Fault Tolerant Formations Control of UAVs Subject to Permanent and Intermittent Faults

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Abstract The paper addresses the formation control of unmanned aerial vehicles (UAVs) in the presence of permanent and intermittent faults in each UAV. A fault tolerant control (FTC) scheme is developed to accommodate the permanent fault. It further shows that for the intermittent fault, the formation stability can be maintained under some conditions of fault appearance and disappearance without requiring to take any FTC action. Simulation results show the efficiency of the proposed method.

Keywords UAVs formation · Permanent fault · Intermittent fault · Fault tolerant control

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1 Introduction

Formation flight of UAVs has attracted a great deal of interest in recent years. The structure of formation can be generally classified as leader-follower, virtual-leader and so on [1]. At present, studies of UAVs formation commonly use simple first-order or second-order kinematic UAV model, which can not describe dynamic behaviors of UAVs in details. Moreover, these models cannot reflect the influences of UAV's own controller on formation flight and the type of fault as well.

On the other hand, a fault is an unpermitted deviation of at least one characteristic property or parameter of the system from the standard condition. The impact of a fault can be a small reduction in efficiency, but could also lead to overall system failure. Thus, an FTC scheme could have been designed to accommodate the fault. Faults can be classified according to their time characteristics as permanent and intermittent. They can also be

classified according to their location of occurrence in the system as actuator faults, sensor faults and component faults [3].

As for the permanent faults, once they occur, they will exist in the system all the time. It is necessary to design a fault tolerant controller to stabilize the faulty system [4–7]. However, unlike permanent faults, the intermittent fault we pay attention to is a kind of fault that may be active at one instant of time causing a malfunction of system or may be inactive at another instant allowing the system to operate correctly. The intermittent fault often exists in electronic equipments and may be caused by noise, wind, magnetic or any other disturbance in the environment. It is well known that FTC takes time and cost. It is often not admissible to apply FTC scheme every time, since intermittent fault may occur frequently.

In this paper we consider the FTC problem of a UAVs formation in leader-follower structure, where each UAV may have both permanent and intermittent faults. Inspired by the idea proposed in [2], we divide the UAVs formation into outer-loop and inner-loop: the outer-loop controls the whole formation; the inner-loop controls UAV's own dynamics and kinematics behavior.

The main contributions of this paper are as follows:

1. As for the permanent fault, a compensation term is added to the nominal controller of the UAV to eliminate the influences caused by the fault such that formation stability is still maintained.
2. The fault tolerance under intermittent faults is analyzed by the switched system approach. It shows that under some conditions of fault appearance and disappearance, the formation stability can be maintained without requiring to take any FTC action.

The rest of this paper is arranged as follows. Section 2 provides some preliminaries. Section 3 discusses the design method of inner-loop and outer-loop. Sections 4 and 5 respectively focus on the FTC design for permanent faults and intermittent faults. Section 6 provides simulation results, followed by conclusions in Section 7.

2 Preliminaries

2.1 Outer-Loop Model

Consider the flight formation consists of $q(q \geq 2)$ UAVs, the topology considered here is leader-follower structure, and each UAV has only one reference UAV. UAV i's kinematic model is

$$\begin{cases} \dot{x}_i = v_i \cos \phi_i \\ \dot{y}_i = v_i \sin \phi_i \\ \dot{\phi}_i = w_i, \quad i = 1, 2, \ldots, q \end{cases} \tag{1}$$

where x_i, y_i represent UAV i's position, v_i is forward speed, ϕ_i is the angle between x-axis and v_i, w_i are angular velocities.

In standard leader-follower formation model, forward error $\tilde{f}_{ij} \triangleq f_{ij} - f_{ij}^d$, lateral error $\tilde{l}_{ij} \triangleq l_{ij} - l_{ij}^d$, $f_{ij}(l_{ij})$, $f_{ij}^d(l_{ij}^d)$ indicate actual and desired distance, as shown in Fig. 1. The information of UAV j are all known. The error model can be described as:

$$\begin{aligned} \tilde{f}_{ij} &= (x_i - x_j) \cos \phi_j + (y_i - y_j) \sin \phi_j \\ &\quad + d \cos (\phi_i - \phi_j) - f_{ij}^d \\ \tilde{f}_{ij} &= (x_i - x_j) \sin \phi_j - (y_i - y_j) \cos \phi_j \\ &\quad - d \sin (\phi_i - \phi_j) - l_{ij}^d \end{aligned} \tag{2}$$
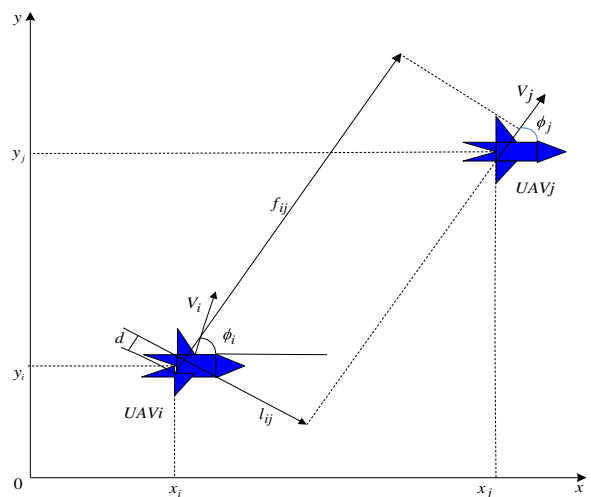

Fig. 1 Formation geometry

Deriving Eq. 2 with regard to time leads to the dynamics of outer-loop model as follows:

$$
\begin{bmatrix} \dot{\tilde{f}}_i \\ \dot{\tilde{l}}_i \end{bmatrix} = \begin{bmatrix} -v_j - l_{ij}w_j \\ f_{ij}w_j \end{bmatrix}
$$

$$
+ \begin{bmatrix} \cos(\phi_i - \phi_j) & -d\sin(\phi_i - \phi_j) \\ -\sin(\phi_i - \phi_j) & -d\cos(\phi_i - \phi_j) \end{bmatrix} \begin{bmatrix} v_i \\ w_i \end{bmatrix}
$$

$$
= \begin{bmatrix} -v_j - l_{ij}w_j \\ f_{ij}w_j \end{bmatrix}
$$

$$
+ \begin{bmatrix} \cos(\phi_i - \phi_j) & -d\sin(\phi_i - \phi_j) \\ -\sin(\phi_i - \phi_j) & -d\cos(\phi_i - \phi_j) \end{bmatrix} \begin{bmatrix} v_i^* \\ w_i^* \end{bmatrix}
$$

$$
+ \begin{bmatrix} \cos(\phi_i - \phi_j) & -d\sin(\phi_i - \phi_j) \\ -\sin(\phi_i - \phi_j) & -d\cos(\phi_i - \phi_j) \end{bmatrix}
$$

$$
\times \begin{bmatrix} v_i - v_i^* \\ w_i - w_i^* \end{bmatrix} \tag{3}
$$

Note that v_i and w_i of UAVi can be regarded as the inputs of outer-loop model, v_i^*, w_i^* in Eq. 3 are the virtual and ideal control laws. Designing v_i^*, w_i^* properly can make the system

$$
\begin{bmatrix} \dot{\tilde{f}}_i \\ \dot{\tilde{l}}_i \end{bmatrix} = \begin{bmatrix} -v_j - l_{ij}w_j \\ f_{ij}w_j \end{bmatrix}
$$

$$
+ \begin{bmatrix} \cos(\phi_i - \phi_j) & -d\sin(\phi_i - \phi_j) \\ -\sin(\phi_i - \phi_j) & -d\cos(\phi_i - \phi_j) \end{bmatrix} \begin{bmatrix} v_i^* \\ w_i^* \end{bmatrix}
$$

stable. This will be discussed in Section 3.

2.2 Inner-Loop Model

Different from outer-loop model, the inner-loop model describes UAVs' own dynamical and kinematical behavior, and can be written as follows

$$
\begin{cases} \dot{m}_i(t) = Am_i(t) + B\delta_i(t) & m_i(0) = m_{i0} \\ n_i(t) = Cm_i(t) = \begin{bmatrix} v_i & \psi_i - \beta_i \end{bmatrix} \\ z_i(t) = D\dot{n}_i(t) = \begin{bmatrix} \dot{\psi}_i - \dot{\beta}_i \end{bmatrix} \end{cases} \tag{4}
$$

where $m_i = \begin{bmatrix} v_i & \alpha_i & q_i & \theta_i & \beta_i & p_i & r_i & \psi_i \end{bmatrix}^T \in \mathbb{R}^8$ are state variables. They are forward velocity, angle of attack, pitch rate, pitch angle, slideslip angle, roll rate, yaw rate and yaw angle. The input variables of inner-loop are $\delta_i = \begin{bmatrix} \delta_{ie} & \delta_{iT} & \delta_{ia} & \delta_{ir} \end{bmatrix}^T \in \mathbb{R}^4$, indicating elevator, throttle, flap and aileron. They are UAV's real physical control variables. $n_i(t)$, $z_i(t)$ are both output variables, and $z_i(t) = \begin{bmatrix} \dot{\psi}_i - \dot{\beta}_i \end{bmatrix} = w_i$. Note that the inputs v_i, w_i of outer-loop are the same as the outputs v_i, $\dot{\psi}_i - \dot{\beta}_i$ of the inner-loop. A, B, C are matrices of certain dimension. $m_i(0)$ is the initial state of the system.

2.3 Models of Faults

2.3.1 Permanent Fault Model

The model of permanent fault discussed here is actuator fault. The type of fault under consideration is the loss of actuator effectiveness. Let $\delta_i^F(t)$ represent the signal from the ith actuator that has failed. Then the permanent fault can be described as follows:

$$
\delta_i^F(t) = \rho_i \delta_i(t) \tag{5}
$$

$$
\rho_i = \text{diag} \begin{bmatrix} \rho_{i1}, & \rho_{i2}, & \rho_{i3}, & \rho_{i4} \end{bmatrix}
$$

where $0 < \rho_{ij} \leq 1$, $j = 1, 2, 3, 4$. If ρ_i equals to a unit matrix, there is no fault.

2.3.2 Intermittent Fault Model

The intermittent fault consider here occurs in the output channels of the onboard control processor, under which the inner-loop model changes into

$$
\dot{m}_i(t) = Am_i(t) + Bf_i\delta_i(t) \tag{6}
$$

where

$$
f_i = \begin{cases} f & otherwise \\ 1 & \text{the system is fault free} \end{cases}
$$

with $0 < f < 1$.

A realistic model to represent the appearance and disappearance property of intermittent fault is continuous-parameter Markov chain [9, 10] as shown in Fig. 2. Mode "0" and "1" represent healthy and faulty situations respectively. The probability for going from 0 to 1 at any time is λ, and the probability for going from 1 to 0 at any time is μ. The equations for these probabilities are [15–17]

$$
\begin{cases} P\{f_i(t + \Delta t) = f \,|\, f_i(t) = 1\} = \lambda \Delta t \\ P\{f_i(t + \Delta t) = 1 \,|\, f_i(t) = 0\} = \mu \Delta t \end{cases} \tag{7}
$$
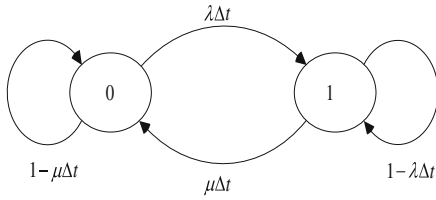
Fig. 2 Continuous two-state model

where $0 \leq \lambda < 1$ represent the fault appearance rates, $0 \leq \mu < 1$ represent the fault disappearance rates. $\Delta t \geq 0$ is the infinitesimal transition time interval. Assume that the initial situation is healthy.

2.4 Problem Formulation

According to the analysis in Section 2.2, one can obtain that $w_i = \dot{\psi}_i - \dot{\beta}_i$. If $\psi_i - \beta_i$ can track the given signal, then $\dot{\psi}_i - \dot{\beta}_i$ can track w_i^* as well. The ultimate goal is to design $\delta_i(t)$ in Eq. 4 to as sure that the error system (3) is asymptotically stable in both normal and faulty situations.

Given a reference signal $y_{ir}(t)$, define

$$\varepsilon_i(t) \triangleq y_{ir}(t) - Sn_i(t) = \begin{bmatrix} v_i - v_i^* \\ \psi_i - \psi_i^* - (\beta_i - \beta_i^*) \end{bmatrix} \tag{8}$$

where $S \in \mathrm{R}^{2 \times 2}$ is a known constant matrix, ψ_i^*, β_i^* are desired values.

Let $\eta_i(t) = \int_0^t \varepsilon_i(\tau) d\tau$, $\overline{m}_i(t) = \begin{bmatrix} \eta_i^T(t) & m_i^T(t) \end{bmatrix}^T$ (4) and (5) can be combined and the following augmented system can be obtained:

$$\begin{bmatrix} \dot{\eta}_i(t) \\ \dot{m}_i(t) \end{bmatrix} = \begin{bmatrix} 0 & -SC \\ 0 & A \end{bmatrix} \begin{bmatrix} \eta_i(t) \\ m_i(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} \delta_i(t) + \begin{bmatrix} I \\ 0 \end{bmatrix} y_{ir}(t) \tag{9}$$

Where the augmented system can be described as:

$$\begin{cases} \dot{\overline{m}}_i(t) = \overline{A}\,\overline{m}_i(t) + \overline{B}\delta_i(t) + Gy_{ir}(t) \\ \overline{m}_i(0) = \overline{m}_{i0} \end{cases} \tag{10}$$

$$\overline{A} = \begin{bmatrix} 0 & -SC \\ 0 & A \end{bmatrix}, \overline{B} = \begin{bmatrix} 0 \\ B \end{bmatrix}, G = \begin{bmatrix} I_{2 \times 2} \\ 0 \end{bmatrix}.$$

$\overline{m}_i(0)$ is the initial state. At the occurrence of a fault, the purpose of FTC scheme is to make $\lim_{t \to \infty} \varepsilon_i(t) = 0$ and variables of $m_i(t)$ bounded.

This work assumes that the appearance and disappearance of the fault can be detected rapidly by using a certain fault diagnosis scheme which is not the main focus of the paper. Interested readers are referred to, e.g., [3, 8, 11, 12] for detailed information.

3 Design of Controller

First of all, we design v_i^*, w_i^* for outer-loop model (3). Then the controller δ_i is designed for the inner-loop system to make sure that the state variables m_i are bounded and v_i, w_i can track v_i^*, w_i^*.

3.1 Outer-Loop Controller Design

For the designed communication topology, UAV i has only one reference vehicle j. The desired v_i^*, w_i^* are as follows

$$\begin{bmatrix} v_i^* \\ w_i^* \end{bmatrix} = \begin{bmatrix} \cos \tilde{\phi}_i & -\sin \tilde{\phi}_i \\ -\frac{1}{d} \sin \tilde{\phi}_i & -\frac{1}{d} \cos \tilde{\phi}_i \end{bmatrix} \begin{bmatrix} -k_1 \tilde{f}_i + v_j + l_{ij}w_j \\ -k_2 \tilde{l}_i - f_{ij}w_j \end{bmatrix} \tag{11}$$

Where $\tilde{\phi}_i \triangleq \phi_i - \phi_j, \tilde{f}_i \triangleq f_{ij} - f_{ij}^d, \tilde{l}_i \triangleq l_{ij} - l_{ij}^d$, $k_1, k_2 > 0$ are feedback gain.

Apply Eqs. 10 to 3, one can get

$$\begin{bmatrix} \dot{\tilde{f}}_i \\ \dot{\tilde{l}}_i \end{bmatrix} = \begin{bmatrix} -k_1 \tilde{f}_i \\ & -k_2 \tilde{l}_i \end{bmatrix} \begin{bmatrix} \tilde{f}_i \\ \tilde{l}_i \end{bmatrix} + \begin{bmatrix} \cos \tilde{\phi}_i & -d \sin \tilde{\phi}_i \\ -\sin \tilde{\phi}_i & -d \cos \tilde{\phi}_i \end{bmatrix} \begin{bmatrix} v_i - v_i^* \\ w_i - w_i^* \end{bmatrix}$$

Obviously, the above system is input-to-state stable with regard to $\begin{bmatrix} v_i - v_i^* \\ w_i - w_i^* \end{bmatrix}$ [19–22]. If the inner-loop controller can ensure that $\begin{bmatrix} v_i - v_i^* \\ w_i - w_i^* \end{bmatrix}$ approaches 0, then the formation error converges to 0.

3.2 Inner-Loop Controller Design

In order to make the inner-loop outputs track the given ideal signal $\left[\, v_i^* \;\; w_i^* \,\right]^T$, we design the following feedback controller of nominal system (10):

$$\delta_{i1}(t) = K_{i1}\overline{m}_i(t) = \begin{bmatrix} K_{i1\eta_i} & K_{i1m_i} \end{bmatrix} \begin{bmatrix} \eta_i(t) \\ m_i(t) \end{bmatrix} \triangleq \delta_{i1}(t) \tag{12}$$

where K_{i1} is the feedback gain of the controller, then the closed-loop system can be described as:

$$\dot{\overline{m}}_i(t) = \left(\overline{A} + \overline{B} K_{i1} \right) \overline{m}_i(t) + G y_{ir}(t) \tag{13}$$

Lemma 1 *For a given constant γ, if there exist symmetric matrices $Z \in \mathbb{R}^{10 \times 10}$ and $W \in \mathbb{R}^{4 \times 10}$ such that the following linear matrix inequality (14) holds*

$$\begin{bmatrix} \overline{A}Z + Z\overline{A}^T + \overline{B}W + W^T\overline{B}^T & G & W^T R^{1/2} & Z Q^{1/2} \\ * & -\gamma I & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -I \end{bmatrix} < 0 \tag{14}$$

Then there exists control law $\delta_{i1}(t) = K_{i1}\overline{m}_i(t)$, $K_{i1} = WZ^{-1}$, such that η_i is asymptotically stable, and m_i is bounded.

Proof The proof is similar to [14], and thus is omitted. □

Lemma 2 (Bellman-Gronwall) *If some numbers $t_b \geq t_a$, some constant $C \geq 0$, $N \geq 0$, and some non-negative, piecewise-continuous-function $g : [t_a, t_b] \to \mathbb{R}$, $w : [t_a, t_b] \to \mathbb{R}$ is a continuous function satisfying*

$$|w(t)| \leq C + N \int_{t_a}^{t} |g(\tau)| \, |w(\tau)| \, d\tau, \quad t \in [t_a, \quad t_b]$$

Then

$$|w(t)| \leq C e^{N \int_{t_a}^{t} |g(\tau)| d\tau} \quad t \in \left[t_a, \, t_b \right]$$

□

Proof The proof is similar to [25], and thus is omitted. □

Theorem 1 *If there exists a symmetric matrix P_i such that the following inequality*

$$A^T P_i + P_i A + (B K_{i1})^T P_i + P_i B K_{i1} < 0$$

holds, then the error η_i is exponential stable and m_i is bounded.

Proof We choose a Lyapunov function for η_i as $V_i = \eta_i^T(t) P_i \eta_i(t)$

The time derivative of V_i along the solution of Eq. 10 with Eq. 13 is

$$
\begin{aligned}
\dot{V}_i(t) &= \dot{\eta}_i^T(t) P_i \eta_i(t) + \eta_i^T(t) P_i \dot{\eta}_i(t) \\
&= \left[\left(A_{\eta_i} + B_{\eta_i} K_{i1} \right) \eta_i(t) + G_{\eta_i} y_r(t) \right]^T P_i \eta_i(t) \\
&\quad + \eta_i^T(t) P_i \left[\left(A_{\eta_i} + B_{\eta_i} K_{i1} \right) \eta_i(t) + G_{\eta_i} y_{ir}(t) \right] \\
&= \eta_i^T(t) A_{\eta_i}^T P_i \eta_i(t) + \eta_i(t) \left(B_{\eta_i} K_{i1} \right)^T P_i \eta_i(t) \\
&\quad + y_{ir}^T G_{\eta_i}^T(t) P_i \eta_i(t) + \eta_i^T(t) P_i A_{\eta_i} \eta_i(t) \\
&\quad + \eta_i^T(t) P_i B_{\eta_i} K_{i1} \eta_i(t) + \eta_i^T P_i G_{\eta_i} y_{ir}(t) \\
&= \eta_i^T(t) \left(A_{\eta_i} P_i + P_i A_{\eta_i} \right) \eta_i(t) \\
&\quad + \eta_i^T(t) \left[\left(B_{\eta_i} K_{i1} \right)^T P_i + P_i B_{\eta_i} K_{i1} \right] \eta_i(t) \\
&\quad + G_{\eta_i}^T y_{ir}^T(t) P_i \eta_i(t) + \eta_i^T(t) P_i G_{\eta_i} y_{ir}(t) \\
&= \eta_i^T(t) \left[A_{\eta_i}^T P_i + P_i A_{\eta_i} + \left(B_{\eta_i} K_{i1} \right)^T P_i \right. \\
&\quad \left. + P_i B_{\eta_i} K_{i1} \right] \eta_i(t) + 2\eta_i^T(t) P_i G_{\eta_i} y_{ir}(t) \\
&\leq \eta_i^T(t) \left[A_{\eta_i}^T P_i + P_i A_{\eta_i} \right. \\
&\quad \left. + \left(B_{\eta_i} K_{i1} \right)^T P_i + P_i B_{\eta_i} K_{i1} \right] \eta_i(t) \\
&\quad + 2 \left| \eta_i^T(t) \right| P_i G_{\eta_i} |y_{ir}(t)| \\
&\leq \eta_i^T(t) \left[A_{\eta_i}^T P_i + P_i A_{\eta_i} + \left(B_{\eta_i} K_{i1} \right)_i^T P \right. \\
&\quad \left. + P_i B_{\eta_i} K_{i1} \right] \eta_i(t)
\end{aligned}
\tag{15}
$$

It follows that

$$A_{\eta_i}^T P_i + P_i A_{\eta_i} + \left(B_{\eta_i} K_{i1}\right)^T P_i + P_i B_{\eta_i} K_{i1} < 0$$

Let

$$A_{\eta_i}^T P_i + P_i A_{\eta_i} + \left(B_{\eta_i} K_{i1}\right)^T P_i + P_i B_{\eta_i} K_{i1} = -Q_i$$

We further have

$$\dot{V}_i \leq -\lambda_{i1} \|\eta_i(t)\|^2 \tag{16}$$

where $\lambda_{i1} > 0$ is the eigenvalue of Q_i, which means that the error $\eta_i(t)$ of $\overline{m}_i(t)$ is exponential stable.

As for $m_i(t)$ of $\overline{m}_i(t)$, the reference signal $y_{ir}(t)$ is not considered at first, then

$$\dot{m}_i(t) = \overline{A}_{m_i} m_i(t) + \overline{B}_{m_i} \delta_i(t) \tag{17}$$

Assuming that the station transition matrix of closed-loop system (17) is $\phi_i(t) = e^{\overline{A}_{m_i} t}$, and the following inequality holds

$$\|\phi_i(t)\| = \left\|e^{\overline{A}_{m_i} t}\right\| \leq m_0 e^{-\alpha t}, \ \alpha > 0, \ \forall t \geq 0$$

the solution of Eq. 17 can be described as:

$$m_i(t) = \phi_i(t,0) m_i(0) + \int_0^t \phi_i(t-\tau) \overline{B}_{m_i} \delta_i(\tau) \, d\tau$$

$$\leq \|\phi_i(t)\| \|m_i(0)\|$$

$$+ \int_0^t \|\phi_i(t-\tau)\| \left\|\overline{B}_{m_i}\right\| \|\delta_i(\tau)\| \, d\tau$$

$$\leq \|\phi_i(t)\| \|m_i(0)\|$$

$$+ \int_0^t \|\phi_i(t-\tau)\| \left\|\overline{B}_{m_i}\right\| \left\|K_{i1_{m_i}} m_i(\tau)\right\| \, d\tau$$

$$\leq \|\phi_i(t)\| \|m_i(0)\|$$

$$+ \int_0^t \|\phi_i(t-\tau)\| \left\|\overline{B}_{m_i} K_{i1_{m_i}}\right\| \|m_i(\tau)\| \, d\tau$$

$$\leq \|\phi_i(t)\| \|m_i(0)\|$$

$$+ \int_0^t \|\phi_i(t-\tau)\| \beta_A \|m_i(\tau)\| \, d\tau$$

$$\|m_i(t)\| \leq m_0 e^{-\alpha t} \|m_i(0)\|$$

$$+ \int_0^t m_0 e^{-\alpha(t-\tau)} \beta_A \|m_i(\tau)\| \, d\tau$$

$$\|m_i(t)\| e^{\alpha t} \leq m_0 \|m_i(0)\|$$

$$+ \int_0^t m_0 e^{\alpha \tau} \beta_A \|m_i(\tau)\| \, d\tau$$

According to Bellman-Gronwall Lemma, we have

$$\|m_i(t)\| e^{\alpha t} \leq m_0 \|m_i(0)\| e^{m_0 \beta_A \int_0^t dt}$$

$$\leq m_0 \|m_i(0)\| e^{m_0 \beta_A t}$$

$$\|m_i(t)\| \leq m_0 \|m_i(0)\| e^{-(\alpha - m_0 \beta_A) t}$$

While $-(\alpha - m_0 \beta_A) < 0$, $\|m_i(t)\|$ is bounded. Then take the reference $y_{ir}(t)$ into consideration, $y_{ir}(t)$ and matrix G are known, they are all bounded, so $\|m_i(t)\|$ is still bounded.

According to the above analysis, one can see that the error η_i of the inner-loop system is exponential stable and the states m_i of inner-loop system are bounded. The proof is completed. □

4 FTC of Permanent Faults

4.1 The Control Strategy

Once a permanent fault occurs, a fault tolerant controller is needed to re-stabilize the system. Actuator fault is one of common permanent faults.

The augmented system (10) with permanent fault can be described as:

$$\begin{cases} \dot{\overline{m}}_i(t) = \overline{A} \overline{m}_i(t) + \overline{B} \rho_i \delta_i(t) + G y_{ir}(t) \\ n_i(t) = C \overline{m}_i(t) \\ z_i(t) = D \dot{n}_i(t) = \left[\dot{\psi}_i - \dot{\psi}_i^* - \left(\dot{\beta}_i - \dot{\beta}_i^*\right)\right] \end{cases} \tag{18}$$

If ψ_i, β_i track the signal $y_{ir}(t)$, $\dot{\psi}_i, \dot{\beta}_i$ track the desired signal as well, then $w_i = \left[\dot{\psi}_i - \dot{\beta}_i\right]$ is tracked.

4.2 Fault Tolerant Controller Design

In order to eliminate the influence of the faults, a compensation controller is designed.

$$\delta_{iad}(t) = K_{i2}\overline{m}_i(t) = \begin{bmatrix} K_{i2\eta_i} & K_{i2m_i} \end{bmatrix} \begin{bmatrix} \eta_i(t) \\ m_i(t) \end{bmatrix} \quad (19)$$

The whole control law $\delta_i(t)$ for the system is

$$\delta_i(t) = \delta_{i1}(t) + c_i\delta_{iad}(t) \quad (20)$$

where c_i is a function that satisfies

$$c_i = \begin{cases} 0 & if\ all\ actuators\ are\ fault\text{-}free \\ 1 & otherwise \end{cases}$$

Once the permanent fault occurs, the system stabilized by controller $\delta_i(t)$ becomes

$$\begin{cases} \dot{\overline{m}}_i(t) = \tilde{A}\overline{m}_i(t) + \tilde{B}\delta_{iad}(t) + Gy_{ir}(t) \\ n_i(t) = C\overline{m}_i(t) \end{cases} \quad (21)$$

$$\tilde{A} = \overline{A} + \overline{B}\rho_i K_{i1}, \quad \tilde{B} = \overline{B}\rho_i$$

The closed-loop system can be described as

$$\begin{cases} \dot{\overline{m}}_i(t) = \left(\tilde{A} + \tilde{B}K_{i2}\right)\overline{m}_i(t) + Gy_{ir}(t) \\ n_i(t) = C\overline{m}_i(t) \end{cases} \quad (22)$$

where $\widehat{A} = \tilde{A} + \tilde{B}K_{i2}$.

Lemma 3 (Schur Complement) *For a partitioned matrix* $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}$, *where* X_{11} *is a block-matrix, then the following three conditions are equivalent:*

a) $X < 0$
b) $X_{11} < 0,\ X_{22} - X_{12}^T X_{11}^{-1} X_{12} < 0$
c) $X_{22} < 0,\ X_{11} - X_{12} X_{22}^{-1} X_{12}^T < 0$

Proof The proof is similar to [14], and thus is omitted. □

Theorem 2 *Given a constant* γ_f, *if there exists matrix* $X = X^T > 0$, Y, *such that the following inequality holds*

$$\begin{bmatrix} \tilde{A}^T X + X\tilde{A} + \tilde{B}Y + Y^T\tilde{B}^T & * & * \\ G & -I & * \\ CX & 0 & -\gamma_f^2 I \end{bmatrix} < 0 \quad (23)$$

Then compensate state feedback controller (19) *exists, such that the error* $\eta_i(t)$ *is asymptotically stable. The gain of controller* (19) *is*

$$K_{i2} = YX^{-1}$$

Proof According to [13], if

$$\begin{bmatrix} \widehat{A}^T P + P\widehat{A} + C^T C & PG \\ G^T P & -\gamma_f^2 I \end{bmatrix} < 0$$

holds, the system satisfies H∞ performance indicators, left-multiply and right-multiply matrix $diag\left(P^{-1}, I\right)$,

$$\begin{bmatrix} \widehat{A}^T P + P\widehat{A} + C^T C & PG \\ G^T P & -\gamma_f^2 I \end{bmatrix} = \begin{bmatrix} P^{-1}\widehat{A}^T PP^{-1} + P^{-1}P\widehat{A}P^{-1} + P^{-1}C^T CP^{-1} & P^{-1}PGP^{-1} \\ P^{-1}G^T PP^{-1} & P^{-1} - \gamma_f^2 IP^{-1} \end{bmatrix} < 0$$

Let $X = P^{-1}$, we have

$$\begin{bmatrix} X\widehat{A}^T + \widehat{A}X + XC^T CX & GX \\ XG^T & -\gamma_f^2 I \end{bmatrix} < 0$$

$$\Rightarrow \begin{bmatrix} X\widehat{A}^T + \widehat{A}X + \gamma_f^2 XC^T CX & G^T \\ X & -I \end{bmatrix} < 0$$

while $\widehat{A} = \tilde{A} + \tilde{B}K_{i2}$,

$$\begin{bmatrix} X\widehat{A}^T + \widehat{A}X + \gamma_f^2 XC^T CX & G^T \\ G & -I \end{bmatrix}$$

$$= \begin{bmatrix} X\left(\tilde{A} + \tilde{B}K_{i2}\right)^T + \left(\tilde{A} + \tilde{B}K_{i2}\right)X + \gamma_f^2 XC^T CX & G^T \\ G & -I \end{bmatrix}$$

$$< 0$$

then

$$
\begin{bmatrix}
X\left(\tilde{A}+\tilde{B}K_{i2}\right)^T+\left(\tilde{A}+\tilde{B}K_{i2}\right)X+\gamma_f^2 XC^T CX & G^T \\
G & -I
\end{bmatrix}
$$
$$
=\begin{bmatrix}
X\tilde{A}^T+\tilde{A}X+XK_{i2}^T\tilde{B}^T+\tilde{B}K_{i2}X+\gamma_f^2 XC^T CX & G^T \\
G & -I
\end{bmatrix}
$$
$$<0$$

Denote $Y=K_{i2}X$

$$
\begin{bmatrix}
X\tilde{A}^T+\tilde{A}X+Y^T\tilde{B}^T+\tilde{B}Y+\gamma_f^2 XC^T CX & G^T \\
G & -I
\end{bmatrix}<0
$$

By using Lemma 3,

$$
\begin{bmatrix}
\tilde{A}^T X+X\tilde{A}+\tilde{B}Y+Y^T\tilde{B}^T & * & * \\
G & -I & * \\
CX & 0 & -\gamma_f^2 I
\end{bmatrix}<0
$$

One can get

$$
\begin{bmatrix}
\tilde{A}^T X+X\tilde{A}+\tilde{B}Y+Y^T\tilde{B} & G^T \\
G & -I
\end{bmatrix}
$$
$$
+\begin{bmatrix}X^T C^T \\ 0\end{bmatrix}\bullet\gamma_f^2 I\bullet\begin{bmatrix}CX & 0\end{bmatrix}<0
$$
$$
\Rightarrow\begin{bmatrix}
\tilde{A}^T X+X\tilde{A}+\tilde{B}Y+Y^T\tilde{B}+\gamma_f^2 X^T C^T CX & G^T \\
G & -I
\end{bmatrix}<0
$$

Then the proof is completed.   □

It can be seen that the proposed fault tolerant control method is decentralized since each UAV just needs to know the information of its neighbor [18], and the FTC scheme is needed only for the faulty UAV rather than the whole formation, this control strategy greatly reduces the amount of computation and the efficiency of achievement is extraordinary.

5 FTC of Intermittent Faults

The following lemma analyzes the UAV's behavior when there is an intermittent fault.

Lemma 4 *If there exists an intermittent fault, even if asymmetric matrix P_i' and a positive define matrix Q_i' exist, the error η_i of the inner-loop system may not be exponential stable.*

Proof At the occurrence of intermittent faults, Eq. 15 changes into

$$
\begin{aligned}
\dot{V}_i\left(t\right)=&\,\dot{\eta}_i^T\left(t\right)P_i'\eta_i\left(t\right)+\eta_i^T\left(t\right)P_i'\eta_i\left(t\right)\\
=&\,2\eta_i^T\left(t\right)P_i'G_{\eta_i}y_{ir}\left(t\right)\\
&+\eta_i^T\left(t\right)\Big[A_{\eta_i}^T P_i'+P_i'A_{\eta_i}+\left(B_{\eta_i}K_{i1_{\eta_i}}f_i\right)^T P_i'\\
&\qquad+P_i'B_{\eta_i}K_{i1_{\eta_i}}f_i\Big]\eta_i\left(t\right)\\
\leq&\,2\left|\eta_i^T\left(t\right)\right|P_i'G_{\eta_i}\left|y_{ir}\left(t\right)\right|\\
&+\eta_i^T\left(t\right)\Big[A_{\eta_i}^T P_i'+P_i'A_{\eta_i}+\left(B_{\eta_i}K_{i1_{\eta_i}}f_i\right)^T P_i'\\
&\qquad+P_i'B_{\eta_i}K_{i1_{\eta_i}}f_i\Big]\eta_i\left(t\right)\\
\leq&\,\eta_i^T\left(t\right)\Big[A_{\eta_i}^T P_i'+P_i'A_{\eta_i}+\left(B_{\eta_i}K_{i1_{\eta_i}}f_i\right)^T P_i'\\
&\qquad+P_i'B_{\eta_i}K_{i1_{\eta_i}}f_i\Big]\eta_i\left(t\right)
\end{aligned}
$$

Then

$$
\begin{aligned}
Q_i'=&\,A_{\eta_i}^T P_{i'}+P_i'A_{\eta_i}+\left(B_{\eta_i}K_{i1_{\eta_i}}f_i\right)^T P_i'\\
&+P_i'B_{\eta_i}K_{i1_{\eta_i}}f_i
\end{aligned}\tag{24}
$$

Equation 16 changes into

$$
\dot{V}_i\leq\lambda_{i2}\left\|\eta_i\left(t\right)\right\|^2\tag{25}
$$

where $\lambda_{i2}>0$ is the eigenvalue of Q_i'. The proof is completed.   □

Let $P_{ij}(t)$ denote the probability for going from state $i(i=0,1)$ to state $j(j=0,1)$. The equations for these probabilities are

$$
\begin{cases}
P_{0,1}\left(t\right)=\dfrac{\lambda}{\lambda+\mu}\left(1-e^{-(\lambda+\mu)t}\right)\leq\dfrac{\lambda}{\lambda+\mu}\\
P_{0,0}\left(t\right)=1-P_{0,1}\left(t\right)=\dfrac{\mu}{\lambda+\mu}+\dfrac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}\\
\qquad\geq\dfrac{\mu}{\lambda+\mu}
\end{cases}\tag{26}
$$

Then during time period $[0,t)$, the time period the system keeps stable is $P_{0,0}(t)t$, and the time period the system becomes unstable is $P_{0,1}(t)t$.

Define $P_{0,1} \triangleq \frac{\lambda}{\lambda+\mu}$, $P_{0,0} \triangleq \frac{\mu}{\lambda+\mu}$. It is clear that at each time instant, the probability of the healthy is more than $\underline{P}_{0,0}$, while the probability of faulty is less than $P_{0,1}$ [23, 24]. $E(\bullet)$ indicates the expectation.

Theorem 3 *Consider system* (10) *with intermittent. There exists an initial condition of* $\overline{m}_i(0)$ *and* δ_{i1}, *such that the origin is asymptotically in probability, if*

$$\lambda_{i2}\lambda < \lambda_{i1}\mu \qquad (27)$$

Proof According to Eqs. 15 and 16,

$$V_i(t) \le e^{-\lambda_{i1}\Delta t_1 + \lambda_{i2}\Delta t_2} V_i(0)$$

while $-\lambda_{i1}\Delta t_1 + \lambda_{i2}\Delta t_2$ can be described as $-\lambda_{i1}P_{0,0}(t)t + \lambda_{i2}P_{0,1}(t)t$

Then

$$E(V_i(t)) \le e^{\int_0^t (-\lambda_{i1}P_{0,0}(\tau) + \lambda_{i2}P_{0,1}(\tau))d\tau} V_i(0)$$
$$\le e^{\left(-\lambda_{i1}\frac{\mu}{\mu+\lambda} + \lambda_{i2}\frac{\lambda}{\mu+\lambda}\right)t} V_i(0) \quad \forall t \ge 0$$

If $-\lambda_{i1}\frac{\lambda}{\lambda+\mu} + \lambda_{i2}\frac{\mu}{\lambda+\mu} < 0$, one can have $\lambda_{i2}\lambda < \lambda_{i1}\mu$. Condition (27) ensures that $E(V_i(t)) < V_i(0)$, which means that the control law (12) is always available in probability. Hence, we can have $\lim_{t\to\infty} E(V_i(t)) = 0$. The proof is completed. $\square$

Condition (27) provides an explicit relation among healthy and faulty situations for the maintenance of the stability, which implies that the healthy situation can compensate for the negative effect of faulty situations provided that λ_{i1} and μ are large enough compared with λ_{i2} and λ. Note that any active FTC design is not needed. Such a result can be combined with other FTC design method to improve the reliability of the flight control system with respect to intermittent faults, and to make the FTC scheme more flexible.

6 Simulation Results

In the simulation, the formation is composed of 5 UAVs as is shown in Fig. 3. Details of such formation model can be found in [1].
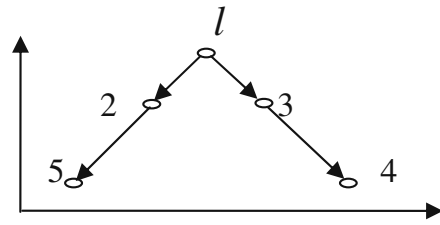

Fig. 3 Topology of 5 UAVs in formation

The desired angle and velocity are $V^* = 30$ m/s, $w^* = 0.5$ rad/s. As for the inner-loop variables, $\psi^* = 32$ deg/s, $\beta^* = 6$ deg/s UAV 2 is faulty.

The system matrices take the form:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} -0.0334 & -2.977 & 0.00 & -9.81 \\ -0.0016 & -4.133 & 0.98 & 0 \\ 0.0077 & -140.2 & -4.435 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -0.732 & 0.0143 & -0.996 & 0.0706 \\ -893 & -9.059 & 2.044 & 0 \\ 101.673 & 0.0186 & -1.283 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} -1.075 & 0.2453 \\ 0.3470 & -4.133 \\ -140.22 & 0 \\ 0 & 0 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 & 0.244 \\ 328.653 & 308.498 \\ 47.5280 & 102.891 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 0 & -SC \\ 0 & A \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ B \end{bmatrix},$$
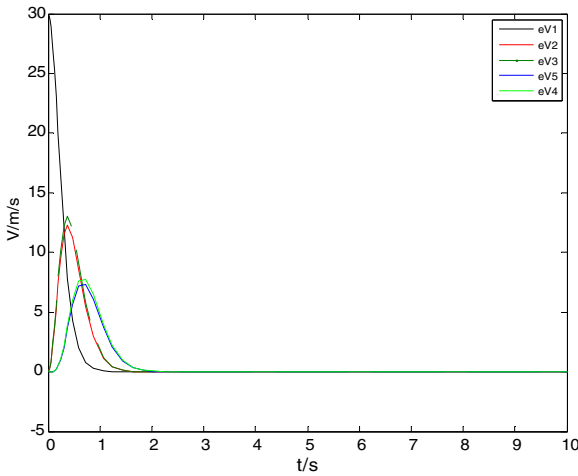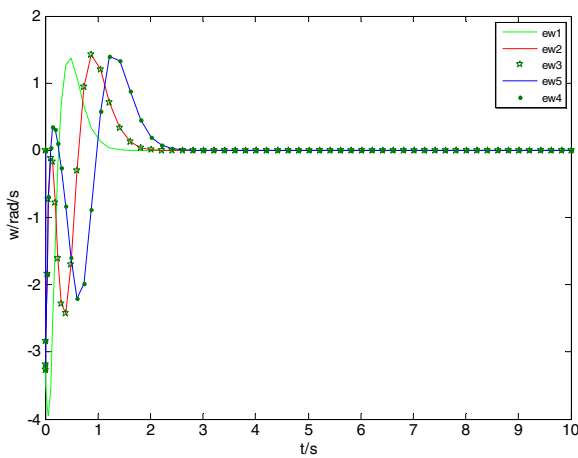
Fig. 4 Velocity error of normal system

Choose

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
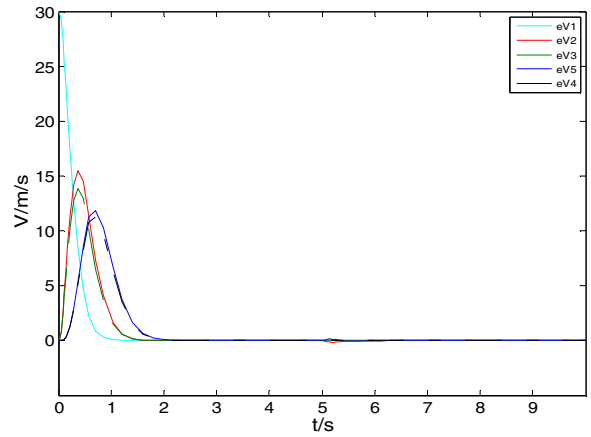
By using LMI, one obtain state feedback control gain of normal system $K_{21} = \begin{bmatrix} K_{21_1} & 0 \\ 0 & K_{21_2} \end{bmatrix}$

$$K_{21_1} = \begin{bmatrix} -1.1350 & 0.1979 & -1.0130 & 0.1079 & 0.5988 \\ 86.7668 & -25.9686 & 4.0673 & 1.1980 & 26.8821 \end{bmatrix}$$

$$K_{21_2} = \begin{bmatrix} 13.5138 & 2.8178 & -0.0577 & 0.4109 & -0.5078 \\ -14.3884 & -0.1090 & 0.0292 & -0.4442 & 0.2494 \end{bmatrix}$$

Fig. 5 Angular velocity error of normal system

Fig. 6 Velocity error under incipient fault

As for the normal system, the velocity error and angular velocity error are showed in Figs. 4 and 5.

6.1 Permanent Faults

The incipient permanent fault happens at 5 s.

$$\rho_2 = \mathrm{diag}\,[0.95,\ 0.99,\ 0.95,\ 0.97]$$

Compensate control gain is $K_{22} = \begin{bmatrix} K_{22_1} & 0 \\ 0 & K_{22_2} \end{bmatrix}$

$$K_{22_1} = \begin{bmatrix} -0.0598 & 0.0104 & -0.0534 & 0.0057 & 0.0315 \\ 0.8764 & -0.2623 & 0.0411 & 0.0121 & 0.2716 \end{bmatrix}$$

$$K_{22_2} = \begin{bmatrix} 0.7112 & 0.1483 & -0.0030 & 0.0216 & -0.0267 \\ -0.1453 & -0.0011 & 0.0003 & -0.0045 & 0.0025 \end{bmatrix}$$
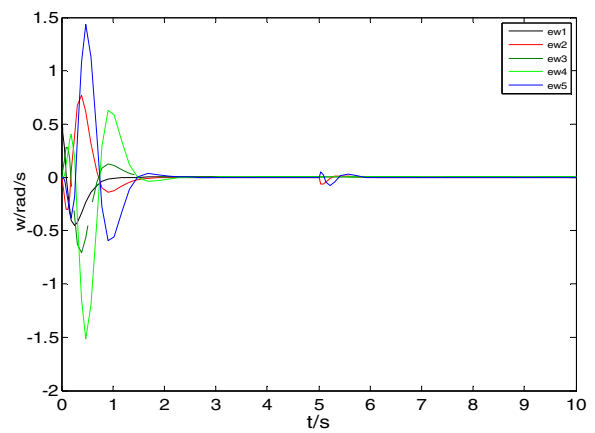
Fig. 7 Angular velocity error under incipient fault
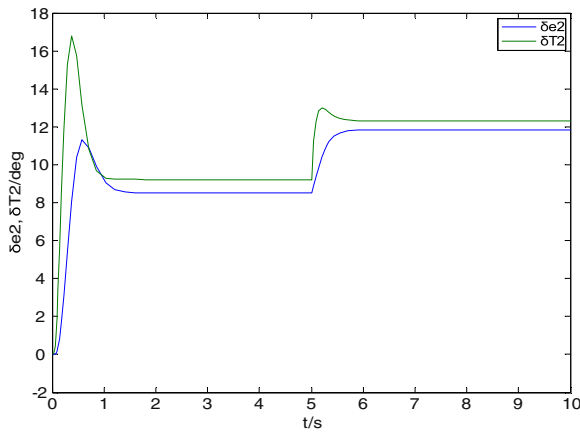
Fig. 8 Change of control surfaces δ_{e2}, δ_{T2} under incipient fault

Figures 6 and 7 show velocity and angular velocity error under incipient fault respectively. Figures 8 and 9 show the changes of control surfaces δ_{2e}, δ_{2T} and δ_{2a}, δ_{2r} under incipient fault respectively.
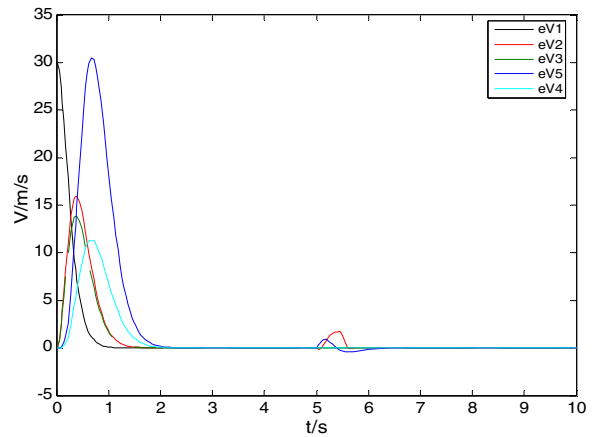
The severe permanent fault happens at 5 s.

$$\rho_2' = \text{diag}\,[0.3, \quad 0.3, \quad 0.3, \quad 0.3]$$

Compensate control gain is $K_{22} = \begin{bmatrix} K_{22_1} & 0 \\ 0 & K_{22_2} \end{bmatrix}$

$$K_{22_1} = \begin{bmatrix} -2.6484 & 0.4598 & -2.3638 & 0.2517 & 1.3971 \\ 202.4558 & -60.5935 & 9.4904 & 2.7953 & 62.7249 \end{bmatrix}$$
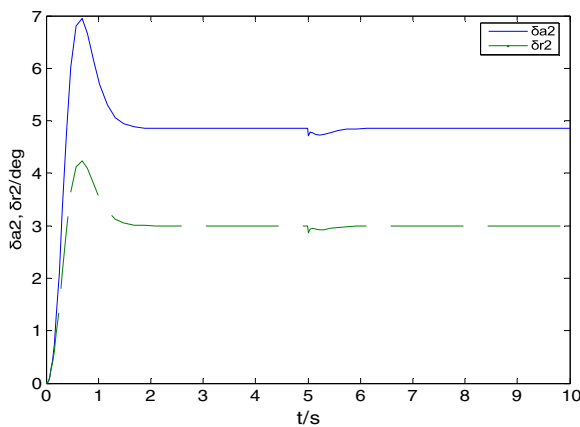
Fig. 9 Change of control surfaces δ_{e2}, δ_{r2} under incipient fault

Fig. 10 Velocity error under severe fault

$$K_{22_2} = \begin{bmatrix} 31.5320 & 6.5748 & -0.1345 & 0.9587 & -1.1848 \\ -33.5729 & -0.2542 & 0.0682 & -1.0366 & 0.5818 \end{bmatrix}$$

Figures 10 and 11 show velocity and angular velocity error of severe fault respectively. Figures 12 and 13 show changes of control surfaces δ_{2e}, δ_{2T} and δ_{2a}, δ_{2r} under severe fault respectively.
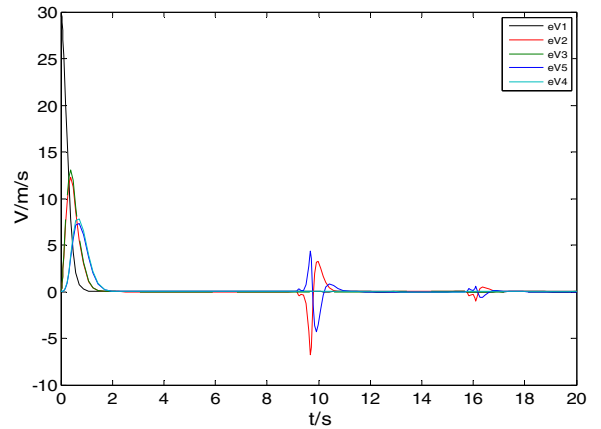
6.2 Intermittent Faults

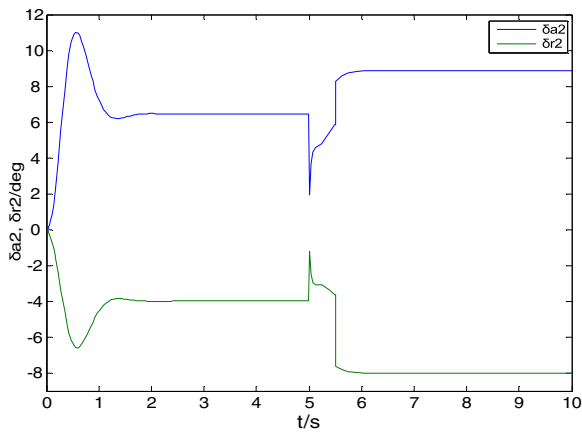At the occurrence of intermittent fault, the fault's appearance and disappearance are showed in
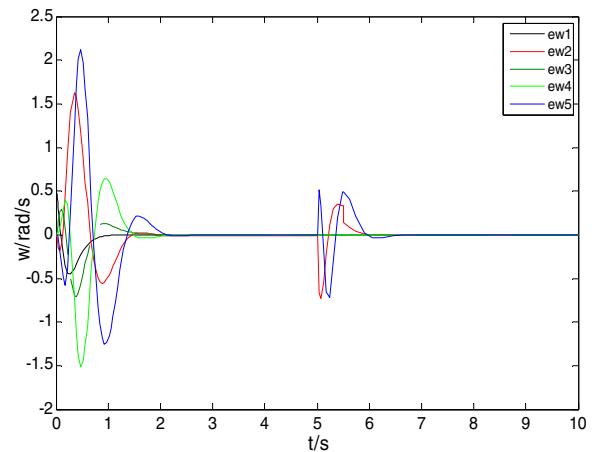
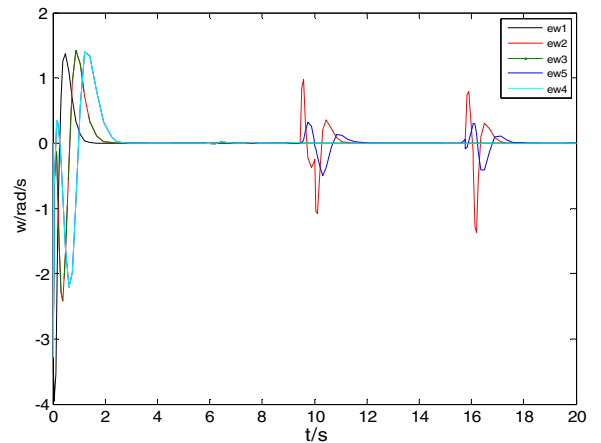**Fig. 11** Angular velocity error under severe fault
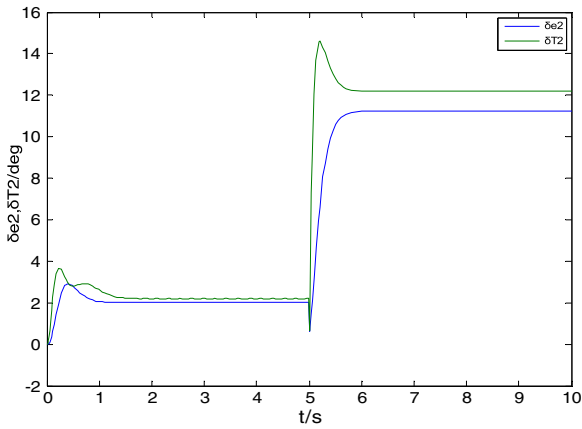
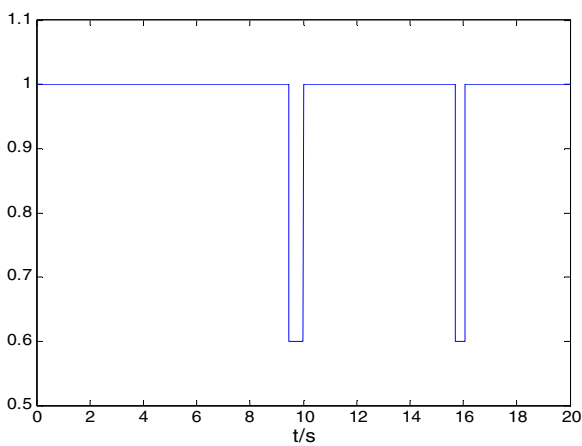**Fig. 12** Change of control surfaces δ_{e2}, δ_{T2} under severe fault

Fig. 13 Change of control surfaces δ_{a2}, δ_{r2} under severe fault

Fig. 14 Intermittent fault model

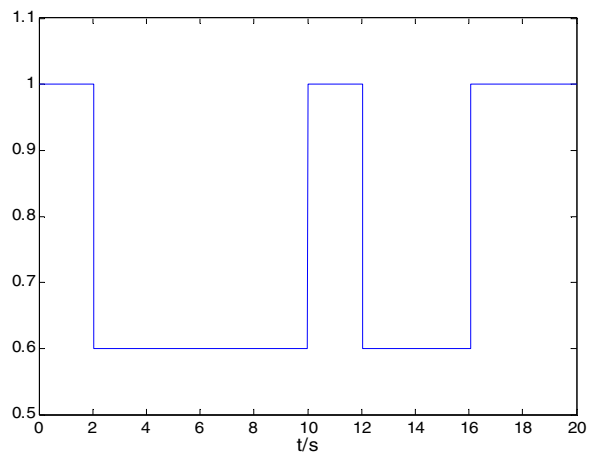**Fig. 15** Velocity error of intermittent fault
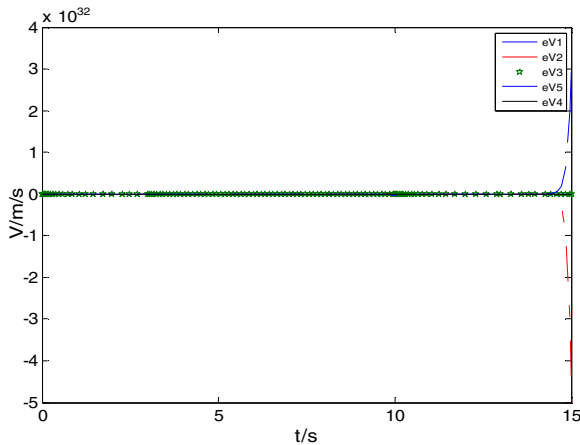

Fig. 16 Angular velocity error of intermittent fault


Fig. 17 Changing fault model

Fig. 18 Velocity error of intermittent fault

Fig. 14. The faults happened at 9 s and 15.5 s respectively and disappeared at 10 s and 16 s respectively.

According to Figs. 14, 15 and 16, one can see that the intermittent model satisfies the probability proposed by Theorem 1. And the faulty system is stable without designing a new controller.

By changing the fault model, one can see from Figs. 17, 18 and 19 that the fault model does not satisfy the probability, so the system becomes unstable. The first time the fault happens at 2 s and disappears at 10 s, the second time the fault happens at 12 s and disappears at 16 s.
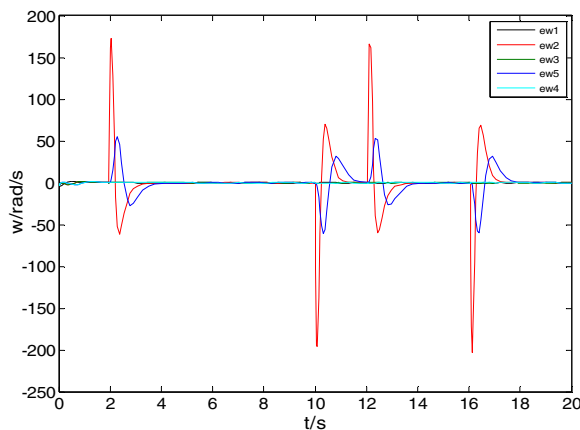
Fig. 19 Angular velocity error of intermittent fault

7 Conclusion

This paper considers the FTC problem of UAVs formation in the presence of permanent and intermittent faults. FTC is achieved in each individual UAV, the future work still focus on cooperative FTC design under which the FTC goal can be achieved by cooperation among UAVs.

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