Fault-Tolerant Routings in Chordal Ring Networks

Lali Barrière
Departament de Matemàtica Aplicada i Telemàtica, Universitat Politècnica de Catalunya, Barcelona, Spain

Josep Fàbrega
Departament de Matemàtica Aplicada i Telemàtica, Universitat Politècnica de Catalunya, Barcelona, Spain

Ester Simó
Departament de Matemàtica Aplicada i Telemàtica, Universitat Politècnica de Catalunya, Barcelona, Spain

Marisa Zaragozá
Departament de Matemàtica Aplicada i Telemàtica, Universitat Politècnica de Catalunya, Barcelona, Spain

This paper studies routing vulnerability in networks modeled by chordal ring graphs. In a chordal ring graph, the vertices are labeled in $\mathbb{Z}_{2n}$ and each even vertex i is adjacent to the vertices $i + a$, $i + b$, $i + c$, where a, b, and c are different odd integers. Our study is based on a geometrical representation that associates to the graph a tile which periodically tessellates the plane. Using this approach, we present some previous results on triple-loop graphs, including an algorithm to calculate the coordinates of a given vertex in the tile. Then, an optimal consistent fault-tolerant routing of shortest paths is defined for a chordal ring graph with odd diameter and maximum order. This is accomplished by associating to the chordal ring graph a triple-loop one. When some faulty elements are present in the network, we give a method to obtain central vertices, which are vertices that can be used to reroute any communication affected by the faulty elements. This implies that the diameter of the corresponding surviving route graph is optimum. © 2000 John Wiley & Sons, Inc.

Keywords: chordal ring networks; fault-tolerance; routing vulnerability; plane tessellations

1. INTRODUCTION

Loop networks have been widely considered in recent years as good models for interconnection or communication computer networks, due to their regularity, simple structure, and symmetry. See Bermond et al. [5] for an exhaustive survey on this topic. Within this context, two problems that have been considered in the literature—because they are closely related to the network bandwidth and the transmission delay—are the minimization of the diameter for a given number of nodes and maximum degree and the maximization of the number of nodes given the diameter and the maximum degree. Authors that have contributed to the study of these problems are, among others, Aguiló and Fiol [1], Aguiló et al. [2], Bermond and Tzvieli [7], Chen and Jia [8], Du et al. [11], Erdös and Hsu [12], Esqué et al. [14], Fiol et al. [17], Hsu and Shapiro [19, 20], Tzivieli [29], and Yebra et al. [31]. See also the survey of Hwang [21].

Chordal ring graphs of degree 3 constitute a family of generalized loop networks that, although they were first introduced by Coxeter [9] more than 50 years ago, were first considered from the point of view of interconnection networks by Arden and Lee in [3]. These graphs are simply obtained by adding chords in a regular manner to an undirected cycle. Yebra et al. [31] improved the number of vertices (in relation to its diameter) in the construction of Arden and Lee. Moreover, several optimal constructions of generalized chordal ring graphs were presented in this reference. Following this work, and also Morillo [25], our study is based on a geometrical representation that associates to a generalized chordal ring graph a tile which periodically tessellates the plane. This geometrical approach leads us to the study of lattices and congruences in $\mathbb{Z}^2$. Within this framework, see the contributions of Fiol [16] and Morillo [26]. It is worth

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Correspondence to: J. Fàbrega; e-mail: matjfc@mat.upc.es
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mentioning that the study of graphs and digraphs associated to plane tessellations was first initiated by Wong and Coppersmith [30] and Fiol et al. [17]. Since then, this geometrical approach has been successfully applied to optimize the order and the diameter of different kinds of networks, as well as to define routing schemes with good properties with respect to routing vulnerability.

In fact, one of the most important features to be taken into account in the design of an interconnection network is the existence of efficient algorithms for routing messages. Let us consider the network modeled by a connected graph $G = (V, E)$. A *routing* ρ in G is a function $\rho : V \times V \to E^*$, where E^* is the set of all paths of G. We say that ρ is a *routing of shortest paths* when $\rho(x, y)$ is a shortest path for every $x, y \in V$. The routing ρ is *bidirectional* if $\rho(x, y) = \rho(y, x)$ for every $x, y \in V$, and ρ is said to be *consistent* when $\rho(x, y) = x, u_1, \ldots, z, \ldots, u_m, y$ implies that $\rho(x, z) = x, u_1, \cdots, z$ and $\rho(z, y) = z, \cdots, u_m, y$. To give a measure of the routing vulnerability, the *surviving route graph* was introduced in [10] by Dolev et al. Given a connected graph G, a routing ρ, and a set $F = V_F \cup E_F$ of faulty vertices and edges, the surviving route graph $R(G, \rho)/F$ is the directed graph with a set of vertices $V \setminus V_F$ and where a vertex x is adjacent to vertex y if $\rho(x, y)$ does not contain any element of F. Note that if the routing is bidirectional then $R(G, \rho)/F$ is a graph. The diameter of $R(G, \rho)/F$ gives the maximum number of rerouting steps needed to send a message, affected by the faulty elements of the set F, along a sequence of surviving routes. Hence, this diameter is a measure of the worst-case behavior of the faulty network. Thus, given a network, an interesting problem is to find routings that can tolerate many faults without increasing too much the diameter of the corresponding surviving route graph. A vertex c adjacent in $R(G, \rho)/F$ to and from any other vertex of the surviving route graph will be called a (ρ, F)-*central* vertex. The existence of such a vertex is interesting since, in the network modeled by G, any communication from node x to node y could always be carried out through c with only one rerouting step. Thus, when (ρ, F)-central vertices exist, the diameter of $R(G, \rho)/F$ is at most two. See Bermond et al. [5] for a survey of known results on the diameter of the surviving route graph of different networks. See also Fraigniaud and Lazard [18] for a survey on different methods and problems of communication in usual networks.

We are interested in the construction of reliable and fault-tolerant routings of shortest paths in networks modeled by chordal ring graphs. Authors that have contributed to the study of these problems, for double- and triple-loop graphs are, among others, Aguiló and Fiol [1], Arden and Lee [4], Escudero et al. [13], Fàbrega and Zaragozá [15], Hwang and Wright [22], Liestman et al. [23], Manabe et al. [24], Mukhopadhyaya and Sinha [27], Narayanan and Opatrny [28], and Zaragozá [32]. As mentioned above, a tile that periodically tessellates the plane will be associated to the graph under study. Using this approach, we present in Section 2 some previously known results on triple-loop networks. Moreover, an algorithm to calculate the coordinates of a given vertex of an optimal triple-loop graph, in the tile that represents it, is also presented in this section. In Section 3, chordal ring graphs of degree 3 are studied. A fault-tolerant routing of shortest paths is defined in Section 4 for a chordal ring graph of degree 3 with odd diameter and maximum order. This is accomplished by associating to the chordal ring graph a triple-loop one and taking into account the coordinates of the vertices. Finally, the vulnerability of this routing is studied in Section 5. When some faulty elements are present in the network, we give a method to obtain central vertices. This implies that the diameter of the corresponding surviving route graph is optimum.

2. TRIPLE-LOOP GRAPHS

Let us review some previous results on triple-loop networks that will be used in the following in connection with chordal ring graphs:

In this paper, a *triple-loop graph* $T_N(A, B, C)$ is a graph with vertex set $\mathbb{Z}_N$ and set of edges constituted by all the pairs $\{i, i \pm A\}$, $\{i, i \pm B\}$, $\{i, i \pm C\}$, where A and B are different integers such that $1 \leq A, B \leq \lfloor N/2 \rfloor$, and $C = -(A + B)$. The graph is vertex-transitive and the translations $i \mapsto i + \alpha$, $\alpha \in \mathbb{Z}_N$, are automorphisms. Moreover, $T_N(A, B, C)$ is connected if and only if $(A, B, N) = 1$. The adjacency pattern shown in Figure 1 leads to a geometrical representation of the graph in which vertices correspond to hexagonal cells and $T_N(A, B, C)$ corresponds to a tile that periodically tessellates the plane. See the references Bermond et al. [6], Morillo et al. [26], Yebra et al. [31], and [15, 32] for a detailed study of the use of this geometrical representation to solve metric problems in loop graphs.

It has been proved in [25] that the triple-loop graphs $T_{N_k}((3k+1)B, B, -(3k+2)B)$, where $N_k = 3k^2 + 3k + 1$ and $B \in \mathbb{Z}^*_{N_k}$, are the ones with maximum order for a given diameter k, and all of them are isomorphic to $T_{N_k}(3k+1, 1, -(3k+2))$. In the tessellation associated to these optimal triple-loop graphs, the hexagons corresponding to vertex 0 are distributed in the plane according to an integer lattice satisfying the following equa-

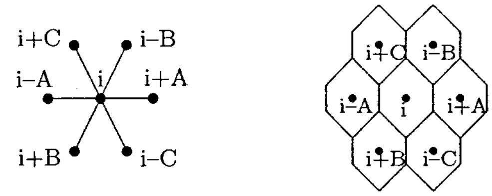

FIG. 1. Adjacency pattern.

tions (in $\mathbb{Z}_{N_k}$), as shown in Figure 2:

$$(k+1)A - kB = 0$$
$$kC - (k+1)B = 0$$

To derive a routing algorithm in a network modeled by a graph associated to a plane tessellation, it will be useful to have a method to calculate the coordinates of a given vertex in the tile. Let $T_{N_k}(A,B,C)$ be a triple-loop graph with diameter k and maximum order N_k. For any $z \in \mathbb{Z}_{N_k}$, we define $[z] = \{(m', n', p') : m'A + n'B + p'C \equiv z \pmod{N_k}\}$. The triple (m', n', p') corresponds in $T_{N_k}(A,B,C)$ to a walk from vertex 0 to vertex z formed by m' edges labeled A, n' edges labeled B, and p' edges labeled C. In the hexagonal tessellation, by doing m' steps A, n' steps B, and p' steps C, we reach from the center of any tile a cell corresponding to vertex z which is not necessarily located in the same tile. Let us call the *coordinates of* a vertex z the triple of integers (m,n,p) such that $mA + nB + pC \equiv z \pmod{N_k}$ and $|m| + |n| + |p| = d(0,z)$. In $T_{N_k}(A,B,C)$ this triple gives the shortest paths from vertex 0 to vertex z. Note that, due to the 2-dimensional nature of our representation, the integers m, n, p will have at least one of them equal to 0. Furthermore, if two of them are different from 0, then their signs must be different.

The purpose of the rest of this section is to derive an algorithm to find the coordinates of a given vertex z in a given tile centered at 0. To this end, we will find first a triple $(M,N,0) \in [z]$ satisfying one of the following conditions: (a) $0 \le M \le k$ and $0 \le N \le k$; (b) $0 \le -M \le k$ and $0 \le -N \le k$; (c) $M > 0$, $N < 0$ and $M - N \le k$; or (d) $M < 0$, $N > 0$ and $N - M \le k$. These conditions will assure that all the cells determined by $(M,N,0)$, corresponding to the different vertices of $T_{N_k}(A,B,C)$, will belong to the same tile, as shown in Figure 3. However, the path determined by $(M,N,0)$ is not necessarily a shortest one. So, the algorithm must finish obtaining the coordinates (m,n,p) from $(M,N,0)$.

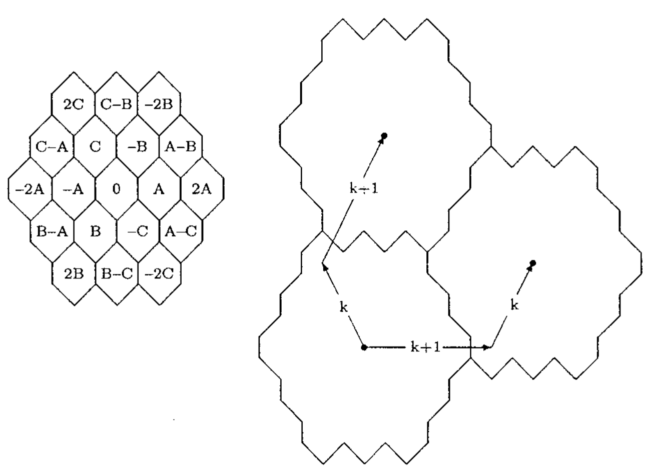

FIG. 2. Tile and hexagonal tessellation corresponding to diameter $k = 2$.

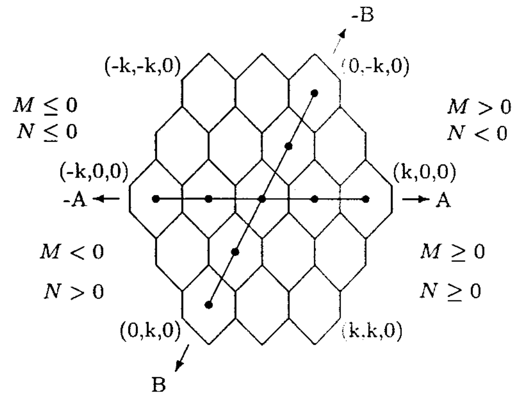

FIG. 3. The four zones, according to $(M,N,0)$.

We can restrict our study to the case $A = 3k + 1$, $B = 1$, and $C = -(3k + 2)$. Let $z \in \mathbb{Z}_{N_k}$ and divide z by $A = 3k + 1$ to obtain $z = qA + r$. Since $B = 1$, we can write $z = qA + rB$. See a detailed example in Figure 4, in which the labeled cells are the ones reached from 0 according to the different values of q and r.

Thereby, the algorithm that we propose is the following:

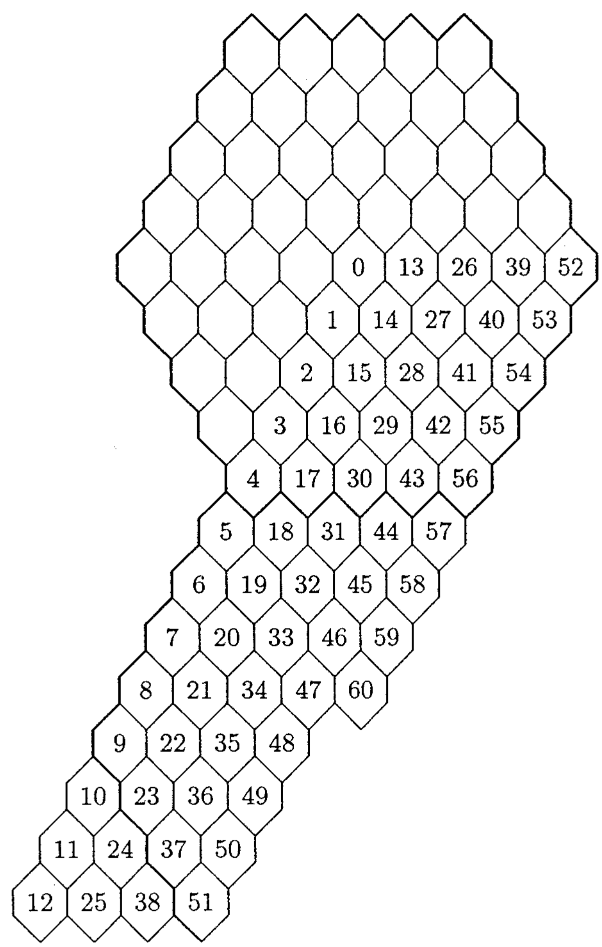

FIG. 4. First step of the algorithm, for $k = 4$, $N_4 = 61$, $A = 13$, $B = 1$.

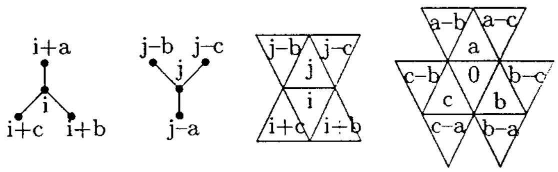

FIG. 5. Plane representation of adjacencies.

ALGORITHM

1. Divide z by $A = 3k + 1$.
   Integers $i = \left\lfloor \dfrac{z}{3k+1} \right\rfloor$ and $j = z \pmod{3k+1}$ verify $(i, j, 0) \in [z]$, $0 \le i \le k$, $0 \le j \le 3k$.
2. Obtain $(M, N, 0)$.

   a. If $j \le k$, then $(M, N, 0) = (i, j, 0)$.
   b. If $j > k$, then $(i_1, j_1, 0) = (i - k, j - 2k - 1, 0)$. Since $kA + (2k+1)B = N_k$, we have $(i_1, j_1, 0) \in [z]$. Moreover, $-k \le i_1 \le 0$, $-k \le j_1 < k - 1$.

   (i) If $j_1 \le 0$ or $j_1 - i_1 \le k$, then $(M, N, 0) = (i_1, j_1, 0)$.
   (ii) If $j_1 - i_1 > k$, then $(i_2, j_2, 0) = (i_1 + k + 1, j_1 - k, 0)$.
   Since $(k+1)A - kB = N_k$, $(i_2, j_2, 0) \in [z]$. Moreover, $0 < i_2 \le k$, $-k < j_2 < 0$, and $j_2 - i_2 \ge k$.
   Thereby, $(M, N, 0) = (i_2, j_2, 0)$.

3. Obtain (m, n, p) from $(M, N, 0)$.

   a. If $M \cdot N \le 0$, then $(m, n, p) = (M, N, 0)$.
   b. If $M \cdot N > 0$, then:

   (i) If $|N| \ge |M|$, then $(m, n, p) = (0, N - M, -M)$.
   (ii) If $|N| < |M|$, then $(m, n, p) = (M - N, 0, -N)$.

This method to calculate the coordinates of a vertex in the tile will be used in Sections 4 and 5 to define a fault-tolerant routing in a chordal ring network. In Liestman et al. [23], an algorithm to calculate the coordinates in triple-loop graphs is also presented. Although both algorithms have the same order of complexity, the one given in [23] depends on a previous method to calculate coordinates in double-loop graphs, while our algorithm is based on simple integer division.

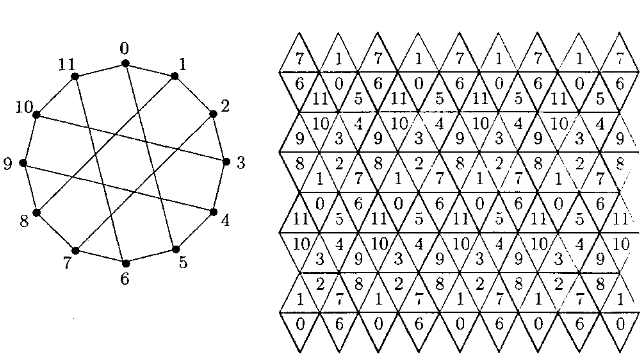

FIG. 6. The graph $C_{12}(1, 5, -1)$ and the associated tessellation.

3. CHORDAL RING GRAPHS OF DEGREE 3

Let $2n$ be an even integer and let a, b, c be different odd integers between 0 and $2n$. A *chordal ring* graph of order $2n$ and steps a, b, c is a graph $C_{2n}(a, b, c)$ in which the set of vertices is $\mathbb{Z}_{2n}$ and any even vertex i is adjacent to the odd vertices $i + a$, $i + b$, $i + c$ (consequently, any odd vertex j is adjacent to the even vertices $j - a$, $j - b$, $j - c$.) The graph $C_{2n}(a, b, c)$ is 3-regular, bipartite, vertex-transitive and it is connected if and only if $(a - b, b - c, 2n) = 2$.

As triple-loop graphs, chordal ring graphs can be represented by a periodic tessellation of the plane. The adjacency pattern shown in Figure 5 defines such a tessellation, in which vertices are now represented by triangular cells. An example is given in Figure 6.

We can consider the vertices of $C_{2n}(a, b, c)$ grouped into n pairs $P = (i, i + a)$ linked by the double steps $2A = b - c$, $2B = c - a$, and $2C = a - b$. This allows us to consider each pair P adjacent to the pairs $P \pm 2A$, $P \pm 2B$, and $P \pm 2C$, so we have a new graph of order n; see Figure 7. In fact, the obtained graph is isomorphic to $T_n(A, B, C)$, but with even vertices and steps.

It is proved in [25] that $C_{m_k}(-1, 3k, 1)$, with $k = 2l + 1$ and $m_k = (3k^2 + 1)/2$, is the chordal ring of maximum order for diameter k. The associated triple-loop graph is $T_{N_l}(3l + 1, 1, -(3l + 2))$, which is also optimal. For the case $k = 2l$, the maximum order attainable is at most $3k^2/2$ and the associated triple-loop graph is not optimal. In the following sections, we concentrate on chordal ring graphs with maximum order and odd diameter.

4. ROUTING

In this section, we define a consistent routing of shortest paths in the graph $C_{m_k}(a, b, c)$ with odd diameter k and steps a, b, and c chosen as above. The routing that we propose has good properties because it makes use of the graph symmetries.

Because of the vertex-transitivity of the graph, it suffices to define the routing only for the pairs $(0, z)$. To this end, let us consider the spanning tree shown in Figure 8, which will be called the *routing tree associated* to $\rho(0, z)$, such that for any vertex z the unique path in

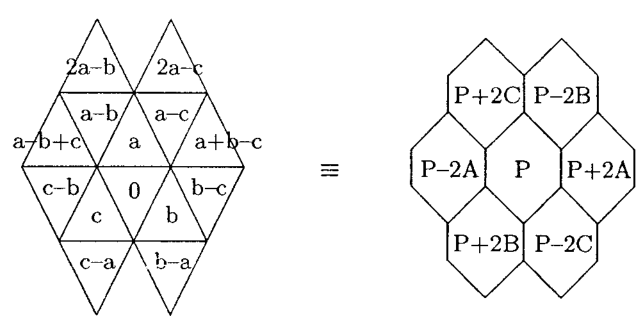

FIG. 7. Adjacency pattern for $C_{2n}(a, b, c)$ and its associated triple-loop graph.

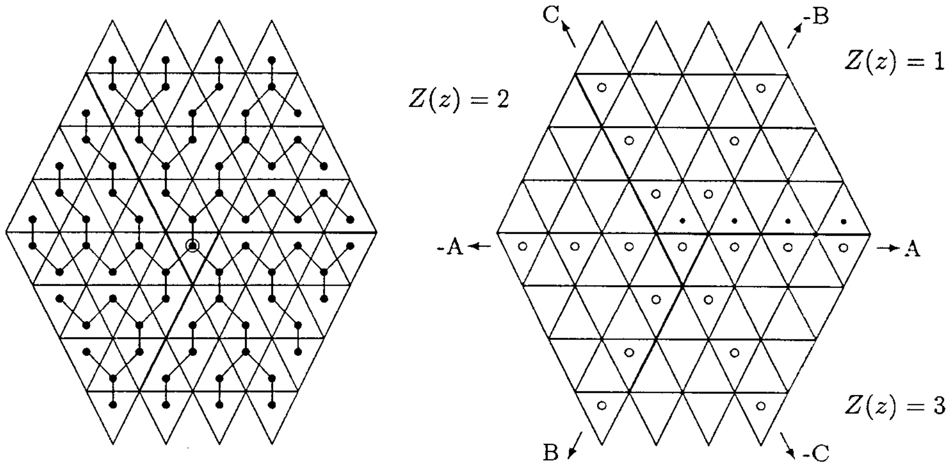

FIG. 8. Routing tree associated to $\rho(0,z)$ and situation of vertices for $k = 7$.

this tree is precisely $\rho(0,z)$. The tile is divided into three zones which give us three subtrees. We only need to define the paths in the first zone and obtain the others by a rotation of $2\pi/3$ or $-2\pi/3$, depending on the situation of the vertex in the tile. Rotating a path is equivalent to permuting the steps, as will be detailed in what follows.

Even vertices are distributed in the tile according to their coordinates in the corresponding triple-loop graph. In the first zone, we find vertices with negative second coordinates: $z = (0,-n,p)$, $n,p \geq 0$ and $(m,-n,0)$, $m,n > 0$. By a $(2\pi/3)$-rotation, we obtain the vertices of the second zone, whose first coordinate is negative: $z = (-m,n,0)$, $m,n \geq 0$ and $z = (-m,0,p)$, $m,p > 0$. The third zone, corresponding to vertices with a negative third coordinate, can be obtained by a $(2\pi/3)$-rotation from the second one or by a $(-2\pi/3)$-rotation from the first one. Odd vertices are all in the same zone as their even mates, except vertices $z = i2A + a$, which are in the first zone, while vertices $z = i2A$ are in the third one.

Let α denote the $(2\pi/3)$-rotation. If $z = (m,n,p)$ is an even vertex, then $\alpha(z) = (n,p,m)$. On the other hand, if z is odd, then $\alpha(z) = \alpha(z-a)+\alpha(a) = \alpha(z-a)+c$. Similarly, if $z = (m,n,p)$ is an even vertex, then $\alpha^{-1}(z) = (p,m,n)$, and if z is odd, then $\alpha^{-1}(z) = \alpha^{-1}(z-a) + \alpha^{-1}(a) = \alpha^{-1}(z-a) + b$.

For the given vertex z, we define $Z(z)$ as 1, 2 or 3, according to the zone of the tile that it belongs to. For $Z(z) = 1$, we calculate the path $\rho(0,z)$. For $Z(z) = 2$, the vertex $\alpha^{-1}(z)$ is in the first zone, so we can calculate $\rho(0,\alpha^{-1}(z))$ and rotate this path by α to obtain $\rho(0,z)$. Finally, for $Z(z) = 3$, the vertex $\alpha(z)$ is in the first zone; thus, we can calculate $\rho(0,\alpha(z))$ and rotate this path by α^{-1} to obtain $\rho(0,z)$. Since we will give a path as a succession of steps, we can also observe how steps a, b and c are changed under α and α^{-1}. To do a $(2\pi/3)$-rotation (or $(-2\pi/3)$-rotation) is the same as permuting the steps a, b, c. In fact, changing a into c, b into a, and c into b we obtain the α-rotation. On the other hand, changing a into b, b into c, and c into a, we have α^{-1}. (See Fig. 8.)

At this point, we finish by giving $\rho(0,z)$ for $Z(z) = 1$. Four possible cases have to be considered, depending on the parity and the coordinates of z:

1. Even vertex, $z = (0,-n,p)$, $n,p \geq 0$
   $\rho(0,z) = \underbrace{a,\ -b}_{1}, \underbrace{a,\ -b}_{2}, \ldots, \underbrace{a,\ -b}_{p}, \underbrace{a,\ -c}_{1}, \underbrace{a,\ -c}_{2}, \ldots, \underbrace{a,\ -c}_{n}$
2. Even vertex, $z = (m,-n,0)$, $m,n > 0$
   $\rho(0,z) = \underbrace{a,\ -c}_{1}, \underbrace{a,\ -c}_{2}, \ldots, \underbrace{a,\ -c}_{n}, \underbrace{b,\ -c}_{1}, \underbrace{b,\ -c}_{2}, \ldots, \underbrace{b,\ -c}_{m}$
3. Odd vertex, $z - a = (0,-n,p)$, $n,p \geq 0$
   $\rho(0,z) = a, \underbrace{-b,\ a}_{1}, \underbrace{-b,\ a}_{2}, \ldots, \underbrace{-b,\ a}_{p}, \underbrace{-c,\ a}_{1}, \underbrace{-c,\ a}_{2}, \ldots, \underbrace{-c,\ a}_{n}$
4. Odd vertex, $z - a = (m,-n,0)$, $m > 0$, $n \geq 0$
   $\rho(0,z) = a, \underbrace{-c,\ a}_{1}, \underbrace{-c,\ a}_{2}, \ldots, \underbrace{-c,\ a}_{n}, \underbrace{-c,\ b}_{1}, \underbrace{-c,\ b}_{2}, \ldots, \underbrace{-c,\ b}_{m}$

For any two vertices x, y, we define $\alpha_{x,y} : \mathbb{Z}_{2n} \to \mathbb{Z}_{2n}$ as follows: If $x - y \equiv 0 \pmod 2$, then $\alpha_{x,y}(i) = y - x + i$;

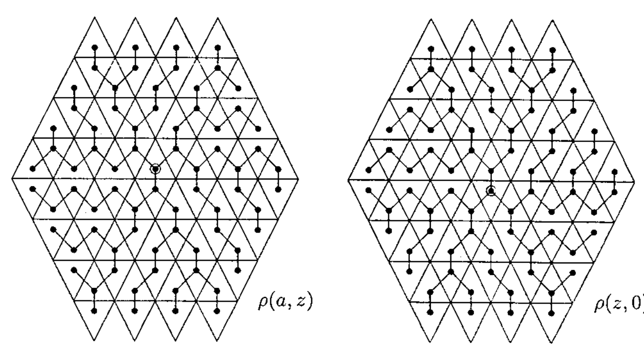

FIG. 9. The routing trees $\rho(a,z)$ and $\rho(z,0)$.

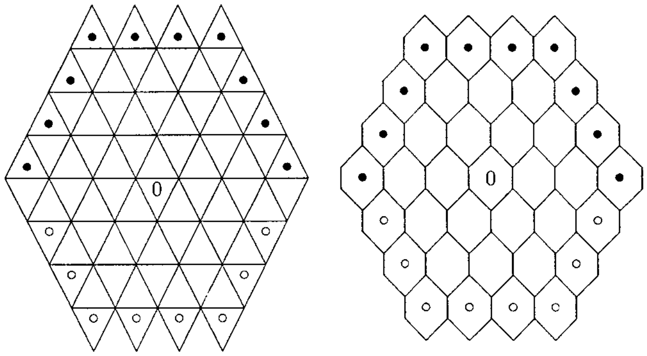

FIG. 10. The boundary of the tile.

else if $x - y \equiv 1 \pmod 2$, then $\alpha_{x,y}(i) = y + x - i$. In both cases, $\alpha_{x,y}$ is an automorphism and it satisfies $\alpha_{x,y}(x) = y$.

If x is even, $\alpha_{0,x}(i) = x + i$ is a translation, so in this case, the representation of $\rho(x, y)$ is the representation of $\rho(0, \alpha_{0,x}^{-1}(y))$. But when x is odd, $\alpha_{0,x}(i) = x - i$ is a π-rotation followed by a translation, while $\alpha_{a,x}(i) = x - a + i$ is a translation. So, we must study the routing tree associated to $\rho(a, z)$. Moreover, since the routing is not bidirectional, it is necessary to study the paths from one vertex to the tile center, which depend on the parity of this center. We will have then two more routing trees associated to $\rho(z, 0)$ and $\rho(z, a)$, respectively.

We will show that the trees given in Figure 9 exactly represent the paths defined above. Observe that to obtain $\rho(a, z)$ from $\rho(0, z)$ we make a π-rotation of the tile, while to pass from $\rho(0, z)$ to $\rho(z, 0)$, the transformation applied is a symmetry with respect to the central vertical axis.

First, we study the representation of $\rho(a, z)$. The automorphism $A(x) = a - x$ transforms $\rho(0, z)$ into $\rho(a, z)$. We must show that A is a π-rotation of the tile. But this is easy to prove because A is a π-rotation centered at the 0 vertex, followed by a translation with step a (in the plane tessellation, step a is represented by a vector). Such a transformation equals a π-rotation with the center the middle point of vertices 0 and a, and this point is the center of the tile. Second, let us study the representation of $\rho(z, 0)$. If the coordinates of z are $(0, -n, p)$, the path $\rho(z, 0)$ depends on the parity of z. For $z = -n\,2B + p\,2C$ this path can be obtained from $\rho(0, -z)$ by applying the

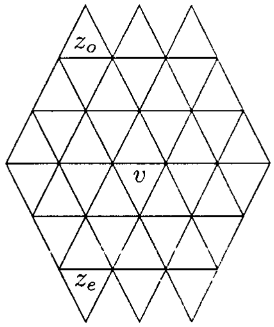

FIG. 11. Only one vertex fails.

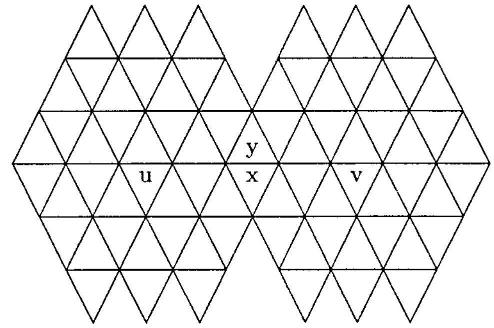

FIG. 12. The special case $y = x + a$.

translation $i \mapsto i + z$. In this way, we obtain $\rho(z, 0) = \underbrace{b,\, -a}_{1},\, \underbrace{b,\, -a}_{2},\, \ldots,\, \underbrace{b,\, -a}_{p},\, \underbrace{c,\, -a}_{1},\, \underbrace{c,\, -a}_{2},\, \ldots,\, \underbrace{c,\, -a}_{n}$. On the other hand, if $z = -n\,2B + p\,2C + a$, the path $\rho(z, 0)$ is obtained now from $\rho(0, z)$ by applying the automorphism $i \mapsto z - i$. Hence, $\rho(z, 0)$ is just $\rho(0, z)$ traversed in the opposite direction. In both cases, this corresponds to the routing tree given in Figure 9. For other vertex coordinates, the determination of $\rho(z, 0)$ can be reduced to the above-considered cases by a permutation of the steps. Finally, note that $\rho(z, a)$ can be determined either from $\rho(a, z)$ in the same way as $\rho(z, 0)$ is determined from $\rho(0, z)$, or from $\rho(z, 0)$ in the same way as $\rho(a, z)$ is determined from $\rho(0, z)$.

5. ROUTING VULNERABILITY: CENTRAL VERTICES

In this last section, the vulnerability of the routing ρ defined in the preceding one is studied. Let us recall that, given a set F of faulty vertices, a vertex v is (ρ, F)-central if and only if, for any vertex u, the paths $\rho(u, v)$ and $\rho(v, u)$ include no elements of F. When (ρ, F)-central vertices exist, the diameter of the surviving route graph is optimum, that is, it is at most two.

Let L_1 be the set of leaves of the routing tree $\rho(0, z)$. For any vertex $v \notin L_1$ the path $\rho(0, v)$ contains no vertices of L_1. Similarly, if L_2 is the set of leaves of $\rho(z, 0)$, then the path $\rho(v, 0)$ does not contain vertices of L_2, for

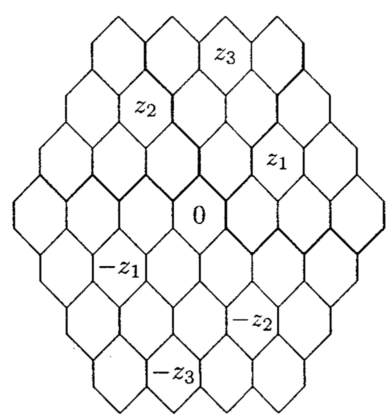

FIG. 13. Three different cases, according to the location of z^*.

any $v \notin L_2$. So, vertex 0 is (ρ, L)-central, for $L = L_1 \cap L_2$. The set L contains the odd vertices $i2A - (l-i)2B + a$ for $0 < i \leq l$, $-i2B + (l-i)2C + a$ for $0 \leq i \leq l$, and $-i2A + (l-i)2C + a$ for $0 < i \leq l$; and the even vertices $-i2A + (l-i)2B$ for $0 < i < l$, $i2B - (l-i)2C$ for $0 \leq i \leq l$, and $i2A - (l-i)2C$ for $0 < i < l$. This set is represented in Figure 10, where we show also the corresponding hexagonal tile for the associated triple-loop graph. Odd vertices of L correspond to the boundary hexagons located at the upper half of the tile, and even vertices of L correspond to lower-half ones. To prove that a vertex v is a (ρ, F)-central, it suffices to show that when centering the tile at v the elements of F are disposed on the boundary of the tile, that is, that the automorphism $\alpha_{v,0}$ transforms F into a subset of L.

In the case $F = \{z\}$, the existence of a (ρ, F)-central vertex v is clear as the following reasoning shows: If $z = z_e$ is even, consider $v = z_e - l2B$. Then, the automorphism $\alpha_{v,0}(i) = i - v$ transforms z_e into $l2B$, which is on the boundary L of the tile. So, v is a (ρ, F)-central vertex. Otherwise, if $z = z_o$ is odd, then we define $v = z_o - a - l2C$. Now, if $\alpha_{v,0}(i) = v - i$, then $\alpha(z_o) = -a - l2C$, which is again on the tile boundary. Hence, v is again a (ρ, F)-central vertex.

The following result proves the existence of (ρ, F)-central vertices in the case $|F| = 2$. In a certain sense, this is the best we can do because the degree of the graph is three and the failure of three or more vertices could disconnect it.

Theorem 1. *For any pair of vertices x, y of $C_{m_k}(a, b, c)$, there exists $(\rho, \{x, y\})$-central vertices.*

Proof. A special case is when $|y - x| = a$. We can assume without loss of generality that $y = x + a$. We cannot place both vertices on the tile boundary. But the vertices $u = x + lA$ and $v = x - lA$ are both $(\rho, \{x, y\})$-central (see Figure 12), because y is a leaf in the routing trees associated to $\rho(u, z)$ and $\rho(z, u)$, x is a leaf in the routing tree associated to $\rho(u, z)$, and x is also a leaf in the tree obtained by removing y from the routing tree associated to $\rho(z, u)$. Also, similarly for v.

In general, $|x - y| \neq a$.

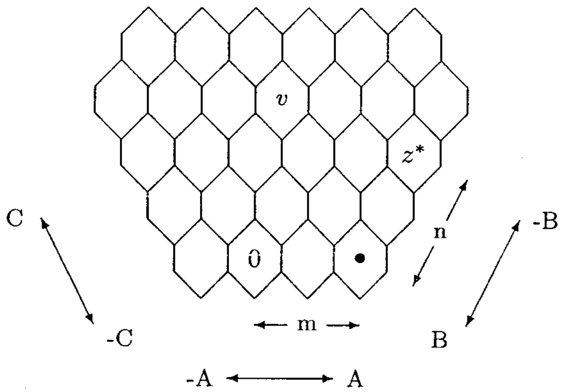

FIG. 14. First case, $z^* = z_1$.

(a) First, we consider two vertices x, y with $x - y$ even. Suppose that $z = \alpha_{x,0}(y)$ and v is a vertex $(\rho, \{0, z\})$-central. Then, $\alpha_{0,x}(v)$ is $(\rho, \{x, y\})$-central. Thus, without loss of generality, we can reduce our study to the case in which the faulty set is $\{0, z\}$. Moreover, the automorphism $\alpha_{x,0}$ transforms x into 0 and y into $z = \alpha_{x,0}(y)$, and the automorphism $\alpha_{y,0}$ transforms y into 0 and x into $-z = \alpha_{y,0}(x)$. Hence, we can further reduce our study to even vertices from half the tile. To work in the associated triple-loop graph, let us define $z^* = z/2$. Now, to calculate a $(\rho, \{0, z\})$-central vertex, we must place the vertices 0 and z^* in the boundary hexagons of the lower half of the triple-loop tile. Then, the vertex in the center of this tile gives us a pair of vertices in the chordal ring graph. At least one of them is $(\rho, \{0, z\})$-central.

Now, we have to consider three subcases according to the coordinates of z^*, namely: $z^* = z_1 = mA - nB$ with coordinates $(m, -n, 0)$, $0 < m \leq l$, $0 \leq n < l$; $z^* = z_2 = -mA + pC$ with coordinates $(-m, 0, p)$, $0 \leq m < l$, $0 < p \leq l$; and $z* = z_3 = -nB + pC$ with coordinates $(0, -n, p)$, $0 < n \leq l$, $0 \leq p < l$. For each of these cases, we will give a $(\rho, \{0, z\})$-central vertex.

If $z^* = z_1$, let us center the tile at $v = mA + lC$. Then, vertices 0 and z^* are in the lower tile boundary, as shown in Figure 14. Let $u = 2v$ and consider the automorphism $\alpha_{u,0}$ of $C_{m_k}(a, b, c)$ defined by $\alpha_{u,0}(i) = i - m2A - l2C$. Then, $\alpha_{u,0}(0) = -m2A - l2C = m(2B + 2C) - l2C = m2B - (l-m)2C$ and $\alpha_{u,0}(z) = z - m2A - l2C = m2A - n2B - m2A - l2C = -n2B - l2C =$

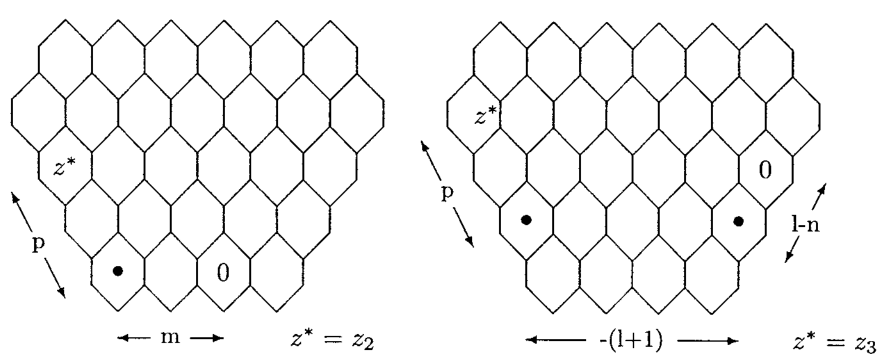

FIG. 15. Second case, $z^* = z_2$, and third case, $z^* = z_3$.

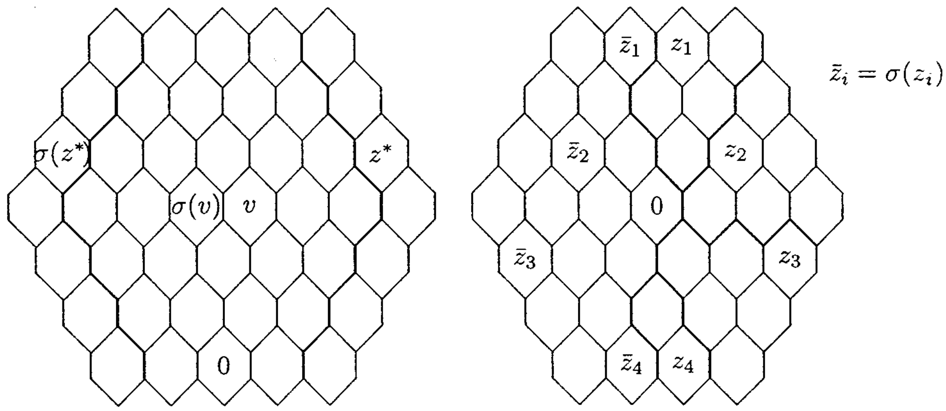

FIG. 16. Four different cases, according to the situation of z^*.

$n(2A+2C) - l2C = n2A - (l-n)2C$ belong both to L. Thereby, vertex u is $(\rho, \{0, z\})$-central. Notice that vertex $u + a$ could have been taken instead of u.

If $z^* = z_2$, centering the tile at $v = -mA - lB$ leaves vertices 0 and z^* in the lower boundary, as shown in Figure 15. Now, let $u = 2v + a$ and consider the automorphism of the chordal ring $\alpha_{u,0}(i) = -m2A - l2B + a - i$. As in the previous case, it is easily checked that $\alpha_{u,0}(0)$ and $\alpha_{u,0}(z)$ belong both to L. Thereby, vertex u is $(\rho, \{0, z\})$-central. By taking the automorphism $\alpha_{u-a,0}$, it is easy to see that vertex $u - a$ is also $(\rho, \{0, z\})$-central, for $p \neq l$.

In the last case, when $z^* = z_3$, it is useful to consider z^* in a convenient adjacent tile. To this end, let us transform the coordinates of z^* by adding $-(l+1)A + lB \equiv 0$. In this way, the new coordinates of vertex z^* are $(-(l+1), l-n, p)$. If we center the tile at $v = (l-n+1)B + lC$, then 0 and z^* are in the lower tile boundary; see Figure 15. Now, let us consider the vertex $u = 2v + a$ of the chordal ring and the automorphism $\alpha_{u,0}(i) = (l-n+1)2B + l2C + a - i$. Again, it is easily checked that $\alpha_{u,0}(0)$ and $\alpha_{u,0}(z)$ belong both to L and that vertex u is $(\rho, \{0, z\})$-central. By considering the automorphism $\alpha_{u-a,0}$, it is also verified that vertex $u - a$ is $(\rho, \{0, z\})$-central, for $p \neq l - 1$ and $n \neq 1$.

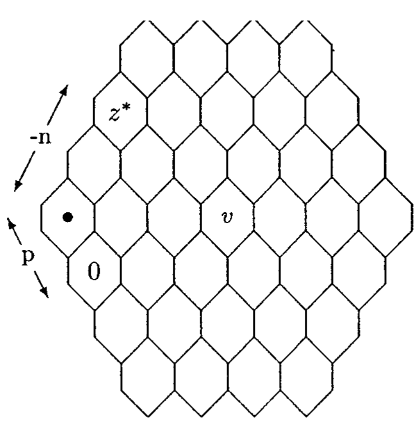

FIG. 17. First case, $z^* = z_1$.

(b) Finally, we study the case in which $x - y$ is odd. As in case (a), it is only necessary to consider $F = \{0, z\}$. The automorphism $\alpha_{x,0}$ transforms vertices x, y into vertices 0, $z = \alpha_{x,0}(y)$, but now $\alpha_{y,0}(x) = \alpha_{x,0}(y)$. To work in the associated triple-loop tile, let us define $z^* = (z-a)/2$. To obtain a $(\rho, \{0, z\})$-central vertex, we must locate vertex 0 in the boundary hexagons of the lower half of the tile and vertex z^* in the boundary hexagons of the upper half. Then, the vertex in the center of the tile gives us a pair of vertices in the chordal ring graph. At least one of them is $(\rho, \{0, z\})$-central.

To further reduce the number of cases to consider, we proceed in the following way: Let σ be the symmetry with respect to the vertical axis passing through vertex 0. It is easy to see that if 0 is in a lower-boundary hexagon and z^* is in an upper-boundary one of the tile centered at v then 0 is in a lower-boundary hexagon and $\sigma(z^*)$ is in an upper-boundary one of the tile centered at $\sigma(v)$; see Figure 16. Thus, the only cases to consider are $z^* = z_1 = -nB + pC$ with coordinates $(0, -n, p)$, $0 < n \leq l$, $0 \leq p < l$, $p \leq n$; $z^* = z_2 = mA - nB$ with coordinates $(m, -n, 0)$, $0 < m \leq l$, $0 \leq n < l$; $z^* = z_3 = mA - pC$ with coordinates $(m, 0, -p)$, $0 \leq m < l$, $0 < p < l$; and $z^* = z_4 = nB - pC$ with coordinates $(0, n, -p)$, $0 < n < l$, $0 < p \leq l$, $n \leq p$.

If $z^* = z_1$, let us center the tile at $v = lA + pC$. Then, vertex 0 is in the lower tile boundary and z^* is in the upper one; see Figure 7. To come back to the chordal ring graph, let $u = 2v + a$ and consider the automorphism $\alpha_{u,0}(i) = l2A + p2C + a - i$. In this way, $\alpha_{u,0}(0) = l2A + p2C + a = l2A - p2(A+B) + a = (l-p)2A - p2B + a \in L$, and $\alpha_{u,0}(z) = l2A + p2C + a - z = l2A + p2C + a - 2z^* - a = l2A + p2C + n2B - p2C = l2A + n2B = l2A - n2(A+C) = (l-n)2A - n2C \in L$. Thereby, vertex u is $(\rho, \{0, z\})$-central. By considering the automorphism $\alpha_{u-a,0}$, it can also be proved that, for $p \neq 0$, vertex $u - a$ is also $(\rho, \{0, z\})$-central.

In the second case $z^* = z_2$, we consider z^* in a convenient adjacent tile, by adding $-A + (l-1)B + 2lC \equiv 0$; see Figure 18. In this way, the coordinates of z^* are $(m-$

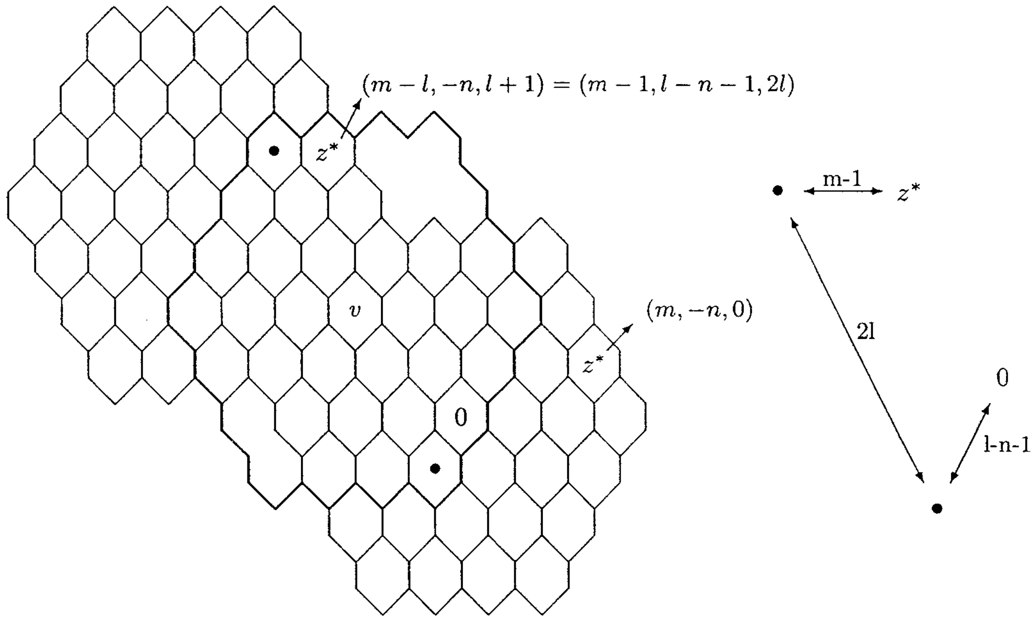

FIG. 18. Second case, $z^* = z_2$.

$1, l-n-1, 2l)$. If we center the tile at $v = (l-n-1)B + lC$, then vertex 0 is in the lower tile boundary and z^* is in the upper one. Now, letting $u = 2v$ and considering the automorphism $\alpha_{u,0}(i) = i - (l-n-1)\,2B - l\,2C$ it can be proved that $\alpha_{u,0}(0)$ and $\alpha_{u,0}(z)$ both belong to L. Thereby, the vertex u is $(\rho, \{0, z\})$-central. In a similar way, it can be proved that another possible $(\rho, \{0, z\})$-central vertex is $u + a$.

If $z^* = z_3$, let us again consider z^* in an adjacent tile, by adding $(1-l)A - 2lB + (1-p)C \equiv 0$. Thus, $z^* = (-l+m+1, -2l, -p+1)$; see Figure 19. As in the previous cases, center the tile at $v = -lB + (1-p)C$, let $u = 2v$, and consider the automorphism $\alpha_{u,0}(i) = i + l\,2B + (p-1)\,2C$. Again, $\alpha_{u,0}(0)$ and $\alpha_{u,0}(z)$ belong to L and vertex u is $(\rho, \{0, z\})$-central. We can also consider vertex $u + a$ instead of u.

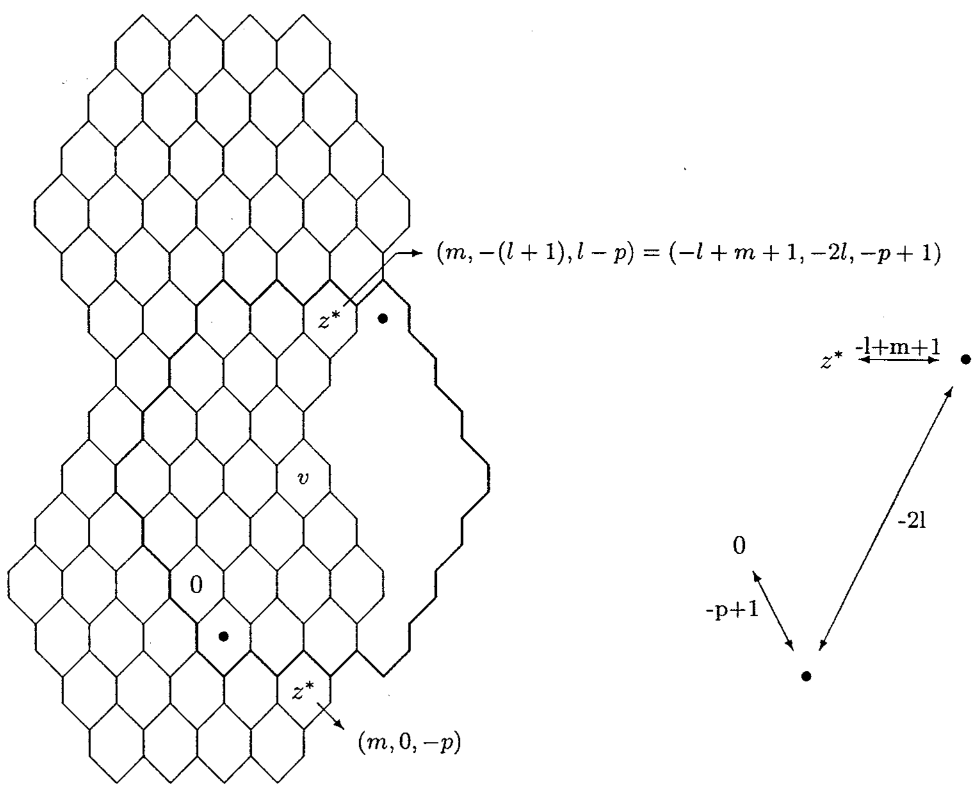

FIG. 19. Third case, $z^* = z_3$.

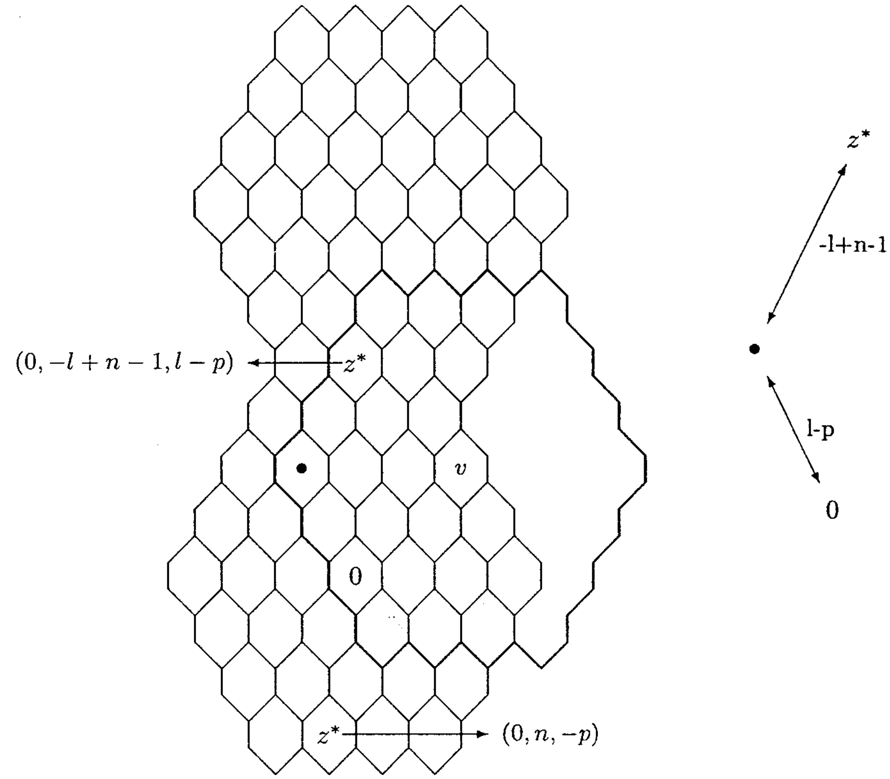

FIG. 20. Fourth case, $z^* = z_4$.

Finally, the case $z^* = z_4$ is shown in Figure 20. Now, the coordinates of z^* that we consider are $(0, -l+n-1, l-p)$ and the center of the triple-loop tile is $v = lA + (l-p)C$. Vertex $u = 2v$ is $(\rho, \{0, z\})$-central in the chordal ring graph, and this is proved by considering the automorphism $\alpha_{u,0}(i) = i + l\,2B + (p-1)\,2C$. Vertex $u + a$ is also $(\rho, \{0, z\})$-central.

6. CONCLUSIONS

The main purpose of the paper was the study of fault-tolerant routings in chordal ring networks. This study was accomplished by using a geometrical approach that associates to the graph under consideration a tile that periodically tessellates the plane. This approach has shown some interesting relations between chordal ring and triple-loop networks. First, an algorithm to calculate the coordinates of the vertices in an optimal triple-loop graph was presented, and this result has been used to define an optimal and consistent routing function of shortest paths in an optimal chordal ring network with maximum order and odd diameter. Finally, the vulnerability of this routing was studied. The techniques used in the paper can also be applied to deal with nonoptimal cases, which include chordal ring networks with maximum order and even diameter. Moreover, the geometrical approach can also be useful in analyzing other communication problems in chordal ring networks, such as broadcasting, gossiping, and WDM routings.

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