

# Fast Quantizing and Decoding Algorithms for Lattice Quantizers and Codes

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**Abstract**—For each of the lattices  $A_n (n \geq 1)$ ,  $D_n (n \geq 2)$ ,  $E_6$ ,  $E_7$ ,  $E_8$ , and their duals a very fast algorithm is given for finding the closest lattice point to an arbitrary point. If these lattices are used for vector quantizing of uniformly distributed data, the algorithm finds the minimum distortion lattice point. If the lattices are used as codes for a Gaussian channel, the algorithm performs maximum likelihood decoding.

## I. INTRODUCTION

THE SO-CALLED *root lattices* are the  $n$ -dimensional lattices  $A_n (n \geq 1)$ ,  $D_n (n \geq 2)$ , and  $E_n (n = 6, 7, 8)$  defined in Section II. These lattices and their duals give rise to the densest known sphere packing and coverings in dimensions  $n \leq 8$ , and they can be used as the basis for efficient block quantizers for uniformly distributed inputs and to construct codes for a band-limited channel with Gaussian noise (see [6], [9], [11], [16]). Around each lattice point is its *Voronoi region*, consisting of all points of the underlying space which are closer to that lattice point than to any other. (Voronoi regions are also called Dirichlet regions, Brillouin zones, Wigner-Seitz cells, or nearest neighbor regions.) If the lattice is used as a quantizer, all the points in the Voronoi region around the lattice point  $x$  are represented by  $x$ ; while if the lattice is used as a code for a Gaussian channel, all the points in the Voronoi region around  $x$  are decoded as  $x$ . In the preceding paper [6] we found the Voronoi regions for most of the root lattices and their duals, as well as the mean-squared quantization error when these lattices are used to quantize uniformly distributed data.

In the present paper we give very fast and simple algorithms which, for any of the lattices  $A_n$ ,  $D_n$ ,  $E_n$ , and their duals (as well as many other lattices), find the closest lattice point to an arbitrary point of the space. In other words the algorithms find which Voronoi region the given point belongs to. The algorithms can therefore be used either for vector quantizing or for channel decoding. The running time of the algorithms for  $D_n$ ,  $D_n^*$ ,  $E_n$ , and  $E_n^*$  is proportional to  $n$ , while for  $A_n$  and  $A_n^*$  it is proportional to  $n \log n$  and  $n^2 \log n$ , respectively.

Although considerable work has been done in the past on  $n$ -dimensional quantizers and codes (see [1], [3], [8], [9],

[14], [16]–[18], and the references given there), these algorithms appear to be new.

## II. DEFINITION OF ROOT LATTICES AND THEIR DUALS

If  $a_1, \dots, a_n$  are linearly independent vectors in  $m$ -dimensional real Euclidean space  $\mathbb{R}^m$  with  $m \geq n$ , the set of all vectors

$$x = u_1 a_1 + \dots + u_n a_n,$$

where  $u_1, \dots, u_n$  are arbitrary integers, is called an  $n$ -dimensional *lattice*  $\Lambda$ . The *dual* (or *reciprocal*) lattice  $\Lambda^*$  consists of all points  $y$  in the subspace of  $\mathbb{R}^m$  spanned by  $a_1, \dots, a_n$  such that the inner product  $x \cdot y = x_1 y_1 + \dots + x_n y_n$  is an integer for all  $x \in \Lambda$ . Most of the lattices considered here are contained in their duals, so that there are coset representatives  $r_0, \dots, r_{d-1}$  such that

$$\Lambda^* = \bigcup_{i=0}^{d-1} (r_i + \Lambda). \quad (1)$$

The number  $d$  is called the *determinant* of  $\Lambda$ . The *norm* (or *energy*) of a vector  $x = (x_1, \dots, x_m) \in \mathbb{R}^m$  is

$$\|x\| = \sqrt{x \cdot x} = \left( \sum x_i^2 \right)^{1/2}.$$

For further information about lattices see for example [2], [4]–[7], [11], [12], [15], [16].

For  $n \geq 1$ ,  $A_n$  is the  $n$ -dimensional<sup>1</sup> lattice consisting of the points  $(x_0, x_1, \dots, x_n)$  having integer coordinates that sum to zero. (Thus  $A_n$  is an  $n$ -dimensional lattice described by  $(n+1)$ -dimensional coordinates.) If two lattices  $A$  and  $B$  differ only by a rotation and change of scale, we say they are equivalent and write  $A \cong B$ . Then  $A_1$  is equivalent to the one-dimensional lattice of integer points,  $\mathbb{Z}$ , and  $A_2$  is equivalent to the familiar two-dimensional hexagonal lattice (see Fig. 1).

The dual lattice  $A_n^*$  consists of the union of  $n+1$  cosets of  $A_n$ :

$$A_n^* = \bigcup_{i=0}^n (r_i + A_n), \quad (2)$$

<sup>1</sup>The subscript gives the dimension of the lattice.

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where, for  $i = 0, 1, \dots, n$  and  $j = n + 1 - i$ ,

$$r_i = \left( \frac{-j}{n+1}, \frac{-j}{n+1}, \dots, \frac{-j}{n+1}, \frac{i}{n+1}, \dots, \frac{i}{n+1} \right) \\ = \left( \left( \frac{-j}{n+1} \right)^i, \left( \frac{i}{n+1} \right)^j \right). \tag{3}$$

Then  $A_1^* \cong A_1 \cong \mathbb{Z}$  and  $A_2^* \cong A_2$ . Also  $A_3$  ( $\cong D_3$ ) is the face-centered cubic lattice and  $A_3^*$  ( $\cong D_3^*$ ) is the body-centered cubic lattice. It is known that  $A_2$  is the optimal two-dimensional quantizer for a uniformly distributed input, and it is conjectured that  $A_3^*$  is optimal in three dimensions.

For  $N \geq 2$ ,  $D_n$  consists of the points  $(x_1, x_2, \dots, x_n)$  having integer coordinates with an even sum. The dual  $D_n^*$  is the union of four cosets of  $D_n$ :

$$D_n^* = \bigcup_{i=0}^3 (r_i + D_n), \tag{4}$$

where

$$r_0 = (0^n), \quad r_1 = \left( \frac{1}{2}^n \right), \\ r_2 = (0^{n-1}, 1), \quad r_3 = \left( \frac{1}{2}^{n-1}, -\frac{1}{2} \right).$$

Also  $D_2 \cong A_1 \oplus A_1 \cong D_2^*$ ,  $D_3 \cong A_3$ , and  $D_4^* \cong D_4$ .

For example the vectors of small norm in  $D_4$  consist of all permutations and sign changes of the following:

type	number	norm <sup>2</sup>
(0, 0, 0, 0)	1	0
(±1, ±1, 0, 0)	24	2
(±2, 0, 0, 0)	8	4
(±1, ±1, ±1, ±1)	16	4
(±2, ±1, ±1, 0)	96	6
(±2, ±2, 0, 0)	24	8
...	...	...

(5)

(See also [16, Table V].)

There is a second, equivalent definition of  $D_n^*$  which is sometimes easier to use:  $D_n^*$  consists of the points of the  $n$ -dimensional integer lattice  $\mathbb{Z}^n$ , together with the translate of  $\mathbb{Z}^n$  by the vector  $(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$ , i.e.,

$$D_n^* = \mathbb{Z}^n \cup \left( \left( \frac{1}{2}^n \right) + \mathbb{Z}^n \right). \tag{6}$$

The most convenient definitions of  $E_6$ ,  $E_7$ , and  $E_8$  are the following.<sup>2</sup>  $E_8$  is the union of  $D_8$  and the coset

$$\left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) + D_8.$$

In other words  $E_8$  consists of the points  $(x_1, \dots, x_8)$  with  $x_i \in \mathbb{Z}$  and  $\sum x_i$  even, together with the points  $(y_1, \dots, y_8)$  with  $y_i \in \mathbb{Z} + \frac{1}{2}$  and  $\sum y_i$  even. Also  $E_8^* = E_8$ .

$E_7$  is a subspace of dimension 7 in  $E_8$ , consisting of the points  $(u_1, \dots, u_8)$  in  $E_8$  with  $\sum u_i = 0$ . Equivalently

$$E_7 = A_7 \cup \left( \left( -\frac{1}{2}^4, \frac{1}{2}^4 \right) + A_7 \right). \tag{7}$$

The dual  $E_7^*$  is given by

$$E_7^* = E_7 \cup \left( \left( -\frac{3}{4}^2, \frac{1}{4}^6 \right) + E_7 \right) \\ = \bigcup_{i=0}^3 (s_i + A_7), \tag{8}$$

where

$$s_i = \left( \left( \frac{-j}{4} \right)^{2i}, \left( \frac{i}{4} \right)^{2j} \right), \quad i + j = 4. \tag{9}$$

Finally  $E_6$  is a subspace of dimension 6 in  $E_8$ , which we may take for example to consist of the points  $(u_1, \dots, u_8)$  in  $E_8$  with  $\sum u_i = 0$  and  $u_1 + u_8 = 0$ . However this lattice and its dual do not appear to be as important as the others for the applications considered here, and we shall not discuss them in as much detail.

### III. FINDING THE CLOSEST POINT OF THE $n$ -DIMENSIONAL INTEGER LATTICE $\mathbb{Z}^n$

The algorithm for finding the closest point of the integer lattice  $\mathbb{Z}^n$  to an arbitrary point  $x \in \mathbb{R}^n$  is particularly simple, and serves to introduce the notation. For a real number  $x$ , let

$$f(x) = \text{closest integer to } x.$$

In case of a tie, choose the integer with the smallest absolute value. For  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ , let

$$f(x) = (f(x_1), \dots, f(x_n)).$$

For later use we also define  $g(x)$ , which is the same as  $f(x)$  except that the *worst* component of  $x$ —that furthest from an integer—is rounded the *wrong way*. In case of a tie, the component with the lowest subscript is rounded the wrong way.

More formally, for  $x \in \mathbb{R}$  we define  $f(x)$  and the function  $w(x)$  which rounds the wrong way as follows. (Here  $m$  is an integer.)

If $x = 0$ ,	then $f(x) = 0$ , $w(x) = 1$ .
If $0 < m \leq x \leq m + \frac{1}{2}$ ,	then $f(x) = m$ , $w(x) = m + 1$ .
If $0 < m + \frac{1}{2} < x < m + 1$ ,	then $f(x) = m + 1$ , $w(x) = m$ .
If $-m - \frac{1}{2} \leq x \leq -m < 0$ ,	then $f(x) = -m$ , $w(x) = -m - 1$ .
If $-m - 1 < x < -m - \frac{1}{2}$ ,	then $f(x) = -m - 1$ , $w(x) = -m$ .

(10)

<sup>2</sup>A slightly different (although equivalent) definition of  $E_7$  was used in [6]. The present definition leads to a simpler algorithm in Section VII.

(Ties are handled so as to give preference to points of

smaller norm.) We also write

$$x = f(x) + \delta(x),$$

so that  $|\delta(x)| \leq 1/2$  is the distance from  $x$  to the nearest integer.

Given  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ , let  $k$  ( $1 \leq k \leq n$ ) be such that

$$|\delta(x_k)| \leq |\delta(x_i)| \quad \text{for all } 1 \leq i \leq n$$

and

$$|\delta(x_k)| = |\delta(x_i)| \Rightarrow k \leq i.$$

Then  $g(x)$  is defined by

$$g(x) = (f(x_1), f(x_2), \dots, w(x_k), \dots, f(x_n)).$$

*Algorithm 1—To Find the Closest Point of  $\mathbb{Z}^n$  to  $x$ :* Given  $x \in \mathbb{R}^n$ , the closest point of  $\mathbb{Z}^n$  is  $f(x)$ . (If  $x$  is equidistant from two or more points of  $\mathbb{Z}^n$ , this procedure finds the one with the smallest norm.)

To see that the procedure works, let  $u = (u_1, \dots, u_n)$  be any point of  $\mathbb{Z}_n$ . Then

$$\|u - x\|^2 = \sum_{i=1}^n (u_i - x_i)^2,$$

which is minimized by choosing  $u_i = f(x_i)$  for  $i = 1, \dots, n$ . Because of (10) ties are broken correctly, favoring the point with the smallest norm.

#### IV. FINDING THE CLOSEST POINT OF $D_n$

*Algorithm 2—To Find the Closest Point of  $D_n$  to  $x$ :* Given  $x \in \mathbb{R}^n$ , the closest point of  $D_n$  is whichever of  $f(x)$  and  $g(x)$  has an even sum of components (one will have an even sum, the other an odd sum). If  $x$  is equidistant from two or more points of  $D_n$  this procedure produces a nearest point having the smallest norm.

This procedure works because  $f(x)$  is the closest point of  $\mathbb{Z}^n$  to  $x$  and  $g(x)$  is the next closest.  $f(x)$  and  $g(x)$  differ by one in exactly one coordinate, and so precisely one of  $\sum f(x_i)$  and  $\sum g(x_i)$  is even and the other is odd. Again (10) implies that ties are broken correctly.

*Example:* Find the closest point of  $D_4$  to  $x = (0.6, -1.1, 1.7, 0.1)$ . We compute

$$f(x) = (1, -1, 2, 0)$$

and

$$g(x) = (0, -1, 2, 0),$$

since the first component of  $x$  is the furthest from an integer. The sum of the components of  $f(x)$  is  $1 - 1 + 2 + 0 = 2$ , which is even, while that of  $g(x)$  is  $0 - 1 + 2 + 0 = 1$ , which is odd. Therefore  $f(x)$  is the point of  $D_4$  closest to  $x$ .

To illustrate how ties are handled, suppose

$$x = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}).$$

In fact (see (5))  $x$  is now equidistant from eight points of  $D_4$ , namely  $(0, 0, 0, 0)$ , any permutation of  $(1, 1, 0, 0)$ , and

$(1, 1, 1, 1)$ . The algorithm computes

$$f(x) = (0, 0, 0, 0), \quad \text{sum} = 0, \quad \text{even},$$

$$g(x) = (1, 0, 0, 0), \quad \text{sum} = 1, \quad \text{odd},$$

and selects  $f(x)$ . Indeed  $f(x)$  does have the smallest norm of the eight neighboring points.

#### V. FINDING THE CLOSEST POINT OF A COSET OR OF A DUAL LATTICE

A procedure  $\Phi$  for finding the closest point of a lattice  $\Lambda$  to a given point  $x$  can be easily converted to a procedure for finding the closest point of a coset  $r + \Lambda$  to  $x$ . For if  $\Phi(x)$  is the closest point of  $\Lambda$  to  $x$ ,

$$\Phi(x - r) + r$$

is the closest point of  $r + \Lambda$  to  $x$ .

Suppose further that  $L$  is a lattice (or in fact any set of points) which is a union of cosets of  $\Lambda$ :

$$L = \bigcup_{i=0}^{d-1} (r_i + \Lambda).$$

Then  $\Phi$  can be used as the basis for the following procedure for finding the closest point of  $L$ .

*Algorithm 3—To Find the Closest Point of  $L$  (A Union of Cosets of a Lattice) to a Given Point  $x$ :* Given  $x$ , compute

$$y_i = \Phi(x - r_i) + r_i$$

for  $i = 0, 1, \dots, d - 1$ . Compare each of  $y_0, \dots, y_{d-1}$  with  $x$  and choose the closest.

In view of (1), (2), (4), (8) this algorithm reduces the problem of finding the closest point of the dual lattices  $A_n^*$ ,  $D_n^*$  and  $E_6^*$ ,  $E_7^*$ ,  $E_8^*$  to that of finding the closest point of the original lattices. Alternatively, a faster algorithm for  $D_n^*$  is obtained from (6) and Algorithm 1. We illustrate Algorithm 3 by applying it first to  $D_5^*$  using definition (4), and then to  $D_5^*$  using definition (6).

For the first example we observe from (4) that  $D_5^*$  is the union of four cosets of  $D_5$  with coset representatives

$$r_0 = (0, 0, 0, 0, 0),$$

$$r_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}),$$

$$r_2 = (0, 0, 0, 0, 1),$$

$$r_3 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}).$$

To find the closest point of  $D_5^*$  to

$$x = (0.1, 0, -0.3, 0.4, 0.8)$$

we proceed as follows:

$$f(x) = (0, 0, 0, 0, 1), \quad \text{sum} = 1, \quad \text{odd},$$

$$g(x) = (0, 0, 0, 1, 1), \quad \text{sum} = 2, \quad \text{even},$$

therefore

$$y_0 = (0, 0, 0, 1, 1).$$

Also

$$\begin{aligned} \mathbf{x} - \mathbf{r}_1 &= (-0.4, -0.5, -0.8, -0.1, 0.3), \\ f(\mathbf{x} - \mathbf{r}_1) &= (0, 0, -1, 0, 0), \quad \text{sum} = -1, \quad \text{odd}, \\ g(\mathbf{x} - \mathbf{r}_1) &= (0, -1, -1, 0, 0), \quad \text{sum} = -2, \quad \text{even}, \end{aligned}$$

therefore

$$\begin{aligned} \mathbf{y}_1 &= g(\mathbf{x} - \mathbf{r}_1) + \mathbf{r}_1 \\ &= (0.5, -0.5, -0.5, 0.5, 0.5); \\ \mathbf{x} - \mathbf{r}_2 &= (0.1, 0, -0.3, 0.4, -0.2), \\ f(\mathbf{x} - \mathbf{r}_2) &= (0, 0, 0, 0, 0), \quad \text{sum} = 0, \quad \text{even}, \\ g(\mathbf{x} - \mathbf{r}_2) &= (0, 0, 0, 1, 0), \quad \text{sum} = 1, \quad \text{odd}, \end{aligned}$$

therefore

$$\begin{aligned} \mathbf{y}_2 &= f(\mathbf{x} - \mathbf{r}_2) + \mathbf{r}_2 \\ &= (0, 0, 0, 0, 1); \end{aligned}$$

and

$$\begin{aligned} \mathbf{x} - \mathbf{r}_3 &= (-0.4, -0.5, -0.8, -0.1, 1.3), \\ f(\mathbf{x} - \mathbf{r}_3) &= (0, 0, -1, 0, 1), \quad \text{sum} = 0, \quad \text{even}, \\ g(\mathbf{x} - \mathbf{r}_3) &= (0, -1, -1, 0, 1), \quad \text{sum} = -1, \quad \text{odd}, \end{aligned}$$

therefore

$$\begin{aligned} \mathbf{y}_3 &= f(\mathbf{x} - \mathbf{r}_3) + \mathbf{r}_3 \\ &= (0.5, 0.5, -0.5, 0.5, 0.5). \end{aligned}$$

The final step is to see which of  $\mathbf{y}_0, \dots, \mathbf{y}_3$  is closest to  $\mathbf{x}$ . We compute

$$\begin{aligned} \|\mathbf{x} - \mathbf{y}_0\|^2 &= 0.5, \\ \|\mathbf{x} - \mathbf{y}_1\|^2 &= 0.55, \\ \|\mathbf{x} - \mathbf{y}_2\|^2 &= 0.3, \\ \|\mathbf{x} - \mathbf{y}_3\|^2 &= 0.55. \end{aligned}$$

Thus

$$\mathbf{y}_2 = (0, 0, 0, 0, 1)$$

is the point of  $D_3^*$  closest to  $\mathbf{x}$ .

For the second example we use (6) to define the body-centered cubic lattice  $D_3^*$  as the union of two cosets of  $\mathbb{Z}^3$  with coset representatives

$$\begin{aligned} \mathbf{r}_0 &= (0, 0, 0), \\ \mathbf{r}_1 &= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right). \end{aligned}$$

To find the closest point of  $D_3^*$  to

$$\mathbf{x} = (0.2, 0.5, 0.8),$$

we compute

$$\begin{aligned} \mathbf{y}_0 &= f(\mathbf{x}) = (0, 0, 1), \\ \mathbf{x} - \mathbf{r}_1 &= (-0.3, 0, 0.3), \\ f(\mathbf{x} - \mathbf{r}_1) &= (0, 0, 0), \\ \mathbf{y}_1 &= \mathbf{r}_1 + f(\mathbf{x} - \mathbf{r}_1) = (0.5, 0.5, 0.5). \end{aligned}$$

The final step is to find which of  $\mathbf{y}_0$  and  $\mathbf{y}_1$  is closer to  $\mathbf{x}$ :

$$\|\mathbf{x} - \mathbf{y}_0\|^2 = 0.33, \quad \|\mathbf{x} - \mathbf{y}_1\|^2 = 0.18.$$

Thus  $\mathbf{y}_1 = (0.5, 0.5, 0.5)$  is the closest point of  $D_3^*$  to  $\mathbf{x}$ .

Obviously the second definition of  $D_n^*$  leads to the faster algorithm. For any  $n$  it is only necessary to compute  $f(\mathbf{x})$ ,  $f(\mathbf{x} - \mathbf{r}_1)$ , and to calculate the squared norms  $\|\mathbf{x} - \mathbf{y}_0\|^2$  and  $\|\mathbf{x} - \mathbf{y}_1\|^2$ .

## VI. FINDING THE CLOSEST POINT OF $E_8$

Since  $E_8$  is the union of two cosets of  $D_8$  the discussion in the previous section leads to the following procedure.

*Algorithm 4—To Find the Closest Point of  $E_8$  to  $\mathbf{x}$ :* Given  $\mathbf{x} = (x_1, \dots, x_8) \in \mathbb{R}^8$ .

Compute  $f(\mathbf{x})$  and  $g(\mathbf{x})$ , and select whichever has an even sum of components; call it  $\mathbf{y}_0$ .

Compute  $f(\mathbf{x} - \frac{1}{2})$  and  $g(\mathbf{x} - \frac{1}{2})$ , where

$$\frac{1}{2} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right),$$

and select whichever has an even sum of components; add  $\frac{1}{2}$  and call the result  $\mathbf{y}_1$ .

Compare  $\mathbf{y}_0$  and  $\mathbf{y}_1$  with  $\mathbf{x}$  and choose the closest.

For example, to find the closest point of  $E_8$  to

$$\mathbf{x} = (0.1, 0.1, 0.8, 1.3, 2.2, -0.6, -0.7, 0.9),$$

we compute

$$\begin{aligned} f(\mathbf{x}) &= (0, 0, 1, 1, 2, -1, -1, 1), \quad \text{sum} = 3, \quad \text{odd}, \\ g(\mathbf{x}) &= (0, 0, 1, 1, 2, 0, -1, 1), \quad \text{sum} = 4, \quad \text{even}, \end{aligned}$$

and take  $\mathbf{y}_0 = g(\mathbf{x})$ . Also

$$\mathbf{x} - \frac{1}{2} = (-0.4, -0.4, 0.3, 0.8, 1.7, -1.1, -1.2, 0.4),$$

$$f(\mathbf{x} - \frac{1}{2}) = (0, 0, 0, 1, 2, -1, -1, 0), \quad \text{sum} = 1, \quad \text{odd},$$

$$g(\mathbf{x} - \frac{1}{2}) = (-1, 0, 0, 1, 2, -1, -1, 0), \quad \text{sum} = 0, \quad \text{even},$$

and so

$$\begin{aligned} \mathbf{y}_1 &= g(\mathbf{x} - \frac{1}{2}) + \frac{1}{2} \\ &= (-0.5, 0.5, 0.5, 1.5, 2.5, -0.5, -0.5, 0.5). \end{aligned}$$

Finally,

$$\|\mathbf{x} - \mathbf{y}_0\|^2 = 0.65, \quad \|\mathbf{x} - \mathbf{y}_1\|^2 = 0.95$$

and we conclude that  $\mathbf{y}_0 = g(\mathbf{x})$  is the closest point to  $\mathbf{x}$ .

## VII. FINDING THE CLOSEST POINT OF $A_n, A_n^*, E_7$ , AND $E_7^*$

*Algorithm 5—To Find the Closest Point of  $A_n$  to  $\mathbf{x}$ :*

*Step 1:* Given  $\mathbf{x} \in \mathbb{R}^{n+1}$ , compute  $s = \sum x_i$  and replace  $\mathbf{x}$  by

$$\mathbf{x}' = \mathbf{x} - \frac{s}{n+1}(1, 1, \dots, 1).$$

*Step 2:* Calculate  $f(\mathbf{x}') = (f(x'_0), \dots, f(x'_n))$  and the deficiency  $\Delta = \sum f(x'_i)$ .

*Step 3:* Sort the  $x'_i$  in order of increasing value of  $\delta(x'_i)$  (defined in Section III). We obtain a rearrangement of the numbers  $0, 1, \dots, n$ , say  $i_0, i_1, \dots, i_n$ , such that

$$-\frac{1}{2} \leq \delta(x'_{i_0}) \leq \delta(x'_{i_1}) \leq \dots \leq \delta(x'_{i_n}) \leq \frac{1}{2}.$$

*Step 4:* If  $\Delta = 0$ ,  $f(x')$  is the closest point of  $A_n$  to  $x$ .  
 If  $\Delta > 0$ , the closest point is obtained by subtracting 1 from the components  $f(x'_{i_0}), \dots, f(x'_{i_{\Delta-1}})$ .  
 If  $\Delta < 0$ , the closest point is obtained by adding 1 to the components  $f(x'_{i_n}), f(x'_{i_{n-1}}), \dots, f(x'_{i_{n-\Delta+1}})$ .

*Remarks:* We know from Section III that  $f(x)$  is the closest point of  $\mathbb{Z}^{n+1}$  to  $x$ . The procedure described here finds the closest point of  $A_n$  because it makes the smallest changes to the norm of  $f(x')$  needed to make  $\Sigma f(x'_i)$  vanish.

Step 1 projects  $x$  onto  $x'$ , the closest point of the hyperplane  $\Sigma x_i = 0$ . Since  $A_n$  is by definition contained in this hyperplane it may be possible to assume that  $x$  already lies there, in which case Step 1 can be omitted.

The only substantial amount of computation needed is for the sort in Step 3, which takes  $O(n \log n)$  steps [10], [13]. However Step 3 can be omitted if  $x$  is expected to be close to  $A_n$ . In this case  $\Delta$  will be small, and Steps 3 and 4 can be replaced by the following:

*Step 3':* If  $\Delta = 0$ ,  $f(x')$  is the closest point of  $A_n$  to  $x$ .

If  $\Delta > 0$ , find the  $\Delta$  components of  $x'$ , say  $x'_{i_0}, \dots, x'_{i_{\Delta-1}}$ , for which  $\delta(x'_i)$  is as small (i.e., as close to  $-\frac{1}{2}$ ) as possible. The closest point of  $A_n$  is obtained by subtracting one from the components  $f(x'_{i_0}), \dots, f(x'_{i_{\Delta-1}})$  of  $f(x)$ .

If  $\Delta < 0$ , find the  $|\Delta|$  components of  $x'$ , say  $x'_{i_n}, x'_{i_{n-1}}, \dots, x'_{i_{n-\Delta+1}}$ , for which  $\delta(x'_i)$  is as large (i.e., as close to  $\frac{1}{2}$ ) as possible. The closest point of  $A_n$  is obtained by adding 1 to the components of  $f(x'_{i_n}), \dots, f(x'_{i_{n-\Delta+1}})$  of  $f(x)$ .

In any case  $|\Delta|$  cannot exceed  $n/2$ . However if  $\Delta$  is expected to be large the first version of the algorithm is preferable.

The closest point of  $A_n^*$  can be found by Algorithm 3, using the fact that  $A_n^*$  is the union of  $n + 1$  cosets of  $A_n$  (see (2)).

For example the hexagonal lattice  $A_2$  is shown in Fig. 1, together with ordinary two-dimensional coordinates  $(u_1, u_2)$  for the points. The three-dimensional coordinates  $(x_0, x_1, x_2)$  with  $x_0 + x_1 + x_2 = 0$  that we have used are obtained by multiplying  $(u_1, u_2)$  on the right by the matrix

$$M = \begin{pmatrix} 1 & 0 & -1 \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

Conversely the  $u$ -coordinates may be obtained from the  $x$ -coordinates by

$$(u_1, u_2) = (x_0, x_1, x_2) \cdot \frac{1}{2} M^T.$$

For example the points  $(0, 0), (1, 0), (1/2, \sqrt{3}/2), (-1/2, \sqrt{3}/2)$  have  $x$ -coordinates  $(0, 0, 0), (1, 0, -1), (1, -1, 0), (0, -1, 1)$ , respectively. To find the closest point of  $A_n$  to the point  $P$  with coordinates

$$(u_1, u_2) = (0.4, -0.4)$$

we first find the  $x$ -coordinates of  $P$ , which are

$$x = (x_0, x_1, x_2) = (0.169, 0.462, -0.631).$$

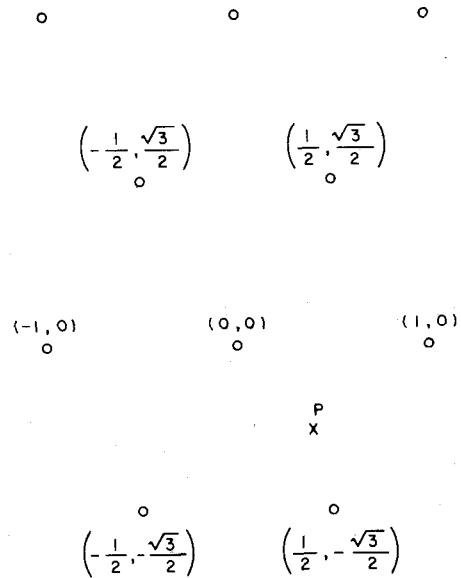


Fig. 1. The hexagonal lattice  $A_2$ .

Step 1 of the algorithm can be omitted, since  $x_0 + x_1 + x_2 = 0$  holds automatically. Step 2 produces

$$f(x) = (0, 0, -1),$$

with deficiency  $\Delta = -1$ . At Step 3 we find

$$\delta(x_0) = 0.169 < \delta(x_2) = 0.369 < \delta(x_1) = 0.462.$$

At Step 4 we add 1 to  $f(x_1)$ , obtaining

$$(0, 1, -1)$$

which is the closest point of  $A_2$ . The  $u$ -coordinates for this point are

$$(0, 1, -1) \cdot \frac{1}{2} M^T = \left( \frac{1}{2}, -\frac{\sqrt{3}}{2} \right)$$

(see Fig. 1).

Since  $A_3 \cong D_3$ , Algorithm 2 is preferable to Algorithm 5 for finding the closest point of the face-centered cubic lattice. Finally  $E_7$  and  $E_7^*$  can be handled via the algorithm for  $A_7$ , using (7), (8), and Algorithm 3.

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# The Design of Two-Dimensional Quantizers using Prequantization

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**Abstract**—The theoretical advantages of two-dimensional quantization over univariate quantization have been studied in the literature. However, in many cases there is no known implementation for the two-dimensional quantizer that can operate in real time. A new approach to the design of two-dimensional quantizers is presented. This technique, called prequantization, is used to design two-dimensional quantizers that operate in real time. The importance of prequantization is demonstrated by the design of the optimum uniform two-dimensional (hexagonal) quantizer. Additional examples are given to illustrate the flexibility of this design approach.

## I. INTRODUCTION

THE USE OF two-dimensional quantizers for encoding analog sources has been of increasing interest in recent years. Two-dimensional quantizers can offer advantages in the design of both optimum and suboptimum quantizers. These advantages may be offset by the difficulty in implementing many two-dimensional quantizers. In this paper we present a new approach to the design of two-dimensional quantizers called prequantization. We show that for a number of examples prequantization simplifies the quantizer implementation and/or improves the quantizer performance.

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The design of two-dimensional quantizers for optimum quantization is one area of interest. Consider the random sequence  $x_1, x_2, x_3, \dots$  where the  $x_i$  are all independent and identically distributed. The traditional approach to quantizing this sequence is to perform the quantization one sample at a time using a one-dimensional quantizer. Much of the early work in quantization theory has addressed this problem. As a result the design and implementation of optimum one-dimensional quantizers is straightforward. In addition these quantizers are often able to operate at high source rates. These properties make one-dimensional quantization an attractive choice for quantizing the above sequence. The advantage of quantizing the independent identically distributed (i.i.d.) sequence in two or more dimensions is discussed by Zador [1]. Simply stated, these results indicate that the minimum obtainable per sample distortion decreases as the quantizer dimension is increased. Therefore, the potential exists to improve the performance of digital encoders by replacing one-dimensional quantizers with two-dimensional quantizers.

Zador's results include derivations of both the upper and lower bounds on the distortion obtained when using an optimum quantizer. Unfortunately, these results do not provide insight into the structure of the quantizer. The design and implementation of optimum two-dimensional quantizers remains a largely unsolved problem. Recently the design of two-dimensional quantizers has been addressed. Computer algorithms for designing optimum quantizers of two or more dimensions have been presented