

Fast Fourier Transform for Discrete Asian Options

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Abstract

This paper presents an efficient methodology for the discrete Asian options consistent with different types of underlying densities, especially non-normal returns as suggested by the empirical literature (Mandelbrot (1963) and Fama (1965)). The interest of this method is its flexibility compared to the more standard ones. Based on Fast Fourier Transform, the method is an enhanced version of the algorithm of Caverhill and Clewlow (1992). The contribution of this paper is to improve their algorithm and to adapt it to non-lognormal densities. This enables us to examine the impact of fat-tailed distributions on price as well as on delta. We find evidence that fat tails lead to wider jumps in the delta.

1 Introduction

First introduced in Tokyo, Asian options are options based on any type of average of underlying equity prices, interest rates or indices. They are among the most popular path-dependent derivatives, since their characteristics capture partially the trajectory of the underlying, with often reduced exposure to volatility. In addition, Asian options are less sensitive to possible spot manipulations or extreme movements at settlement and offer flexibility in the way the average is settled. Consequently, they have become very attractive for investors since they provide a customized cheap way to hedge periodic cash-flows (see Longstaff (1995) for a discussion of the efficiency of Asian interest-rate options for corporations with reasonably predictable cash flows)

When pricing an option, one of the first questions that arises concerns the distributional assumptions for the underlying. Very often the distribution of the latter is taken to be lognormal as in the Black Scholes model. However, when it comes to arithmetic Asian options, one is confronted with the problem of the distributions. Indeed, the empirical literature has rejected normality of returns and hence the geometric Brownian motion. It has rather suggested fat-tailed distributions (see Mandelbrot (1963) and Fama (1965) for the early ones).

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The motivation of this paper is to provide an efficient method for the pricing of Asian options consistent with various underlying densities, especially non log-normal ones. Because of the challenge of getting a correct price for Asian option with a widely used option pricing model, previous research has focussed on the Black Scholes model, adopting different strategies. It has first focussed on the geometric Asian option case (Vorst (1992), Turnbull and Wakeman (1991), Zhang (1995)). It has as well looked at the question of the continuous-time Asian options (Geman and Yor (1993), Rogers and Shi (1995), Alziary et al. (1997), He and Takahashi (1996), Forsyth et al. (1998), Nielsen and Sandmann (1998)). However, the type of average for traded Asian options is arithmetic and discrete: daily, weekly or monthly. Approximating these options by their continuous-time limit is inaccurate and misleading for options with a period of time between two fixing dates longer than a day.

To account for the discrete arithmetic averaging, it has been suggested to use different approximations of the density of the sum of lognormal variables leading to various closed-form solutions: approximation via the geometric average (Vorst (1992)), via a lognormal density (Turnbull and Wakeman (1991)), via an Edgeworth expansion (Levy and Turnbull (1992) and Jacques (1996)), via a Taylor expansion (Zhang (1998) and Bouaziz et al. (1998)) or via the reciprocal Gamma distribution (Milesvky and Posner (1997)).

It has also been advocated to use different numerical methods: Monte Carlo (Kemma and Vorst (1990)), tree methods (Hull and White (1997)) and Fast Fourier Transform techniques (Caverhill and Clewlow (1992)). However, none of these works has considered non-lognormal distributions.

When the underlying density is not lognormal, the approximation methods do not hold any more since they heavily rely on the lognormal assumption. Numerical methods like PDE or lattice methods are as well not easy to adapt to the non-lognormal case, since we need to restrict ourselves to certain types of diffusion like stochastic volatility or deterministic volatility models which implies strong assumptions on the underlying diffusion. It is not very straightforward to derive an empirical density from market data, requiring very often a calibration stage. The two methods adaptable to an ad-hoc empirical non lognormal distribution without too much difficulty, are indeed the Monte Carlo and the Fast Fourier Transform method. However, these two methods perform poorly for non-lognormal case as well as for lognormal one. The Monte Carlo has the drawback to be slow. The algorithm of Caverhill and Clewlow (1992) requires large discretization grid and has slow convergence.

In this paper, we offer a solution to improve the method of Caverhill and Clewlow (1992) and to adapt it to the case of non-lognormal densities. To reduce the size of the grid and therefore the computational time, we recenter intermediate densities. We test this algorithm in the lognormal case since it is only in this particular situation that we have benchmarks in the literature. We then examine the impact of non-lognormal densities on the price as well as on the delta.

The remainder of this paper is organized as follows. In section 2, we describe our algorithm in detail. In section 3, we examine numerical results for the lognormal case, using it as a benchmark for the efficiency of our method. Section 4 deals with non-lognormal densities. It examines the impact of various densities on the price of the option as well as on the delta. We conclude briefly in section 5 suggesting further developments.

2 Description of the method

2.1 Framework

We consider a continuous-time trading economy with infinite horizon. The uncertainty in the economy is classically modelled by a complete probability space (Ω, F, Q) . The underlying is denoted by $(S_t)_{t \in \mathbb{R}^+}$. The information evolves according to the natural filtration $(F_t)_{t \in \mathbb{R}^+}$ implied by the underlying process. Following the traditional empirical literature, we assume that returns $(R_{t_i})_{i \in \mathbb{N}}$, defined by $R_{t_i} = \log(S_{t_i}/S_{t_{i-1}})$ for a given sequence of time $(t_i)_{i \in \mathbb{N}}$, are independently distributed and have a well-known density $f_i(\cdot)$, with a well-known mean denoted by μ_i . In the case of the Black Scholes model, each of these densities is a normal distribution with mean $(r - \frac{\sigma^2}{2})(t_i - t_{i-1})$ and variance $\sigma^2(t_i - t_{i-1})$. The underlying price is then calculated as the initial price S_{t_0} increased by the different returns $e^{R_{t_i}}$:

$$S_{t_i} = S_{t_0} e^{R_{t_1} + R_{t_2} + \dots + R_{t_i}}$$

Assuming that we have n fixing dates for the average, denoted by t_1, t_2, \dots, t_n , the arithmetic average A is defined through:

$$A = \frac{1}{n} \sum_{i=1}^n S_{t_i} \quad (2.1)$$

In complete markets with no arbitrage opportunity, there is a unique risk neutral martingale measure denoted by Q . In this framework, the price P of an Asian call, with strike K , expiring at time T , is defined as the expected value of the time- T payoff discounted at the risk-free rate r :

$$P = \mathbb{E}^Q \left[e^{-rT} (A - K)^+ \right] \quad (2.2)$$

where X^+ stands for $\max(X, 0)$. Since the discrete average process has no well-known density, there is no closed formula. However, we show in this paper that we can compute numerically this density, giving a method which converges to the real densities as long as the size of the discretization grid tends to the infinity.

2.2 Why Fast Fourier Transform?

Well known in signal theory, Fast Fourier Transform (FFT) is efficient for the resolution of many numerical problems. More specifically, the FFT is an efficient algorithm for computing the sum:

$$FFT(f(k)) = \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{N-1} e^{-i \frac{2\pi}{N} (j-1)(k-1)} f(j) \quad \text{for } k = 1 \dots N$$

where N is typically a power of 2. This algorithm reduces the number of multiplications in the required N summations from $O(N^2)$ to that of $O(N \log_2(N))$. This suggests that for a grid with 2^p points, the complexity is $p2^p$, which is typically the complexity of a binomial tree.

Recently, this technique has gained popularity in option valuation (Baskhi and Chen (1998), Scott (1997), Chen and Scott (1992), Carr and Madan (1999)) in view of its numerical efficiency. The property of the Fourier transform used here is its efficiency to calculate convolution products. The Fourier transform of such a product is simply the product of the Fourier transforms. This is helpful in getting the density of the sum of two variables since this is just the convolution product of the individual densities as long as the variables are independent. In the case of the Asian option, the expression involved is not a straightforward

sum of independent variables. In the algorithm section, we show how to use independent variables in a recursive scheme.

The interest of this method is its efficiency compared to a straightforward computation of the density. Instead of computing an $n-1$ dimensional integral with a complexity of $O(N^{n-1})$, we reduce this complexity by means of Fast Fourier Transform to $O(N^2 \log(N))$.

The use of FFT method for Asian option valuation was first suggested by Carverhill and Clewlow (1992). However, their work assumes lognormal densities and is not very efficient since it requires large grid and converges rather slowly. To speed up convergence, one needs to reduce the size of the grid required by the FFT algorithm. To cope with smaller grid, we introduce a proxy for the mean of intermediate densities. This enables us to recenter the different variables. We extend as well the FFT method to non-lognormal densities. We look particularly on the Student-density case since the latter is a well-known example of a fat-tailed distribution. We use the FFT algorithm as described in Press et al. (1992). Indeed, the method explained here is very general and can be applied to many other fat-tailed densities, like extreme value, Pareto and generalized Pareto distributions.

2.3 Algorithm

2.3.1 Inefficiency of the Carverhill and Clewlow method

A simple way to calculate the density of a sum of dependent variables is to transform then into independent variables. With our assumptions on the independence of returns, this comes naturally. Notice that when the underlying distribution is lognormal, returns are normal and their Fourier transform has a closed form solution equal to $f(w) = \frac{1}{\sqrt{2\pi}} e^{(iwm - \frac{w^2 \sigma^2}{2})}$, where m stands for the mean and σ^2 the variance. We introduce the sequence $(B_i)_{i=0..n-1}$ defined by its initial condition: $B_1 = R_{t_n}$ and for the recursion $i = 2..n$,

$$B_i = R_{t_{n+1-i}} + \log(1 + \exp B_{i-1}) \quad (2.3)$$

The Steward and Hodges factorization expresses the sum variable A defined by (2.1) in terms of the variable B_n as stated in the following proposition:

Proposition 1 *The sum variable A can be expressed in terms of the last term of the sequence $(B_i)_{i=0..n-1}$: B_{n-1} ; through :*

$$A = \frac{S_{t_0}}{n} e^{B_n}$$

Proof : We decompose the underlying price as a function of the difference of returns: $S_{t_i} = S_{t_0} e^{R_{t_1} + R_{t_2} + \dots + R_{t_i}}$. Factoring terms leads to a multiplicative expression of the sum variable:

$$A = \frac{S_{t_0}}{n} [e^{R_{t_1}} * (1 + e^{R_{t_2}} * (1 + e^{R_{t_3}} (1 + \dots (1 + e^{R_{t_n}}))))]$$

When taking the logarithm of the above equation, we get an additive expression:

$$A = \frac{S_{t_0}}{n} * \exp(R_{t_1} + \log(1 + \exp(R_{t_2} + \log(\dots + \log(1 + R_{t_n}))))))$$

The term inside the outermost exponential can be calculated recursively using the sequence $(B_i)_{i=1..n}$. \square

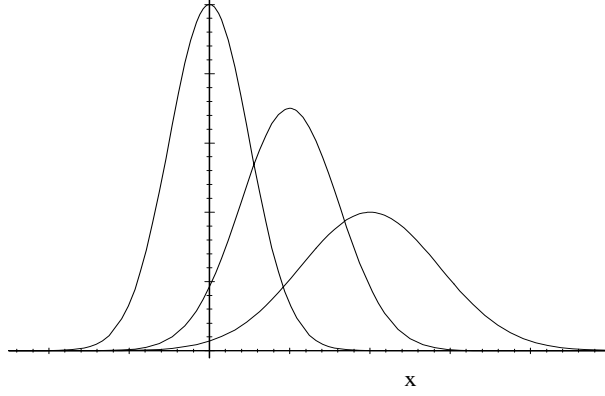


Figure 1: Evolution of the densities

The Proposition 1 together with the recursive equation (2.3) was the starting point of the work of Carverhill and Clewlow (1992). At the i^{th} step of the recursive equation (2.3) the return $R_{t_{n+1-i}}$ is added. The latter is, however, not centred and has often a positive mean which for high volatilities can become negative (see the expression for the mean $\left(r - \frac{\sigma^2}{2}\right) * \frac{T}{n}$). For positive mean, the distribution of B_{i+1} is consequently shifted to the right of the distribution of B_i . If we discretize the distribution of B_{i+1} on the same grid as the one of B_i , this implies that the discretization grid must be large enough to contain the two distributions. When we have n dates in our arithmetic average, this tends to shift more and more in one direction as the order of the distribution increases as shown in figure 1. This is precisely why the algorithm of Carverhill and Clewlow requires a large grid.

2.3.2 Recentering intermediate densities

In order to obtain a smaller grid and therefore to reduce computational time, we can recenter the densities at each step. The difficulty here is that we do not know the exact mean of the variable B_i . Denoting by μ_i the mean of the return R_{t_i} ($\mu_i = \mathbb{E}[R_{t_i}]$), which is supposed to be known, we can approximate the mean of the variable B_i with the following sequence: $(m_i)_{i=1..n}$ initialized with $m_1 = \mu_n$ and for $i = 2..n$

$$m_i = \mu_{n+1-i} + \log(1 + \exp m_{i-1}) \quad (2.4)$$

The term m_i acts as a proxy for the mean of the variable B_i . The approximation of the average is done by taking the lagged B_{i-1} equal to its mean m_{i-1} in the recursive equation (2.3). It is worth noticing that even if we do an approximation on the mean, it does not mean that we approximate the density of B_i . It just means that we do not perfectly center this variable. However, there is no new error implied by the recentering. Indeed, since the function $\log(1 + e^x)$ is convex, we are underestimating some convexity adjustment term as stated by the Jensen inequality for convex functions $f(\mathbb{E}(X)) \leq \mathbb{E}(f(X))$.

The recentered sequence is defined as $(A_i)_{i=1..n}$ with $A_i = B_i - m_i$. Replacing B_{i-1} by its expression in terms of A_{i-1} and m_{i-1} leads to a recursive two dimensional sequence summarized by the following proposition:

Proposition 2 *The sum variable A can be expressed in terms of the last term of the recursive sequence A_n and m_n : as follows:*

$$A = \frac{S_{t_0}}{n} e^{A_n + m_n}$$

where the sequence $(m_i)_{i=1..n}$ is defined as above (2.4) and the sequence $(A_i)_{i=1..n}$ is given by the initial condition $A_1 = R_{t_n} - m_1$ and for $i = 2..n$

$$A_i = R_{t_{n+1-i}} + \log(1 + \exp A_{i-1} \exp m_{i-1}) - m_i$$

To get the density of A_i with respect to the one of A_{i-1} , we use the standard change of variable theorem, which relates the density of a variable $Y = g(X)$, denoted by dq_Y , with the one of the variable X , denoted by $dp_{f^{-1}(y)}$, through the Jacobian of the function $f^{-1}(Y) = X$

$$dq_Y = dp_{f^{-1}(x)} |J_{f^{-1}(y)}| dy$$

leading to the interpolation formula:

$$dp_{\log(1+e^{m_{i-1}+X})-m_i}(y) = \frac{e^{y+m_i}}{e^{y+m_i}-1} p_X(\log(e^{y+m_i}-1) - m_{i-1}) 1_{\{y > -m_i\}} dy \quad (2.5)$$

We can now describe the different steps of the first algorithm. The algorithm is initialized with the value of the two dimensional sequence $m_1 = \mu_n$ and $A_1 = R_{t_n} - m_1$. It finishes when we get m_n and A_n .

The recursive sequence is calculated as follows. Assume that we know the value of the bi-dimensional sequence at step $i-1$, that is m_{i-1} and A_{i-1} .

- We then interpolate the variable A_{i-1} by means of the remark (2.5) to get the density of the variable $\log(1 + e^{m_{i-1}+A_{i-1}}) - m_i$.
- We calculate the density of A_i , which is the sum of the two independent variables $R_{t_{n+1-i}}$ and $\log(1 + e^{m_{i-1}+A_{i-1}}) - m_i$, by calculating the convolution product via FFT.
- Having obtained the density of the average, we calculate the payoff of the option, defined as an expectation, by a numerical integration, using the Simpson rule.

2.3.3 Discussion of the numerical techniques

The FFT algorithm requires the density function to be represented at a sufficient number of equally spaced points. The grid for the discretization of the different densities needs to be sufficiently dense as well as sufficiently large to avoid interference errors implied by the periodisation of the density function in the FFT algorithm. We use the FFT algorithm as described in Press et al. (1992)

Errors in the numerical integration by the Simpson rule (exact for the integration of polynom up to degree 3) are negligible compared to the ones produced by the discretization of the distribution. The error in the Simpson rule for the integral of a function f infinitely differentiable $\int_a^b f(x) dx$ can be shown to be an $O\left(\left(\frac{b-a}{2}\right)^5 f^{(4)}\right)$.

3 Efficiency of the algorithm for the lognormal case

3.1 Black Scholes assumption

The information evolves according to the augmented filtration $\{F_t, t \in [0, T]\}$ generated by a standard one-dimensional standard Brownian motion $(W_t)_{t \in R_+}$. We assume the underlying price process is a geometric Brownian motion, solution of the Black Scholes (1973) diffusion defined by equation (3.6) with initial condition $S_{t=0} = S_0$

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \quad (3.6)$$

In this case the returns R_{t_i} have a normal density with mean $\left(r - \frac{\sigma^2}{2}\right)(t_i - t_{i-1})$ and variance $\sigma^2(t_i - t_{i-1})$.

3.2 Choice of the Grid

The choice of an efficient grid is not easy. The grid is determined by its range as well as its number of points. Choosing a range not correctly leads to interference errors. Taking a grid not dense enough leads as well to inaccurate Fourier Transform computation. We choose a centred grid with 4096 points, that is 2^{12} and with a width of $9n\sigma\sqrt{dt}$, where n stands for the number of fixing dates, σ the volatility, dt the time between two fixings. For a one year weekly Asian option, with fifty fixings, the number of fixing n is equal to 50 and the period of time between two successive prices dt is equal to one week or $1/52$ of a year.

3.2.1 Recentering the densities

The improvement of this paper is to recenter densities at each step. Since we approximate the mean, the recentering is imperfect as figure 2 shows. For low volatilities up to 20%, densities are perfectly recentered for a one-year weekly Asian option. For volatilities higher than 20%, the approximation of the mean is not rigorously correct and leads to a shift of the different densities to the right. Indeed, since the function $\log(1 + e^x)$ is convex, we are underestimating a convexity adjustment term, as stated by the Jensen inequality for a convex function f , $f(\mathbb{E}(X)) \leq \mathbb{E}(f(X))$. However, the bias in our estimation is quite small, since for large values of x , the function $\log(1 + \exp(x))$ is very little convex, roughly equal to x , justifying our method.

In the figure 2, we can see that for small volatility level (20%, figure on the left), the recentering is perfect whereas for higher volatility (30%, figure on the right), we are missing the convexity adjustment term. In the original algorithm of Caverhill and Clewlow, the grid size can be shown to be equal to $9n\sigma\sqrt{dt} + \mu_n$. The gain in our method can be measured by the grid width ratio $(9n\sigma\sqrt{dt} + m_n)/9n\sigma\sqrt{dt}$. For the case of a one year option with 10% volatility and a risk-free rate of 20%, this gain is equal to 1.317. This means that with the old algorithm, we need 1317 points to get the same precision as 1000 points with the new one. This means that the equivalent of 4096 points with the new algorithm is about 5400 points with the old algorithm.

It is worth noticing that the difference between our method and the one of Caverhill and Clewlow is larger for big volatility. This is because the advantage of getting a correct distribution becomes more relevant for high volatility even if the comparative advantage of recentering becomes less and less important for larger grids caused by larger volatilities.

3.2.2 Interference on the FFT algorithm

When the grid is not large enough, interference alters the results' quality as shown in figure 3, where we used a grid width of only $4n\sigma\sqrt{dt}$. This comes from the fact that the FFT algorithm assumes the periodicity of our function. It can cause interference terms when the grid size is too small.

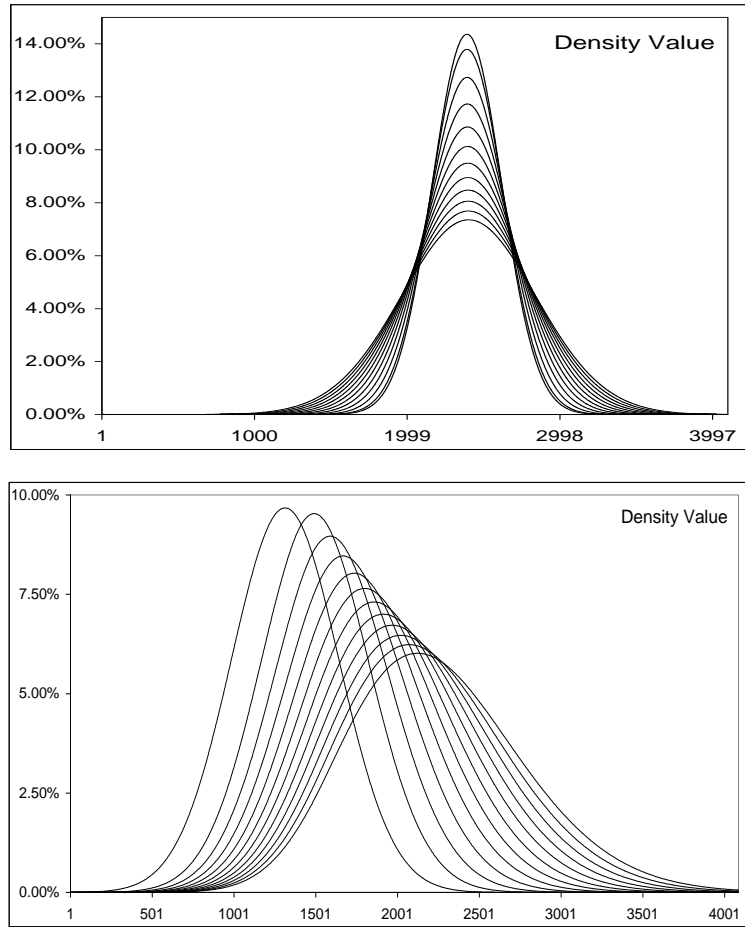


Figure 2: Evolution of the density with recentering at each step. The two graphics concern a one year weekly Asian option. The figure on the top is with 20% of volatility whereas the one on the bottom is for 30% volatility.

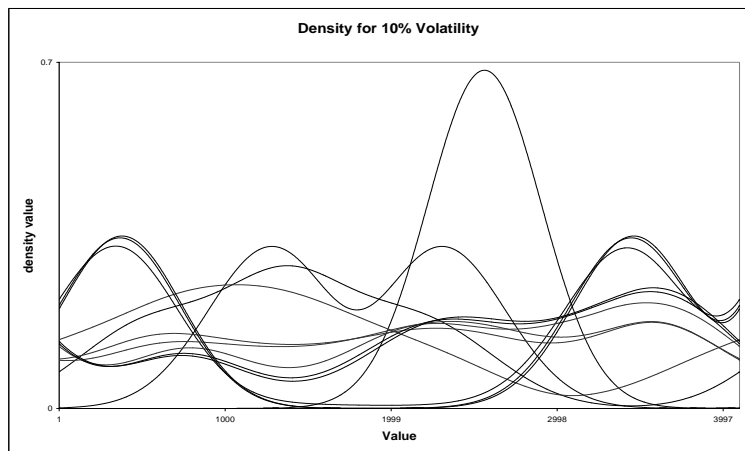


Figure 3: Interferences on the densities

3.3 Comparison of the different methods

Because of no well-known example, we arbitrary decided to use as a benchmark the same option example as in the work of Levy and Turnbull (1992) . We compute the price of a one year Asian option, with the underlying starting at 100 ($S = 100$), with a risk-free interest rate of 10% ($r = 10\%$), and 50 fixings per year (weekly average with two weeks of holidays).

The results, given in the table 1, compare different methods and show that the convolution method is efficient for the pricing of Asian option. Regarding the column titles, MC stands for Monte Carlo with its standard error given in the next column SE. WE means Wilkinson Estimates, E Edgeworth method, RG the reciprocal Gamma approximation, CV the Convolution method of Caverhill and Clewlow, CVR the convolution method with recentering. The reference price is the one of the Monte Carlo simulation. The accuracy of a formula is given by its comparison with this reference price.

We found that recentering the density improves significantly the efficiency of the Fast Fourier Transform method for high volatilities since the estimation of the density becomes more important. Among the traditional approximation methods, we tested Wilkinson estimates, Edgeworth expansion, and the reciprocal gamma approximation. We found that Wilkinson estimates was the most robust method. The Edgeworth expansion formula can blow up when the third and fourth moments are too different from the ones implied by a lognormal. We also got poor results for the reciprocal gamma approximation. This comes from the small number of variables in our Asian options. The density of the average is therefore far from its asymptotic limit, which can be shown to be a reciprocal gamma density (see Milesvky and Posner (1997)).

σ	K	MC	Std Err	WE	E	RG	CV	CVR
10%	80	22.78	0.00	22.78	22.78	21.64	22.78	22.78
	90	13.73	0.00	13.73	13.73	13.1	13.73	13.73
	100	5.24	0.00	5.25	5.25	4.98	5.25	5.25
	110	0.72	0.00	0.72	0.72	0.71	0.72	0.72
	120	0.03	0.00	0.02	0.02	0.02	0.02	0.02
30%	80	23.07	0.01	23.14	23.07	21.92	23.09	23.08
	90	15.22	0.01	15.30	15.16	14.46	15.29	15.26
	100	9.01	0.01	9.08	9.00	8.56	9.08	9.05
	110	4.83	0.01	4.84	4.85	4.58	4.86	4.84
	120	2.35	0.01	2.33	2.40	2.23	2.40	2.33
50%	80	24.83	0.03	25.06	24.10	23.58	25.01	24.88
	90	18.32	0.03	18.57	17.83	17.40	18.50	18.37
	100	13.18	0.03	13.34	13.02	12.52	13.47	13.20
	110	9.23	0.03	9.33	9.36	8.77	9.45	9.19
	120	6.36	0.03	6.37	6.63	6.04	6.68	6.40

Table 1: Comparison of different methods for the Asian option σ stands for the volatility, K for the strike

Our results confirm that the lognormal approximation slightly overprices Asian options as already shown by Levy and Turnbull (1992), Zhang (1998). This can be seen in figure 4 by the skew to the right of the Wilkinson estimates (or lognormal approximation) density. The accuracy of the FFT method is confirmed

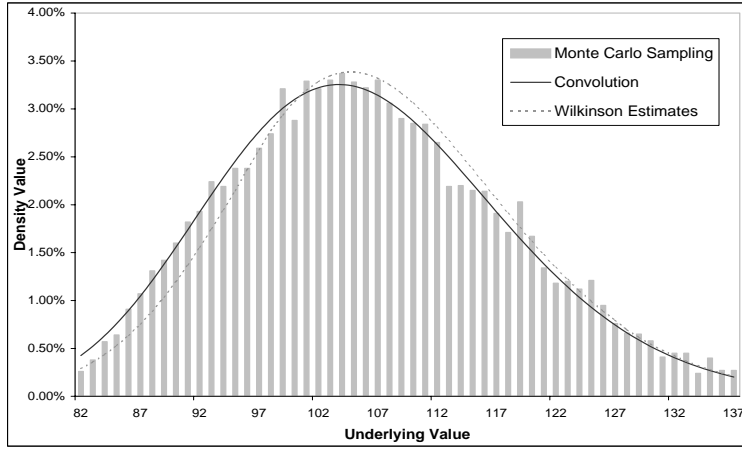


Figure 4: Comparison of the different densities for a one year Asian option for 30 % volatility

by the close fit with the Monte Carlo sampling in figure 4. The Monte Carlo sampling was based on a simulation of a Sobol sequence with 30,000 draws.

It is worth noting that the precision of the method is heavily depending on the type of the options: in, at or out-of-the-money. One should expect little difference in price for options depending on a wide part of the distribution like in or at-the-money options. However, for out-of-the-money options, that are depending mainly on the tails of the distribution, there is a real advantage in terms of precision to use the Fast Fourier Transform method compared to Wilkinson estimates. Indeed, fat tails are the true motivation of this paper. It is already interesting to realize that even in the case of a lognormal underlying, the Fast Fourier Transform method takes better account for fat tails than most standard approximation methods with closed form.

4 Using non-lognormal densities

4.1 Interest of the method

It is now widely accepted that markets differ from the seminal Black Scholes (1973) lognormal model. The empirical literature has extensively reported on these anomalies, especially on two of them, which indeed are closely linked. First, it has been shown that unconditional returns show excess kurtosis and skewness, inconsistent with normality assumptions (see Mandelbrot (1963) and Fama (1965) for the early ones, Kon (1984), Jorion (1988) and Bates (1996)). Second, research has concentrated its attention on the implied volatility smile or skew (see Dumas et al. (1995) for a survey). Interestingly, the second fact is just another hint of the non-normality of returns. However, research has focussed at implied Black Scholes volatility since implied volatility has become a key concept in option pricing. Option prices are often quoted by their implied volatility. A more rigorous justification of the interest in modelling volatility is its less volatile character when compared with prices. Since, corresponding prices fluctuate more than implied volatilities, the trading environment is best captured by a model about the implied volatility.

How to cope with the smile in option pricing has become an extensive field of research. Classically, it is divided into two different approaches: parametric and non-parametric ones.

In the first method, the equation of the evolution of the underlying process is given. This description can consist in a continuous diffusion process with either a so called deterministic volatility (Rubinstein (1994), Dupire (1993) and Derman and Kani (1994)) or a stochastic volatility process (Hull and White (1987), Wiggings (1987), Melino and Turnbull (1990), Stein and Stein (1991), Amin and Ng (1993) and Heston (1992)) or a model with jumps (Aase [1993, Ahn and Thompson (1988), Amin (1993), Bates (1991), Jarrow (1984), Merton (1976)).

Other works close in spirit are assuming constant elasticity of volatility distribution often called power-law (Cox Ross (1976)) or a mapping principle between normal and lognormal distributions (Hagan (1998), Pradier and Lewicki (1999)).

The second type of methods involves inferring the underlying distribution from market data. This has been called the expansion method where one induces the different terms of the expansion and can reconstitute the distribution (Jarrow and Rud (1982), Bouchaud et al.(1998), Abken et al. (1996)).

The interest of our methodology lies in its flexibility on the distributions of returns. We do not assume any specific distribution. The distribution is an input like all other parameters. Therefore, we can use distribution derived from market data, like option prices. In this paper, we decided to illustrate the fat-tailed distribution with the specific case of a Student density. This is because this density is often used in the literature. It has the additional advantage to converge to the normal density when the number of degree of freedom tends to infinity. Indeed, there are many other densities which could have been used, like Pareto, generalized Pareto, power-laws distributions and many more.

4.2 Densities for leptokurtic effect

To account for leptokurtic returns, we assume that centred and pseudo-normalized (with a parameter $\lambda > 1$) returns $\frac{R_{t_i} - (r - \frac{\sigma^2}{2})(t_i - t_{i-1})}{\sqrt{\frac{\sigma^2(t_i - t_{i-1})}{\lambda}}}$ have a density given by a Student distribution with a degree of freedom $n = \frac{2\lambda}{\lambda-1}$ given by $\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})\sqrt{\pi n}} \left(1 + \frac{t^2}{n}\right)^{-\frac{n+1}{2}}$. The cumulative distribution is then given by

$$\Pr(X \leq t) = \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})\sqrt{\pi n}} \int_{-\infty}^t \left(1 + \frac{1}{n}u^2\right)^{-\frac{n+1}{2}} du$$

where $\Gamma(y) = \int_0^\infty e^{-x} x^{y-1} dx$ is the Gamma function at y . Since a Student density has always a variance bigger than one we need to specify this variance by the parameter λ .

4.3 Numerical results

4.3.1 Effect on the price

As expected, fat-tailed distributions hereby illustrated by the Student density lead to a more expensive price of the Asian option. The Fast Fourier method is efficient as confirmed by a comparison with Monte Carlo simulations with 20,000 draws. To simulate the Student density, we simulate uniform distribution and inverse the cumulative distribution by means of the approximation given in the appendix section.

Without any surprise, the discrepancy between the lognormal distribution and a distribution with fat tails increases with the volatility. It also grows for distribution with fatter tails as shown by the increase of price between the Student density with 44 degrees of freedom and the one with only 22. We have chosen

the Student density since its asymptotic distribution is precisely the normal distribution when the degrees of freedom tend to infinity.

Interestingly, practitioners have kept on using the lognormal approximation for the Asian option. We have seen that the approximation of a sum of lognormal by a lognormal distribution is not correct. It tends to overprice the Asian option. However, when assuming a fat-tailed distribution for the underlying, we also found that the price of the option was more expensive than the corresponding one with lognormal individual underlyings. This explains why practitioners have been very keen on using the lognormal approximation since this includes the rise of price due to fat-tailed distributions.

In a sense, over-estimation suits practitioners since the lognormality of the law is not very realistic. However, by using the lognormal distribution, practitioners are confusing getting a correct price with correct assumptions and correct method with getting a correct price with wrong assumptions and wrong method. Because the lognormal approximation method has the advantage of overestimating, practitioners get a price which in a sense include the fat tailed distribution. However, first, it is not easy to know by how much traders need to overprice to obtain market price and second, the hedging position is with the lognormal distribution not including fat tails even it leads to a correct price. In this article, we argue that the convolution method enable to calculate accurately both the price and the hedging strategy. Because actual distributions exhibit fatter tails than normal distributed returns, fat-tailed distributions like Student distribution are more appropriate. The core of this methodology is then to calibrate the Student distribution to know which degree of freedom to apply.

σ	K	Lognormal	Student 44 df	MC Student 44 df	Student 22 df	MC Student 22 df
10%	80	22.7838	22.7911	22.7914	22.8021	22.8028
	90	13.7347	13.7420	13.7425	13.7539	13.7550
	100	5.2438	5.2843	5.2850	5.3278	5.3294
	110	0.7211	0.7642	0.7649	0.8078	0.8094
	120	0.0336	0.0358	0.0362	0.0423	0.0430
30%	80	23.0733	23.2033	23.2050	23.3372	23.3406
	90	15.2231	15.4014	15.4036	15.5808	15.5855
	100	9.0110	9.2238	9.2277	9.4335	9.4416
	110	4.8338	5.0345	5.0379	5.2355	5.2426
	120	2.3545	2.5097	2.5117	2.6682	2.6728
50%	80	24.8324	25.2213	25.2243	25.6099	25.6168
	90	18.3207	18.7471	18.7510	19.1694	19.1779
	100	13.1811	13.6182	13.6239	14.0508	14.0628
	110	9.2300	9.6476	9.6530	10.0610	10.0721
	120	6.3615	6.7363	6.7412	7.1104	7.1209

Table 2: Price of Asian option with Fat-tails distributions. σ stands for the volatility, K for the strike, df for degrees of freedom

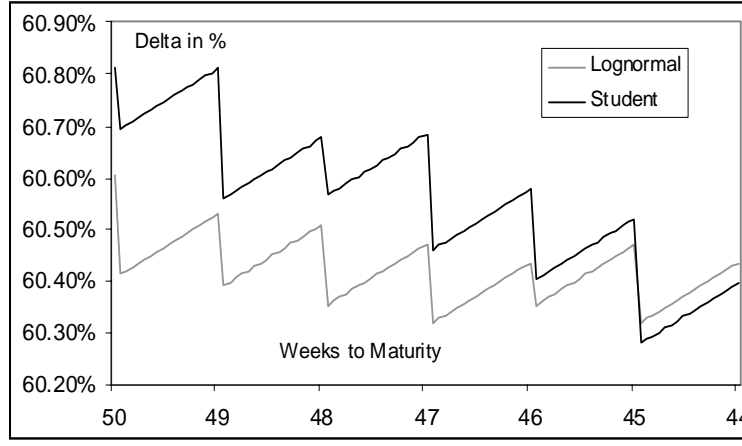


Figure 5: Evolution of the delta with time to maturity under different distributions: (lognormal and Student).

4.3.2 Delta hedging

The motivation for our numerical method is to examine the impact of fat-tailed distributions on the delta. In the case of the discrete Asian options, the delta jumps every time we cross a fixing date.

The comparative study of the delta evolution with lognormal density and the Student density shows that a fat-tailed distributions lead to higher jumps in the delta, a logical consequence of the fact that fat-tailed distributions imply more expensive prices and therefore larger drop of the price with the downfall of a fixing date. The difference in the delta is quite significant as shown by figure 5 and 6. The figure 5 show the evolution of the delta for a weekly Asian option far from the maturity of the option. The Student density taken here is the one with 22 degrees of freedom.

In figure 5 and 6, the sharp drops at integer number is the delta jump at fixing dates. Maturities in the two figures 5 and 6 are in portions of weeks (therefore a maturity of 50 means a 50 weeks maturity).

There is no rule concerning the difference between the delta for lognormal densities and for Student densities. In the figure 5, the option is 50 to 44 weeks before the expiration. In this particular case, the delta implied by the Student density is on overall more expensive. This is not the case when the option is close to the maturity as shown by figure 6 where there are only 10 to 1 week before the maturity of the option. However, it is worth noting that on the average the delta is almost the same for the two densities. This suggests that for a long-run delta hedging, assuming normal returns is not too much inaccurate. However, for short run delta hedging, the assumptions on the return densities lead to very different hedging strategies.

It is worth noting that the difference of delta for the two distribution depends on the maturity. For maturities between 50 to 45 weeks, the delta given by the student distribution is higher than the one of the lognormal distribution while it is lower for maturities of less than 5 weeks.

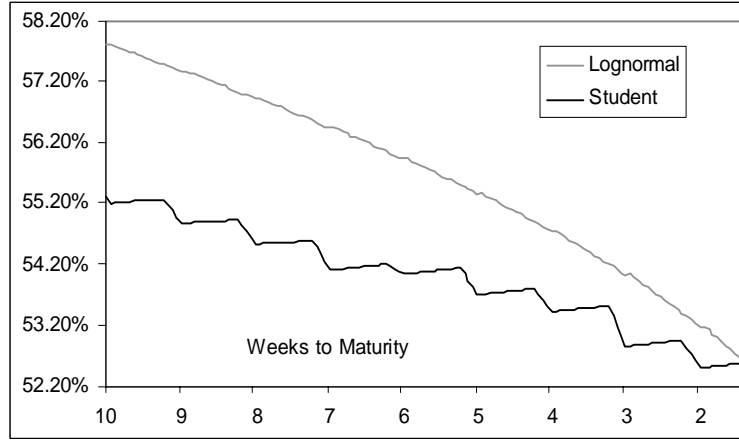


Figure 6: Evolution of the delta with time to maturity under different distributions when closed to the expiry.

5 Conclusion

In this paper, we have seen that Fast Fourier Transform is an efficient way for pricing discrete Asian options with non-lognormal densities. The systematic recentering of intermediate densities enables to reduce the size of the grid so as to speed up the convergence. We show that the price of the Asian option should be more expensive with fat-tailed distributions. This indicates that approximation methods overpricing the Asian option incorporate, in a way, fat-tailed distribution. However, as far as the delta is concerned, fat-tailed distributions lead to very different hedging strategies, especially on the short run.

Our methodology raises many remarks. First, the Fast Fourier Transform technique enables to take into account volatility smile since, as an input, we can take returns' distribution derived by market data incorporating the smile effect. Second, the same approach can be applied with minor changes to basket and multi-asset options. Third, this methodology raises the issue of the way of deriving the density from market data properly.

6 Appendix: Inverse of the cumulative distribution of the student density

The general algorithm for computing the inverse t_p of the cumulative distribution of the Student density, with n degree of freedom is given below with $0 < p < 1$ and with x_p the inverse of the cumulative distribution of the normal density $N(0, 1)$:

$$\begin{aligned}
 t_p &= x_p + \frac{g_1(x_p)}{n} + \frac{g_2(x_p)}{n^2} + \frac{g_3(x_p)}{n^3} + \frac{g_4(x_p)}{n^4} \\
 g_1(x) &= \frac{1}{4}(x^3 + x) \\
 g_2(x) &= \frac{1}{96}(5x^5 + 16x^3 + 3x) \\
 g_3(x) &= \frac{1}{384}(3x^7 + 19x^5 + 17x^3 - 15x) \\
 g_4(x) &= \frac{1}{92160}(79x^9 + 776x^7 + 1482x^5 - 1920x^3 - 945x)
 \end{aligned}$$

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