A^{\uparrow_{a\to I}} = (a \otimes A^{*_X})^{\Uparrow_I}$ (easy to check from the definitions (3) and (4) of $^{\uparrow_{I,a}}$ and $^{\downarrow_{I,a}}$, respectively). Hence $A^{\uparrow_{I,a}\downarrow_{I,a}} = A^{\uparrow_{a\to I}\downarrow_{a\to I}} = a \to (a \otimes A^{*_X})^{\Uparrow\downarrow}$. Since we consider $^{*_X} = \mathrm{id}_L$, we have $A^{\uparrow_{a\to I}\downarrow_{a\to I}} = C_a(A)$. Then, we obtain the following theorem which is analogous to (in fact, it is a generalization of) the crucial theorem of [9].

Theorem 5. *If ^{*_X} is identity on L then for any $\langle X, Y, I\rangle$ and a threshold $a \in L$ we have*

$$\mathcal{B}(X^{*_X}, Y^{*_Y}, I)/^a\!\approx\ \cong\ \mathcal{B}(X^{*_X}, Y^{*_Y}, a \to I).$$

In words, $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)/^a\!\approx$ is isomorphic to $\mathcal{B}(X^{*_X}, Y^{*_Y}, a \to I)$. Moreover, under the isomorphism, $[\langle A_1, B_1\rangle, \langle A_2, B_2\rangle] \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I)/^a\!\approx$ corresponds to $\langle A_2, B_1\rangle \in \mathcal{B}(X^{*_X}, Y^{*_Y}, a \to I)$.

Proof. We proceed the same way as in the proof of the theorem for $\mathcal{B}(X, Y, I)/^a\!\approx$ in [9]. Again, we give only a sketch. First, the operators $^{\uparrow_a}$ and $^{\downarrow_a}$ induced by $a \to I$ are described by terms of operators $^{\uparrow}$ (1) and $^{\downarrow}$ (2) induced by I. We get

$$A^{\uparrow_a} = a \to A^{\uparrow} \quad \text{and} \quad A^{\uparrow_a\downarrow_a} = a \to (a \otimes A^{*_X})^{\Uparrow\downarrow}. \tag{9}$$

The verification of $\mathcal{B}(X, Y^{*_Y}, I)/^a\!\approx\ \cong\ \mathcal{B}(X, Y^{*_Y}, a \to I)$ is then easy and we postpone the proof to full version of the paper.

Remark 6. (1) The blocks of $\mathcal{B}(X, Y^{*_Y}, I)/^a\!\approx$ can be reconstructed from the formal concepts of $\mathcal{B}(X, Y^{*_Y}, a \to I)$:
If $\langle A, B\rangle \in \mathcal{B}(X, Y^{*_Y}, a \to I)$ then $[\langle B^{\downarrow}, B\rangle, \langle A, A^{\Uparrow}\rangle] \in \mathcal{B}(X, Y^{*_Y}, I)/^a\!\approx$.

(2) Computing $\mathcal{B}(X, Y^{*_Y}, a \to I)$ means computing the fuzzy concept lattice with hedges, where the hedge ^{*_X} is identity. This can be done by an algorithm of polynomial time delay complexity, see [6].

First, this shows a way to obtain $\mathcal{B}(X, Y^{*_Y}, I)/{}^a\!\approx$ directly from input data, without computing first the whole $\mathcal{B}(X, Y^{*_Y}, I)$ and then computing the similarity blocks. Second,

$$\mathcal{B}(X, Y^{*_Y}, I)/{}^a\!\approx \;\cong\; \mathcal{B}(X, Y^{*_Y}, a \to I) \;\cong\; \mathcal{B}(X_a, Y_a^{*_Y}, I).$$

In words, $\mathcal{B}(X, Y^{*_Y}, I)/{}^a\!\approx$ is isomorphic to $\mathcal{B}(X_a, Y_a^{*_Y}, I)$. This means, if at least one of the hedges is identity, the factor lattice (by similarity ${}^a\!\approx$) and the thresholded lattice (by threshold a) of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ are the same (up to isomorphism). To sum up, the approaches to reducing the size of a fuzzy concept lattice (with hedges) via factorization by a similarity ${}^a\!\approx$ and via thresholded concept forming operators by a threshold a lead to the same reduction.

4 Illustrative example

	size		distance	
	small (s)	large (l)	far (f)	near (n)
Mercury	1	0	0	1
Venus	0.75	0	0	1
Earth	0.75	0	0	0.75
Mars	1	0	0.5	0.75
Jupiter	0	1	0.75	0.5
Saturn	0	1	0.75	0.5
Uranus	0.25	0.5	1	0.25
Neptune	0.25	0.5	1	0
Pluto	1	0	1	0

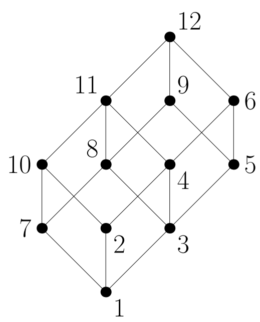

Fig. 1. Data table with fuzzy attributes and factor (and thresholded) lattice of corresponding fuzzy concept lattice.

We illustrate the relationship described in preceding section by a simple example. Take a finite Łukasiewicz chain $\mathbf{L}$ with $L = \{0, 0.25, 0.5, 0.75, 1\}$ as a structure of truth degrees. Consider an input data table $\langle X, Y, I\rangle$ depicted in Fig. 1 (left) which describes properties of planets of our solar system. The set X of objects consists of objects "Mercury", "Venus", ..., set Y contains four (fuzzy) attributes: size of the planet (small / large) and distance from Sun (far / near). Let $*_X$ be identity and $*_Y$ be a hedge defined as follows: for $a \in L$, $a^{*_Y} = 0.5$ if $a = 0.75$ and $a^{*_Y} = a$ otherwise. Finally, to measure the similarity of concepts we can use $\approx$, since it is compatible with $*_Y$ and each $a \in L$ except $a = 0.75$ and satisfies (8).

The whole fuzzy concept lattice $\mathcal{B}(X, Y^{*_Y}, I)$ has 94 formal concepts. We show and compare the factor concept lattices $\mathcal{B}(X, Y^{*_Y}, I)/{}^a\!\approx$ for thresholds $a = 0.25$ and $a = 0.5$ (note that $\approx$ is not compatible with $*_Y$ and $a = 0.75$ and for thresholds 0 and 1 the concept lattice contains only one concept

Table 1. Concepts of suprema of blocks of factor lattice and concepts of thresholded lattice for thresholds 0.25 (top) and 0.5 (bottom).

	$\bigvee$-s of ${}^{0.25}{\approx}$-blocks of $\mathcal{B}(X, Y^{*_Y}, I)/{}^{0.25}{\approx}$													$\mathcal{B}(X_{0.25}, Y^{*_Y}_{0.25}, I)$												
1	.75	.75	.75	.75	.75	.75	1	.75	.75	.25	.25	.25	.25	.75	.75	.75	.75	.75	.75	1	.75	.75	1	1	1	1
2	.75	.75	.75	.75	.75	.75	1	1	.75	.25	.25	.25	0	.75	.75	.75	.75	.75	.75	1	1	.75	1	1	1	.75
3	.75	.75	.75	1	.75	.75	1	.75	.75	.25	0	.25	.25	.75	.75	.75	1	.75	.75	1	.75	.75	1	.75	1	1
4	.75	.75	.75	1	.75	.75	1	1	1	.25	0	.25	0	.75	.75	.75	1	.75	.75	1	1	1	1	.75	1	.75
5	1	1	1	1	.75	.75	1	.75	.75	.25	0	0	.25	1	1	1	1	.75	.75	1	.75	.75	1	.75	.75	1
6	1	1	1	1	.75	.75	1	1	1	.25	0	0	0	1	1	1	1	.75	.75	1	1	1	1	.75	.75	.75
7	.75	.75	.75	.75	1	1	1	.75	.75	0	.25	.25	.25	.75	.75	.75	.75	1	1	1	.75	.75	.75	1	1	1
8	.75	.75	.75	1	1	1	1	.75	.75	0	0	.25	.25	.75	.75	.75	1	1	1	1	.75	.75	.75	.75	1	1
9	1	1	1	1	1	1	1	.75	.75	0	0	0	.25	1	1	1	1	1	1	1	.75	.75	.75	.75	.75	1
10	.75	.75	.75	.75	1	1	1	1	.75	0	.25	.25	0	.75	.75	.75	.75	1	1	1	1	.75	.75	1	1	.75
11	.75	.75	.75	1	1	1	1	1	1	0	0	.25	0	.75	.75	.75	1	1	1	1	1	1	.75	.75	1	.75
12	1	1	1	1	1	1	1	1	1	0	0	0	0	1	1	1	1	1	1	1	1	1	.75	.75	.75	.75

	$\bigvee$-s of ${}^{0.5}{\approx}$-blocks of $\mathcal{B}(X, Y^{*_Y}, I)/{}^{0.5}{\approx}$													$\mathcal{B}(X_{0.5}, Y^{*_Y}_{0.5}, I)$												
1	.5	.5	.5	.5	.5	.5	.75	.5	.5	.5	.5	.5	.5	.5	.5	.5	.5	.5	.5	.75	.5	.5	1	1	1	1
2	.5	.5	.5	.5	.5	.5	.75	.75	.5	.5	.5	.5	.25	.5	.5	.5	.5	.5	.5	.75	.75	.5	1	1	1	.75
3	.5	.5	.5	1	.5	.5	.75	.5	.5	.5	0	.5	.5	.5	.5	.5	1	.5	.5	.75	.5	.5	1	.5	1	1
4	.5	.5	.5	1	.5	.5	.75	.75	1	.5	0	.5	0	.5	.5	.5	1	.5	.5	.75	.75	1	1	.5	1	.5
5	1	1	1	1	.5	.5	.75	.5	.5	.5	0	0	.5	1	1	1	1	.5	.5	.75	.5	.5	1	.5	.5	1
6	1	1	1	1	.5	.5	.75	.75	1	.5	0	0	0	1	1	1	1	.5	.5	.75	.75	1	1	.5	.5	.5
7	.5	.5	.5	.5	1	1	.75	.5	.5	0	.5	.5	.5	.5	.5	.5	.5	1	1	.75	.5	.5	.5	1	1	1
8	.5	.5	.5	1	1	1	.75	.5	.5	0	0	.5	.5	.5	.5	.5	1	1	1	.75	.5	.5	.5	.5	1	1
9	1	1	1	1	1	1	.75	.5	.5	0	0	0	.5	1	1	1	1	1	1	.75	.5	.5	.5	.5	.5	1
10	.5	.5	.5	.5	1	1	1	1	.5	0	.5	.5	0	.5	.5	.5	.5	1	1	1	1	.5	.5	1	1	.5
11	.5	.5	.5	1	1	1	1	1	1	0	0	.5	0	.5	.5	.5	1	1	1	1	1	1	.5	.5	1	.5
12	1	1	1	1	1	1	1	1	1	0	0	0	0	1	1	1	1	1	1	1	1	1	.5	.5	.5	.5

and all 94 concepts, respectively). The factor lattices $\mathcal{B}(X, Y^{*_Y}, I)/^{0.25}\!\approx$ and $\mathcal{B}(X, Y^{*_Y}, I)/^{0.5}\!\approx$ happen to be the same structure of 12 similarity blocks. The lattice of blocks is depicted in Fig. 1 (right) and is isomorphic to thresholded lattices $\mathcal{B}(X_{0.25}, Y^{*_Y}_{0.25}, I)$ and $\mathcal{B}(X_{0.5}, Y^{*_Y}_{0.5}, I)$. The greatest concepts (suprema, $\bigvee$-s) of $^{0.25}\!\approx$-blocks and $^{0.5}\!\approx$-blocks are listed in Table 1, together with concepts of thresholded lattices. We can see from the tables that $\mathcal{B}(X, Y^{*_Y}, I)/^{a}\!\approx$ is isomorphic to $\mathcal{B}(X_a, Y^{*_Y}_a, I)$. Note that according to Theorem 5, extents of the corresponding concepts coincide.

5 Conclusions and future research

We presented a method of factorization of fuzzy concept lattices with hedges. If one of the hedges is identity, the factor lattice can be computed directly from input data, without first computing the whole fuzzy concept lattice. Furthermore, under the same assumption we concluded that reducing the size of a fuzzy concept lattice (with hedges) either via factorization or via thresholds leads to the same (isomorphic) results. Future research will focus on eliminating some restrictions from the assumptions of our methods (e.g., allowing both hedges to be different from identity).

References

1. Bělohlávek R.: Fuzzy Galois connections. *Math. Logic Quarterly* **45**,4 (1999), 497–504.
2. Bělohlávek R.: Similarity relations in concept lattices. *J. Logic and Computation* Vol. **10** No. **6**(2000), 823–845.
3. Bělohlávek R.: *Fuzzy Relational Systems: Foundations and Principles.* Kluwer, Academic/Plenum Publishers, New York, 2002.
4. Bělohlávek R.: Concept lattices and order in fuzzy logic. *Ann. Pure Appl. Logic* **128**(2004), 277–298.
5. Bělohlávek R.: A note on variable precision concept lattices. Draft, 2006.
6. Bělohlávek R.: Algorithms for fuzzy concept lattices. *Proc. RASC 2002.* Nottingham, UK, 12–13 Dec., 2002, pp. 200–205.
7. Bělohlávek R., Chlupová M., Vychodil V.: Implications from data with fuzzy attributes. In: AISTA 2004 in cooperation with IEEE Computer Society Proceedings, 15–18 November 2004, Kirchberg - Luxembourg, 5 pp.
8. Bělohlávek R., Dvořák J., Outrata J.: Fast factorization of concept lattices by similarity: solution and an open problem. In: Proc. CLA 2004, pp. 47–57, Ostrava, Czech Republic.
9. Bělohlávek R., Dvořák J., Outrata J.: Direct factorization in formal concept analysis by factorization of input data. *Proc. 5th Int. Conf. on Recent Advances in Soft Computing, RASC 2004.* Nottingham, United Kingdom, 16–18 December, 2004, pp. 578–583.
10. Bělohlávek R., Funioková T., Vychodil V.: Galois connections with hedges. In: Yingming Liu, Guoqing Chen, Mingsheng Ying (Eds.): *Fuzzy Logic, Soft Computing & Computational Intelligence: Eleventh International Fuzzy Systems Association World Congress* (Vol. II), 2005, pp. 1250–1255. Tsinghua University Press and Springer, ISBN 7–302–11377–7.

11. Bělohlávek R., Outrata J., Vychodil V.: Thresholds and shifted attributes in formal concept analysis of data with fuzzy attributes. In: H. Schärfe, P. Hitzler, and P. Øhrstrøm (Eds.): Proc. ICCS 2006, *Lecture Notes in Artificial Intelligence* **4068**, pp. 117–130, Springer-Verlag, Berlin/Heidelberg, 2006.
12. Bělohlávek R., Sklenář V., Zacpal J.: Crisply Generated Fuzzy Concepts. In: B. Ganter and R. Godin (Eds.): ICFCA 2005, *LNCS* **3403**, pp. 268–283, Springer-Verlag, Berlin/Heidelberg, 2005.
13. Bělohlávek R., Vychodil V.: Reducing the size of fuzzy concept lattices by hedges. In: FUZZ-IEEE 2005, The IEEE International Conference on Fuzzy Systems, May 22–25, 2005, Reno (Nevada, USA), pp. 663–668.
14. Bělohlávek R., Vychodil V.: What is a fuzzy concept lattice? In: Proc. CLA 2005, 3rd Int. Conference on Concept Lattices and Their Applications, September 7–9, 2005, Olomouc, Czech Republic, pp. 34–45, URL: `http://ceur-ws.org/Vol-162/`.
15. Ben Yahia S., Jaoua A.: Discovering knowledge from fuzzy concept lattice. In: Kandel A., Last M., Bunke H.: *Data Mining and Computational Intelligence*, pp. 167–190, Physica-Verlag, 2001.
16. Elloumi S. *et al.*: A multi-level conceptual data reduction approach based in the Łukasiewicz implication. *Inf. Sci.* **163**(4)(2004), 253–264.
17. Fan S. Q., Zhang W. X.: Variable threshold concept lattice. *Inf. Sci.* (to appear).
18. Ganter B., Wille R.: *Formal Concept Analysis. Mathematical Foundations.* Springer-Verlag, Berlin, 1999.
19. Goguen J. A.: The logic of inexact concepts. *Synthese* **18**(1968-9), 325–373.
20. Guigues J.-L., Duquenne V.: Familles minimales d'implications informatives resultant d'un tableau de données binaires. *Math. Sci. Humaines* **95**(1986), 5–18.
21. Hájek P.: *Metamathematics of Fuzzy Logic.* Kluwer, Dordrecht, 1998.
22. Hájek P.: On very true. *Fuzzy Sets and Systems* **124**(2001), 329–333.
23. Krajči S.: Cluster based efficient generation of fuzzy concepts. *Neural Network World* **5**(2003), 521–530.
24. Pollandt S.: *Fuzzy Begriffe.* Springer-Verlag, Berlin/Heidelberg, 1997.