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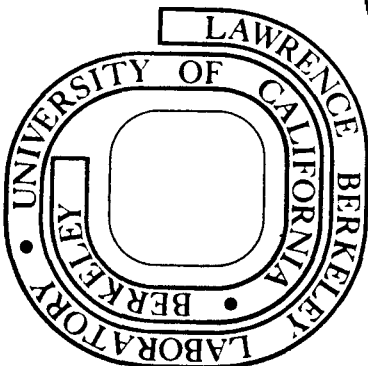
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## FAST POISSON SOLVERS ON GENERAL TWO DIMENSIONAL REGIONS

## FOR THE DIRICHLET PROBLEM

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## ABSTRACT

It is shown that by using the simplest construction of discrete dipoles, the operation count for solving the Dirichlet problem of Poisson's equation by the capacitance matrix method does not exceed constant times  $n^2 \log n$ ,  $n = 1/h$  for certain first and second order schemes of interpolating boundary conditions.

## SIGNIFICANCE AND EXPLANATION

The Dirichlet problem for the Poisson equation is the following: Given a function  $f$  and a function  $g$ , find a function  $u$  such that

$$\begin{aligned} u_{xx} + u_{yy} &= f \text{ on } \Omega, \\ u &= g \text{ on } \partial\Omega. \end{aligned}$$

Here  $\Omega$  is a simply connected domain with boundary  $\partial\Omega$ .

The problem has wide applications in electrostatics, elasticity, temperature distributions and plasma physics. Its solution by finite difference or finite elements methods have received considerable attention. It is known that if  $\Omega$  is a rectangle, then fast Fourier transform methods are very efficient in solving the linear system of equations arising from finite difference or finite element discretizations.

There seems to be no such short cut to the solution of these equations when  $\Omega$  is a general region. In many conventional methods, the operation count is usually proportional to  $N^{3/2}$  ( $N$  is the number of mesh points in  $\Omega$ ) while at least  $N$  computer storage is required. These methods are therefore undesirable when  $N$  is very large. In this paper we describe an algorithm and prove mathematically that the operation count of this algorithm can be proportional to  $N \log N$ . While some versions of our algorithm also require at least  $N$  computer storage, there is one version that requires less than  $N/3$  computer storage.

FAST POISSON SOLVERS ON GENERAL TWO DIMENSIONAL REGIONS  
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A. S. L. Shieh

§1. Introduction

Over the past ten years, very fast numerical methods have been developed to solve Poisson's or Helmholtz's equation on certain simple regions with Dirichlet, Neumann or periodic boundary conditions. See e.g. [2], [3], [8], [9], [12], [19] and [21]. These methods can only be used for regions and boundary conditions that allow for separation of the variables. Typical examples are Poisson's or Helmholtz's equations in Cartesian coordinates on rectangular regions with boundary conditions that do not change type along any of the sides of the rectangle. In these special cases, the operation count for solving the discrete problem is almost proportional to the number of mesh points.

The purpose of this paper is to establish similar results for the Poisson equation on general regions. In this work we are only concerned with finite difference schemes of first and second order accuracy for the Dirichlet problem on simply connected bounded domains with smooth boundaries. A formal discrete potential theory motivated by the classical potential theory is incorporated into the so-called capacitance matrix method. It is shown that by using the simplest construction of discrete dipoles in our Ansatz, it is possible to have an algorithm the operation count of which is proportional to  $N^2 \log N$ , where  $h = 1/N$  is the mesh size. Some numerical results are given in section 9 and a brief survey of past work in this direction is given in section 8.

§2. Certain results from classical potential theory.

We give only a very brief review of a few results of classical potential theory. For a detailed exposition see e.g. [10], [14], [25] and [28]. We define the potential  $V$  resulting from a charge distribution  $\rho$  on a smooth boundary curve  $\partial\Omega$  by

$$V(x) = (1/\pi) \int_{\partial\Omega} \rho(\xi) \log \pi \, ds(\xi) .$$

Here  $x = (x_1, x_2)$ ,  $\xi = (\xi_1, \xi_2)$  and  $\pi^2 = (x_1 - \xi_1)^2 + (x_2 - \xi_2)^2$ . The Green's function  $(1/2\pi) \log \pi$  which we shall denote by  $G^*$  satisfies

$$\Delta(1/2\pi) \log r = \delta(x) ,$$

where  $\delta(x)$  is the delta function. Similarly the potential  $W$  of a dipole density  $\mu$  on  $\partial\Omega$  defined by

$$(2.1) \quad W(x) = (1/\pi) \int_{\partial\Omega} \mu(\xi) \partial G^* / \partial v_\xi \, ds(\xi) .$$

We adopt here the convention that the normal direction of  $\partial\Omega$  is towards the exterior of the region  $\Omega$  in which we want to solve our problem.

The interior Dirichlet problem can be reduced to a Fredholm integral equation of the second kind if we make the double layer Ansatz as follows. Let

$$u(x) = -(1/2\pi) \iint_{\Omega} f(\xi) \log r \, d\xi + (1/\pi) \int_{\partial\Omega} \mu(\xi) \partial G^* / \partial v_\xi \, ds(\xi) = u_s(x) + W(x) ,$$

for the solution of

$$(2.2) \quad \begin{aligned} -\Delta u &= f, \quad x \in \Omega \\ u &= g, \quad x \in \partial\Omega . \end{aligned}$$

It can be shown that the dipole density  $\mu$  satisfy the following integral equation

$$(2.3) \quad \mu + (1/\pi) \int_{\partial\Omega} \mu (\partial G^* / \partial v_\xi) ds = g - u_s \Big|_{\partial\Omega} = \tilde{g} .$$

This is a well posed problem of the form

$$(2.4) \quad (I + K)\mu = \tilde{g} ,$$

where  $K$  is a compact operator defined by the integral above.

If we instead attempt to use a single layer Ansatz for the Dirichlet problem we obtain a Fredholm integral equation of the first kind. It has the form

$$V(x) = g - u_s \Big|_{\partial\Omega}, \quad x \in \partial\Omega ,$$

which is an ill posed problem.

To illustrate the distribution of the eigenvalues of the compact operator  $K$  in equation (2.4), we study the case when  $\Omega$  is an ellipse with

$\gamma = (a-b)/(a+b)$  where  $a$  and  $b$  are the half axes of the ellipse. It is known (see e.g. p. 135 of [25]) that  $K = K^T$  and

$$(2.5) \quad \lambda_i(K+K^T) = 2\gamma^{i-1}, \quad i = 1, 2, \dots$$

On the other hand, both the interior and exterior Neumann problems can also be reduced to Fredholm integralequations of the second kind if we make the single layer Ansatz. The charge density  $\rho$  for the exterior Neumann problem satisfies

$$(I + K^T)\rho = g^*,$$

for some suitably chosen function  $g^*$  defined on  $\partial\Omega$ . The existence and uniqueness problems for the solution of equation (2.3) can therefore be determined from that of equation (2.4) and vice versa. Finally, we remark that the  $G^*$  in equations (2.1) and (2.3) can be replaced by the Green's function on a sufficiently large rectangle with zero Dirichlet boundary conditions.



### §3. The capacitance matrix method

In this section we develop a similar, formal potential theory for the discrete problems arising from the original Dirichlet problem (2.2). See also Sections 3 and 4 of [29] for a similar discussion. We shall assume that uniform mesh sizes in both coordinate directions are used.

We replace the Laplace operator by the five-point formula. The fundamental solution  $(1/2\pi) \log(1/r)$ , used in Section 2, will be replaced by its discrete analogue, the discrete Green's function on the entire plane, which we shall denote by  $G$ . Properties and efficient methods of generating  $G$  and its undivided differences will be studied in Section 4. An efficient method of computing  $Gv$  for arbitrary  $N \times N$  vectors  $v$  is also given in Section 4. We will denote by  $B$  the matrix representing the five-point discrete Laplacian  $h^2 \Delta_h$ , using undivided differences, on the entire plane. We then divide the set of mesh points into three disjoint sets  $\Omega_h, \partial\Omega_h$  and  $(C\Omega)_h$ . The set  $\partial\Omega_h$  contains all the irregular mesh points in  $\Omega$ , i.e. mesh points that do not have all four neighbours within the open set  $\Omega$ .  $\Omega_h$  is the set of regular mesh points inside  $\Omega$  and  $(C\Omega)_h$  contains the remaining, the exterior mesh points.

We then set up the matrix equation

$$(3.1) \quad Au = v$$

that we are solving as follows. We use the same discretization formula for both  $A$  and  $B$  on  $\Omega_h \cup (C\Omega)_h$ . For points in  $\partial\Omega_h$ , a linear combination of the discrete Laplacian and interpolation formulas of first or second order accuracy for the boundary conditions are used. The values of the solution at the exterior mesh points are always eliminated from the discrete Laplacian, centered at an irregular mesh point. This guarantees that  $A$  is a reducible matrix with no couplings to the exterior mesh points from the irregular mesh points. If  $P$  is a suitably chosen permutation matrix, then

$$P A P^T = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

where  $A_{11}$  is the coefficient matrix for our discrete problem on  $\Omega_h \cup \partial\Omega_h$ . It is easily seen that the solution on  $\Omega_h \cup \partial\Omega_h$  will not be influenced by either the solution or the data on  $(C\Omega)_h$ .

The matrix  $A$  differs from  $B$  by only  $m$  rows where  $m$  is the cardinal number of  $\partial\Omega_h$ . We can therefore write

$$A = B + U W^T,$$

where the matrices  $U$  and  $W$  have  $m$  columns. The matrix  $U$  represents an extension operator. It maps any mesh function defined only on  $\partial\Omega_h$  into a function on all mesh points. Its transpose,  $U^T$ , is a trace operator mapping any mesh function defined for all mesh points into its restriction to  $\partial\Omega_h$ . We easily verify that

$$W^T = U^T(A-B).$$

We now describe our method for solving the discrete problem (3.1). Guided by the continuous analog we make the Ansatz,

$$(3.2) \quad u = Gv + GV\mu.$$

The vector  $Gv$  satisfies  $BGV = v$ . The  $m$ -vector  $\mu$  is determined by solving a system of  $m \times m$  linear equations derived below. The mesh function  $V\mu$  should vanish on  $\Omega_h$ . Each column of the matrix  $V$  represents a discrete dipole of unit strength. Let such a column corresponding to  $P \in \partial\Omega_h$  be regarded as a mesh function, denoted by  $V^T(P)$ . We require that  $(1/h_\alpha)V^T(P)u = [\partial u/\partial\nu](P) + O(h)$ . Here  $h_\alpha = h/\cos\alpha$  where  $\alpha$  is the angle between the normal through the irregular mesh point  $P$  and the closest coordinate axis. In particular, if that the western and northwestern neighbours of  $P$  in  $(C\Omega)_h$ , then

$$(3.3) \quad [V^T u](P) = u(P) - (1 - \tan\alpha)u(w) - (\tan\alpha)u(NW).$$

We now use our Ansatz and compute the residual vector,

$$(3.4) \quad \begin{aligned} Au - v &= (B + UW^T)(Gv + GV\mu) - Fv \\ &= (V + UW^T GV)\mu + UW^T GFv. \end{aligned}$$

From the properties of  $U$  and  $V$ , it follows that the residuals are zero for all  $x \in \Omega_h$ . To derive a linear system of equations for the vector  $\mu$  we multiply equation (3.4) by  $U^T$ . It is easy to verify that  $U^T U = I_m$  and  $U^T V = I_m$ . Here  $I_m$  is the  $m \times m$  identity matrix. We thus obtain

$$(3.5) \quad (I_m + W^T GV)\mu = -W^T GFv.$$

This choice of  $\mu$  make the residuals zero for all  $x \in \partial\Omega_h$ . Hence substitution of  $\mu$  in equation (3.2) will provide us a solution on  $\Omega_h \cup \partial\Omega_h$  if equation (3.5) is solvable. Note that the residuals will in general not be equal to zero for all  $x \in (C\Omega)_h$ . The matrix on

the left-hand side of Equation (3.5) is the capacitance matrix  $C$ . We shall refer to Equation (3.5) as the capacitance matrix equation.

In the special case when  $v = U U^T v$ , we can simply make the Ansatz  $u = B^{-1} V \mu$ . It is easily seen that the residual  $Au - v$  will again be zero at  $x \in \Omega_h$ . The capacitance matrix equation now becomes

$$(3.6) \quad C\mu = U^T v.$$

If Equation (3.6) is solvable, then  $Au = v$  will also be zero on  $\partial\Omega_h$ . The solvability of Equations (3.5) and (3.6) will be discussed in Section 6.

We now describe our choices of difference equations at the irregular mesh points. We approximate the boundary conditions by interpolation schemes of first or second order accuracy, which we shall refer to as schemes Ia, Ib and II respectively.

We start with Scheme II. Let  $P \in \partial\Omega_h$  and  $P^*$  be its closest point on  $\partial\Omega$ . Let  $W, E, N$  and  $S$  be the western, eastern, northern and southern neighbours of  $P$  on the mesh respectively. We assume that the local orientation of the boundary is such that either both  $W$  and  $N$  are in  $(C\Omega)_h$  or only  $W$  is in  $(C\Omega)_h$ . Assume that both  $W$  and  $N$  are in  $(C\Omega)_h$ . Let  $d_1$  denote  $h_1/h$  where  $h_1$  is the distance, along a mesh line parallel with the  $x_1$ -axis, between the mesh point  $P$  and the boundary  $\partial\Omega$ . Hence  $d_1 \in (0, 1]$ . The Dirichlet data at this point on  $\partial\Omega$  is denoted by  $u_W$ . The values of  $d_2$  and  $u_N$  are similarly defined. We then approximate  $u_W$  and  $u_N$  by  $(1/2)[(1+d_1)u(W) + (1-d_1)u(E)]$  and  $(1/2)[(1+d_2)u(N) + (1-d_2)u(S)]$  respectively. By combining the above with the five-point formula for the Laplacian and eliminating  $u(W)$  and  $u(N)$  between them, we obtain

$$(3.6) \quad \begin{aligned} 4u(P) - [2d_1/(1+d_1)]u(E) - [2d_2/(1+d_2)]u(S) \\ = h^2 f(P) + [2/(1+d_1)]u_W + [2/(1+d_2)]u_N. \end{aligned}$$

If only  $W$  is in  $(C\Omega)_h$ , then we obtain

$$(3.7) \quad \begin{aligned} 4u(P) - [2d_1/(1+d_1)]u(E) - u(N) - u(S) \\ = h^2 f(P) + [2/(1+d_1)]u_W. \end{aligned}$$

We now describe the two variants of Scheme I, namely Scheme Ia and Ib. In Scheme Ib, if both  $W$  and  $N$  are in  $(C\Omega)_h$ , we obtain

$$(3.8) \quad 4u(P) - u(S) - u(E) = h^2 f(P) + u_W + u_N.$$

If only  $W$  is in  $(\Omega)_h$ , we obtain

$$(3.9) \quad 2[4u(P) - u(S) - u(E) - u(N)] = 2[h^2 f(P) + u_W]$$

The scaling factor 2 is largely artificial and is put in only for the convenience of theoretical estimates in Sections 5 and 6. In Scheme Ia, we seek to eliminate this scaling factor while retaining the theoretical convenience. We require that Equation (3.8) should be used regardless of whether both  $W$  and  $N$  are in  $(\Omega)_h$  or only  $W$  is in  $(\Omega)_h$ . The matrices  $A_{11}$  for all the above three schemes are of positive type. Hence, the results in [4] or [13] apply and all these schemes are convergent.

There is an important alternative to the above approach. Instead of the discrete Green's function of the entire plane, we may use the discrete analog of the Green's function on a sufficiently large square  $S$  with zero boundary conditions as our  $G$  in equations (3.2) and (3.5). In this case  $G = B_D^{-1}$ ;  $A = B_D + UW^T$ . Here  $B_D$  denotes the matrix representing the discrete Laplacian  $h^2 \Delta_h$  on  $S$  and zero boundary values on the grid points of  $\partial S$ . The residual  $Au - v$  will again be zero on  $\Omega_h \cup \partial\Omega_h$  if  $C = U^T A B_D^{-1} V$  is nonsingular.

Finally we come to the central question as to whether the capacitance matrix equation (3.5) is closely related to the Fredholm integral equation (2.2)? It is known (see e.g. [16]) that the conjugate gradient method converges superlinearly for Fredholm integral equations of the second kind. In our experiments we normally fail to observe superlinear convergence. To understand this fully, we split up the matrices  $C$  into two parts as follows.

$$C = B_h + K_h.$$

The matrices  $B_k$  are defined by

$$(3.10) \quad \begin{aligned} B_h(P, Q) &\equiv C(P, Q), & \text{if } d(P, Q) \leq \sqrt{h} \\ &\equiv 0, & \text{otherwise.} \end{aligned}$$

They are therefore the near diagonal parts of  $C$ ; and the matrices  $K_h$  are the remaining parts, the off diagonal parts of  $C$ .

It will be shown in Section 6 that for Schemes Ia and Ib, and, after a suitable scaling, for schemes II, the matrices  $K_h$  are closely related to the compact integral operator  $K$  in Equations (2.2) or (2.3). The matrices  $B_h$ , however, will not in general be formal approximations to the identity operator. In fact, the algebraic row sums of  $B_h$  need not always be equal to one.

It is, however, shown in [16] that for operator equations with symmetric positive definite operators of the form  $B + K$  with  $B$  symmetric positive definite and  $K$  symmetric compact, the conjugate gradient method will converge linearly with asymptotic rate of convergence governed only by the spectral condition number of  $B$ . We therefore proceed to study the special condition numbers of  $B_h$  in Section 5 and the distribution of singular values of  $K_h$  and  $C$  in Section 6. We shall show in Section 7 that the asymptotic convergence of the conjugate gradient method for solving the capacitance matrix equations will depend essentially on the spectral condition number of  $B_h$ .

We now discuss briefly two different methods of implementing our algorithm and the operation count involved. We use the conjugate gradient method to solve

$$C^T C u = C^T b,$$

where  $b$  denotes the right hand side of the capacitance matrix equation. The solution  $u$  is then computed from (3.2). In the first method, we generate the capacitance matrix explicitly. Assume that the  $G$  in equations (3.2) and (3.5) is the discrete Green's function on the entire plane. Because of translational invariance it is only necessary to compute  $G$  with the second parameter fixed at the origin. It is shown in section 4 that only one call of fast Poisson solved on a sufficiently large rectangle is needed to generate  $G$  and only two calls of a similar solver is needed to compute the final solution and the right hand side  $b$ . The operation count of the algorithm is therefore constant  $N^2 \log N + 2 C_0 m^2 + O(m)$ , where  $C_0$  is the number of iterations needed to achieve a certain accuracy. If the  $G$  in equation (3.5) is  $B_D^{-1}$ , it is desirable to use the second method where the solution  $u$  is computed by an iterative implicit method first appearing in [15]. The operation count for computing  $u$  is proportional to  $C_0(m+m_1)N$ , where  $m_1$  is the number of nonzero entries in the matrix  $V$  provided that a special fast solver is used in the process. See section 4 of [31] for details. It will be shown in section 7 that  $C_0$  cannot exceed constant  $\log m$  if  $G = B_D^{-1}$  is used in (3.5) for all domains with sufficiently smooth boundaries and that  $C_0$  is uniformly bounded in some special cases if the discrete Green's function on the entire plane is used in (3.5). The total operation count of our algorithm therefore does not exceed constant  $N^2 \log N$ .

54. Properties and fast generation of  $G$ , the discrete Green's function on the entire plane, and the efficient computations of  $Gv$ ,  $GV\mu$  and  $W^T Gv$ .

A discrete fundamental solution of the five-point Laplacian  $\Delta_h$  with respect to the origin is a mesh function  $\gamma$  that satisfies

$$(4.1) \quad \Delta_h \gamma(\mu h) = \begin{cases} h^{-2} & \text{if } \mu = 0, \\ 0 & \text{if } \mu \neq 0, \end{cases}$$

where  $\mu$  has integer components  $\mu_1$  and  $\mu_2$ .

Clearly  $\gamma$  is unique up to an arbitrary linear function. The constants involved will be chosen so that we have a proper discrete analog of the logarithmic potential. The resulting discrete fundamental solution will then be our discrete Green's function  $G$ .

It is established in [27] that if  $g(r,s)$  denotes

$$(4.2) \quad (2/\pi) \int_0^\pi [1 - \cos(s\lambda) \exp(-|r|\mu)] / \sinh \mu \, d\lambda$$

where

$$(4.3) \quad \cos \lambda + \cosh \mu = 2$$

$$(4.4) \quad \mu/\lambda \rightarrow 1 \text{ as } \mu \rightarrow 0,$$

then the function  $G$  defined by

$$(4.5) \quad G(rk, sh) = (1/4) g(r,s) + (1/2\pi) \log h - (1/4\pi) (\log 8 + 2\gamma_1)$$

is the desired Green's function of the entire plane. Here  $\gamma_1$  is the Euler's constant;  $r$  and  $s$  integers and  $h$  is the mesh size.

It is shown in [27] that

$$(4.6) \quad g(0,0) = 0, \quad g(0,1) = 1$$

$$(4.7) \quad g(r,s) = g(s,r) = g(-s,r) = g(s,-r),$$

$$(4.8) \quad g(r,r) = (4/\pi) [1 + 1/3 + \dots + 1/(2r-1)],$$

$$(4.9) \quad g(r,s) = (1/\pi) \log(s^2 + r^2) + (1/\pi) (\log 8 + 2\gamma_1) + o(1/r),$$

$$(4.10) \quad g(r,s) - g(r,t) = (1/\pi) \log[(r^2 + s^2)/(r^2 + t^2)] + o(1/r).$$

We have found it necessary to obtain sharper estimates for the remainder term in (4.10) when  $t = s+1$  and a similar estimate for  $g(r+1,s) - g(r,s)$ .

Theorem 4.1. Let  $r, s$  and  $t$  be nonnegative integers with  $r \geq 8$ ;  $s = t-1$ . Then

$$(4.11a) \quad G(rh, sh) - G(rh, th) = (1/4\pi) \log[(s^2 + r^2)/(t^2 + r^2)] + R_1(r,s) + R_2(r,s) + R_3(r,s),$$

where

$$(4.11b) \quad R_1(r,s) \equiv (1/24\pi) \int_0^\infty (\lambda^2 - r\lambda^3) \sin[(s + 1/2)\lambda]$$

$$(4.11c) \quad R_2(r,s) \equiv -(1/2\pi) \int_0^\infty \{[(1/30)\lambda^4 + (7/96)r\lambda^5 + (1/288)r^2\lambda^6]\} \cdot e^{-r\lambda} \sin[(2s+1)\lambda/2] d\lambda,$$

$$(4.11d) \quad |R_3(r,s)| \leq (0.37)r^{-1}e^{-1.35r^{2/3}} + (167)r^{-7}.$$

Proof. We have

$$(\pi/2)[g(r,s) - g(r,t)] = \int_0^\pi [\cos(t\lambda) - \cos(s\lambda)] e^{-r\lambda} [\sinh\mu]^{-1} d\lambda,$$

and it is known that

$$\int_0^\infty [\cos(t\lambda) - \cos(s\lambda)] e^{-r\lambda} \lambda^{-1} d\lambda = (1/2) \log[(s^2 + r^2)/(t^2 + r^2)].$$

Since

$$(4.12) \quad \int_\pi^\infty |\cos(t\lambda) - \cos(s\lambda)| e^{-r\lambda} \lambda^{-1} d\lambda < (2/\pi)r^{-1}e^{-r\pi},$$

it suffices to estimate

$$\int_0^\pi [\cos(s\lambda) - \cos(t\lambda)] [e^{-r\lambda} \lambda^{-1} - e^{-r\mu} (\sinh\mu)^{-1}] d\lambda.$$

The integrand in the above expression will be denoted by  $J$ . Let  $\epsilon = (1.5)r^{-1/3}$ ,  $r \geq 8$ . We have

$$(4.13) \quad \int_0^\epsilon J d\lambda = \int_0^\epsilon [\cos(s\lambda) - \cos(t\lambda)] e^{-r\lambda} [2 \sin(\lambda/2)]^{-1} F(\lambda) d\lambda,$$

where

$$(4.14) \quad F(\lambda) \equiv 2[\sin(\lambda/2)] \lambda^{-1-2} \sin(\lambda/2) [\sinh\mu]^{-1} e^{(\lambda-\mu)r}.$$

By (4.3) and (4.4), we have

$$(4.15) \quad \sinh \mu = 2[\sin(\lambda/2)][1 + \sin^2(\lambda/2)]^{1/2},$$

$$(4.16) \quad e^{-\mu} = 2 - \cos \lambda - 2[\sin(\lambda/2)][1 + \sin^2(\lambda/2)]^{1/2}.$$

It is easily verified that for  $0 \leq \lambda \leq \epsilon$ ,

$$(4.17) \quad e^{(\lambda-\mu)r} = 1 + (1/12)(\lambda^3 + \lambda^5) + (1/288)\lambda^6 + C_1(\lambda)\lambda^7,$$

with  $|C_1(\lambda)| \leq 0.035$ ,  $0 \leq \lambda \leq 1$ . Hence,

$$e^{(\lambda-\mu)r} = 1 + (1/12)(r\lambda^3 + r\lambda^5) + (1/288)r^2\lambda^6 + C_2(r,\lambda).$$

Here  $|C_2(r,\lambda)| \leq (0.04)r\lambda^7 + (0.01)r^2\lambda^8 + (0.001)r^3\lambda^9$ . Therefore,

$$F(\lambda) = (1/12)[\lambda^2 - r\lambda^3 - (7/8)r\lambda^5] - (1/30)\lambda^4 - (1/288)r^2\lambda^6 + C_3(r,\lambda).$$

Here  $|C_3(r,\lambda)| \leq (0.01)\lambda^6 + (0.051)r\lambda^7 + (0.0104)r^2\lambda^8 + (0.001)r^3\lambda^9$ .

Hence,

$$(4.18) \quad \int_0^\epsilon J d\lambda = 2\pi[R_1(r,s) + R_2(r,s)] + Z_1(r,s) + Z_2(r,s),$$

where (4.19)

$$(4.19) \quad |Z_1(r,s)| < \int_0^\infty e^{-r\lambda} C_3(r,\lambda) d\lambda < 1047 r^{-7}.$$

(4.20)  $|z_2(r,s)| < (1/6)r^{-1} e^{-1.5r^{2/3}}$ .

It remains to estimate  $\int_{\epsilon}^{\pi} J d\lambda$ .

Clearly,

(4.21)  $|\int_{\epsilon}^{\pi} J d\lambda| \leq \int_{\epsilon}^{\pi} (e^{-r\lambda} + e^{-r\mu}) d\lambda$ .

It can be shown that  $e^{-\mu}$  is a decreasing function of  $\lambda$  for  $0 < \lambda < \pi$ . See e.g. p. 33 of

[33]. It is also shown there that

(4.22)  $e^{-\mu} < e^{-0.76}, \quad \lambda = 0.8$   
 $< e^{-0.91}, \quad \lambda = 1$   
 $< e^{-1.07}, \quad \lambda = 1.2$   
 $< e^{-1.31}, \quad \lambda = \pi/2$ .

By (4.17),

$e^{-\mu} = e^{-\lambda} (1 + c^*(\lambda)\lambda), |c^*(\lambda)| < 0.1, 0 < \lambda < 0.8,$

so that

$e^{-r\mu} \leq e^{-0.9\lambda r}, \quad 0 < \lambda < 0.8.$

Hence,

$\int_{0.8}^{\pi} e^{-r\mu} d\lambda < (0.32)e^{-0.76r},$

and

(4.23)  $\int_{\epsilon}^{\pi} [e^{-r\mu} + e^{-r\lambda}] d\lambda < (2.15)r^{-1} e^{-1.35r^{2/3}} - r^{-1} e^{-\pi r}.$

By combining (4.12), (4.19), (4.20) and (4.22), we see that

(4.24)  $2\pi |R_3(r,s)| \leq 2.3 r^{-1} e^{-1.35r^{2/3}} + 1047 r^{-7}.$

The theorem then follows from (4.5), (4.13), (4.18)-(4.21) and (4.23)-(4.24).

Theorem 4.2. Let  $r, s$  and  $t$  be nonnegative integers with  $r \geq 8; t = r+1$ . Then

(4.25a)  $G(rh,sh) - G(th,sh) = (1/4\pi) \log[(r^2+s^2)/(t^2+s^2)] + S_1(r,s) + S_2(r,s) + S_3(r,s),$

where

(4.25b)  $S_1(r,s) = \int_0^{\infty} (1/24\pi) [\lambda^2 - r\lambda^3 + \lambda^2 e^{-\lambda/2}] e^{-(r+1/2)\lambda} \cos s\lambda (s\lambda)$

(4.25c)  $S_2(r,s) = - \int_0^{\infty} (1/2\pi) [(1/30)\lambda^4 + (7/96)r\lambda^5 + (1/288)r^2\lambda^6 + (1/12)\lambda^4 e^{-\lambda/2}] \cdot e^{-(r+1/2)\lambda} \cos s\lambda (s\lambda)$

(4.25d)  $|S_3(r,s)| \leq (1.9)r^{-6} + (206)r^{-7} + (0.5)r^{-1} e^{-1.35r^{2/3}}.$



Proof. We have

$$(\pi/2) [g(r,s) - g(t,s)] = \int_0^\pi [e^{-t\lambda} - e^{-r\lambda}] \lambda^{-1} \cos(s\lambda) d\lambda + \int_0^\pi J^* d\lambda,$$

where

$$J^* \equiv \cos(s\lambda) [e^{-t\mu} (\sinh\mu)^{-1} - e^{-t\lambda} \lambda^{-1} - e^{-r\mu} (\sinh\mu)^{-1} + e^{-r\lambda} \lambda^{-1}].$$

It is known that

$$(4.26) \quad \int_0^\infty [e^{-t\lambda} - e^{-r\lambda}] \lambda^{-1} \cos(s\lambda) d\lambda = (1/2) \log[(r^2 + s^2)/(t^2 + s^2)].$$

Clearly,

$$(4.27) \quad \int_0^\infty |e^{-t\lambda} - e^{-r\lambda}| \lambda^{-1} d\lambda \leq (1/\pi) r^{-1} e^{-r\pi}.$$

It therefore suffices to estimate  $\int_0^\pi J^* d\lambda$ . Let

$$(4.28) \quad J^* = J_1^* + J_2^*,$$

where

$$(4.29) \quad J_1^* \equiv \cos(s\lambda) [e^{-r\lambda} \lambda^{-1} - e^{-r\mu} (\sinh\mu)^{-1}] (1 - e^{-\mu}),$$

$$(4.30) \quad J_2^* \equiv \cos(s\lambda) [e^{-\mu} - e^{-\lambda}] e^{-r\lambda} \lambda^{-1}.$$

Let  $\varepsilon \equiv (1.5)r^{-1/3}$ . By (4.16),

$$(1 - e^{-\mu}) [2 \sin(\lambda/2)]^{-1} = -\sin(\lambda/2) + [1 + \sin^2(\lambda/2)]^{1/2} \leq 1.$$

But

$$-\sin(\lambda/2) + [1 + \sin^2(\lambda/2)]^{1/2} = e^{-\lambda/2} + b_1(\lambda) \lambda^3,$$

where

$$|b_1(\lambda)| \leq 1/12, \quad 0 \leq \lambda \leq \varepsilon.$$

Hence, by (4.14)

$$(4.31) \quad \int_0^\varepsilon J_1^* = \int_0^\varepsilon \cos(s\lambda) e^{-(r+1/2)\lambda} F(\lambda) + E_1(r,s),$$

where

$$(4.32) \quad |E_1(r,s)| \leq \int_0^\infty (1/12) \lambda^3 F(\lambda) e^{-r\lambda} d\lambda \leq (5.84)r^{-6} + 364r^{-8} + 10^5 r^{-10}.$$

By (4.17),

$$(4.33) \quad \int_0^\varepsilon J_2^* = \int_0^\varepsilon [(1/12)(\lambda^2 + \lambda^4) + (1/288)\lambda^5 + C_1(\lambda)\lambda^6] \cos(s\lambda) e^{-t\lambda} d\lambda,$$

with  $|C_1(\lambda)| \leq 0.035$ . It is easily seen (see also p. 38 of [30]) that

$$(4.34) \quad \left| \int_0^\pi J^* d\lambda \right| \leq r^{-1} e^{-(1.5)r^{2/3}} + (2.15)r^{-1} e^{-1.35r^{2/3}} - r^{-1} e^{-r\pi}.$$

The theorem easily follows from (4.19), (4.20) and (4.26)-(4.34).

Theorems 4.1 and 4.2 provide accurate estimates for distant values of the undivided differences of  $G$ . One can use these estimates as Dirichlet conditions for a fast Poisson solver to generate all the values that are needed to set up the capacitance matrix  $C$ . One can also use integer arithmetic as in [27] to construct a table of values of  $G$  for  $r, s \leq 7$ . See e.g. Table II on p. 292 of [27] or Table I on p. 41 of [30]. By using the above tables and Theorems 4.1 and 4.2, we obtain the following.

Theorem 4.3. Let  $r$  and  $t = s+1$  be positive integers. Then

$$(4.35) \quad G(sh, rh) - G(th, sh) = (1/4\pi) \log[(s^2+r^2)/(t^2+r^2)] + R(a, r)$$

where

$$(4.36) \quad |R(s, r)| < (0.34) \min \{s^{-3}, r^{-3}\}.$$

Moreover,

$$(4.37) \quad \max_{r=2}^7 \left\{ \max_{s \leq r} |R(s, r)|, \sum_{s=2}^7 \max_{r \leq s} |R(s, r)|, \sum_{s=r}^7 \max_{r \leq s} |R(s, r)| \right\} \leq 0.01.$$

We next investigate the monotone behavior of the undivided differences of  $G$  in certain directions. Let

$$G_x(i, j) \equiv G((i+1)h, jh) - G(ih, jh)$$

$$G_y(i, j) \equiv G(ih, (j+1)h) - G(ih, jh).$$

By the five-point formula and symmetry,

$$(4.38) \quad G_{xx}(i-1, j) = G_{yy}(i, j-1) \quad i \neq 0 \text{ or } j \neq 0;$$

$$(4.39) \quad 2G_x(0, j) = -G_{yy}(0, j-1) \quad j \neq 0.$$

Theorem 4.4 Let  $r$  and  $s$  be nonnegative integers. Then  $G_x(s, r)$ ,  $G_y(s, r)$ ,  $-G_{xy}(s, r)$  and  $G_{yy}(1, r) - G_{yy}(0, r)$  are always positive; and  $G_{xx}(s-1, r)$  is always nonnegative for  $r \geq s$ .

Proof. Except for the result on  $G_{xx}(s-1, r)$ , the proof for all the other results are similar. We first estimate the values of the expression for  $s = 0$ ,  $r > 8$  using Theorem 4.1 or 4.2. By symmetry, the results hold for  $s \geq 8$ ,  $r = 0$  as well. We then verify with the aid of Table II on p. 292 of [27] that the same results hold for  $s = 0$ ,  $r \leq 8$  and  $s \geq 8$ ,  $r = 0$ . Since the five point formula is satisfied at all points  $rh$  and  $sh$  with  $r > 0$  and  $s > 0$ , an application of discrete maximum principle immediately yields the desired result.

The proof for  $G_{xx}(s-1, r)$  is as follows. By symmetry and (4.38), we note that

$$G_{xx}(s, r) \equiv 0 \text{ for } s = r.$$

By symmetry,

$$G_{xx}(s-1, r) = 2G_x(0, r) \quad \text{for } s = 0.$$

An application of the discrete maximum principle therefore completes the proof of the theorem.

This concludes our discussion for undivided differences of  $G$ . We now proceed to obtain better estimates for the remainder terms in (4.8) and (4.9).

The following theorem is an immediate consequence of (4.8) and some well known result of asymptotic series. See also p. 325 of [5].

**Theorem 4.5** Let  $r$  be any positive integer. Then

$$(4.40) \quad G(rh, rh) = (1/4\pi)\log(2r^2h^2) + (1/48\pi)r^{-2} + R^7,$$

where

$$(4.41) \quad |R^7| \leq (7/1920\pi)r^{-4}.$$

**Theorem 4.6** Let  $r > s$ ;  $r$  and  $s$  are nonnegative integers. Then

$$(4.42) \quad G(rh, sh) = (1/4\pi)\log[(s^2+r^2)h^2] - (1/24\pi)(r^2+s^2)^{-1} + (1/3\pi)r^2s^2(r^2+s^2)^{-3} + L(r, s)$$

where

$$(4.43) \quad |L(r, s)| \leq (2.51/\pi)r^{-4} + (154/\pi)r^{-6} + (1/\pi)\log r e^{-1.35r^{2/3}}.$$

**Proof.** As in the proof of Theorem 4.1, we have

$$\int_0^\varepsilon J \, d\lambda = \int_0^\varepsilon [\cos(s\lambda) - \cos(t\lambda)] e^{-r\lambda} [2 \sin(\lambda/2)]^{-1} F(\lambda) \, d\lambda.$$

Here  $\varepsilon = (1.5)r^{-1/3}$ ,  $t$  may be any nonnegative integer. It can be shown that

$$[2 \sin(\lambda/2)]^{-1} F(\lambda) = (1/12)(\lambda - r\lambda^2) - (11/144)r\lambda^4 - (43/1440)\lambda^3 - (1/288)r^2\lambda^5 + L_1(r, \lambda).$$

Here  $|L_1(r, \lambda)| \leq (0.012)\lambda^5 + (0.06)r\lambda^6 + (0.012)r^2\lambda^7 + (0.0012)r^3\lambda^8$ . But

$$\int_0^\infty e^{-ax} x^n \cos(bx) \, dx = n! [(a-ib)^{n+1} + (a+ib)^{n+1}] / 2(a^2+b^2)^{n+1}.$$

Hence,

$$(4.44) \quad \int_0^\infty e^{-r\lambda} \cos(s\lambda) (1/12)(\lambda - r\lambda^2) \, d\lambda = -(1/12)(r^2+s^2)^{-1} + (2/3)r^2s^2(r^2+s^2)^{-3} = (1/24)r^{-2},$$

when  $r = s$ .

We also have

$$(4.45) \quad (1/72) \int_0^\infty [11r\lambda^4 + (43/10)\lambda^3 + (1/2)r^2\lambda^5 + 144|L_1(r, \lambda)|] e^{-r\lambda} \, d\lambda \leq 5r^{-4} + 308r^{-6}.$$

Since  $|\sinh\mu|^{-1} \leq \lambda^{-1}$  for  $\lambda \leq 1$ , we have

$$\left| \int_\varepsilon^\pi J \, d\lambda \right| \leq \int_\varepsilon^\pi \lambda^{-1} (e^{-r\lambda} + e^{-r\mu}) \, d\lambda.$$

But  $e^{-\mu} \leq e^{-0.9\lambda}$ ,  $0 < \lambda \leq \varepsilon$ . Hence,

$$(4.46) \quad \left| \int_{\varepsilon}^{\pi} \tau d\lambda \right| \leq 2(e^{-1.5r^{2/3}} + e^{-1.35r^{2/3}}) \left| \int_{\varepsilon}^{\pi} \lambda^{-1} d\lambda \right| \\ \leq (2/3) \log r [e^{-15r^{2/3}} + e^{-1.35r^{2/3}}].$$

By combining (4.40), (4.41), (4.44)-(4.46) and using the technique in the proof of Theorem 4.1 to estimate the remaining remainder terms, we see that Theorem 4.6 holds.

Theorem 4.6 also provides a means of generating  $G(rh, sh)$  by means of a fast Poisson solver on a rectangle using the first three terms on the right hand side of Equation (4.42) as approximations for distant values of the Dirichlet data.

We now describe an efficient method developed in [24] of computing  $Gv$  for any vector  $v$  defined on a square mesh  $S_h \supset \Omega_h$ , with boundary mesh  $\partial S_h$ . Let  $U_S$  and  $U_{\partial S}$  denote the extension operators from  $S_h$  and  $\partial S_h$  respectively to all mesh points that are defined the same way as  $U$ . We are actually computing  $U_S^T G U_S v$ . We first solve the system of equations

$$U_S^T B U_S \phi = v \quad \text{on } S \\ \phi = 0 \quad \text{on } \partial S$$

for the potential  $\phi$ . We then extend  $\phi$  by zero to all mesh points. We represent  $\phi$  as

$$(4.47) \quad \phi = G U_S v + G U_{\partial S} \rho$$

where  $\rho$  is an unknown vector defined on the mesh points on  $\partial S$  to be determined. It is easy to see that

$$\rho = U_{\partial S}^T B \phi.$$

The vector  $U_S^T G U_{\partial S} \rho$  can easily be computed by one fast Poisson solver on  $S$  with  $U_{\partial S}^T G U_{\partial S} \rho$  as the Dirichlet data on  $\partial S$ . Because of the sparsity of the vector  $U_{\partial S}$ , the Dirichlet data can be computed at a cost of constant  $N^2$ .  $U_S^T G U_S v$  is then computed from (4.47).

We now describe a method of computing all three of the vectors  $Gv$ ,  $Gv_{\mu}$  and  $W^T Gv$  using only two calls of fast Poisson solvers. This may appear to be impossible since  $W^T Gv$ , the right hand side of the capacitance matrix equation, must be determined first and the computation of  $Gv$  alone requires two calls of fast Poisson solvers. We can, however, first compute and store the vectors  $\phi$  and  $\rho$  in equation (4.47). Clearly,

$$W^T Gv = W^T \phi - W^T G U_{\partial S} \rho.$$

But we need only to compute  $G U_{\partial S} \rho$  at those mesh points that  $W^T Gv$  is defined. Hence, the computation of  $W^T G U_{\partial S} \rho$  requires only constant  $n^2$  operations. On the other hand,  $G U_{\partial S} \rho$  and  $G v_{\mu}$  can be computed simultaneously with one call of fast Poisson solver with  $U_{\partial S}^T G (U_{\partial S} \rho + v_{\mu})$  as

the Dirichlet data on  $S$ . Hence our algorithm of computing all three of the vectors  $W^T G v$ ,  $G U$  and  $G v$  requires only calls of fast Poisson solvers plus constant times  $n^2$  operations.

In the methods described so far, a fast solver on  $2N \times 2N$  mesh points is needed to generate the discrete Green's function or its undivided differences on a  $N \times N$  mesh  $S_h$ . An alternative method is to first generate the undivided differences of  $G$  on a  $N/2 \times N/2$  mesh using a fast solver on  $N \times N$  mesh points. The values on the remaining mesh points of  $S_h$  are computed by using (4.11) and (4.25). An accuracy of eight decimal digits is guaranteed by Theorems 4.1 and 4.2 if  $N \geq 60$ . A somewhat less accurate but easier to program method is to generate the values of  $G$  on a  $N/2 \times N/2$  mesh and compute the values on the rest of  $S_h$  by using (4.42) and (4.43). An accuracy of five decimal digits is guaranteed by Theorem 4.6 if  $N \geq 60$ .

We shall assume in the next two sections that the  $G$  used in the capacitance matrix equation is the discrete Green's function on the entire plane. The main results in sections 5 and 6, however, will also hold if  $G = B_D^{-1}$  is used in equations (3.2) and (3.5). See section 5 of [31] for a discussion in this respect.

5. Spectral bounds of  $B_h$ 

We shall show in this section that  $B_h$  is uniformly well conditioned in the spectral norm as  $h \rightarrow 0$ . The following well known lemma is crucial to the proof of our main result.

Lemma 5.1. Let the symmetric part of a matrix  $A$  satisfy

$$(A+A^T)/2 \geq \delta I, \quad \delta > 0.$$

Then

$$A^T A \geq \delta^2 I.$$

Theorem 5.1. (0.25)  $I \leq B_h^T B_h \leq (7.29)I$  for scheme I.a

(0.25)  $I \leq B_h^T B_h \leq (13.7)I$  for scheme I.b

(0.04)  $I \leq B_h^T B_h \leq (5.57)I$  for scheme II

for all sufficiently small  $h > 0$ .

Proof. We shall first prove that the following holds for scheme Ib.

$$(5.1) \quad I \leq B_h + B_h^T.$$

Let  $B_s \equiv B_h + B_h^T$ . We shall show that

$$(5.2) \quad \min_{P \in \partial\Omega_h} \{B_s(P,P) - \sum_{Q \in \partial\Omega_h, Q \neq P} |B_s(P,Q)|\} \geq 1$$

so that (5.1) holds because of a well known Gerschgorin theorem. The inequality

$$I \leq B_h^T B_h$$

will then follow from Lemma 5.1.

Let  $P \in \partial\Omega_h$ . Assume that the local orientation of the boundary near  $P$  is such that for any point  $P' \in \partial\Omega_h$  in that neighbourhood, either  $W'$  and  $N'$ , the western and northern neighbours of  $P'$ , are both in  $(C\Omega)_h$  or  $W'$  alone is in  $(C\Omega)_h$ . Let  $\alpha_p$  be the angle  $\leq \pi/4$  that the normal through  $P'$  makes with the  $x_1$  axis in the east-west direction. By (3.5) and (3.9), we have, for  $P \neq Q$ ,

$$(5.3) \quad B_h(P,Q) = 2[G(W_P;W_Q) - G(W_P;Q)] + 2 \tan\alpha_p [G(W_P;NW_Q) - G(W_P;W_Q)]$$

if  $P$  has only one neighbour in  $(C\Omega)_h$ . Here  $Y_P$  denotes the immediate neighbour on the mesh for any point  $P$  in the  $Y$  direction. Similar expressions to (5.3) are easily obtained when  $N_P$  is also in  $(C\Omega)_h$  or when  $P \equiv Q$ . If  $P \equiv (jh,kh)$ ,  $Q \equiv (mh,nh)$ , then because of translational invariance,  $G(P;Q) \equiv G(|j-m|h, |k-n|h)$ .

Assume that  $0 < \alpha_p < \pi/4$  and that  $(\partial\Omega_h)_{loc}$ , which denotes a subset of  $\partial\Omega_h$  that contains a  $\sqrt{h}$  neighbourhood of  $P$ , can be partitioned into blocks as follows. Let

$$I_0 = \{(0, h), \dots, (0, M_1 h)\}$$

$$I_k = \{(kh, M_k h+h), \dots, (kh, M_{k+1} h)\}, \quad k = 1, \dots, K_1,$$

$$I_k = \{(-kh, -M_{-(k-1)} h), \dots, (-kh, -M_{-k} h+h)\}, \quad k = 1, \dots, K_2.$$

Then,

$$(\partial\Omega)_h \text{ loc} = \bigcup_{k=-K_2}^{K_1} I_k, \quad P \in I_0.$$

Note that  $M_k - M_{k-1}$  gives the number of points in  $I_{k-1}$ ,  $k = 1, \dots, K_1$  while  $M_{-k} - M_{-(k-1)}$  gives the number of points in  $I_{-k}$ ,  $k = 1, \dots, K_2$ ;  $M_0 \equiv 0$ .

Let  $P_j$  denote the point with  $x_2$ -coordinate  $jh$ . From (5.3) and Theorem 4.3, it is easily verified that  $\sum_{Q \neq P} |B_s(P, Q)|$  will remain essentially unchanged for sufficiently smooth  $\partial\Omega$  if  $\tan \alpha_Q$  is replaced throughout by  $\tan \alpha_P$ . Let  $a \equiv \tan \alpha_P$ . Let  $P \equiv P_i$  and  $G(i, j) \equiv G(ih, jh)$ . We shall assume that  $P \neq P_{M_1}$  unless otherwise stated. We easily verify that for  $P_j \in I_0$ ,

$$(5.4) \quad B_s(P, P) = 3+a,$$

$$(5.5) \quad B_s(P, P_j) = -4(1+a) G_x(0, |i-j|), \quad j \neq i, j \neq M_1,$$

$$(5.6) \quad B_s(P, P_{M_1}) = -2(1+a) G_x(0, M_1-i) + (1-a) G_{yx}(0, M_1-i).$$

By (4.39), we see that for  $P_j \in I_0$ ,

$$(5.7) \quad \sum_{i \neq j} B_s(P, P_j) = -2(1+a) [2 G_y(0, 0) - G_y(0, M_1-i) - G_y(0, i-1)] \\ + 2(1+a) G_x(0, M_1-i) + (1-a) G_{yx}(0, M_1-i).$$

For  $P_j \in I_k$ ,  $k = 1, \dots, K_1$ , we have

$$(5.8) \quad B_s(P, P_j) = 2(1+a) G_{yy}(k, j-i-1), \quad j \neq M_{k+1}$$

$$(5.9) \quad B_s(P, P_j) = (1+a) G_{yy}(k, j-i-1) + (1-a) G_{yx}(k, j-i), \quad j = M_{k+1}.$$

$$(5.10) \quad \sum_{P_j \in I_k} B_s(P, P_j) = -2(1+a) [G_y(k, M_{k+1}-i) - G_y(k, M_{k+1}-i)] - (1+a) G_{yy}(k, M_{k+1}-i-1) \\ + (1-a) G_{yx}(k, M_{k+1}-i).$$

By Theorem 4.4, we see that each  $B_s(P, P_j)$ ,  $j \neq i$ ,  $j < 1$ , is negative. Hence

$$(5.11) \quad \sum_{k=1}^{K_1} \sum_{P_j \in I_k} B_s(P, P_j) = 2(1+a) [G_y(1, M_1-i) + \sum_{k=1}^{K_1} G_{yx}(k, M_{k+1}-i)] \\ - (1-a) \sum_{k=1}^{K_1} G_{yx}(k, M_{k+1}-i) + (1+a) \sum_{k=1}^{K_1} G_{yy}(k, M_{k+1}-i-1) \\ < 2(1+a) G_y(1, M_1-i).$$

Similarly,

$$(5.12) \quad \sum_{k=1}^{K_1} \sum_{P_j \in I_{-k}} |B_s(P, P_j)| < 2(1+a) G_y(1, i-1) + (1-a) G_y(0, i-1).$$

By combining (5.7), (5.11) and (5.12), we see that

$$(5.13) \quad \sum_{i \neq j} |B_s(P, P_j)| < 4(1+a) G_y(0,0) + (1-a) G_y(0, i-1) < (5+3a) G_y(0,0) = 5/4 + 3/4a.$$

Here we have used (4.5), (4.6) and Theorem 4.4. Hence, by (5.4) and (5.13), we see that

$$(5.2) \text{ holds for } P \in I_0, P \neq P_{M_1}.$$

The proof for  $P \in P_{M_1}$  is quite similar and is sketched as follows.

$$(5.14) \quad B_s(P, P) = 2+2(1-a) G_{xy}(0,0).$$

$$(5.15) \quad \sum_{P_j \in I_0, j \neq 1} |B_s(P, P_j)| = (1+a)[G_y(0,0) - G_y(0, M_1-1)] + (1-a)[G_x(0,1) - G_x(0, M_1)];$$

$$(5.16) \quad \sum_{k=1} \sum_{P_j \in I_k} |B_s(P, P_j)| = (1-a)[G_x(0,0) + \sum_{k=1} \{G_{xx}(k-1, M_{k+1}-M_1) - G_{yx}(k, M_{k+1}-M_1)\}] \\ + (1+a)[G_y(1,0) + \sum_{k=1} \{G_{yx}(k, M_{k+1}-M_1) + G_{yy}(k, M_{k+1}-M_1-1)\}];$$

$$(5.17) \quad \sum_{k=1} \sum_{P_j \in I_{-k}} |B_s(P, P_j)| = (1+a)[G_y(1, M_1) + \sum_{k=1} \{G_{yx}(k, M_1+M_k) + G_{yy}(k, M_1+M_k-1)\}] \\ + (1-a)[G_x(1, M_1) + \sum_{k=1} \{G_{xx}(k, M_1+M_k) - G_{yx}(k-1, M_1+M_k-(k-1)-1)\}].$$

By (4.38) and Theorem 4.4, we see that

$$(5.18) \quad \sum_{k=1} \sum_{P_j \in I_k} |B_s(P, P_j)| \leq (1-a) G_x(0,0) + (1+a) G_y(1,0).$$

Similarly, by using the identity (5.20), we have

$$(5.19) \quad \sum_{k=1} \sum_{P_j \in I_{-k}} |B_s(P, P_j)| \leq (1+a) G_y(1, M_1) + (1-a)[G_x(1, M_1) - G_{yx}(0, M_1-1)].$$

$$(5.20) \quad G_{yy}(k, M_1+M_k-1) \equiv G_{yy}(k+1, M_1+M_k-1) + G_{yx}(k, M_1+M_k-1) - G_{yx}(k, M_1+M_k).$$

Hence,

$$(5.21) \quad \sum_{k=1} |B_s(P, P_j)| \leq 2G_x(0,0) + 2G_x(0,1) + aG_y(1, M_1).$$

By combining (5.14) and (5.21), we have

$$(5.22) \quad B_s(P, P) - \sum_{i \neq j} |B_s(P, P_j)| \geq 2 + 2(1-a)G_{xy}(0,0) - 2G_x(0,0) - 2G_x(0,1) - aG_y(1, M_1).$$

Clearly, the right hand side of (5.22) attains its maximum at  $a = 0$ . Hence, (5.2) holds for

$P \in P_{M_1}$ . The proof for other choices of  $P$  is similar and will not be repeated. We note

that we have assumed that  $B_s(P, Q) \neq 0$  for any  $Q \in (\partial \Omega)_{h \text{ loc}}$ . This assumption will not

affect our estimate (5.2) because each  $B_s(P, Q)$  is either zero or negative for  $P \neq Q$ .

Finally we remark that the schemes Ia and Ib described in this work are essentially dual to the schemes I.N.a and I.N.b described in [31] in the following sense. If we maintain that  $a \equiv \tan \alpha_p$  does not change its value for the entire row or column of  $B_h$



corresponding to  $P$ , then  $B_h(P, Q)$  is the same for both schemes Ia and I.N.a. or for both schemes Ib and I.N.b. Therefore we refer the reader to [31] for the proof of Theorem 5.1 for scheme I.a. We now proceed to prove the following inequality for scheme II.

$$(5.23) \quad \min_P \{B_s(P, P) - \sum_{Q \neq P} |B_s(P, Q)|\} \geq 0.41.$$

Let  $P \in \partial\Omega_h$ . We assume the same local configuration of irregular mesh points near  $P$  as before. By (3.5) and (3.7), we have for  $d_2 \geq 1$ ,  $P \neq Q$ ,

$$(5.24) \quad B_h(P, Q) = -G_x(W_P; W_Q) - e_1 G_x(E_P; W_Q) + a[G_y(W_P; W_Q) + e_1 G_y(E_P; W_Q)]$$

where

$$(5.25) \quad e_1 \equiv (1-d_1)/(1+d_1), \quad a \equiv \tan \alpha_P.$$

Here  $G_x(\cdot; \cdot)$  and  $G_y(\cdot; \cdot)$  denote the forward undivided differences in the  $x_1$  and  $x_2$  direction with respect to the second variable of any mesh function  $G$  respectively. A similar expression involving  $e_2$  which is similarly defined or a constant one should be added respectively to the right hand side if  $d_2 < 1$  or if  $P \equiv Q$ . Let  $e_{1j}$  and  $e_{2j}$  denote the corresponding  $e_1$  and  $e_2$  respectively.

Let  $P \equiv P_i$ . We first estimate  $\sum_{|i-j| \geq 3} \{|B_h(P, P_j)| + |B_h^T(P, P_j)|\}$ .

Let  $B_h^*(P, Q)$  be defined by Equation (5.24) with  $G$  replaced by its continuous analog which we shall denote by  $G^*$ .  $B_h^{T*}(P, Q)$  is similarly defined. By (5.24) and (5.25), we have for  $d_2 > 1$  and  $P \neq Q$

$$(5.26) \quad \begin{aligned} B_h^*(P, Q) &= [2/(1+d_1)] \{ -G^*(W^*; Q) + (1-a)G^*(W^*; W_Q) + aG^*(W^*; NW_Q) \} + R_{WE} \\ &= [2/(1+d_1)] \{ G^*(W^*; W') - G^*(W^*; Q) \} + R_{WE} + R_Q, \end{aligned}$$

where

$$(5.27) \quad \begin{aligned} R_{WE} &\equiv [2/(1+d_1)] \{ \sum_{k=2}^{n-1} (h^k/k!) [(d_1-1)^k + (d_1+1)^k] \\ &\quad \cdot [(\partial/\partial x_1)^k \{ -G^*(\cdot; Q) + (1-a)G^*(\cdot; W_Q) + aG^*(\cdot; NW_Q) \}] (W^*) \} \\ &\quad + \text{nth order remainder term.} \end{aligned}$$

$$(5.28) \quad R_Q \equiv [h^2 a(1-s)/(1+d_1)] \{ a[(\partial/\partial x_2)^2 G^*(W^*; \cdot)](NW_0) + (1-a)[(\partial/\partial x_2)^2 G^*(W^*; \cdot)](NW_1) \}.$$

Here  $W^*$  is the point where the Dirichlet data  $U_W$  is given.  $W'$  is a point on the mesh line connecting  $W_Q$  and  $NW_Q$  and at a distance  $ah$  from  $W_Q$ .  $NW_0$  and  $NW_1$  are respectively points that can be anywhere on the mesh lines between  $W'$  and  $W_Q$  and between  $W'$  and  $NW_Q$ . We shall assume that  $d_{20}$ , the analogue of  $d_2$  for the point  $Q$ , is also greater than 1. In that case, we have a similar expression for  $B_h^{T*}(P, Q)$  as that in (5.26).

We now proceed to estimate  $B_h^{T^*}(P, Q) + B_h^*(P, Q)$ . It is easily verified that if  $P, Q$  are two points on  $\partial\Omega$  with  $d(P, Q) \leq h^\gamma$ ,  $0 < \gamma \leq 1$  and  $ta$  is the tangent at  $Q$  to  $\partial\Omega$ , then

$$d(P, ta) \leq (K_{\max} + o(1))h^{\partial\gamma}.$$

Here  $K_{\max}$  is the maximum absolute value of the curvatures of  $\partial\Omega$ . Hence, without loss of generality, we may assume that  $W^*$  lies on the tangent to  $\partial\Omega$  through  $Q^*$ , the point where the normal through  $Q$  intersects with  $\partial\Omega$  and vice versa. Let  $r$  denote  $d(W^*, Q)$  and  $r'$  denote  $d(W^*, W')$ . It is easily seen that  $r' \geq r$  if  $d(Q^*, Q) \leq h(1+a^2)/2$ . We separate our discussion into four cases. The first case is when  $r' > r$  and  $d(P, P^*) < h(1+a^2)/2$ . The maximum of  $\log(r'/r)$  then occurs when  $Q$  coincides with  $Q^*$ . In this case,  $r'^2 - r^2 = (1+a^2)h^2$ . Hence, if  $P \equiv P_i, Q \equiv P_j, i \neq j$ , then

$$0 \leq G^*(W^*; W') - G^*(W^*; Q) \leq (1/4\pi)|j-i|^{-2}.$$

By (5.28),

$$\begin{aligned} -(1/8\pi)|i-j|^{-2} \leq R_Q \leq 0; & \quad \text{if } j > i \\ -(1/8\pi)|i-j-1|^{-2} \leq R_Q \leq [a(1-a^2)/\sqrt{2}(1+d_1)\pi](1+a^2)^{-3/2}|j-i-1|^{-3}; & \quad \text{if } i > j. \end{aligned}$$

Similarly, by (5.27),

$$(1+d_1)|R_{WE}| \leq (2/\pi)|j-i|^{-4}.$$

Hence, for  $|j-i| \geq 3$ ,

$$(5.29) \quad \begin{aligned} (1+d_1)|B_h^*(P, Q)| & \leq (1/2\pi)|j-i|^{-2} + (1/5\pi)|j-i-1|^{-3} + (2/\pi)|j-i|^{-4}, \quad \text{if } i > j; \\ & \leq (1/2\pi)|j-i|^{-2} + (2/\pi)|j-i|^{-4}, \quad \text{if } j > i. \end{aligned}$$

Since  $d(P, P^*) < h(1+a^2)/2$ , the estimate for  $B_h^{T^*}(P, Q)$  is the same as that given in (5.29). Hence,

$$(5.30) \quad |B_h^{T^*}(P, Q)| + |B_h^*(P, Q)| \leq (1/\pi)|j-i|^{-2} + (4/\pi)|j-i|^{-4} + (1/5\pi)|j-i-1|^{-3}.$$

The second case is when  $r \leq r'$  and  $d(P, P^*) < h(1+a^2)/2$ . Let  $d_{1Q}$  denote  $d(Q^*, Q)/h$ . Then  $d_{1Q} \geq 1/2$ . Hence by (5.29),

$$\begin{aligned} |B_h^{T^*}(P, Q)| & \leq (1/3\pi)|j-i|^{-2} + (2/15\pi)|j-i-1|^{-3} + (4/3\pi)|j-i|^{-4}, \quad \text{if } j > i; \\ & \leq (1/3\pi)|j-i|^{-2} + (4/3\pi)|j-i|^{-4}, \quad \text{if } i > j. \end{aligned}$$

On the other hand, the maximum of  $\log r/r'$  occurs when  $Q^*$  coincides with  $W_2$ . In this case,  $r^2 - r'^2 = (1-a^2)h^2$ . Hence,

$$\begin{aligned} G^*(W^*; W') - G^*(W^*; Q) & \leq (1/4\pi)[(1-a^2)/(1+a^2)]|j-i|^{-2}, \quad \text{if } j > i; \\ & \leq (1/4\pi)[(1-a^2)/(1+a^2)]|j-i-a|^2, \quad \text{if } i > j; \end{aligned}$$

and

$$(5.31) \quad |B_h^*(P,Q)| \leq [(1/2\pi) + (1/8\pi)] |j-i|^{-2} + (2/\pi) |j-i|^{-4} \quad \text{if } j > i; \\ \leq (1/2) |j-i|^{-2} + (1/8\pi) |j-i-1|^{-2} + (2\pi) |j-i|^{-4}, \quad \text{if } i > j.$$

Therefore,

$$(5.32) \quad |B_h^{T^*}(P,Q)| + |B_h^*(P,Q)| \leq [(1/3\pi) + (1/2\pi) + (1/8\pi)] |j-i|^{-2} + (4/3\pi) |j-i|^{-4} \\ + (2/15\pi) |j-i-1|^{-3}, \quad \text{if } j > i; \\ \leq [(1/2\pi) + (1/3\pi)] |j-i|^{-2} + (1/8\pi) |j-i-1|^{-2} \\ + [(4/3\pi) + (2/\pi)] |j-i|^{-4}, \quad \text{if } i > j.$$

The third case is when  $r < r'$  and  $d(P,P^*) \geq h(1+a^2)/2$ . The estimate for  $|B_h^*(P,Q)| + |B_h^{T^*}(P,Q)|$  is the same as that for the second case.

The fourth case is when  $r > r'$  and  $d(P,P^*) \geq h(1+a^2)/2$ . Both  $d_1$  and  $d_{1Q}$  are not less than  $1/2$ . Therefore by (5.31) and the above observation,

$$(5.33) \quad |B_h^{T^*}(P,Q)| + |B_h^*(P,Q)| \leq (2/3\pi) |j-i|^{-2} + (1/12\pi) |j-i|^{-2} \\ + (1/12\pi) |j-i|^{-2} + (8/3\pi) |j-i|^{-4}.$$

By comparing (5.30), (5.32) and (5.33), we see that

$$\max |B_h^{T^*}(P,Q)| + |B_h^*(P,Q)| \leq (1/\pi) |j-i|^{-2} + (4/\pi) |j-i|^{-4} + (1/5\pi) |j-i-1|^{-3} \quad \text{if } j > i \\ \leq (1/\pi) |j-i|^{-2} + (1/8\pi) |j-i|^{-2} + (1/5\pi) |j-i-1|^{-3} + (4/\pi) |j-i|^{-4} \\ \text{if } i > j.$$

Hence, if  $d_2, d_{2Q} > 1$ , then

$$\sum_{|j-i| \geq 3} |B_h^{T^*}(P,P_j)| + |B_h^*(P,P_j)| \leq (1/\pi) \sum_{k=3} \{2/k^2 + 8/k^4 + 2/5k^3 + 1/8k^2\} \\ \leq 0.328.$$

Similarly, it can be shown that if both  $d_2$  and  $d_{2Q}$  are not greater than 1, then

$$(5.34) \quad \sum_{|j-i| \geq 3} |B_h^{T^*}(P,P_j)| + |B_h^*(P,P_j)| \leq 0.677.$$

By Theorems 4.1-4.3 and Table I on p. 41 of [30],

$$\sum_{|i-j| \leq 3} \{|B_h(P,P_j) - B_h^*(P,P_j)| + |B_h^T(P,P_j)|\} \leq 0.04.$$

Hence

$$(5.35) \quad \sum_{|j-i| \geq 3} \{|B_h(P,P_j)| + |B_h^T(P,P_j)|\} \leq 0.717.$$

It remains to estimate  $B_s(P,P) - \sum |B_s(P,P_j)|$ ,  $|i-j| \leq 2$ ,  $i \neq j$ . Without loss of generality, we may assume that both  $d_1$  and  $d_2$  are less than 1. We shall assume that  $P \equiv P_i \equiv P_{M_1}$  with  $M_1 \geq 3$  and  $M_2 \geq 5$ . The case when  $i \neq M_1$  can be treated in a similar manner. We have

$$(5.36) \quad B_s(P,P) = 2 + 2(1-a) G_{xy}(0,0) + g_0(a),$$

where

$$g_0(a) \equiv 2 e_1 [G_x(1,0) + G_y(2,0)] + 2 e_2 [G_x(0,1) + a G_y(1,1)].$$

For any  $P_j \in I_0$ ,  $j \neq M_1$ , we have

$$B_s(P,P_j) = g_1(a,j) + g_2(a,j),$$

where

$$g_1(a,j) \equiv G_{xy}(0,i-j) + a G_{yy}(0,i-j-1) - a G_y(1,i-j) - G_x(0,i-j),$$

$$g_2(a,j) \equiv e_1 [G_x(1,i-j) - a G_y(2,i-j-1)] + e_{ij} [G_x(1,i-j) + a G_y(2,i-j)] \\ + e_2 [G_x(0,i-j-1) - a G_y(1,i-j-2)].$$

For any  $P_j \in I_1$ , we have

$$B_s(P,P_j) = g_3(a,j) + g_4(a,j),$$

where

$$g_3(a,j) \equiv -G_x(0,j-i-1) + a G_{yy}(1,j-i-1) + a G_y(0,j-i-1)$$

$$g_4(a,j) \equiv e_1 [G_x(0,j-i) + a G_y(1,j-i)] + e_2 [-G_x(0,j-i+1) + a G_y(0,j-i+1)] \\ + e_{ij} [G_x(2,j-i) - a G_y(3,j-i-1)].$$

By Theorem 4.4,  $g_1(a,j)$  is negative. It is easily verified that  $g_2(a,j)$  is nonnegative for  $0 \leq i-j \leq 2$ ;  $g_3(a,j)$  is negative for  $a \leq 1/2$ ; and  $g_4(a,j)$  is positive. Moreover, for  $a \leq 1/2$ ,

$$(5.37) \quad \sum_{i-j=1}^2 |g_1(a,j) + g_2(a,j)| + \sum_{j-i=1}^2 |g_3(a,j) + g_4(a,j)| \\ < g_0(a) + \sum_{i-j=1}^2 |g_1(a,j)| + \sum_{j-i=1}^2 |g_3(a,j)|.$$

Hence, for  $a < 1/3$ ,  $M_1 \geq 3$ ,  $M_2 \geq 5$ , we have from (5.37) that the following holds.

$$(5.38) \quad B_s(P,P) - \sum \{ |B_s(P,P_j)|, i \neq j, |i-j| \leq 2 \} \\ \geq 2+2(1-a)G_{xy}(0,0) + \sum_{i-j=1}^2 g_1(a,j) + \sum_{j-i=1}^2 g_3(a,j) \\ \geq 2+2(1-a)G_{xy}(0,0) - G_x(0,0) - 3G_x(0,1) - G_x(0,2) + G_x(0,3) \\ \geq 1.17.$$

By considering all possible configurations of  $P_j$ ,  $i-j \leq 2$ ,  $a \geq 1/3$ , it can be shown that the constant 1.17 is always majorized by the left hand side of (5.38). It is easy to see that (5.38) also holds when  $P_i \neq P_{M_1}$ . Hence by (5.35), we see that (5.23) holds.

This established the lower spectral bounds of  $B_h$  for all schemes. To complete the

proof of the theorem, we note that the spectral norm of  $B_h^T B_h$  is majorized by  $(1/2)(\|B_h^T\|_\infty + \|B_h\|_\infty)^2$ . It therefore suffices to prove the following two inequalities

$$(5.39) \quad \|B_h^T\|_\infty + \|B_h\|_\infty \leq 7.4 \quad \text{if scheme I.b is used}$$

$$(5.40) \quad \|B_h^T\|_\infty + \|B_h\|_\infty \leq 4.72 \quad \text{if scheme II is used.}$$

We first prove that (5.39) holds. Without loss of generality we may assume that  $P \in \partial\Omega_h$  has only one western neighbour  $W$  in  $(C\Omega)_h$ . Let  $P \equiv (0,0)$  and  $P_j \equiv (x,y)$ , be in  $\partial\Omega_h$ . Then  $x = ay + b$ ,  $|b| \leq (1+o(1))h$  if  $d(P, P_1) < \sqrt{h}$ . And

$$\begin{aligned} B_h^*(P, P_j) &= -(1/2\pi) \log[(x+h)^2 + y^2] + (a/2\pi) \log[x^2 + (y+h)^2] + [(1-a)/2\pi] \log(x^2 + y^2) \\ &= (1/2\pi) [2h(x-ay) + (a-1)h^2] / r^2 + R, \end{aligned}$$

where  $r = d(P, P_j)$  and  $|R| \leq (1/2\pi) [(2x+h)^2 + (2y+h)^2] h^2 / 2r^4$ . It is easily verified that

$$(5.41) \quad \sum_{P \neq P_j} |B_h^*(P, P_j)| \leq 1.25 \quad \text{if } |y| \geq 2h$$

By Theorems 4.1-4.3 and the Table I on p. 41 of [30],

$$(5.42) \quad B_h(p, P) = 2 - (1/2)(1-a);$$

$$(5.43) \quad \sum_{|y| < 2h} |B_h(P, P_j)| \leq 0.28 + 0.55a/2;$$

$$(5.44) \quad \sum_{|y| \geq 2h} |B_h(P, P_j) - B_h^*(P, P_j)| < 0.06.$$

By (5.41)-(5.44),

$$\sum |B_h(P, P_j)| < 3.7.$$

It is easily seen that the above inequality also holds when  $B_h$  is replaced by  $B_h^T$ . We have therefore completed the proof of (5.39).

Let  $P \equiv P_i$ . By (5.36),

$$(5.45) \quad |B_s(P, P)| \leq 2+2(1-a)G_{xy}(0,0) + g_0(a).$$

By (5.37) and (5.45),

$$(5.46) \quad \sum_{|i-j| \leq 2} |B_h(P, P_j)| + |B_h^T(P, P_j)| < 2+2g_0(a) + 2G_x(0,0) + 6G_x(0,1) \leq 4.$$

By (5.35) and (5.46), we see that (5.40) holds.

## 6. Singular values of $K_h$ and $C$ .

We shall show that all except a few singular values of  $C$  lie in the interval  $[d_1 - \epsilon, d_2 + \epsilon]$ ,  $\epsilon > 0$ , where  $d_1$  and  $d_2$  are the spectral bounds of  $B_h$ . This is accomplished by first proving that the singular values of  $K_h$  cluster around that of a compact operator  $K$ . Our main result then follows as an immediate consequence of a well known result in matrix theory which will be stated below as Lemma 6.8. We first need some definitions from modern analysis. Let  $X$  denote a Banach space throughout this section.

Definition. A subset  $S \subset X$  is sequentially compact if any sequence in  $S$  contains a convergent subsequence with limit in  $X$ .

Definition. A family of operators  $K_n$  on  $X$  is collectively compact if the set  $\{K_m f: \|f\| < 1, f \in X, m = 1, 2, \dots\}$  is sequentially compact in  $X$ .

We shall first assume that either scheme I.a or scheme I.b is used. We start by constructing a family of operators  $\{K_m\}$  from  $\{K_h\}$  in the same way that is done in Section 5 of [32]. For completeness, we briefly sketch this construction in the following. Define

$K_m: C[0,1] \rightarrow C[0,1]$  by

$$(6.1) \quad [K_m]f(t) = \sum_{j=1}^m k(t, t_j) f(t_j), \quad t_j \in [0,1]; f \in C[0,1],$$

where

$$(6.2) \quad k(t, t_j) = K_h(P_i, P_j) + [(t-t_i)/(t_{i-1}-t_i)][K_h(P_{i-1}, P_j) - K_h(P_i, P_j)], \quad t_{i-1} \leq t \leq t_i.$$

$C[0,1]$  is the Banach space of continuous functions on  $[0,1]$ . The  $t_i, i = 1, \dots, m$  are defined as follows. Let  $\phi, \psi$  be a smooth parametrization of  $\partial\Omega$ . Then  $(\phi(t_i), \psi(t_i))$  is the closest point on  $\partial\Omega$  to  $P_i \in \partial\Omega_h$  which is on the normal through  $P_i$ . When  $t$  is very close to 0 or 1, the  $k(t, t_j)$  in (6.2) should be adjusted slightly. See [32] for the details. We can construct by the same procedure a family of operators  $\{K'_m\}$  from  $\{K_h^T\}$ . Let  $K_s \equiv K'_m K_m$ .

Lemma 6.1. The nonzero eigenvalues of  $K_h, K_h + K_h^T$  and  $K_h^T K_h$  coincides with that of  $K_m, K_m + K'_m$  and  $K_s$  respectively.

Proof: See e.g. Lemma 5.2 in [32].

Lemma 6.2. Let  $P$  and  $Q$  be two points in  $\partial\Omega_h$  with  $d(P, Q) = h^\beta, \beta \leq 1/2$ . Let  $P^*$  and  $Q^*$  the closest points on  $\partial\Omega$  to  $P$  and  $Q$  respectively. Then

$$K_h(P, Q) = 2[\partial G^* / \partial v_{Q^*}](P^*; Q^*) h \sec \alpha_Q + O(h^{2-2\beta}).$$

Proof: Essentially the same as that of Lemma 5.4 in [31].

Lemma 6.3. The families of operators  $\{K_m\}, \{K'_m\}$  and  $\{K_s\}$  are collectively compact on  $C[0,1]$ .

Proof: Essentially the same as that of Lemma 5.5 in [32].

Lemma 6.4.  $K_m f \rightarrow Kf, K'_m f \rightarrow K^T f$  and  $K_s f \rightarrow K^T Kf$  for each  $f \in C[0,1]$  where  $K$  is the compact integral operator defined by

$$(Kf)(t_p) = 2 \int_{\Omega} [\partial G^* / \partial v_Q](P; Q) f ds_Q$$

where  $P \equiv (\phi(t_p), \psi(t_p))$ .

Proof: Essentially the same as that of Lemma 5.6 in [32].

In order for the above theorems to apply in the case when scheme II is used, we scale the matrix  $K_h$  in that case as follows. The rows of  $K_h$  that correspond to irregular mesh points that have one or two neighbours in  $(\Omega)_h$  are multiplied with  $(1+d_1)$  or  $(1+d_1)(1+d_2)(1+d_1+d_2)^{-1}$  respectively. It is easily verified that Lemmas 6.3 and 6.4 hold for scheme II if  $K_m$  is constructed from the scaled  $K_h$ . It will be shown after theorem 6.1 that such a scaling is not essential and our main results will hold even without it.

Lemma 6.5. Let  $\{K_n\}$  be collectively compact on  $X$ ;  $K_n f \rightarrow Kf$  for each  $f \in X$ . Given  $\epsilon > 0$ , let  $\mu_i$ , with algebraic multiplicities  $m_i, i = 1, \dots, N$  be the eigenvalues of  $K$  with absolute values greater than or equal to  $\epsilon > 0$ . Then there exist positive numbers  $N^*$  and  $\epsilon^* < \epsilon$  such that for all  $n \geq N^*$  each  $\epsilon^*$  neighbourhood of  $\mu_i$  contains exactly  $m_i$  eigenvalues of  $K_n$  while all the other eigenvalues of  $K_n$  lie in an  $\epsilon$ -neighbourhood of zero.

Proof: This is an immediate consequence of Theorem 4.8 on p. 65 of [1]. See also Chapter 4 of [30]. By combining Lemmas 6.1, 6.3, 6.4 and 6.5, we easily have the following.

Theorem 6.1 Given  $\epsilon > 0$ , there exists a positive integer  $N$  such that for all  $h > 0$ , all except  $N$  singular values of  $K_h$  lie in  $[0, \epsilon]$ .

Lemma 6.6. Let  $C = AB$ , where  $A, B$  and  $C$  are arbitrary matrices with singular values

$\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_m \geq 0, \beta_1 \geq \beta_2 \geq \dots \geq \beta_m \geq 0$  and  $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_m \geq 0$  respectively, then

$$\gamma_{i+j+1} \leq \alpha_{i+1} \beta_{j+1}, \quad i, j \text{ positive integers.}$$

Proof: See e.g. Exercise 28 on p. 89 of [23]. An immediate consequence of Lemma 6.6 is that Theorem 6.1 holds in the case when scheme II is used even if the matrices  $C$  or  $K_h$  are not scaled by the scaling described just before Lemma 6.5.

Lemma 6.7. If  $D = A+B$ , where  $A$  and  $B$  are as in Lemma 6.6, and  $\delta_1 \geq \delta_2 \geq \dots \geq \delta_m \geq 0$  are the singular values of  $D$ , then

$$\delta_{i+j+1} \leq \alpha_{i+1} + \beta_{j+1}, \quad i, j \text{ positive integers.}$$

Proof: See e.g. Exercise 30 on p. 89 of [23].

Theorem 6.2 Let  $d_1$  and  $d_2$  be the spectral bounds of  $B_h$ . Then given  $\epsilon > 0$ , there exists a positive integer  $N$  independent of  $h$  such that all except  $N$  singular values of  $C$  lie in  $[d_1 - \epsilon, d_2 + \epsilon]$ .

Proof: An immediate consequence of Lemma 6.7 and Theorem 6.1. See also Theorem 5.3 in [32]. In the following,  $\|\cdot\|$  shall denote either the spectral norm of a matrix or the Euclidean norm of a vector.

Lemma 6.8 Let  $U^*$  be the extension operator from  $\Omega_h \cup \partial\Omega_h$  to all mesh points that is defined the same way as  $U$ . Suppose that  $U^{*T} G V \mu \neq 0$  for any nonzero  $m$ -vector  $\mu$  defined on  $\partial\Omega_h$ . Then  $C$  is nonsingular. Moreover, if  $\|U^* G V \mu\| \geq C_1 \|\mu\| / \|A_{11}\|$  for any  $m$ -vector  $\mu$  then  $\|C^{-1}\| \leq \kappa(A_{11}) / C_1$ , where  $\kappa(A_{11})$  is the spectral condition number of  $A_{11}$  with respect to the norm  $\|\cdot\|$ .

Proof: Let  $Au = v = UU^T v$  with  $U^T v$  in the range of  $C$  be the equation we are solving.

From Section 3, we see that  $u = GV\mu$  is a solution of  $Au = v$  if  $\mu$  satisfies Equation (3.6).

Suppose  $C$  is singular so that there exist two distinct solutions  $\mu_1$  and  $\mu_2$  of Equation (3.6).

Let  $\mu_0 \equiv \mu_1 - \mu_2$ . Then  $AGV\mu_0 = 0$ . Because of the reducible structure of  $A$ ,  $A_{11} U^{*T} GV\mu_0 = 0$ .

This contradicts the assumptions that  $A_{11}$  is nonsingular and  $U^{*T} GV\mu_0 \neq 0$ . Moreover, if

$\|U^{*T} GV\mu\| \geq C_1 \|\mu\| / \|A_{11}\|$ , then

$$\|A_{11}^{-1}\| \|U^T v\| \geq \|U^{*T} GV\mu\| \geq C_1 \|\mu\| / \|A_{11}\|.$$

The lemma easily follows.

Definition. A scheme of interpolating boundary conditions is said to be admissible if its corresponding coefficient matrix  $A_{11}$  of the discrete problem is nonsingular and  $\kappa(A_{11}) \leq$  constant  $h^{-2}$ .

Lemma 6.9 Let  $C^*$  and  $A^*$  denote respectively the capacitance matrix and the coefficient matrix of the discrete problem for a certain scheme of interpolating boundary conditions. Suppose that both  $C^*$  and  $A^*$  are nonsingular. Then  $C$  is nonsingular for any admissible scheme of interpolating boundary conditions. Moreover, if  $\|C^{*-1}\| \leq c_2$  and  $\|A^*\| \leq c_3 \|A_{11}\|$ , then  $\|C^{-1}\| \leq c_2 c_3 \kappa(A_{11})$ .



Proof: We first claim that if both  $C^*$  and  $A^*$  are nonsingular, then there exists no  $u \neq 0$  such that  $U^{*T}GV\mu = 0$ . Suppose this is not so, then there exists a  $\mu^* \neq 0$  such that  $U^{*T}GV\mu^* = 0$ . Let  $v^* = UU^T v^*$  where  $C^* \mu^* = U^T v^*$ . Since  $C^*$  is nonsingular,  $U^{*T} v^* \neq 0$ . But  $U^{*T} v^* = U^{*T} A G V \mu^* = A^* U^{*T} G V \mu^* = 0$ .

This proves our claim. By Lemma 6.8,  $C$  is nonsingular. Suppose now that  $\|C^{*-1}\| \leq c_2$  and  $\|A^*\| \leq c_3 \|A_{11}\|$ . Let  $Au = v = UU^T v$ . Clearly,

$$\|\mu\| \leq \|C^{*-1}\| \|U^T v\| \leq \|C^{*-1}\| \|A^*\| \|U^{*T} G V \mu\|.$$

Hence,

$$\|U^{*T} G V \mu\| \geq \|\mu\| / c_2 c_3 \|A_{11}\|.$$

The lemma easily follows from Lemma 6.8.

Definition  $\Omega$  is said to be in  $\mathcal{F}(\beta)$  if the associated integral operator  $K$  defined by Equations (2.2)-(2.3) is such that  $K + K^T > -\beta I$ .

Lemma 6.10. All ellipses with thickness  $b/a > 1/3$  are in  $\mathcal{F}(1)$ . Here  $a$  and  $b$  are respectively the major and minor axes of the ellipses.

Proof: An immediate consequence of (2.5).

Theorem 6.3 Let  $G = B_D^{-1}$  be the discrete Green's function used in equations (3.2) and (3.5). Then the capacitance matrix  $C$  is nonsingular and  $\|C^{-1}\| \leq \text{constant } h^{-q}$  for some positive integer  $q$  independent of  $h$ .

Proof: It suffices to find a pair  $\langle C^*, A^* \rangle$  that satisfies the hypothesis of lemma 6.9. Assume that the difference equations are already preordered in such a way so that

$$B = \begin{pmatrix} B_{11} & B_{12} & 0 \\ B_{21} & B_{22} & B_{23} \\ 0 & B_{32} & B_{33} \end{pmatrix},$$

where the first, second and last rows of  $B$  in block form correspond to the coefficient matrices of the difference equations on  $\Omega_h, \partial\Omega_h$  and  $(C\Omega)_h$  respectively.

Suppose that in forming  $A$  we use a zero order interpolation of Dirichlet data on  $\partial\Omega$  at  $\partial\Omega_h$  to obtain the equations on  $\partial\Omega_h$ . Partition  $V, A$  and  $G$  in the same way as  $B$ . We obtain

$$(6.3) \quad V = \begin{pmatrix} 0 \\ V_2 \\ V_3 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} B_{11} & B_{12} & 0 \\ 0 & I & 0 \\ 0 & B_{32} & B_{33} \end{pmatrix}$$

so that the capacitance matrix which we now denote by  $C_D$  satisfies

$$(6.4) \quad C_D = G_{22} V_2 + G_{23} V_3 .$$

Let

$$A_N = \begin{pmatrix} B_{11} & B_{12} & 0 \\ 0 & V_2^T & V_3^T \\ 0 & B_{32} & B_{33} \end{pmatrix}, \quad A_{1N} = \begin{pmatrix} V_2^T & V_3^T \\ B_{32} & B_{33} \end{pmatrix} .$$

It is easily seen that  $A_{1N}$  is the coefficient matrix of a discrete exterior Neumann problem, with the normal derivative approximated by the first order scheme described on p. 203 of [13].

Hence  $A_N$  is nonsingular since both  $B_{11}$  and  $A_{1N}$  are nonsingular and  $A_N$  is reducible.

Clearly,

$$(6.5) \quad A_N = B + UV^T - UV^T B .$$

Let  $u_N$  be the solution of

$$(6.6) \quad A_N u_N = f_N ,$$

where  $f_N$  is any mesh function that vanishes outside  $\partial\Omega_h$ . Suppose we make the Ansatz that

$$(6.7) \quad u_N = B^{-1} U\rho$$

where  $\rho$  satisfies

$$(6.8) \quad U^T A_N B^{-1} U\rho = U^T f_N .$$

By (6.5), (6.7) and (6.8), it is easily seen that (6.6) is satisfied. Let  $C_N \equiv U^T A_N B^{-1} U$ . It is clear that  $C_N$  is nonsingular. By (6.4) and (6.5), we have

$$C_N^T = D_D$$

so that  $C_D$  is nonsingular. Moreover, using an argument similar to the proof of lemma 6.8, we have

$$\|C_D^{-1}\| = \|C_N^{-1}\| \leq h^{-q}$$

for some positive integer  $q$  independent of  $h$ . The Theorem easily follows.

**Theorem 6.4** Let  $\Omega \in \mathfrak{F}(1)$ . Assume that the  $G$  in equation (3.5) is the discrete Green's function of the entire plane. Then  $\|C^{-1}\| \leq \text{constant } h^{-2}$  as  $h \rightarrow 0$  for any admissible scheme of interpolating boundary conditions. Moreover,  $\|C^{-1}\| \leq \text{constant}$  as  $h \rightarrow 0$  if either scheme I.a or scheme I.b is used or if scheme II is used and  $\Omega \in \mathfrak{F}(0.4)$ .

**Proof:** Let  $\Omega \in \mathfrak{F}(1)$ . Assume that either scheme I.a or scheme I.b is used for interpolating the boundary conditions. By (5.1),  $B_h + B_h^T \geq I$ . By assumption, there exists an  $\varepsilon > 0$  such that  $K + K^T \geq -I + \varepsilon$ . By lemmas 5.1 and 6.3-6.5, we see that for sufficiently small  $h$ ,

$K_h + K_h^T \geq -I + \epsilon/2$ . Hence,  $C + C^T \geq \epsilon/2$  and  $\|C^{-1}\| \leq \text{constant}$  as  $h \rightarrow 0$ . Similarly, it can be shown that  $\|c^{-1}\| \leq \text{constant}$  if scheme II is used and  $\Omega \in \mathfrak{F}(0,4)$ . By lemma 6.9,  $\|c^{-1}\| \leq \text{constant} \cdot h^{-2}$  for any admissible scheme of interpolating boundary conditions.

57. Convergence of conjugate gradient iteration.

Let  $b$  denote the right hand side of the capacitance matrix equation multiplied by  $C^T$ . Let  $Q$  denote  $C^T C$ . We are concerned with solving  $Q\rho = b$  by the conjugate gradient method. Detailed exposition of the method can be found e.g. in [11], [16], [17], [18] and [26]. A brief description of the method plus a simple extension of the known results in the above references can be found in Section 6 of [32]. It will be assumed that the readers are familiar with the results in [32].

Let  $\rho_k$  denote the vectors approximating the solution  $\rho$  generated by the conjugate gradient process. Let  $R$  denote the set of real numbers and  $L_m$  denote the set of  $m$  vectors. Let  $Z = R^2 \rightarrow R$  and  $E : L_m \rightarrow R$  be defined respectively by

$$Z(a,b) = \{(1 - \sqrt{a})/(1 + \sqrt{a})\}^b;$$

$$E(\rho_k) \equiv (1/2)(\rho_k - \rho)^T Q (\rho_k - \rho).$$

It is shown in [32] that the following holds.

Theorem 7.1 Let  $K$  and  $K_1$  be the spectral condition numbers of  $Q$  and  $B_h^T B_h$  respectively. Let  $d^*$  and  $d'$  denote the smallest and largest eigenvalues of  $B_h^T B_h$  respectively. Then given  $\epsilon > 0$ , there exists a positive integer independent of  $k$  and  $h$  such that

$$E(\rho_k)/E(\rho_0) \leq \min\{4Z(K, 2k), 4Z(K_1 - 2\epsilon/d', 2k - 2N)\chi(\lambda)\}.$$

Here  $\chi(\lambda) = \max_{d^* < \lambda < d'} \prod |1 - \lambda/\lambda_i|$ ,  $i = 1, \dots, N$ , where  $\lambda_i$ ,  $i = 1, \dots, N$  are the  $N$  eigenvalues of  $Q$  that lie outside of  $[d^* - \epsilon, d' + \epsilon]$ .

Corollary 7.1 Let  $G = B_D^{-1}$  be used in equation (3.5). The number of iterations needed to reduce  $E(\rho_k)/E(\rho_0)$  to a given accuracy can grow no faster than constant  $\cdot \log m$  as  $h \rightarrow 0$ .

Proof. By Theorem 6.3,  $|\chi(\lambda)| \leq h^{-kN}$ , where  $k$  is a constant independent of  $h$ . The corollary is therefore an easy consequence of Theorem 7.1.

Corollary 7.2 Let  $\Omega \in \mathcal{F}^*(\beta)$ ,  $\beta = 1$  if either scheme I.a or I.b is used;  $\beta = 0.4$  if scheme II is used. Then the number of iterations needed to reduce  $E(\rho_k)/E(\rho_0)$  to a given accuracy stays constant as  $h \rightarrow 0$  if the  $G$  in (3.5) is the discrete Green's function on the entire plane.

Proof. By Theorem 5.1 and Theorem 6.4,  $C$  is uniformly well conditioned in the spectral norm. The corollary is therefore an immediate consequence of Theorem 7.1.

§8 Survey of previous work on capacitance matrix methods.

R. W. Hockney in [20] and [21] described a method of this type which can be used for the solution of the interior Dirichlet problem for Laplace's equation. His capacitance matrices are always positive definite symmetric. His method thus corresponds to a single layer Ansatz for the Dirichlet problem. Buzbee, Dorr, George and Golub used a similar method in [8]. They made the Ansatz

$$u = B^{-1}v + B^{-1}UWp ,$$

when  $B$  is nonsingular. Here  $W$  is a  $m \times m$  nonsingular matrix. The choice  $W = I$  gives the Woodbury formula.

Proskurowski and Widlund introduced the double layer Ansatz in [29]. The algorithm used in their work differs from the one used here only in the discrete Green's function  $G$  and the  $W^T$  matrix. No theoretical analysis was presented in [29]. In [2] the author analyzed the method for the Neumann problem. The algorithm used in [32] is similar to the one used by George in [15] which corresponds to solving the single layer Ansatz of the Dirichlet problem in an iterative imbedding fashion.

59 Numerical experiments

The results in this section were obtained on the CDC 7600 at Lawrence Berkeley Laboratory. The model problem is the Laplace equation on ellipses with  $\gamma = (a-b)/(a+b)$ , where  $a, b$  are the half axes.  $E(u) = \|u - u^*\|_0$ , where  $u^*$  is the true solution of  $-\Delta u = 0$  on  $\Omega$ ,  $u = 1$  on  $\partial\Omega$ . The mesh size  $h = 1/32$ . The number of iterations of the conjugate gradient method is denoted by  $n$ .  $N(R)$  denotes the normalized norm of the residuals which is the  $L_2$  norm of the residual divided by the square root of points in  $\Omega$ . The numbers given for  $E(u)$  are actually upper bounds that describe the number of accurate digits only. The capacitance matrix is generated explicitly and the discrete Green's function on the plane is used in (3.5).

TABLE I

n	$\gamma$	Scheme	I.a	Scheme	I.b	Scheme	II
		N(R)	E(u)	N(R)	E(u)	N(R)	E(u)
4	0.2	---	---	---	---	3.9-04	1.0-03
5	0.2	---	---	---	---	2.1-04	1.0-03
4	1	1.5-04	1.0-03	8.7-03	1.0-02	3.5-04	1.0-03
5	1	1.0-04	1.0-03	4.2-03	1.0-02	1.6-04	1.0-03

In Table I we see that typically it takes four iterations to achieve three digits accuracy. The operation count of the conjugate gradient routine is therefore approximately  $64n^2$ . The total operation count (not counting that of setting up the matrix  $c$ ) is therefore approximately  $5n^2 \log n + 80n^2$  for the Laplace's equation and  $10n^2 \log n + 120n^2$  for the Poisson equation.

#### References

1. P. M. Anselone, Collectively compact operator, Prentice Hall, New Jersey, 1971.
2. R. E. Bank, Marching algorithms for elliptic boundary value problems, Ph.D. Thesis, 1975, Harvard University.
3. R. E. Bank and D. J. Rose, Extrapolated fast direct algorithms for elliptic boundary value problems, TR-7-16, Harvard University.
4. J. H. Bramble and B. E. Hubbard, Approximation of derivatives by finite difference analogues of the Dirichlet problem for elliptic equations, Contr. Diff. Eqns., 2(1963).
5. A. Bromwich, An introduction to the theory of infinite series, McMillian & Co., 1926.
6. O. Buneman, Analytic inversion of the five-point Poisson operator, J. of Comp. Phys. 8, (1971), 500-505.
7. B. L. Buzbee and F. W. Dorr, The direct solution of the biharmonic equation on rectangular regions and the Poisson equation on irregular regions, SIAM J. Numer. Anal., 11 (1974), 753-763.
8. B. L. Buzbee, F. W. Dorr, J. A. George and G. H. Golub, The direct solution of the discrete Poisson equation on irregular regions, SIAM J. Numer. Anal., 8 (1971), 722-736.
9. B. L. Buzbee, G. H. Golub and C. W. Nielson, On direct methods for solving Poisson's equation, SIAM J. Numer. Anal., 7 (1970), 627-656.
10. R. Courant and D. Hilbert, Methods of mathematical physics, Interscience, 1953.
11. J. W. Daniel, The conjugate gradient method for linear and nonlinear operator equations, SIAM J. Numer. Anal.,
12. D. Fischer, G. Golub, O. Hald, C. Leiva and O. Widlund, On Fourier-Toeplitz methods for separable elliptic problems, Math. Comp., 28 (1974), 349-368.
13. G. E. Forsythe and W. R. Wasow, Finite differences methods for partial differential equations, Wiley, 1960.
14. P. R. Garabedian, Partial differential equations, Wiley, 1964.
15. J. A. George, The use of direct methods for the solution of the discrete Poisson equation on nonrectangular regions. Comp. Sci. Dept. Report 159, Stanford University.
16. R. M. Hayes, Iterative methods of solving linear problems on Hilbert space, Nat. Bur. of Standards, Applied Math. Series, 39 (1954), 71-103.

17. M. R. Hestenes, The conjugate gradient method for solving linear systems, Proc. Symp. Appl. Math., 6 (1956), 83-102.
18. M. R. Hestenes and E. Stiefel, Method of conjugate gradients for solving linear systems, J. Res. Nat. Bur. Standards, 49 (1952), 409-436.
19. R. W. Hockney, A fast direct solution of Poisson's equation using Fourier analysis, J. Assoc. Comp. Mach., 12 (1965), 95-113.
20. R. W. Hockney, Formation and stability of virtual electrodes in a cylinder, J. Appl. Phys., 39 (1968), 4166-4170.
21. R. W. Hockney. POT4- A fast direct Poisson solver for the rectangle allowing some mixed boundary conditions and internal electrodes, IBM Research, R.C. 28\*, 1970.
22. R. W. Hockney, The potential calculation and some applications, Methods in computational physics, 9 (1970), Academic Press.
23. A. S. Householder, The theory of matrices in numerical analysis, Blaisdell, 1964.
24. R. A. James, The solution of Poisson's equation for isolated source distribution, to appear.
25. L. V. Kantorovich and V. I. Krylov, Approximate methods of higher analysis, Interscience, 1958.
26. D. G. Luenberger, Introduction to linear and nonlinear programming, Addison-Wesley, 1973.
27. W. H. McCrea and F. J. W. Whipple, Random paths in two or three dimensions, Proc. Roy. Soc. Edinburg, Sect. A, 60, 281-298.
28. I. G. Petrovsky, Partial differential equations, Interscience, 1954.
29. W. Proskurowski and O. Widlund, On the numerical solution of Helmholtz equation by the capacitance matrix method, Math. Comp., 30 (1976), 433-168.
30. A. Shieh, Fast Poisson solvers on nonrectangular regions, Ph.D. Thesis, N.Y.U., 1976
31. A. Shieh, On the convergence of the conjugate gradient method for singular capacitance matrix equations, MRC TSR #1730, Math. Research Center, University of Wisconsin, Madison Wisconsin, 1977.



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