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Fast parameterized algorithms for graphs on surfaces: Linear kernel and exponential speed-up*

Fedor V. Fomin[1] and Dimitrios M. Thilikos[2]

[1] Department of Informatics, University of Bergen, N-5020 Bergen, Norway, fomin@ii.uib.no

[2] Departament de Llenguatges i Sistemes Informàtics, Universitat Politècnica de Catalunya, Campus Nord – Mòdul C5, c/Jordi Girona Salgado 1-3, E-08034, Barcelona, Spain, sedthilk@lsi.upc.es

Abstract. Preprocessing by data reduction is a simple but powerful technique used for practically solving different network problems. A number of empirical studies shows that a set of reduction rules for solving Dominating Set problems introduced by Alber, Fellows & Niedermeier leads efficiently to optimal solutions for many realistic networks. Despite of the encouraging experiments, the only class of graphs with *proven* performance guarantee of reductions rules was the class of planar graphs. However it was conjectured in that similar reduction rules can be proved to be efficient for more general graph classes like graphs of bounded genus. In this paper we (i) prove that the same rules, applied to any graph G of genus g, reduce the k-dominating set problem to a kernel of size $O(k+g)$, i.e. *linear kernel*. This resolves a basic open question on the potential of kernel reduction for graph domination. (ii) Using such a kernel we improve the best so far algorithm for k-dominating set on graphs of genus $\leq g$ from $2^{O(g\sqrt{k}+g^2)}n^{O(1)}$ to $2^{O(\sqrt{gk}+g)} + n^{O(1)}$. (iii) Applying tools from the topological graph theory, we improve drastically the best so far combinatorial bound to the branchwidth of a graph in terms of its minimum dominating set and its genus. Our new bound provides further exponential speed-up of our algorithm for the k-dominating set and we prove that the same speed-up applies for a wide category of parameterized graph problems such as k-vertex cover, k-edge dominating set, k-vertex feedback set, k-clique transversal number and several variants of the k-dominating set problem. A consequence of our results is that the non-parameterized versions of all these problems can be solved in *subexponential time* when their inputs have sublinear genus.

Keywords: Dominating set, branch-width, Σ-embedded graphs, parameterized algorithms, subexponential algorithms

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1 Introduction

The theory of fixed-parameter algorithms and parameterized complexity has been thoroughly developed over the past few years [9]. Dominating Set is one of the basic problems in parameterized complexity belonging to the complexity class W[2] and it is not surprising that it was investigated intensively. In the last three years, there was a breakthrough in understanding the parameterized complexity of Dominating Set on planar graph and different generalizations. The first fixed-parameter algorithm for k-dominating set in planar graphs [9] has running time $O(11^k n)$; subsequently, the first subexponential parameterized algorithm with running time $O(4^{6\sqrt{34k}} n)$ have been obtained by Alber et al. [1]. The development in the area of subexponential parameterized algorithms has proceeded in several directions:

Direction (i): *Reduction to linear kernel.* Let $\mathcal{L}$ be a parameterized problem, i.e. $\mathcal{L}$ consists of pairs (I, k) where k is the *parameter* of the problem. *Reduction to linear problem kernel* is the replacement of problem inputs (I, k) by a reduced problem with inputs (I', k') (linear kernel) with constants c_1, c_2 such that $k' \leq c_1 k$, $|I'| \leq c_2 k'$ and $(I, k) \in \mathcal{L} \Leftrightarrow (I', k') \in \mathcal{L}$. The existence of linear kernel for Dominating Set is highly unlikely because existing of such a kernel would imply the collapsing of the W-hierarchy. For planar graphs Alber, Fellows & Niedermeier [2] proved that Dominating Set has a linear kernel. This kernel is obtained by repetitively applying on the input graph G a set of *reduction rules.* We call this reduction *AFN-reduction.* It was also conjectured in [2] that the AFN-reduction provide linear kernels not only for class of planar graphs but for more general classes, like graphs of bounded genus. This was one of the biggest remaining challenges in the field.

Direction (ii): *The generality of graph classes to which these algorithms apply.* Ellis, Fan & Fellows [10] claimed that Dominating Set is fixed parameter tractable for graphs of bounded genus. Demaine et al. [5] recently proved this result by obtaining a subexponential parameterized algorithm that requires $2^{O(g\sqrt{k}+g^2)} n^{O(1)}$ steps on graphs of genus g. Subexponential parameterized algorithms are also known for graphs excluding a fixed graph as a minor [5], map graphs [4] and graphs of bounded local treewidth [6].

Direction (iii): *Optimization of the constants in the exponents of the running time.* The running time of Alber et al. algorithm [1] was improved to an algorithm of $O(2^{27\sqrt{k}} n)$-time by Kanj & Perković in [13], and finally to the $O(2^{15.13\sqrt{k}} k + n^3 + k^4)$-time algorithm of [11].

Direction (iv): *Extensions to other parameters.* In [1, 7, 14] it was observed that dominating set number is related to several graph parameters in a way that implies the existence of subexponential parameterized algorithms for all of them. This observation has been generalized in [5] to the general family of the *bidimensional* parameters. Examples of such parameters are: vertex cover, r-domination, edge-dominating set, weighted vertex dominating set, feedback set, maximal marching, clique transversal number, perfect code, and total dominating set.

Our contribution. Our results span all the research directions that we just mentioned. We enumerate them in the same order:

(i) We answer affirmatively the conjecture of [2]. More precisely, we prove that the application of the AFN-reduction on any graph G reduce it to a graph G' of size $O(k+g)$ where k and g are the dominating set number and the Euler genus of G respectively (see Section 3).

(ii) The kernel existence implies combinatorial bounds that are able to improve the best so far $2^{O(g\sqrt{k}+g^2)}n^{O(1)}$-time algorithm given in [5] to one of $2^{O(\sqrt{kg}+g)}+n^{O(1)}$-time (see Section 4).

(iii) All our algorithms have small hidden constants in the "O"-notation of their exponential part. We stress that this is not a straightforward consequence of the kernel existence and for this we need to prove better combinatorial bounds using elements of the Graph Minor Theory (see Section 5).

(iv) Using the above combinatorial bounds we can design $2^{O(\sqrt{kg}+g)}n^{O(1)}$-time algorithms for the majority of the parameters examined in direction (iii) (see Section 6).

The main graph theoretic tool of this paper is the representativity of a graph embedded in a surface Σ that is the minimum number of vertices met by an edge-avoiding non-contractible cycle of Σ. Very roughly, we implement the following "trick" several times: For graphs of representativity more than 6 we prove that they are enough "locally planar" and that certain arguments about planar graphs can be extended to graphs that are embedded that way on a surface. If representativity is at most 6, we can "cut" the surface, "split" the graph, decrease its genus, and apply certain inductive arguments.

We note that the contribution of the genus in the time complexity of our algorithms has some more general consequences. The first, is that the k-dominating set problem can be solved by a subexponential parameterized algorithm when restricted to graphs of genus $o(\log n)$. The second is that the algorithm remains subexponential on k even when $g = o(k)$. Therefore, for graphs with genus $o(n)$ the dominating set problem admits a subexponential *exact* algorithm. The same holds for a number of other problems discussed in Section 6.

2 Preliminaries

We denote by G a finite, undirected and simple graph with $|V(G)| = n$ vertices and $|E(G)| = m$ edges. For any non-empty subset $W \subseteq V(G)$, the subgraph of G induced by W is denoted by $G[W]$. The *neighbourhood* of a vertex v is $N(v) = \{u \in V(G) : \{u,v\} \in E(G)\}$ and for a vertex set $S \subseteq V(G)$ we put $N[S] = \bigcup_{v\in S} N[v]$ and $N(S) = N[S] \setminus S$.

A set $D \subseteq V(G)$ is a *dominating set* in a graph G if every vertex in $V(G) \setminus D$ is adjacent to a vertex in D. Graph G is D*-dominated* if D is a dominating set in G. We denote by $\gamma(G)$ the minimum size of dominating set in G.

Given an edge $e = \{x, y\}$ of a graph G, the graph G/e is obtained from G by contracting the edge e; that is, to get G/e we identify the vertices x and y and remove all loops and duplicate edges. A graph H obtained by a sequence of edge-contractions is said to be a *contraction* of G. H is a *minor* of G if H is the subgraph of a contraction of G.

Graphs on surfaces. A *surface* Σ is a compact 2-manifold without boundary. We will always consider connected surfaces. We denote by $\mathbb{S}_0$ the sphere $(x, y, z \mid x^2 + y^2 + z^2 = 1)$. A *line* in Σ is subset homeomorphic to $[0, 1]$. An O-arc is a subset of Σ homeomorphic to a circle. Let G be a graph 2-cell embedded in Σ. To simplify notations we do not distinguish between a vertex of G and the point of Σ used in the drawing to represent the vertex or between an edge and the line representing it. We also consider G as the union of the points corresponding to its vertices and edges. That way, a subgraph H of G can be seen as a graph H where $H \subseteq G$. We call by *region* of G any connected component of $\Sigma - E(G) - V(G)$. (Every region is an open set.) We use the notation $V(G)$ and $E(G)$, for the set of the vertices and edges of G. For $\Delta \subseteq \Sigma, \overline{\Delta}$ is the *closure* of Δ. The boundary of Δ is $\mathbf{bor}(\Delta) = \overline{\Delta} \cap \overline{\Sigma - \Delta}$.

A subset of Σ meeting the drawing only in vertices of G is called G-*normal*. If an O-arc is G-normal then we call it *noose*. The length of a noose is the number of its vertices. For a D-dominated Σ-embedded graph G we define a D-noose on G as a noose meeting exactly two vertices x, y of D, two neighbors of x and two neighbors of y. A D-noose N is *consecutive* is any two vertices of G that are met consecutively in N are adjacent.

Representativity [16] is the measure how dense a graph is embedded on a surface. The *representativity* (or *face-width*) $\mathbf{rep}(G)$ of a graph G embedded in surface $\Sigma \neq \mathbb{S}_0$ is the smallest length of a non-contractible noose in Σ. In other words, $\mathbf{rep}(G)$ is the smallest number k such that Σ contains a non-contractible (non null-homotopic in Σ) closed curve that intersects G in k points.

It is more convenient to work with Euler genus. The *Euler genus* $\mathbf{eg}(\Sigma)$ of a surface Σ is equal to the non-orientable genus $\tilde{g}(\Sigma)$ (or the crosscap number) if Σ is a non-orientable surface. If Σ is an orientable surface, $\mathbf{eg}(\Sigma)$ is $2g(\Sigma)$, where $g(\Sigma)$ is the orientable genus of Σ. Given a graph G its Euler genus $\mathbf{eg}(G)$ is the minimum $\mathbf{eg}(\Sigma)$ where Σ is a surface where G can be embedded.

Let N be a noose in a Σ-embedded graph G. We need to define *cutting along* the noose N. The formal definition can be found in [15], here we prefer to give a more intuitive one. We suppose that G is embedded in Σ such that for any $v \in N \cap V(G)$, there exists an open disk Δ containing v and such that for every edge e adjacent to v,$e \cap \Delta$ is connected. We also assume that $\Delta - N$ has two connected components Δ_1 and Δ_2. Thus we can define partition of $N(v) = N_1(v) \cup N_2(v)$, where $N_1(v) = \{u \in N(v) \colon \{u, v\} \cap \Delta_1 \neq \emptyset\}$ and $N_2(v) = \{u \in N(v) \colon \{u, v\} \cap \Delta_2 \neq \emptyset\}$. Now for each $v \in N \cap V(G)$ we do the following: (a) remove v and its incident edges (b) introduce two new vertices v^1, v^2 and (c) connect v^i with the vertices in $N_i, i = 1, 2$. The resulting graph is obtained from Σ-embedded graph G by cutting along N.

The following lemma is very useful in proofs by induction on the genus. The first part of the lemma follows from Proposition 4.2.1 (corresponding to surface separating cycle) and the second part follows from Lemma 4.2.4 (corresponding to non-separating cycle) in [15].

Lemma 1. *Let G be a Σ-embedded graph and let G' be a graph obtained from G by cutting along a non-contractible noose N on G. Then one of the following holds*
• *G' is the disjoint union of graphs G_1 and G_2 that can be embedded in surfaces Σ_1 and Σ_2 such that $\mathbf{eg}(\Sigma) = \mathbf{eg}(\Sigma_1) + \mathbf{eg}(\Sigma_2)$ and $\mathbf{eg}(\Sigma_i) > 0$, $i = 1, 2$.*
• *G' can be embedded in a surface with Euler genus strictly smaller than $\mathbf{eg}(\Sigma)$.*

Branch-width. A *branch decomposition* of a graph (or a hyper-graph) G is a pair (T, τ), where T is a tree with vertices of degree 1 or 3 and τ is a bijection from the set of leaves of T to $E(G)$. For a subset of edges $X \subseteq E(G)$ let $\delta_G(X)$ be the set of all vertices incident to edges in X and $E(G) \setminus X$. For each edge e of T, let $T_1(e)$ and $T_2(e)$ be the sets of leaves in two components of $T \setminus e$. The *order* of an edge e in T is $|\bigcup_{v \in T_1(e)} \delta_G(\tau(v))|$. In other words, the order of e is the number of vertices $v \in V(G)$ such that there are leaves t_1, t_2 in T in different components of $T(V(T), E(T) \setminus e)$ with $\tau(t_1)$ and $\tau(t_2)$ both containing v as an endpoint. The *width* of (T, τ) is the maximum order over all edges of T, and the *branch-width* of G, $\mathbf{bw}(G)$, is the minimum width over all branch decompositions of G.

The following relation was obtained in [11].

Theorem 1 ([11]). *For any planar D-dominated graph G, $\mathbf{bw}(G) \leq 3\sqrt{4.5}\sqrt{|D|}$.*

The following lemma that is based on Theorem 1 and the results of Djidjev & Venkatesan on planarizing sets in [8].

Lemma 2. *For any Σ-embedded graph G on n vertices, $\mathbf{bw}(G) \leq (\sqrt{4.5} + 2\sqrt{2 \cdot \mathbf{eg}(\Sigma)})\sqrt{n}$.*

3 Kernelization

Alber et al. [2] introduce reduction rules for the dominating set problem. Let us call these rules *AFN-reduction*. AFN-reduction can be applied to any graph G and the domination number of the reduced graph is equal to the domination number of G. As it was proved in [2], when G is planar, the reduced graph has at most $335\gamma(G)$ vertices, i.e. is a *linear* kernel. Also it was conjectured in [2] that AFN-reduction produces a kernel for graphs embedded in a surface of bounded genus. In this section we give an affirmative answer to this conjecture by proving that for any Σ-embedded graph G the size of the reduced graph is $O(\gamma(G) + \mathbf{eg}(\Sigma))$.

In fact, the rules of AFN-reduction are not important for our proofs. The only facts we need are the properties of the reduced graph proved in [2] and due to space restriction we move the description of the rules to Appendix. We also call a graph *reduced* if none of these rules can be applied to it.

The rules are based on a partition of the open neighbourhood of vertices or pair of vertices into three categories of sets.

For every vertex $v \in V(G)$ we partition $N(v)$ into:
- $N_{\text{exit}} = \{u \in N(v) \mid N(u) - N[v] \neq \emptyset\}$
- $N_{\text{guard}} = \{u \in N(v) - N_{\text{exit}} \mid N(u) \cap N_{\text{exit}}(v) \neq \emptyset\}$
- $N_{\text{prison}} = N(v) - (N_{\text{exit}}(v) \cup N_{\text{guard}}(v))$

For every pair v, w we partition $N(v, w) = N(v) \cup N(w)$ into:
- $N_{\text{exit}}(v, w) = \{u \in N(v, w) \mid N(u) - N[v, w] \neq \emptyset\}$
- $N_{\text{guard}}(v, w) = \{u \in N(v, w) - N_{\text{exit}}(v, w) \mid N(u) \cap N_{\text{exit}}(v, w) \neq \emptyset\}$
- $N_{\text{prison}}(v, w) = N(v, w) - (N_{\text{exit}}(v, w) \cup N_{\text{guard}}(v, w))$.

Lemma 3. *Let G be an n-vertex Σ-embedded graph. Then AFN-reduction can be performed in $O(n^2 \cdot \mathbf{eg}(\Sigma))$ steps.*

The main result of [2] it is the following:

Theorem 2. *For any reduced planar graph G, $|V(G)| \leq 335 \cdot \gamma(G)$.*

The next proof is a generalization of Theorem 2 for graphs embedded in arbitrary surfaces of representativity at least 6. Such graphs are "locally planar" in whatever the AFN-reduction is concerned. In particular, the machinery of the reduction and the proof of its correctness in [2] are applied on planar discs with boundary of length ≤ 6. This gives an opportunity to reproduce the arguments from [2] for Σ-embedded graphs of representativity at least 6.

Theorem 3. *Let G be a reduced Σ-embedded graph where $\mathbf{rep}(G) > 6$. Then $|V(G)| \leq 335 \cdot \gamma(G) + 333 \cdot \mathbf{eg}(\Sigma)$.*

Let G be a Σ-embedded graph. For a noose N in Σ we define the graph G_N as follows. First we take the graph G' obtained from G after cutting along N. Then for every $v \in N \cap V(G)$ if v^i, $i = 1, 2$, is not adjacent to a vertex u which is pendant in G, we add to G' a pendant vertex u^i adjacent to v^i. Thus in G_N each new vertex obtained from splitting of vertices $N \cap V(G)$ is adjacent to exactly one pendant vertices. Clearly, G_N has the same genus as G'. Since every dominating set D in G can be turned into dominating set of G_N by adding all new vertices to D, we have that $\gamma(G_N) \leq \gamma(G) + 2|N \cap V(G)|$.

According to [2], a graph G is reducible iff it satisfies the following properties:
(i) For every $v \in V(G)$, the set $N_{\text{prison}}(v)$ is empty with only one exception: $N_{\text{prison}}(v)$ can contain one "gadget" pendant vertex.
(ii) For all $v, w \in V(G)$ there exist a single vertex $v \in N_{\text{guard}}(v, w) \cup N_{\text{prison}}(v, w)$ where $N_{\text{prison}}(v, w) \subseteq N[v]$ (i.e. v dominates all vertices in $N_{\text{prison}}(v, w)$).

By construction, every vertex v^i, $i = 1, 2$, $v \in N \cap V(G)$, is not a prison vertex (it is adjacent to pendant vertex) and every vertex vertex has no more than one pendant neighbor. So we conclude that if G is reducible then G_N is also reducible.

Theorem 4. *For any reduced Σ-embedded graph G, $|V(G)| \leq 335(\gamma(G) + 24 \cdot \mathbf{eg}(\Sigma))$.*

Proof. If $\Sigma = \mathcal{S}_0$, the result follows from Theorem 2. Suppose then that $\mathbf{eg}(G) > 0$. We prove a stronger inequality: $|V(G)| \leq 335(\gamma(G) + 24\mathbf{eg}(\Sigma) - 12)$ by induction on $\mathbf{eg}(\Sigma)$. For $\mathbf{eg}(\Sigma) = 1$ and $\mathbf{rep}(G) > 6$ the result follows from

Theorem 3. For $\mathbf{eg}(\Sigma) = 1$ and $\mathbf{rep}(G) \leq 6$, Lemma 1 implies that the graph G' obtained from G by cutting along N is planar, and hence the graph G_N is also planar. By Theorem 2 $|V(G_N)| \leq 335 \cdot \gamma(G_N)$ and thus (the length of N is at most 6), $|V(G)| \leq |V(G_N)| \leq 335 \cdot \gamma(G_N) \leq 335 \cdot (\gamma(G) + 12)$.

Assume now that $|V(G)| \leq 335(\gamma(G) + 24 \cdot \mathbf{eg}(\Sigma) - 12)$ for any Σ-embedded graph where $1 \leq \mathbf{eg}(\Sigma) < g$ and let G be a Σ-embedded graph where $\mathbf{eg}(\Sigma) = g \geq 2$. Again by Theorem 3, it is enough to examine the case where $\mathbf{rep}(G) \leq 6$. Let N be a non-contractible noose of minimum length in Σ. Then the length of N is at most 6.

By Lemma 1, either G_N is the disjoint union of graphs G_1 and G_2 that can be embedded in surfaces Σ_1 and Σ_2 such that $\mathbf{eg}(\Sigma) = \mathbf{eg}(\Sigma_1) + \mathbf{eg}(\Sigma_2)$ and $\mathbf{eg}(\Sigma_i) > 0$, $i = 1, 2$ (this is the case when N is surface separating curve), or G_N can be embedded in a surface with Euler genus strictly smaller than $\mathbf{eg}(\Sigma)$ (this holds when N is not surface separating).

Let us consider first the case when G_N is the disjoint union of graphs G_1 and G_2 that can be embedded in surfaces Σ_1 and Σ_2. As we discussed above, G_N is a reduced graph and thus G_1 and G_2 are also reduced graphs. The conditions $\mathbf{eg}(\Sigma) = \mathbf{eg}(\Sigma_1) + \mathbf{eg}(\Sigma_2)$ and $\mathbf{eg}(\Sigma_i) > 0$, $i = 1, 2$, imply that $1 \leq \mathbf{eg}(\Sigma_i) \leq \mathbf{eg}(\Sigma) - 1 < g$. Therefore we can apply the induction hypothesis on G_i and get that $|V(G_i)| \leq 335(\gamma(G_i) + 24 \cdot \mathbf{eg}(\Sigma_i) - 12), i = 1, 2$. Thus $|V(G)| \leq |V(G_N)| = |V(G_1)| + |V(G_2) \leq 335(\gamma(G_1) + 24 \cdot \mathbf{eg}(\Sigma_1) - 12) + 335(\gamma(G_2) + 24 \cdot \mathbf{eg}(\Sigma_2) - 12) = 335(\gamma(G_1) + \gamma(G_2) + 24 \cdot \mathbf{eg}(\Sigma_1) + 24 \cdot \mathbf{eg}(\Sigma_2) - 24) = 335(\gamma(G') + 24 \cdot (\mathbf{eg}(\Sigma_1') + \mathbf{eg}(\Sigma_2')) - 24) \leq 335(\gamma(G) + 12 + 24 \cdot \mathbf{eg}(\Sigma) - 24) = 335(\gamma(G) + 24 \cdot \mathbf{eg}(\Sigma) - 12)$. For the second case, when G_N can be embedded in a surface Σ' with Euler genus strictly smaller than $\mathbf{eg}(\Sigma)$, we have that $1 \leq \mathbf{eg}(\Sigma') \leq \mathbf{eg}(\Sigma) - 1 < g$ and therefore we can apply the induction hypothesis on G_N. Thus $|V(G)| \leq |V(G_N)| \leq 335(\gamma(G_N) + 24 \cdot \mathbf{eg}(\Sigma') - 12) \leq 335(\gamma(G) + 12 + 24 \cdot (\mathbf{eg}(\Sigma) - 1) - 12) \leq 335(\gamma(G) + 24 \cdot \mathbf{eg}(\Sigma) - 24) < 335(\gamma(G) + 24 \cdot \mathbf{eg}(\Sigma) - 12)$ □

Lemma 3 and Theorem 4 imply the main result of this section.

Theorem 5. *Let G be a graph that can be embedded in Σ. AFN-reduction constructs in $O(n^3 \cdot \mathbf{eg}(\Sigma))$ steps a graph G' of size $\leq 335(\gamma(G) + 24 \cdot \mathbf{eg}(\Sigma))$ such that $\gamma(G) = \gamma(G')$.*

4 Direct consequences of the kernel construction

As far as we have kernel reduction we can improve the algorithms given in [5, 11, 12] for the dominating set problem. The key observation is that after the AFN-reduction, the size of the remaining kernel depends only on the genus and the minimum dominating set of the initial graph and, because of Lemma 2, the same will hold for its branchwidth as well.

Theorem 6. *For a given graph G and constants k, g, there is an $2^{O(\sqrt{kg}+g)}$ $\mathsf{poly}(k, g) + O(n^3)$ algorithm that either computes a dominating set in G of size $\leq k$, or concludes that at least one of the following holds: (a) $\gamma(G) > k$, (b) G can not be embedded in a surface of Euler genus g.*

Theorem 6 improves asymptotically the algorithm for dominating set in [5] that requires $2^{O(g\sqrt{k}+g^2)}n^{O(1)}$ steps. However, we should admit that the hidden constants in the big-O notation are quite big. Even using the smallest factor approximation algorithm of [3], for $k=1$ and $\mathbf{eg}(\Sigma)=1$ the algorithm requires more than 2^{200} steps, which makes this result interesting only from theoretical point of view. In the next section we explain how the combinatorial bound to the branchwidth of G' in step 3 can be improved. Such an improvement immediately accelerates steps 2 and 3 that dominate the exponential part of the running time of the algorithm.

5 Better combinatorial bounds – faster algorithms

We call a D-dominated graph G *uniquely dominated* if there is no path of length < 3 connecting two vertices of D. Notice that this implies that each vertex $x \in V(G) \setminus D$ has exactly one neighbor in D (i.e. is uniquely dominated). The proof of the following normalization lemma is omitted because of lack of space.

Lemma 4. *For every D-dominated Σ-embedded graph G without multiple edges, there exists a Σ-embedded graph H such that (a) G is a minor of H, (b) H is uniquely D-dominated, (c) If $x, y \in D$ have distance 3 in H then there exist at least two internally disjoint (x,y)-paths in H, and (d) Any D-noose of Σ is consecutive.*

Let G be a connected D-dominated Σ-embedded graph satisfying properties (b) – (d) of Lemma 4. We call such graphs *nicely* D-dominated Σ-embedded graphs.

Let G be a nicely D-dominated Σ-embedded graph. We say that a cycle of length 6 is a D-cycle if it contains exactly two vertices from D. If $\mathbf{rep}(G) > 6$, every D-cycle C is contractible and thus one of the components of $\Sigma \setminus C$ is homeomorphic to $\{(x,y)\colon x^2+y^2 \leq 1\}$. We denote such a disk by $\mathbf{disk}(C)$. Clearly, $G \cap \mathbf{disk}(C)$ is a planar graph.

A D-cycle C of a nicely D-dominated Σ-embedded graph G is *maximal* if there is no D-cycle of G where $\mathbf{disk}(C) \subset \mathbf{disk}(C')$. We denote as $\mathcal{C}(G)$ the set of all the maximal cycles of G.

For a nicely D-dominated Σ-embedded graph G and the set $\mathcal{C}(G)$ of all maximal D-cycles of G, we define hypergraph $\mathcal{H}(G) = (V(G), E(G) \cup \{V(C) \mid C \in \mathcal{C}(G)\})$, i.e. $\mathcal{H}(G)$ is obtained from G by adding hyperedges corresponding to maximal D-cycles of G. Clearly, $\mathbf{bw}(G) \leq \mathbf{bw}(\mathcal{H}(G))$.

If representativity of G is more than 6, for every D-maximal cycle C (which is of length 6), the hypergraph $\mathcal{H}(C) = \mathcal{H}(G) \cap \mathbf{disk}(C)$ is a hypergraph that can be obtained from a planar graph $H(C)$ by adding one hyperedge of cardinality 6. Since the planar graph H is D'-dominated for some $D' \subseteq D$, we have that by Theorem 1, $\mathbf{bw}(\mathcal{H}(C)) \leq 3\sqrt{4.5}\sqrt{|D|} + 6$.

We also define a hypergraph $\mathcal{S}(G)$ as the hypergraph obtained by removing from $\mathcal{H}(G)$ all edges of graphs $G \cap \mathbf{disk}(C)$, $C \in \mathcal{C}(G)$. Using properties (c) and

(d) one can prove that the hyperedges of $\mathcal{S}(G)$ are exactly the maximal D-cycles of G (all edges of G will be removed).

We need the following technical Lemma from [11]

Lemma 5. *If $\mathcal{G}_1$ and $\mathcal{G}_2$ are hypergraphs where $V(\mathcal{G}_1) \cap V(\mathcal{G}_1) = f$ and $\{f\} = E(\mathcal{G}_1) \cap E(\mathcal{G}_2)$, then $\mathbf{bw}(\mathcal{G}_1 \cup \mathcal{G}_2) \leq \max\{\mathbf{bw}(\mathcal{G}_1), \mathbf{bw}(\mathcal{G}_1), |f|\}$.*

For every $C \in \mathcal{C}(G)$ hypergraphs $\mathcal{H}(C)$ and $\mathcal{S}(G)$ have only hyperedge C in common and Theorem 1 and Lemma 5 imply the following result.

Lemma 6. *Let G be a nicely D-dominated Σ-embedded graph of representativity > 6. Then $\mathbf{bw}(G) \leq \mathbf{bw}(\mathcal{H}(G)) \leq \max\{3\sqrt{4.5}\sqrt{|D|} + 6, \mathbf{bw}(\mathcal{S}(G))\}$.*

Thus to obtain the upper bound for branch-width of nicely dominated graphs we need to estimate the branch-width of $\mathcal{S}(G)$.

Lemma 7. *Let G be a nicely D-dominated Σ-embedded graph of representativity > 6. Then $\mathbf{bw}(\mathcal{S}(G)) \leq 3(\sqrt{4.5} + 2\sqrt{2 \cdot \mathbf{eg}(\Sigma)})\sqrt{|D|}$.*

Proof (Sketch). Let us show first that for any two distinct maximal cycles $C_1, C_2 \in \mathcal{C}(G)$ **(i)**: *For each $u \in V(C_1) \cap V(C_2)$, $u \in N_G[v]$ for some $v \in D$.*

In other words, for any two distinct maximal cycles $C_1, C_2 \in \mathcal{C}(G)$ the set $C_1 \cap C_2$ is either empty, or a vertex of D, or a set of vertices adjacent to one vertex of D. In fact, if $(V(C_1) \cap V(C_2)) \cap D = \emptyset$ then every vertex $u \in V(C_1) \cap V(C_2)$ is not uniquely dominated. If $|V((C_1) \cap V(C_2)) \cap D| = 2$ then cycles are not maximal. If $(V(C_1) \cap V(C_2)) \cap D = v$, we again have that every vertex $u \in (V(C_1) \cap V(C_2)) \setminus N_G[v]$ is not uniquely dominated. In all three cases we obtain a contradiction either to the definition of maximal cycle, or to the property (b) of nicely D-dominated graphs.

To estimate the value of $\mathbf{bw}(\mathcal{S}(G))$ we need the following notion. Let D' be the set of vertices of D that are also vertices of some maximal cycles, i.e. $D' = D \cap \bigcup_{C \in \mathcal{C}(G)} V(C)$. For a nicely D-dominated Σ-embedded graph G and the set of its maximal D-cycles $\mathcal{C}$ we define *concise* graph, $\mathbf{con}(G)$, as the graph with vertex set D' and where two vertices $x, y \in D'$ are adjacent in $\mathbf{con}(G)$ if and only if the distance x and y in G is 3. There is a natural bijection π correspondence between hyperedges of $\mathcal{S}(G)$ and $\mathbf{con}(G)$. Every cycle $C \in \mathcal{C}(G)$ (which is edge in $\mathcal{S}(G)$) π maps to an edge of $\mathbf{con}(G)$ with endpoints $D \cap V(C)$. By property (c) of nicely dominated graphs, π is surjection. Because cycles in $\mathcal{C}$ are maximal, π is injection.

By making use of **(i)** one can prove that $\mathbf{con}(G)$ is also Σ-embedded graph. Then by Lemma 2, $\mathbf{bw}(\mathbf{con}(G)) \leq (\sqrt{4.5}+2\sqrt{2 \cdot \mathbf{eg}(\Sigma)})\sqrt{|D|}$ **(ii)** which implies the lemma if $\mathbf{bw}(\mathcal{S}(\mathcal{C})) \leq 3 \cdot \mathbf{bw}(\mathbf{con}(G))$ **(iii)**.

Let us prove **(iii)** first for the case when the maximum vertex degree in $\mathbf{con}(G)$ is at most 3. Let A, B be a partition of $\mathcal{C}(G)$. We claim that $|\delta_{\mathcal{S}(\mathcal{C})}(A)| \leq 3|\delta_{\mathbf{con}(G)}(\pi(A))|$ **(iv)**. Let $v \in D$. By **(i)**, every $u \in N_G(v)$ is contained in at most two hyperedges of $\mathcal{S}(G)$ and both these edges contain v. Also for every vertex $u \in N_G(v)$, $u \in \delta_{\mathcal{S}(G)}(A)$ if and only if $u \in V(C_1) \cap V(C_2)$ for some $C_1 \in A$ and $C_2 \in B$. The degree of v in $\mathbf{con}(G)$ is ≤ 3. Thus v is contained in at most three maximal cycles and therefore at most two neighbors of v in G can be in $\delta_{\mathcal{S}(G)}(A)$. Hence For each $v \in \delta_{\mathbf{con}(G)}(\pi(A))$, $|N_G[v] \cap \delta_{\mathcal{S}(G)}(A)| \leq 3$ **(v)**. Now **(iv)** follows from **(i)** and **(v)**. Finally, **(iv)** implies **(iii)** when the maximum vertex degree of $\mathbf{con}(G)$ is at most 3.

To prove **(iii)** in general case we need the following deep result following from Theorem (4.3) of [17] and (6.6) of [18]: *for any Σ-embedded graph G of branch-width ≥ 2, the branch-width of G is equal to the branch-width of its dual.*

A Σ-embedded graph G is *multiply triangulated* if all its regions are of length 2 or 3. A graph is $(2,3)$-regular if all its vertices have degree 2 or 3. Notice that the dual of a multiply triangulated graph is $(2,3)$-regular and vice versa. The proof of the following claim is similar to the proof for planar graphs (Lemma 3.3 in [11]) and we omit it here. *Every 2-connected Σ-embedded graph G has a weak triangulation H such that* $\mathbf{bw}(H) = \mathbf{bw}(G)$.

We claim now that *every 2-connected Σ-embedded graph G is the contraction of a (2,3)-regular Σ-embedded graph H such that* $\mathbf{bw}(H) = \mathbf{bw}(G)$. In fact, let G^d be the dual graph of G. By Robertson & Seymour theorem, $\mathbf{bw}(G^d) = \mathbf{bw}(G)$. There is a weak triangulation H^d of G^d such that $\mathbf{bw}(H^d) = \mathbf{bw}(G^d)$. The dual of H^d, we denote it by H, contains G as a contraction (each edge removal in a Σ-embedded graph corresponds to an edge contraction in its dual and vice versa). Applying Robertson & Seymour the second time, we obtain that $\mathbf{bw}(H) = \mathbf{bw}(H^d)$. Hence, $\mathbf{bw}(H) = \mathbf{bw}(G)$. Since H^d is multiply triangulated, we have that H is $(2,3)$-regular.

Suppose that now that $\mathbf{con}(G)$ is 2-connected. For $\mathbf{con}(G)$ we construct (2,3)-regular Σ-embedded graph H such that $\mathbf{con}(G)$ is the contraction of H and $\mathbf{bw}(H) = \mathbf{bw}(\mathbf{con}(G))$. Then one can construct a hypergraph $\mathbf{ext}(H)$ such that $\mathbf{bw}(\mathcal{S}(G)) \leq \mathbf{bw}(\mathbf{ext}(H))$ and H is the concise graph of $\mathbf{ext}(H)$. Such a construction is similar to the case of planar graphs (see [11]) and we omit it here. Since **(iii)** is already proved for concise graphs of degree ≤ 3, we have that $\mathbf{bw}(\mathcal{S}(G)) \leq \mathbf{bw}(\mathbf{ext}(H)) \leq 3 \cdot \mathbf{bw}(H) = 3 \cdot \mathbf{bw}(\mathbf{con}(G))$ and **(iii)** follows.

So we proved that **(iii)** holds when $\mathbf{con}(G)$ is 2-connected. To finish the proof we use induction on the number of 2-connected components of $\mathbf{con}(G)$. □

Theorem 7. *For any Σ-embedded graph G,*
$\mathbf{bw}(G) \leq 3(\sqrt{4.5} + 2\sqrt{2 \cdot \mathbf{eg}(\Sigma)})\sqrt{\gamma(G) + 6 \cdot \mathbf{eg}(G)}$.

Proof. We use induction on the Euler genus of Σ. For $\Sigma = \mathbb{S}_0$ the result follows from Theorem 1. Suppose that the theorem is correct for all graphs that can be embedded in surfaces of Euler genus $< g$ for some $g > 0$. Let G be a D-dominated Σ-embedded graph where $\mathbf{eg}(\Sigma) = g$. If representativity of G is more than 6, By Lemma 4, there is a nicely D-dominated graph H such that G is a minor of H. Thus $\mathbf{bw}(G) \leq \mathbf{bw}(H)$ and by Lemmata 6 and 7, $\mathbf{bw}(G) \leq \mathbf{bw}(H) \leq 3(\sqrt{4.5} + 2\sqrt{2 \cdot \mathbf{eg}(\Sigma)})\sqrt{|D|}$.

If representativity of G is ≤ 6, let G' be the graph obtained from G by cutting along a non-contractible noose N of length ≤ 6. Let $G_1, \ldots, G_q$ be the connected components of G'. Clearly, each of the components G_i has a dominating set of size at most $|D| + 6$. By Lemma 5, $\mathbf{bw}(G) \leq \max_{1 \leq i \leq q} \mathbf{bw}(G_i) + 6$

and by Lemma 1, every component G_i of G' can be embedded in a surface Σ_i of Euler genus $\leq g-1$. Thus $\mathbf{bw}(G) \leq \max_{1 \leq i \leq q} \mathbf{bw}(G_i) + 6 \leq 3(\sqrt{4.5} + 2\sqrt{2 \cdot (g-1)})\sqrt{|D| + 6 + 6 \cdot (g-1)} + 6 \leq 3(\sqrt{4.5} + 2\sqrt{2 \cdot g})\sqrt{|D| + 6 \cdot g}$. $\square$

A simplification of the formula in Theorem 7 gives that any graph with dominating set $\leq k$ and Euler genus $\leq g$ has branchwidth at most $(7+9\sqrt{g})\sqrt{k+6g}$. Applying Theorem 7 to the reduced graph G' in the second step of the algorithm of Theorem 6 we have that $\mathbf{bw}(G') \leq (7+9\sqrt{g})\sqrt{k+6g}$. Therefore, it is enough to apply Amir's algorithm for $\omega = \frac{3}{2}(7+9\sqrt{g})\sqrt{k+6g}$ and get a tree decomposition of width $\leq (3+\frac{2}{3})\frac{3}{2}(7+9\sqrt{g})\sqrt{k+6g} = 5.5 \cdot (7+9\sqrt{g})\sqrt{k+6g}$. This improves significantly the constants of the exponential part in the time of the algorithm in Theorem 6. As we will see in the next section, Theorem 7 has consequences to the design of subexponential parameterized algorithms for more parameters.

6 Generalizations

The combinatorial and algorithmic results of the previous two sections can be generalized to a general family of parameters. Due to lack of space we just mention the results and leave the proofs for the full version. We describe a general class of parameterized problems C including minimum vertex cover, the minimum edge dominating set, the minimum clique transversal set, the minimum vertex feedback set, the minimum maximal matching, variations of domination like minimum independent dominating set, the total minimum dominating set, the minimum perfect dominating set, the minimum perfect code, the minimum weighted dominating set, and the minimum total perfect dominating set, and prove that for any graph G every problem in P can be solved in $2^{O(\sqrt{k \cdot \mathbf{eg}(G)} + \mathbf{eg}(G))} n^{O(1)}$ steps. This implies that for $\mathbf{eg}(G) = o(\log n)$ all these problems can be solved in subexponential parameterized time (i.e. in $2^{o(k)} n^{O(1)}$-time) and for $\mathbf{eg}(\Sigma) = o(n)$ all these problems can be computed in subexponential time (i.e. in $2^{o(n)}$-time).

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