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TITLE: **FAST RANDOMIZED POINT LOCATION WITHOUT PREPROCESSING IN TWO- AND THREE-DIMENSIONAL DELAUNAY TRIANGULATIONS**

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Fast Randomized Point Location Without Preprocessing in Two- and Three-dimensional Delaunay Triangulations

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Abstract

This paper studies the point location problem in Delaunay triangulations without preprocessing and additional storage. The proposed procedure finds the query point simply by "walking through" the triangulation, after selecting a "good starting point" by random sampling. The analysis generalizes and extends a recent result for $d = 2$ dimensions by proving this procedure to take expected time close to $O(n^{1/(d+1)})$ for point location in Delaunay triangulations of n random points in $d = 3$ dimensions. Empirical results in both two and three dimensions show that this procedure is efficient in practice.

1 Introduction

Point location is one of the classical problems in computational geometry and has various applications of practical relevance, e.g., in the areas of geographic information systems (GIS) or computer-aided design and engineering (CAD/CAE). The problem is well studied in the computational geometry literature and several theoretically optimal algorithms have been proposed. Unfortunately, algorithms that are optimal in theory do not necessarily yield to good practical performance. This is also true in the case of point location, mainly because of the necessary preprocessing time and additional storage requirements.

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The best known practical algorithm uses "bucketing" and is due to Asano et al. [AEI+85]. It achieves optimal logarithmic time complexity; however, it, too, requires some extra preprocessing, especially within each bucket, and additional storage. Actual engineering implementations also often use tree structures to guide the point location, e.g., the "alternating digital tree" described in Bonet and Peraire [BP91]. Obviously they, as well, require building and maintaining additional data structures. Here, we will discuss a technique that is efficient in practice, uses *no* preprocessing time, *no* additional storage, and, as a bonus, could not be easier to implement.

The point location problem in its full generality deals with locating query points in arbitrary subdivisions. This work, however, focuses on point location in triangulations (in fact, the analysis is even further restricted to Delaunay triangulations of random points). This is justified because regions of arbitrary subdivisions can be triangulated; moreover, the query problem in triangulations itself occurs quite frequently in practice, e.g., in mesh generation and finite-element analysis (FEA).

Simple walk-through. The basic idea is straightforward and not at all new; it goes back to early papers on constructing Delaunay triangulations in 2D and 3D [GS78, Bow81]. The underlying assumption is that the Delaunay triangulation $\mathcal{D}$ of a set $X \in \mathbb{R}^d$ of n points is given by an internal representation such that constant-time access between neighboring simplices (i.e., triangles for $d = 2$, tetrahedra for $d = 3$) is possible. This can be achieved by using, e.g., the 2D quad-edge data structure [GS85], the edge-facet structure in 3D [DL89], its specialization and compactification to the domain of 3D triangulations [E93], or its generalization to d dimensions [Bri93]. Now, in order to locate a query point q, select some simplex of $\mathcal{D}$, consider the line segment L, from a vertex of the initial simplex to the query point q, and simply "walk towards" q by traversing all simplices intersected by L.

This method has been ignored by most theoreticians in computational geometry since not much can be said about its performance theoretically, other than it is "expected" to take time proportional to $n^{1/d}$ when the points are randomly distributed [GS78, Bow81]. However, because of its exceptional simplicity, the method is indeed used by practitioners in the geometric computing community, in particular, in FEA, e.g., [GH92].

Improved jump-and-walk. We can improve the simple walk-through by "jumping" to a "good starting point" via random sampling on the point set $\{X_1, X_2, \ldots, X_n\}$. Given the Delaunay triangulation $\mathcal{D}$ of these n points, and a query point q, the following procedure locates the simplex of $\mathcal{D}$, if any, which contains q.

(1) Select m points $Y_1, \ldots, Y_m$ at random and without replacement from $X_1, \ldots, X_n$.
(2) Determine the index $j \in \{1, \ldots, m\}$ minimizing the distance $d(Y_j, q)$. Set $Y = Y_j$.
(3) Locate the simplex containing q by traversing all simplices intersected by the line segment (Y, q).

Step (3), i.e., the simple walk-through, can be implemented in constant time per simplex visited, once the initial simplex, intersected by L and incident to "starting point" Y, is determined.

Motivated by the positive empirical results of [E93], where the jump-and-walk is used to implement the randomized incremental flip algorithm to construct 3D Delaunay triangulations, this procedure was recently analyzed for $\mathbb{R}^2$, with the result that the expected query time is $O(n^{1/3})$ when the points are randomly distributed [DMZ95]. This result, in turn, builds on the work of Bose and Devroye [BD95] who prove that for any line segment L the expected number of intersected triangles in proportional to $|L|n^{1/2}$.

In the following, we extend both results to $\mathbb{R}^3$, showing that jump-and-walk point location in spatial Delaunay triangulations of n random points has an expected running time of $O\left(\delta(n)^{1/4}\, n^{1/4}\, (\log n/\log\log n)^{3/4}\right)$, where $\delta(n)$ denotes the *expected* degree of a Delaunay vertex. A result of Bern et al. [BEY91] on the expected *maximum* degree would give $\delta(n) = O(\log n/\log\log n)$. On the other hand, Dwyer [Dwy91] shows that $\delta(n) = O(1)$ for any fixed dimension d, assuming that the points are chosen uniformly at random in a d-dimensional ball. In any case, it is always a fair assumption that Delaunay triangulations occurring in problems of practical relevance are only of linear size (rather than worst-case quadratic size), and we can immediately argue that $\delta(n)$ is constant for all practical purposes, yielding an expected running time close to $O(n^{1/4})$. This compares well to the theoretically optimal $O(\log n)$ bound, at least for practical sizes of input data; e.g., $n^{1/4}/\log_2 n < 2.5$, for n in the range up to 10^7.

On a theoretical side, our work addresses and solves two difficult issues. First, when proving "probabilistic impossibility results" for Delaunay triangulations one is naturally led to define volumes and to argue that these volumes are likely to contain some Delaunay vertices. One must be careful though to define (as much as possible) these volumes *independently* from the vertices. We achieve this difficult task in 3D. Second, the perturbing effect of the boundary is very well-known. The probabilistic model of [BEY91], for instance, was designed to analyze typical properties of Delaunay triangulations away from the boundary. Here, we provide a specific estimate of the range of this perturbation. Our methods seem well suited to bring even more precise results.

Outline. The paper is organized as follows. In Sections 2 and 3, we first generalize the result of [BD95] regarding the intersection of a line segment with a random Delaunay triangulation to 3D. Then, we generalize the proof of [DMZ95] to 3D. In Sections 4 and 5, we present empirical results over randomly generated point sets ranging from $n = 1000$ to 50000. Our tests confirm that the method is efficient in practice, and is also comparable with the optimal $O(\log n)$, at least in the above range, which seems to be of most relevance for practitioners in GIS and CAD.

2 Statement of Results

Let C be a convex domain of $\mathbb{R}^3$ and let α and β be two reals such that $0 < \alpha < \beta$. We say that a probability measure P is an (α, β)-*measure* over C if $P[C] = 1$ and if we have $\alpha\,\lambda(S) \le P[S] \le \beta\,\lambda(S)$ for every measurable subset S of C, where λ is the usual Lebesgue measure.[1] An $\mathbb{R}^3$-valued random variable X is called an (α, β)-random variable over C if its probability law $\mathcal{L}(X)$ is an (α, β)-measure over C. A particular and important example of an (α, β)-measure P is when P is a probability measure with density $f(x)$ such that $\alpha \le f(x) \le \beta$ for all $x \in C$. One of the advantages of our more general notion is that it allows for a probability measure charging only points with rational coordinates: this is the case for most computer simulations. This probabilistic model was introduced in [BD95]. The Poisson model of [BEY91] is related to ours in the sense that, conditioned on the number n of points observed over a finite volume, the probability distribution is uniform, i.e., an (α, α)-measure.

Below is our main result on the expected running time of the jump-and-walk algorithm, when applied on $\mathcal{D}$, the Delaunay triangulation of n random points $X_1, X_2, \ldots, X_n$ in $\mathbb{R}^3$.

Theorem 1. Let C be a bounded convex set of $\mathbb{R}^3$ having small curvature. Let $X_1, \ldots, X_n$ be n points drawn independently in C from an (α, β)-measure. Then there exist constants c_1, c_2 and c_3 depending only upon α, β and C such that the following holds. Assume that $m \ge n^{1/5}$ and that the query point is selected independently of $X_1, \ldots, X_n$ and is at distance of at least

[1] Note that the relation $\lambda(C) \le 1/\alpha < \infty$ implies that C has finite area. The convexity of C then implies that C is bounded (i.e., that C is included in some finite ball.)

$c_1/n^{1/18}$ from the boundary ∂C. Then the expected time of the jump-and-walk algorithm is bounded by

$$c_2 m\delta(n) + c_3(n/m)^{1/3}\ \log n/\log\log n\,,$$

where $\delta(n)$ is the expected vertex degree of the Delaunay triangulation. In particular, the expected time is optimized to $O\left(\delta(n)^{1/4}\, n^{1/4}\, (\log n/\log\log n)^{3/4}\right)$ with the choice of $m = \Theta\left(n^{1/4}/\delta(n)^{3/4}\, (\log n/\log\log n)^{3/4}\right)$.

The proof of Theorem 1 rests on the following theorem.

Theorem 2. Let C be a bounded convex set of $\mathbb{R}^3$ having small curvature. Let $X_1, \ldots, X_n$ be n points drawn independently in C from an (α, β)-measure. Then there exist constants c_4 and c_5 depending only upon α, β and C such that the following holds. Let L be a segment in C being at distance of at least $c_4(\log n/n)^{1/3}$ from the boundary ∂C. Let N be the number of intersections between L and $\mathcal{D}$. Then:

$$E[N] \le c_5\,(1 + |L|)\, n^{1/3}\ \log n/\log\log n\,.$$

We can easily extend Theorem 2 to the case where L is a random segment *independent* of the n points $X_1, \ldots, X_n$. For this, define the event $B = \{d(L, \partial C) \ge c_4(\log n/n)^{1/3}\}$. We then have $E\left[N \mid B\right] \le c_5\left(1 + E[|L| \mid B]\right) n^{1/3}\ \log n/\log\log n$. In Section 3, we first prove Theorem 2 following the same ideas as [BD95]; however, we would like to point out that the technical details are quite different in 3D and more difficult. Given Theorem 2, it is easy to generalize the result of [DMZ95] to obtain Theorem 1.

3 Probabilistic Analysis

For every bounded domain D of $\mathbb{R}^3$ let $\mathcal{B}(D)$ denote the smallest 3-dimensional ball containing D; we will say that $\mathcal{B}(D)$ is the *canonical* ball circumscribed to D. Also, for every point x and $r \ge 0$ let $B(x, r)$ denote the ball of radius r centered at point x. A plane xyz, passing through the center of a ball $\mathcal{B}$, cuts $\mathcal{B}$ into two hemispheres. We will sometimes for emphasis speak of xyz-hemispheres and refer to xyz as the equator of $\mathcal{B}$. The following property of a Delaunay triangulation which will be used in the sequel; if follows straightforwardly from the definitions.

Lemma 3. Consider the Delaunay triangulation of n points in $\mathbb{R}^3$ and a triangle F defined by three of these points. Then, F cannot be a Delaunay face if the open ball $\mathcal{B}(F)$ contains (at least) one point in both F-hemispheres above and below F.

We select now two positive numbers k_1 and k_2 with $k_1 < k_2$ such that the following holds. Consider the situation described in Figure 1. We have a circle of radius $Oq = 1/2$ centered at a point O. The two spokes qr and qs are of length k_1 and form a 45^o angle. A ball A_1 of diameter k_2 is attached at the end of qr; a similar ball A_2 is attached at the end of qs. The ball A_1 is tangent to the line Oq. We begin with assigning to k_1 and k_2 two values such that $k_1 + k_2 < 1/2$ and such that the ball A_2 lies in the *interior* of the ball $B(O, 1/2)$. We will add further restrictions to k_1 and k_2 when necessary below.

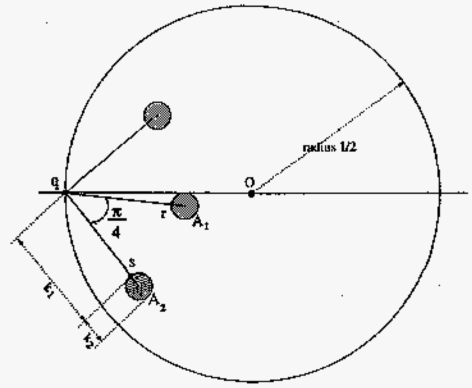

Figure 1: Defining k_1 and k_2.

Let l be a positive number. We define an *l-spindle* to be a geometric object composed of an axle surrounded by 8 concentric little balls:

- The axle is a segment of length l.
- At the middle of the axle are attached 8 spokes of length $k_1 l$. The spokes are in a plane perpendicular to the axle and are placed in a regular octagonal fashion (i.e., at 45 degrees from each other).
- At the end of each spoke is attached a little ball of diameter $k_2 l$. Hence the distance from the axle to the outer side of a little ball is $(k_1 + k_2)l$.

The collection of the 8 little balls is called the *wheel* of the spindle.

Lemma 4. There exists a number $k_3, 0 < k_3 < 1$ such that the following holds. Let $x, y, x_1, \ldots, x_n$ be points of $\mathbb{R}^3$. We let $l = d(x, y)$ denote the distance between x and y. Let S be any l-spindle whose axle is xy. Let $A_1, \ldots, A_8$ be the 8 balls of the associated wheel. Consider the Delaunay triangulation based on the points $x, x_1, \ldots, x_n$. Then no Delaunay triangulation face incident to x touches or crosses the ball $B(y, k_3 l)$ when all 8 balls $A_1, \ldots, A_8$ each contain a point x_j.

Proof: **Base case:** We consider first the case where y is also a point x_i and show that there is no Delaunay face xyz incident to both x and y (i.e., for which xy is an edge) when all 8 balls $A_1, \ldots, A_8$ each contain a point x_j.

Working by contradiction, assume the existence of such a Delaunay face xyz. Consider the canonical ball $\mathcal{B}(xyz)$ circumscribed to xyz. The center of $\mathcal{B}(xyz)$ is equidistant to x and y and therefore lies in the plane $\mathcal{P}_0$ per-

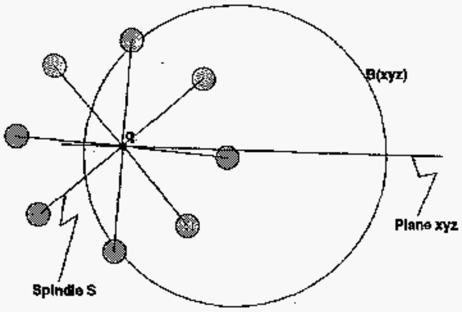

Figure 2: The base case for Lemma 4.

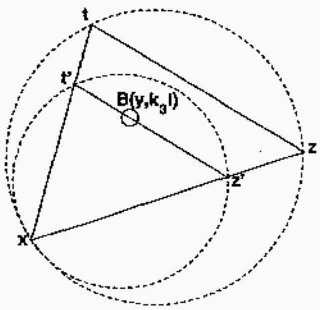

Figure 3: The edge $t'z'$ intersects $\mathcal{B}(y, k_3 l)$

pendicular to xy and going through the middle point q of x and y: this plane contains also the centers of the 8 balls $A_1, \ldots, A_8$. We can therefore visualize the situation within $\mathcal{P}_0$ as represented in Figure 2.

By definition, the point q is the center of the spindle and lies in the plane xyz *inside* $\mathcal{B}(xyz)$. The radius of $\mathcal{B}(xyz)$ is at least equal to the distance $d(x,y)/2 = l/2$. By our choice of k_1 and k_2, $\mathcal{B}(xyz)$ contains (at least) one ball A_i in each $\mathcal{P}_1$-hemisphere.

An argument by continuity (omitted here) shows that there exists a value $k_3 > 0$ such that, for every $y' \in \mathcal{B}(y, k_3 l)$, and for every $z \in \mathbb{R}^3$, the canonical ball $\mathcal{B}(xy'z)$ similarly contains (at least) one ball A_i in each of the hemispheres defined by $xy'z$.

General case: We fix k_3 to be (one of) the value(s) just found. Let xtz be a triangle incident to x and crossing the ball $\mathcal{B}(y, k_3 l)$. Let $\mathcal{P}_1$ denote the plane defined by xtz. We now set out to prove that the canonical ball $\mathcal{B}(xtz)$ contains at least one ball A_i in each of its $\mathcal{P}_1$-hemispheres.

(a) Assume that the edge tz does not intersect $\mathcal{B}(y, k_3 l)$. Consider a sub-triangle $xt'z'$ of xtz obtained from xtz by homothetie (i.e., scaling) through x and such that $t'z'$ intersects $\mathcal{B}(y, k_3 l)$; see Figure 3. As the canonical ball $\mathcal{B}(xt'z')$ is included in $\mathcal{B}(xtz)$ it suffices to show that $\mathcal{B}(xt'z')$ contains at least one ball A_i in each of its $\mathcal{P}_1$-hemispheres. This allows us to reduce the analysis to when tz intersects $\mathcal{B}(y, k_3 l)$.

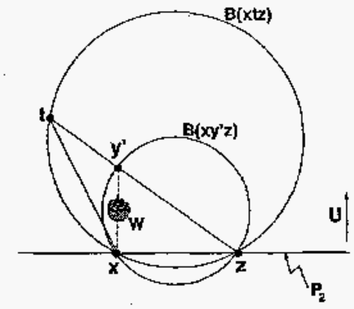

Figure 4: The canonical balls $\mathcal{B}(xtz)$ and $\mathcal{B}(xy'z)$, the wheel $\mathcal{W}$ of the spindle surrounded by $\mathcal{B}(q, (k_1 + k_2)l)$, the plane $\mathcal{P}_2$, the half-space U.

(b) Assume now that neither t nor z are contained in $\mathcal{B}(y, k_3 l)$ and consider the case where the ball $\mathcal{B}(q, (k_1 + k_2)l)$ intersects both xt and xz.[2] This happens only if one of the three angles $\widehat{xtz}, \widehat{tzx}$ or $\widehat{zxt}$ is "small." This implies that $\mathcal{B}(xtz)$ contains a "big" cap of each $\mathcal{P}_1$-hemisphere of $\mathcal{B}(\mathcal{W})$, and more specifically, that, in this case, the center o of $\mathcal{B}(xtz)$ is either close to q, in which case $\mathcal{B}(xtz)$ fully contains $\mathcal{B}(q, (k_1 + k_2)l)$, or is very far, in which case $\mathcal{B}(xtz)$ cuts close to half of $\mathcal{B}(q, (k_1 + k_2)l)$. (The details are omitted here.)

(c) We can therefore restrict ourselves to the case where $\mathcal{B}(q, (k_1 + k_2)l)$ does not intersect either with xt or with xz. Assume without loss of generality that $\mathcal{B}(q, (k_1 + k_2)l)$ does not intersect with xz; see Figure 4. Consider the plane $\mathcal{P}_2$ containing xz and perpendicular to the plane xtz. The fact "$\mathcal{B}(q, (k_1 + k_2)l)$ does not intersect xz" means that $\mathcal{B}(q, (k_1 + k_2)l)$ (and hence the spindle $\mathcal{S}$) is fully contained in one of the two half-spaces defined by $\mathcal{P}_2$. Let U denote that half-space: $\mathcal{B}(q, (k_1 + k_2)l) \subseteq U$. For every point y' in the segment tz one verifies that $\mathcal{B}(xy'z) \cap U \subseteq \mathcal{B}(xtz) \cap U$. Therefore a ball A_i of the spindle $\mathcal{S}$ is included in the northern (resp. southern) $\mathcal{P}_1$-hemisphere of $\mathcal{B}(xy'z)$ only if it is included in the northern (resp. southern) $\mathcal{P}_1$-hemisphere of $\mathcal{B}(xtz)$.

We now specialize y' to be a point in the intersection of tz and $\mathcal{B}(y, k_3 l)$. By the discussion in the base case above, $\mathcal{B}(xy't)$ contains at least one ball A_i in each of its $\mathcal{P}_1$-hemispheres. We conclude that $\mathcal{B}(xtz)$ similarly contains at least one ball A_i in each of its $\mathcal{P}_1$-hemispheres.

In summary, we have established that, for every triangle xtz incident to x and crossing the ball $\mathcal{B}(y, k_3 l)$, the canonical ball $\mathcal{B}(xtz)$ contains at least one ball A_i in each of its $\mathcal{P}_1$-hemispheres. The following application of Lemma 3 closes the argument: no triangle xtz incident to x and crossing the ball $\mathcal{B}(y, k_3 l)$ is a Delaunay face when all 8 balls $A_1, \ldots, A_8$ each contain a point x_j. ■

[2] Recall that $\mathcal{B}(q, (k_1 + k_2)l)$ is the smallest 3-dimensional ball $\mathcal{B}(\mathcal{W})$ containing the spindle wheel $\mathcal{W}$.

Lemma 5. Let X_1 be a random variable drawn from an (α, β)-measure over a bounded convex set C. Then there exist constants $r_0 > 0$ and $\gamma > 0$ such that for every $r \leq r_0$

$$\inf_{y \in C} P\left[\, d(X_1, y) \leq r \,\right] \geq \gamma r^3 .$$

Proof: This follows easily from the (α, β)-measure hypothesis and from the convexity of C. ■

Lemma 6. Consider n points $X_1, \ldots, X_n$ drawn independently from an (α, β)-measure over a bounded convex set C with small curvature. Then there exist positive constants a, b, c and d, depending upon α, β and C only, such that the following holds. Let y be any point in C at distance $a(\log n/n)^{1/3}$ from the boundary ∂C. Let $r, r \leq 1/2\, d(y, \partial C)$ be a positive quantity. Let N_1 denote the number of X_i's with the property that one of its incident Delaunay faces intersects $\mathcal{B}(y, r)$. Then

$$E[N_1] \leq b + c r^2 n^{2/3} + d n r^3 .$$

Proof: By linearity of the expectation, $E[N_1] = np$ where p is the probability that X_1 has one of its Delaunay faces intersecting the ball $\mathcal{B}(y, r)$. Let $L = d(X_1, y)$; L is itself a random variable. In the following we condition on the value of X_1. The points X_1 and y are then fixed, and we let $\mathcal{S}_{X_1}$ be any L-spindle whose axle is $X_1 y$. As before, $A_1, \ldots, A_8$ denote the 8 little balls of the wheel. Remark that the event $\{\mathcal{S}_{X_1} \subseteq C\} = \{$the 8 balls of the spindle $\mathcal{S}_{X_1}$ are included in $C\}$. To simplify notation let event $B_1 = \{k_3 L \geq r,\ \mathcal{S}_{X_1} \subseteq C\}$.

The first inequality of the following derivation is a direct consequence of Lemma 4. In the third inequality, $\mathbf{1}_{k_2 L \leq r_0}$ denotes the random variable equal to 1 when $k_2 L \leq r_0$ and 0 else.

$$\begin{aligned}
& P\,[\, X_1 \text{ has one of its incident Delaunay faces} \\
& \quad \text{intersecting}\, \mathcal{B}(y, r) \;\big|\; X_1,\, B_1 \,] \\
& \leq\; P\,[\,\text{one of } A_1, \ldots, A_8 \text{ contains} \\
& \qquad \text{no point } X_2, \ldots, X_n \;\big|\; X_1,\, B_1 \,] \\
& \leq\; \sum_{j=1}^{8} P\left[\, A_j \text{ contains none of } X_2, \ldots, X_n \;\big|\; X_1,\, B_1 \,\right] \\
& =\; \sum_{j=1}^{8} \prod_{i=2}^{n} P\left[\, X_i \notin A_j \;\big|\; X_1,\, B_1 \,\right] \qquad (1) \\
& \leq\; 8E\left[\, \left(1 - \gamma (k_2 L)^3\right)^{n-1} \mathbf{1}_{k_2 L \leq r_0} \;\big|\; X_1,\, B_1 \,\right] \\
& \quad + 8E\left[\, \left(1 - \gamma r_0^3\right)^{n-1} \mathbf{1}_{k_2 L > r_0} \;\big|\; X_1,\, B_1 \,\right] .
\end{aligned}$$

Equation (1) is a consequence of the (conditional) independence of the events $X_i \notin A_j$. This independence comes both from the fact that the random variables X_i are independent and the fact that the 8 balls $A_1, \ldots, A_8$ are defined *independently* of the points $X_2, \ldots, X_n$. The necessity of this last independence is not always recognized and leads to frequent mistakes in the literature.

We now justify the last inequality. Note first that, by Lemma 5, $\gamma(k_2 L)^3 \leq 1$ when $k_2 L \leq r_0$. The expression $(1 - \gamma(k_2 L)^3)^{n-1}$ is therefore well-defined. The conditioning on $\mathcal{S}_{X_1} \subseteq C$ ensures that each ball A_i is fully contained in C. This implies in particular that the center y_i of each ball A_i is in C. We can then apply Lemma 5 (which determines the values γ and r_0) using the fact that the random variables X_i are all drawn independently according to an (α, β)-measure.

Integrating the previous inequality with respect to X_1 therefore gives:

$$\begin{aligned}
p \;&\leq\; P\,[\, k_3 L < r \,] \\
& \quad + P\,[\, \mathcal{S}_{X_1} \not\subseteq C,\ X_1 \text{ has one of its incident} \\
& \qquad \text{Delaunay faces intersecting}\, \mathcal{B}(y, r) \,] \\
& \quad + 8E\left[\, \left(1 - \gamma (k_2 L)^3\right)^{n-1} \mathbf{1}_{k_2 L \leq r_0}\, \mathbf{1}_{k_3 L \geq r} \,\right] \\
& \quad + 8\left(1 - \gamma r_0^3\right)^{n-1} P\,[\, k_2 L > r_0;\ k_3 L \geq r \,] \\
& \overset{\text{def}}{=}\; I + II + III + IV .
\end{aligned}$$

The fact that X_1 is drawn from an (α, β)-measure implies that $I = P\,[\, k_3 L < r \,] \leq (4/3)\, \beta\, \pi (r/k_3)^3$. Also $IV \leq 8 e^{-(n-1)\gamma r_0^3}$, which is exponentially small with n sufficiently large. We now turn to III. Note first that $\left(1 - \gamma (k_2 L)^3\right)^{n-1} \mathbf{1}_{k_2 L \leq r_0} \leq e^{-(n-1)\gamma (k_2 L)^3}$ and therefore, $III \leq 8E\left[e^{-(n-1)\gamma (k_2 L)^3}\, \mathbf{1}_{k_3 L \geq r} \right]$. To estimate this expression we use spherical coordinates and obtain $III \leq 16\pi^2 \beta \left(\frac{2}{3(n-1) k_2^3 \gamma} + \frac{2r^2}{k_2 k_3^2 (\gamma(n-1))^{1/3}} \frac{1}{3} \Gamma(\frac{1}{3}) \right)$.

We now turn to expression $II = P\,[\, \mathcal{S}_{X_1} \not\subseteq C,\ X_1$ has one of its incident Delaunay faces intersecting $\mathcal{B}(y, r)\,]$. The majoration of II will involve showing that only "local" vertices X_1 have a Delaunay face extending to y.[3] We will use this general fact in the vicinity of the boundary ∂C of C. Part of our assumptions is that C has low curvature. Therefore, at the very small distances that we consider, ∂C appears flat. We will take advantage of this fact and model locally the boundary ∂C to be a plane $\mathcal{P}$: C appears locally like a half-plane U.[4]

To simplify we set $K = 2(k_1 + k_2)$ and recall that $K < 1$. Recall also that we defined $\mathcal{S}_{X_1}$ to be any arbitrary, externally fixed, spindle whose axle is $y X_1$: $\mathcal{S}_{X_1}$ is not uniquely determined by X_1. For every X_1 define $\mathcal{S}'_{X_1}$ to be the 3-dimensional "tire" span by $\mathcal{S}_{X_1}$ when rotating around its axle. $\mathcal{S}'_{X_1}$ is *uniquely* determined by X_1 (along with y, K) and contains $\mathcal{S}_{X_1}$ so that, clearly, $\{\mathcal{S}_{X_1} \not\subseteq U\} \subseteq \{\mathcal{S}'_{X_1} \not\subseteq U\}$. To further simplify introduce $\mathcal{S}''_{X_1}$ to be the following (simpler) object. $\mathcal{S}''_{X_1}$ is composed of (i) the axle $X_1 y$, and (ii) a circle of diameter K, the *wheel*, perpendicular to $X_1 y$ whose center is the mid-point q of $X_1 y$. We sometimes write $\mathcal{S}''_{X_1}(K)$ to emphasize the value of K. Furthermore, let $\mathcal{P}$ be a plane,

[3] Here, "local" means being within distance $O((\log n/n)^{1/3})$ of y: a quantity asymptotically small.

[4] This is valid only locally. Recall: C is bounded.

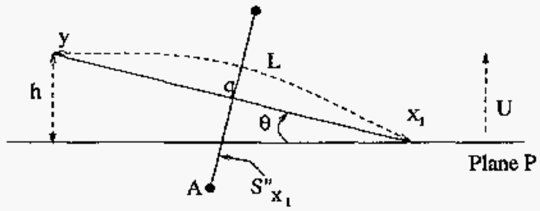

Figure 5: A spindle $\mathcal{S}''_{X_1}$ crossing $\mathcal{P}$.

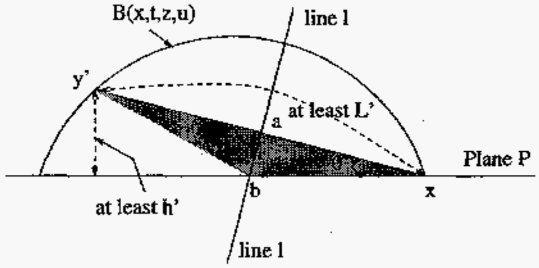

Figure 6: The cone $\mathcal{C}$ contained in every $\mathcal{B}(x,t,z,u)$ containing y and such that $d(x,y) \geq L_0(h)$.

($y \notin \mathcal{P}$), let U denote the $\mathcal{P}$-half-space to which y belongs, and let h denote the distance $d(y, \mathcal{P})$ from y to $\mathcal{P}$. We only consider points X in U and say that the spindle $\mathcal{S}''_{X_1}$ *crosses* $\mathcal{P}$ if its wheel crosses $\mathcal{P}$. As before, we set $L = d(X_1, y)$.

Claim: The following is true:
(a) $\mathcal{S}''_{X_1}(K)$ crosses $\mathcal{P}$ only if $L \geq L_0(h) \stackrel{\text{def}}{=} \sqrt{1+1/K^2}h$.
(b) $\{\mathcal{S}'_{X_1}(K) \not\subseteq U\} \subseteq \{\mathcal{S}''_{X_1}(1.02K) \not\subseteq U\}$ for $K \leq 0.2$.

We just show (a); see Figure 5. Assertion (b) implies that only a minute adjustment in K allows to consider $\{\mathcal{S}''_{X_1} \not\subseteq U\}$ in place of $\{\mathcal{S}'_{X_1} \not\subseteq U\}$.

Consider a given value of L. It is clear that, if an L-spindle $\mathcal{S}_{x_0}$ crosses $\mathcal{P}$, then every L-spindle $\mathcal{S}_x$ with $x \in \mathcal{P}$ does also cross $\mathcal{P}$. To prove impossibility results we can therefore consider only $x \in \mathcal{P}$. Consider such an x. As the spindle crosses $\mathcal{P}$, $d(q, A) = (K/2)L \geq L/2 \tan\theta$ where $\theta = \arcsin(h/L)$. This immediately implies $L^2 \geq \frac{1+K^2}{K^2}h^2$.

We say that a ball $\mathcal{B}(X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4})$ is Delaunay if it contains no other point X_i in its interior, and are now ready for the majorations below.

$$\begin{aligned}
&P\left[\mathcal{S}''_{X_1} \not\subseteq U,\ X_1 \text{ has one of its incident Delaunay faces intersecting } \mathcal{B}(y,r)\right] \\
&= P\left[\mathcal{S}''_{X_1} \not\subseteq U, \exists X_{i_2}, X_{i_3} \text{ s.t. } (X_1, X_{i_2}, X_{i_3}) \text{ is a Delaunay face intersecting } \mathcal{B}(y,r)\right] \\
&\leq \max_{y' \in \mathcal{B}(y,r)} P\left[\exists X_{i_2}, X_{i_3}, X_{i_4};\ d(X_1, y) \geq L_0(h),\ \mathcal{B}(X_1, X_{i_2}, X_{i_3}, X_{i_4}) \text{ is Delaunay and contains } y'\right] \\
&\leq \max_{y' \in \mathcal{B}(y,r)} \binom{n-1}{3} P\left[d(X_1, y) \geq L_0(h),\ \mathcal{B}(X_1, X_2, X_3, X_4) \text{ is Delaunay and contains } y'\right] \\
&\leq \max_{y' \in \mathcal{B}(y,r)} n^3/6 \int_{T_{y'}} (P[X_5 \notin \mathcal{B}(x,t,z,u)])^{n-4}\, dP_{X_1,X_2,X_3,X_4}(x,t,z,u),
\end{aligned}$$

where $T_{y'} \stackrel{\text{def}}{=} \{(x,t,z,u) \in U^4; d(x,y) \geq L_0(h), \text{ and } y' \in \mathcal{B}(x,t,z,u)\}$. P_{X_1,X_2,X_3,X_4} denotes the probability law of the random variable (X_1, X_2, X_3, X_4).

We now compute an upper-bound for the expression $P[X_5 \notin \mathcal{B}(x,t,z,u)]$ when the points x, t, z, u of U are such that $d(x,y) \geq L_0(h)$, and such that $y' \in \mathcal{B}(x,t,z,u)$. We define $K' \stackrel{\text{def}}{=} \sqrt{1+1/K^2}$. Hence $L_0(h) = K'h$. We also set $K'' \stackrel{\text{def}}{=} 2K' - 1$. We need the following claim.

Claim: If $d(x,y) \geq L_0(h)$ and $y' \in \mathcal{B}(x,t,z,u) \cap \mathcal{B}(y,r)$ then $\text{Vol}(\mathcal{B}(x,t,z,u) \cap U) \geq \frac{\pi}{24}(\frac{K''^2}{1-1/K''^2})^{1/2} h^3 \stackrel{\text{def}}{=} c'h^3$.

The situation is presented in Figure 6. The ball $\mathcal{B}(x,t,z,u)$ cuts minimally U only when (i) x is on the boundary $\mathcal{P}$, (ii) when y' is on the boundary of $\mathcal{B}(x,t,z,u)$, and (iii) when the center o of $\mathcal{B}(x,t,z,u)$ is such that the plane xoy' is perpendicular to $\mathcal{P}$. We therefore consider the situation within plane xoy', as in Figure 6.

The center o is located on the line l perpendicular to $y'x$ and going through the mid-point a of $y'x$. Let b be the intersection of l with $\mathcal{P}$. Consider the cone $\mathcal{C}$ issued from b and whose base is the circle with diameter $y'x$. We claim that $\mathcal{C}$ is included in the ball $\mathcal{B}(x,t,z,u)$ cutting minimally U. (i) This ball $\mathcal{B}(x,t,z,u)$ must have its center o below a: if not, at least half of $\mathcal{B}(x,t,z,u)$ is in U. (ii) If o is under b, then b is in the convex hull of the 3 points y', x, o. All 3 points are in $\mathcal{B}(x,t,z,u)$ and hence so is b. $\mathcal{B}(x,t,z,u)$ then clearly contains $\mathcal{C}$. (iii) Every point c belonging to the segment ab verifies $d(c,b) \leq d(c,y')$. Thus, if $o \in [a,b]$ then $b \in \mathcal{B}(x,t,z,u)$ and $\mathcal{B}(x,t,z,u)$ contains $\mathcal{C}$ as before.

This shows that $v \stackrel{\text{def}}{=} \text{Vol}(\mathcal{B}(x,t,z,u) \cap U) \geq \text{Vol}(\mathcal{C})$. Using the fact that $d(y,y') \leq r \leq h/2$, we find that v is at least equal to $\frac{\pi}{3}(L'/2)^3 \tan(\theta'/2)$, where $h' \stackrel{\text{def}}{=} h - h/2 = h/2$, $L' \stackrel{\text{def}}{=} L_0(h) - h/2 = K''h'$ and where $\theta' = \arcsin(h'/L')$. We obtain $v \geq c'h^3$, where $c' = \frac{\pi}{24}(\frac{K''^2}{1-1/K''^2})^{1/2}$, and have thus proven:

$$\begin{aligned}
&P\left[\mathcal{S}''_{X_1} \not\subseteq C,\ X_1 \text{ has one of its incident Delaunay faces intersecting } \mathcal{B}(y,r)\right] \\
&\leq \frac{n^3}{6}(1 - c'\alpha h^3)^{n-4} \leq \frac{n^3}{6} e^{-(n-4)c'\alpha h^3}.
\end{aligned}$$

Selecting $h(n) = \left(\frac{4}{c'\alpha}\frac{\log n}{n}\right)^{1/3}$ gives $II = O(1/n)$, as needed. Then, multiplying by n the bounds found for

I, II, III and IV and summing establishes the result, and concludes the proof of Lemma 6 ∎

Lemma 7. Under the hypothesis of Lemma 6

$$E[N_2] \leq e(b + cr^2 n^{2/3} + dnr^3) \log n / \log\log n,$$

where N_2 is the number of Delaunay tetrahedra that intersect $B(y, r)$ and where e is a constant depending solely on C, α and β.

To prove this, we need the following result, very similar to the result derived by [BEY91] in their Theorem 7.

Lemma 8. There exists a constant c'' such that $P[d^o(X_1) > c'' \log n / \log\log n] \leq 1/n^4$. Therefore

$$P[\exists i,\ d^o(X_i) > c'' \log n / \log\log n] \leq 1/n^3.$$

Proof: The proof of this result follows very closely the proof for the Poisson model given in Lemmas 8 and 9 of [BEY91]. The only technical difference is that they bound the first probability by $1/n^2$ instead of $1/n^4$. We show here that $1/n^4$ is similarly valid. A careful reading of their proof shows that we only need to establish that, with probability at least $1 - 1/n^4$, the maximum Delaunay edge length is $O\left((\log n/n)^{1/3}\right)$.[5] We compute:

$$\begin{aligned} & P\left[\, d(X_1, X_2) \geq h \text{ and } X_1X_2 \text{ is a Delaunay edge}\right] \\ & \leq \quad P\left[\, d(X_1, X_2) \geq h \text{ and } \right. \\ & \qquad \left. \exists i_3, i_4 \text{ s.t. } B(X_1X_2X_{i_3}X_{i_4}) \text{ is a Delaunay ball}\right] \\ & \leq \quad n^2(1 - \gamma(h/2)^3)^{n-d-1} \ \leq \ O(n^2 e^{-\gamma(h/2)^3 n}). \end{aligned}$$

Hence $P\left[\exists X_{i_1}X_{i_2} \text{ s.t. } d(X_{i_1}, X_{i_2}) \geq h \text{ and } X_{i_1}X_{i_2} \text{ is a Delaunay edge}\right] \leq O(n^4 e^{-\gamma(h/2)^3 n})$. This is $o(1/n^4)$ if $h > \left(\frac{32}{\gamma}\frac{\log n}{n}\right)^{1/3}$. ∎

Proof of Lemma 7: By Euler's formula, there is a constant K such that the tetrahedron-degree is equal to K times the edge-degree. Let B_2 denote the event $\{\forall i,\ d^o(X_i) \leq c'' \log n / \log\log n\}$. Then B_2 is also equal to $\{\forall i,\ \text{tetrahedron-}d^o(X_i) \leq Kc'' \log n / \log\log n\}$ and Lemma 8 implies $P[\overline{B_2}] \leq 1/n^3$. We have: $E[N_2] = E[N_2; B_2] + E[N_2; \overline{B_2}]$. The two terms $E[\,N_2\,;\,B_2\,]$ and $E[\,N_2\,;\,\overline{B_2}\,]$ need to be bounded separately. We begin with $E[N_2; B_2]$:

$$\begin{aligned} & E[\,N_2\,;\,B_2\,] \\ & \quad = \quad P[B_2]\, E[\,N_2 \,|\, B_2\,] \\ & \quad \leq \quad Kc''\, P[B_2]\, E[\,N_1 \,|\, B_2\,] \log n / \log\log n \\ & \quad = \quad Kc''\, E[N_1; B_2] \log n / \log\log n \\ & \quad \leq \quad Kc''\, E[N_1] \log n / \log\log n \\ & \quad \leq \quad Kc''\,(b + cr^2 n^{2/3} + dnr^3) \log n / \log\log n. \end{aligned}$$

On the other hand, $E[N_2; \overline{B_2}] \leq O(n^2) P[\overline{B_2}] \leq O(n^2)\, 1/n^3 = O(1/n)$. ∎

Corollary 9. Consider n points $X_1, \ldots, X_n$ drawn independently from an (α, β)-measure over a bounded convex set C with small curvature. Then there exist positive constants a, b, c, d and e, depending upon α, β and C only, such that the following holds. Let y be any point in C at distance $a(\log n/n)^{1/3}$ from the boundary ∂C. Let $r, r \leq \frac{a}{2n^{1/3}}$ be a positive quantity. Let N_2 denote the number of Delaunay tetrahedra that intersect $B(y, r)$. Then

$$E[N_2] \leq e(b + c(a/2)^2 + d(a/2)^3) \log n / \log\log n.$$

Proof of Theorem 2: We set $c_1 = a$, where a is the constant of Lemma 6 and Corollary 9. The segment L may be covered by $\left\lceil \frac{|L|+1}{a} n^{1/3} \right\rceil$ circles of radius $\frac{a}{2n^{1/3}}$ each and centered on points y_i of L. The number N of intersections between L and the Delaunay triangulation is bounded by the sum of the number of intersections with these circles. By Corollary 9, the expected number of intersections with each of these circles is bounded by $K \log n / \log\log n$ for some constant K. Hence:

$$\begin{aligned} E[N] \quad & \leq \quad K \left\lceil \frac{|L|+1}{a} n^{1/3} \right\rceil \log n / \log\log n \\ & \leq \quad K \left(1 + \frac{|L|+1}{a} n^{1/3}\right) \log n / \log\log n. \end{aligned}$$ ∎

Proof of Theorem 1: We have in mind to apply Theorem 2 to the segment $L = qY$. We are faced with two difficulties. First, both Y and L are defined in terms of $Y_1, \ldots, Y_m$ and are therefore not independent of $X_1, \ldots, X_n$. Second, Y can possibly be within distance $c_4(\log n/n)^{1/3}$ from the boundary ∂C. We will solve the first difficulty by considering a slightly different Delaunay triangulation with respect to which L is independent. We will solve the second difficulty by showing that Y is with high probability at distance of at least $c_4(\log n/n)^{1/3}$ from ∂C.

Let us first recall that q and Y are defined in very different ways. The condition that they be "far enough" from the boundary must therefore be handled differently. The query point q is not in the control of the algorithm. It is instead decided externally and the algorithm is claimed to perform well for all admissible choices of q. Thus, the assumption "q is at distance of at least $n^{1/18}$ from ∂C" is merely a restriction on the set of query points against which the algorithm has to measure. On the other hand, the point Y is chosen randomly, as described in the algorithm on page 2. The fact that "Y is at distance of at least $c_4(\log n/n)^{1/3}$ from ∂C" cannot therefore be imposed externally.

Let us relabel $\{X_1, \ldots, X_n\} - \{Y_1, \ldots, Y_m\}$ into $\{X'_1, \ldots, X'_{n-m}\}$. As usual, let $\mathcal{D}$ denote the Delaunay triangulation associated to the n points $X_1, \ldots, X_n$,

[5]Theorem 1 of [BEY91] establishes that the maximum edge length is at most $O\left((\log n)^{1/3}\right)$ with high probability. Our additional factor $1/n^{1/3}$ comes from the fact that they consider a cube of variable side length $1/n^{1/3}$.

and let $\mathcal{D}_m$ denote the Delaunay triangulation associated to the $n-m$ points $X'_1, \ldots, X'_{n-m}$. The random variables $X'_1, \ldots, X'_{n-m}$ are independent from the random variables $Y_1, \ldots, Y_m$. This implies that, for every query point q, $(X'_1, \ldots, X'_{n-m})$ is independent from Y, which allows us to make the following two conclusions. First, $L = (Y, q)$, the line segment connecting Y and q, is independent of the $n-m$ data points defining $\mathcal{D}_m$. Second, the probabilistic behavior of $X'_1, \ldots, X'_{n-m}$ is unaffected when conditioning on the event $B_3 \stackrel{\text{def}}{=} \{d(Y, \partial C) \geq c_4(\log n/n)^{1/3}\}$. In formal terms, the probabilistic law $\mathcal{L}(X'_1, \ldots, X'_{n-m})$ is equal to the conditional law $\mathcal{L}\left((X'_1, \ldots, X'_{n-m}) \,|\, B_3\right)$. In particular, the random variables $X'_1, \ldots, X'_{n-m}$ are independent identically distributed (α, β)-random variables under the conditional probability distribution $P[\,\cdot\,|B_3]$.

Let N denote the number of tetrahedra in $\mathcal{D}_m$ crossed by L. We have $E[N] = E[N; B_3] + E[N; \overline{B_3}]$ where $\overline{B_3}$ denotes the complement of B_3. We provide upper bounds for the two terms $E[N; B_3]$ and $E[N; \overline{B_3}]$.

We begin with $E[N; \overline{B_3}]$. It is well known that $N = O(n^2)$. Hence $E[N; \overline{B_3}] \leq O(n^2) P[\overline{B_3}]$. We fix c_1 to be a constant such that, for every n, $c_4(\log n/n)^{1/3} \leq \frac{c_1}{2}(1/n)^{1/18}$. Recall that $P[\overline{B_3}] = P[d(Y, \partial C) < c_4(\log n/n)^{1/3}]$ and that, by assumption, $d(q, \partial C) \geq c_1(1/n)^{1/18}$. By triangular inequality, this implies that:

$$
\begin{aligned}
P[\overline{B_3}] &\leq P\left[d(Y,q) \geq \frac{c_1}{2}\frac{1}{n^{1/18}}\right] \\
&= \left(P\left[d(Y_1,q) \geq \frac{c_1}{2}\frac{1}{n^{1/18}}\right]\right)^m \\
&= \left(1 - P\left[d(Y_1,q) \leq \frac{c_1}{2}\frac{1}{n^{1/18}}\right]\right)^m \\
&\leq e^{-m\,P\left[d(Y_1,q) \leq \frac{c_1}{2}\frac{1}{n^{1/18}}\right]} \\
&\leq e^{-n^{1/5}\alpha\frac{4\pi}{3}(\frac{c_1}{2})^3\frac{1}{n^{1/6}}} \\
&= o(1/n^2).
\end{aligned}
$$

This shows that $E[N\,;\,\overline{B_3}] = o(1)$. We now turn to $E[N; B_3] = E[N\,|\,B_3]\,P[B_3]$. Theorem 2 (see the remark after Theorem 2), along with the fact that $X'_1, \ldots, X'_{n-m}$ are (α, β)-random variables, independently identically distributed under the measure $P[\,\cdot\,|B_3]$, implies that:

$$
\begin{aligned}
&E[N\,|\,B_3] \\
&\quad\leq c_5\,(1 + E\,[\,d(Y,q)\,|\,B_3\,])\;(n-m)^{1/3}\,\log n/\log\log n\,.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&E[N; B_3] \\
&\quad\leq c_5\,(P[B_3] + E\,[\,d(Y,q)\,;\,B_3\,]) \\
&\qquad\qquad (n-m)^{1/3}\,\log n/\log\log n \\
&\quad\leq c_5\,(1 + E\,[\,d(Y,q)\,])\;n^{1/3}\,\log n/\log\log n\,.
\end{aligned}
$$

The estimation of $E\,[\,d(Y,q)\,]$ is done as in [DMZ95]. The beginning of the argument is similar to the estimation of $P[\,\overline{B_3}\,]$ above. Lemma 5 is then used. We let $\text{diam}(C)$ denote the diameter of C. Note that Y and q are in C so that $P\,[\,d(Y,q) > t\,] = 0$ if $t > \text{diam}(C)$.

$$
\begin{aligned}
E\,[\,d(Y,q)\,] &= \int_0^\infty P\,[\,d(Y,q) > t\,]\,dt \\
&= \int_0^{diam(C)} P\,[\,d(Y,q) > t\,]\,dt \\
&\leq \int_0^{diam(C)} e^{-mP[d(Y_1,q)\leq t]}\,dt \\
&\leq \int_0^{r_0} e^{-m\gamma t^3}\,dt + \int_{r_0}^{diam(C)} e^{-m\gamma r_0{}^3}\,dt \\
&\leq \int_0^\infty e^{-m\gamma t^3}\,dt + \text{diam}(C)e^{-m\gamma r_0{}^3} \\
&= O(\frac{1}{m^{1/3}})\,.
\end{aligned}
$$

We have therefore shown that

$$
E[N] = O\left((n/m)^{1/3}\,\log n/\log\log n\right)\,.
$$

N_{total}, the total number of tetrahedra in $\mathcal{D}$ crossed by L is not more than that for $\mathcal{D}_m$, i.e., above N, plus the sum S of the tetrahedra degrees of (i.e., the number of tetrahedra adjacent to) $Y_1, \ldots, Y_m$ in the Delaunay triangulation $\mathcal{D}$. To see this, note that L either crosses a tetrahedron without one of the Y_i's as a vertex (in which case the tetrahedron is both in $\mathcal{D}$ and $\mathcal{D}_m$) or one for which Y_i is a vertex (in which case the tetrahedron is in $\mathcal{D}$ but not in $\mathcal{D}_m$). The total number of the latter kind of tetrahedra does not exceed S. The expected value of S is, by linearity of expectation, $3m$ times the expected (vertex) degree $\delta(n)$ of Y_1, where the constant 3 results from Euler's formula. Combining all this we have:

$$
E[N_{\text{total}}] = O\left((n/m)^{1/3}\,\log n/\log\log n + m\delta(n)\right)\,.
$$

The time complexity T of the jump-and-walk algorithm on page 2 is proportional to $m + N_{\text{total}}$; the sample size m comes into play because of steps (1) and (2), N_{total} is due to step (3). $E[T]$ can thus be optimized to $O\left(\delta(n)^{1/4} n^{1/4}\,(\log n/\log\log n)^{3/4}\right)$ with the choice of $m = \Theta\left(n^{1/4}/\delta(n)^{3/4}\,(\log n/\log\log n)^{3/4}\right)$. ■

4 Empirical Results in 2D

This section presents some empirical results on the planar jump-and-walk, or better, a variation of it. To ease the implementation even further, we sample $n^{1/3}$ edges of the Delaunay triangulation $\mathcal{D}$, rather than points. Then, we choose the edge whose midpoint has minimum distance to the query point q. We find the triangle containing q by traversing the triangles intersected by $L = (y, q)$, where y is the midpoint of the initially chosen edge.

We tested this procedure for random point sets of size $n = 1000, 2000, \ldots, 50000$; the coordinates were chosen randomly out of the unit square. In Figures 7 and 8, M_n denotes the sample mean of the number of triangles visited, over a sample of 999 queries, and for a point set of size n; the coordinates of q (and the point set) are again chosen by random out of the unit square. Thus, M_n corresponds to the $E[N_{\text{total}}]$ in the analysis. Since $\Theta(\log n)$ is the best known theoretical bound for planar point locations, see, e.g., [PS85], Figure 9 plots the ratio $M_n/\log_2 n$ to give a measure for the efficiency of the method. Note that the best known planar point location algorithm [DL76] is obtained by two binary searches, one horizontally and one vertically, thus has at least a constant of 2 in front of the $\log_2 n$.

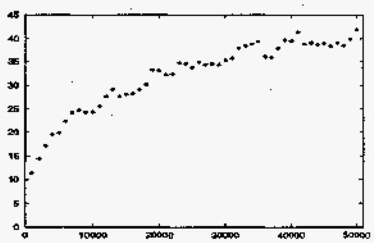

Figure 7: The sample mean M_n of the number of triangles visited, for a sample of 999 random query points q for each data set of size n.

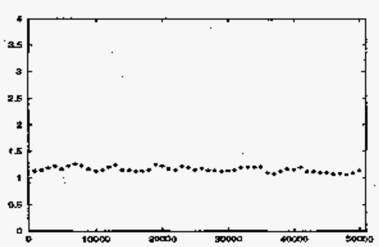

Figure 8: The ratio $M_n/n^{1/3}$.

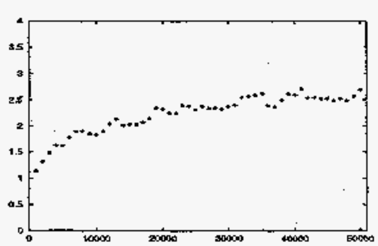

Figure 9: The ratio $M_n/\log_2 n$.

It should be noted that it might be difficult to compare our algorithm with that of the bucketing algorithm of [AEI+85]. Although the latter algorithm takes an average constant query time, the constant depends on the size of the buckets, hence depends on the amount of preprocessing performed in the buckets. We believe that when n is significantly big, e.g., n is greater than a million, the bucketing method might be the best solution for planar point locations. In any case, remember that bucketing requires preprocessing and additional data structures.

5 Empirical Results in 3D

Let us now see how well the jump-and-walk performs in 3D. For convenience, we again implement only a slight variation of the original procedure; cf. [E93]. First, we sample triangles rather than vertices. The size of the sample is set to $m = 2n^{1/4}$, for Delaunay triangulations of n points. The "distance" of a triangle to the query point q is calculated simply as the minimum distance of its three vertices to q. The triangle τ that scores with the minimum distance is selected. We adjust its orientation such that q is on its positive side, i.e., $q \in \tau^+$.

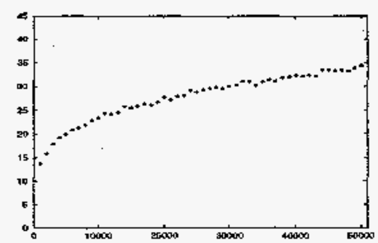

Figure 10: The sample mean M_n of the number of tetrahedra visited, for a sample of 999 random query points q for each data set of size n.

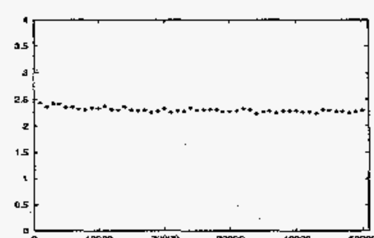

Figure 11: The ratio $M_n/n^{1/4}$.

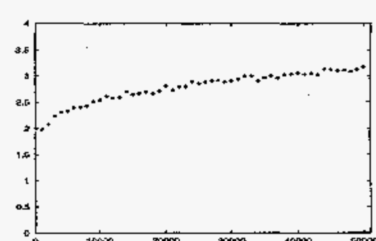

Figure 12: The ratio $M_n/\log_2 n$.

Second, we do a *jump-and-stroll* rather than a jump-and-walk: namely, for each visited (oriented) triangle τ we select some other face σ of the tetrahedron incident to τ (and in τ^+) such that σ has the same orientation than τ and $q \in \sigma^+$. If no such σ exists, the tetrahedron containing q has been found. If the new triangle σ is a convex hull triangle, then we know that q lies outside the Delaunay triangulation.

In terms of the number of faces visited, jump-and-stroll can only be worse than jump-and-walk. However, since intersection tests in 3D are computationally more expensive than just plain orientation tests, it is not clear whether the jump-and-walk is actually faster in terms of CPU seconds or when counting geometric primitive operations. (Empirical tests to this regard will be included in the full-paper version of this text.)

Analogous to Section 4, we ran the jump-and-stroll for Delaunay triangulations of random 3D point sets of size n, for $n = 1000, 2000, \ldots, 50000$; one random set for each n. Each data set was then queried with 999 random points, and the number of tetrahedra visited is counted, yielding 999 numbers for each n. Figure 10 plots their sample means M_n. The corresponding confidence intervals were consistently smaller than $\pm 2.45\%$.[6] Figure 11 plots the ratio $M_n/n^{1/4}$. It indicates that the constants in our analysis are low, i.e., less than 2.4.[7] Moreover, the method compares well with the theoretically best possible $O(\log n)$, which assumes both preprocessing and additional storage. Figure 12 plots $M_n/\log_2 n$ and shows that, for the observed range of n, the number of visited tetrahedra stays well under $3.5 \log_2 n$.

6 Closing Remarks

The full version of this text will include the complete proofs with all the details omitted here. We will also provide empirical evidence on jump-and-walk's efficiency in Delaunay triangulations of practical (but not random) point sets. Another question is, how does the method perform on non-Delaunay triangulations? Can we give a rigorous expected case analysis for arbitrary triangulations of random points?

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[6] We say a sample mean has a *confidence interval* of $\pm H\%$ if the interval $[\overline{Y} - H\%, \overline{Y} + H\%]$ contains the real mean value μ with a probability of 90%.

[7] B.t.w., our above choice of $m = 2n^{1/4}$ simply aimed to balance the constants in front of the $n^{1/4}$ for both m and M_n.

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