# Feature allocations, probability functions, and paintboxes

Tamara Broderick
UC Berkeley
(MIT starting 2015)

# Clustering/Partition

$$\pi_7 = \{\{1, 2, 7\}, \{3, 5\}, \{4\}, \{6\}\}$$

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Document 1

Document 2

Document 3

Document 4

Document 5

Document 6

Document 7

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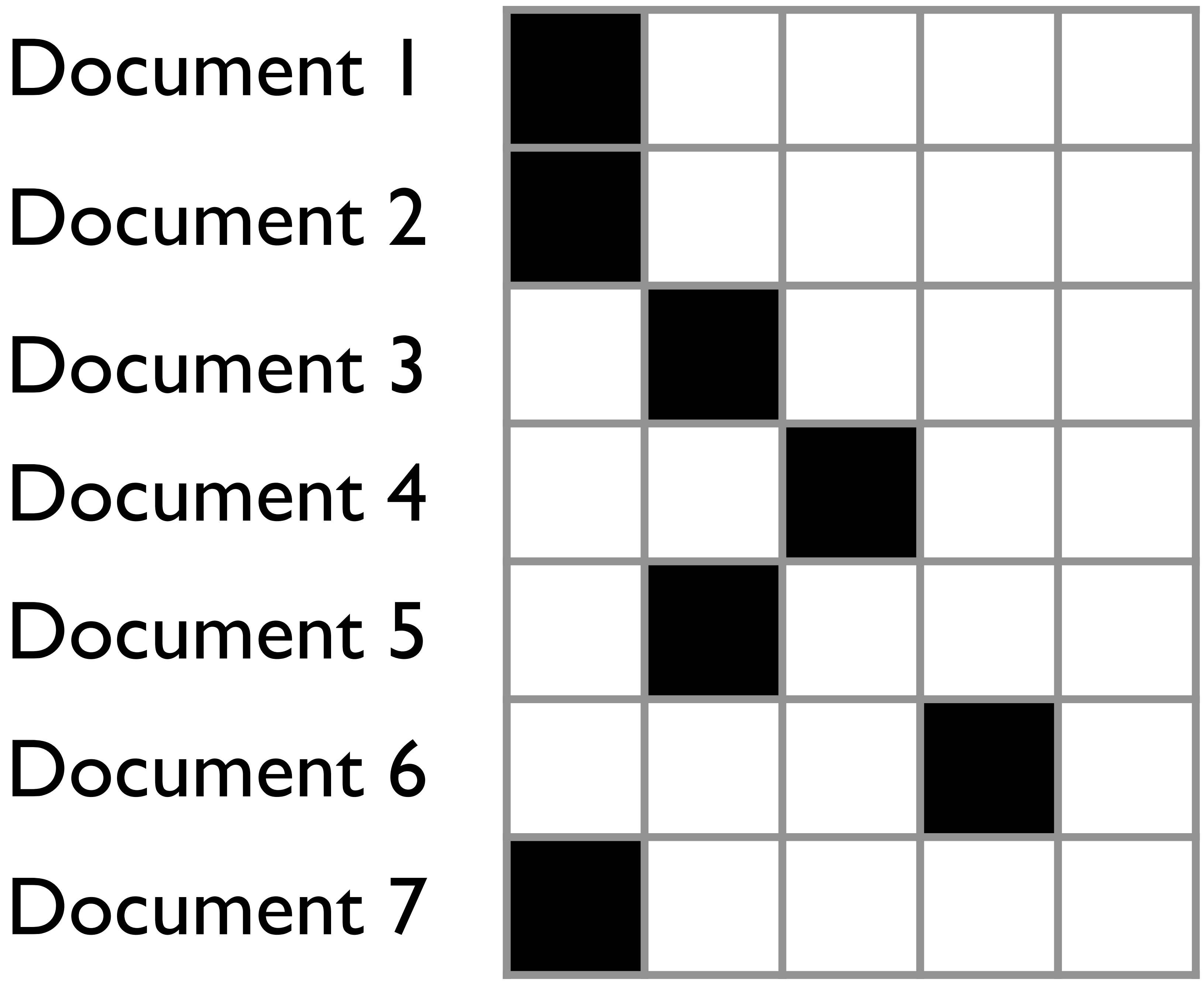

# Clustering/Partition

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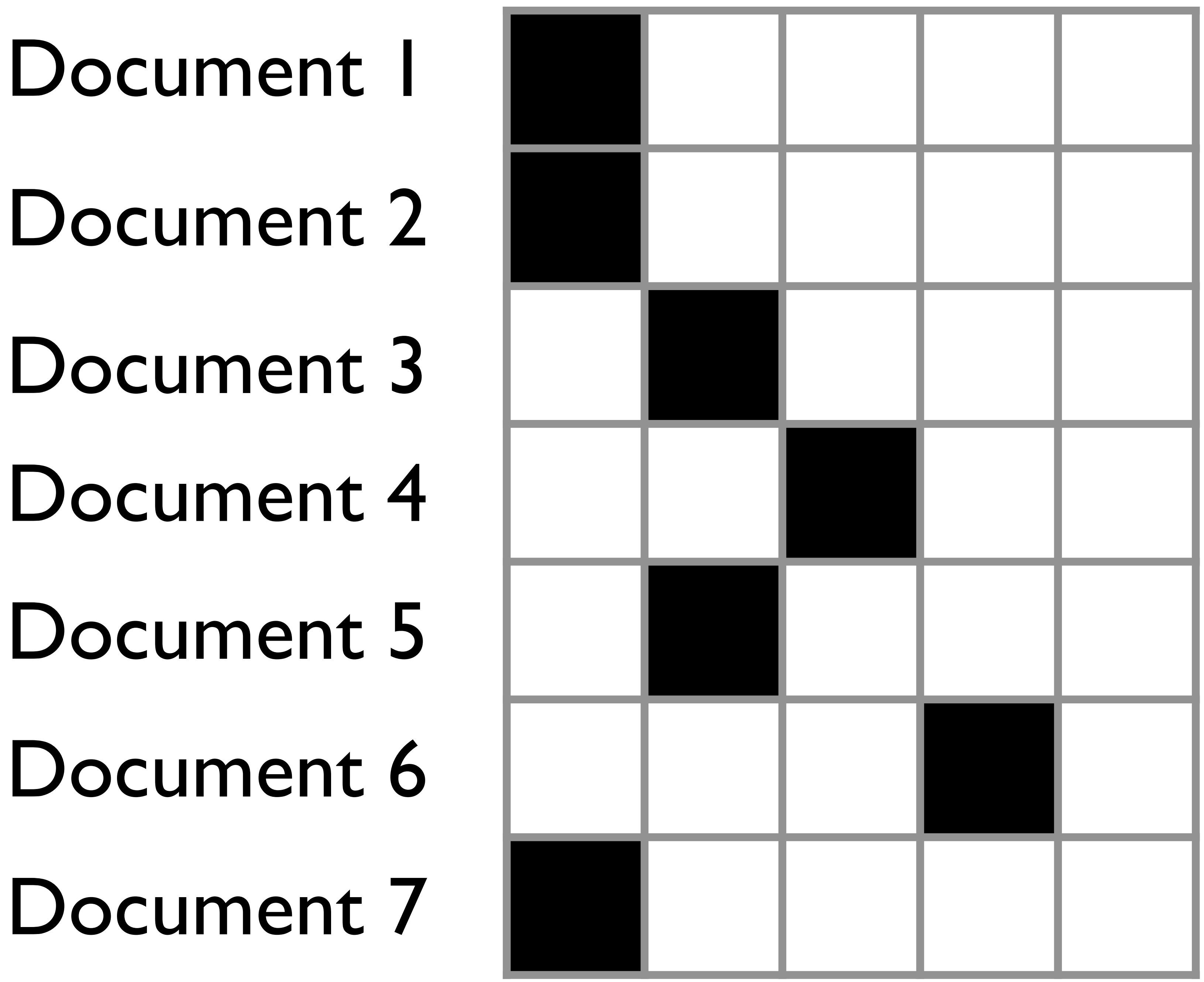


"clusters", "blocks (of a partition)"

# Clustering/Partition

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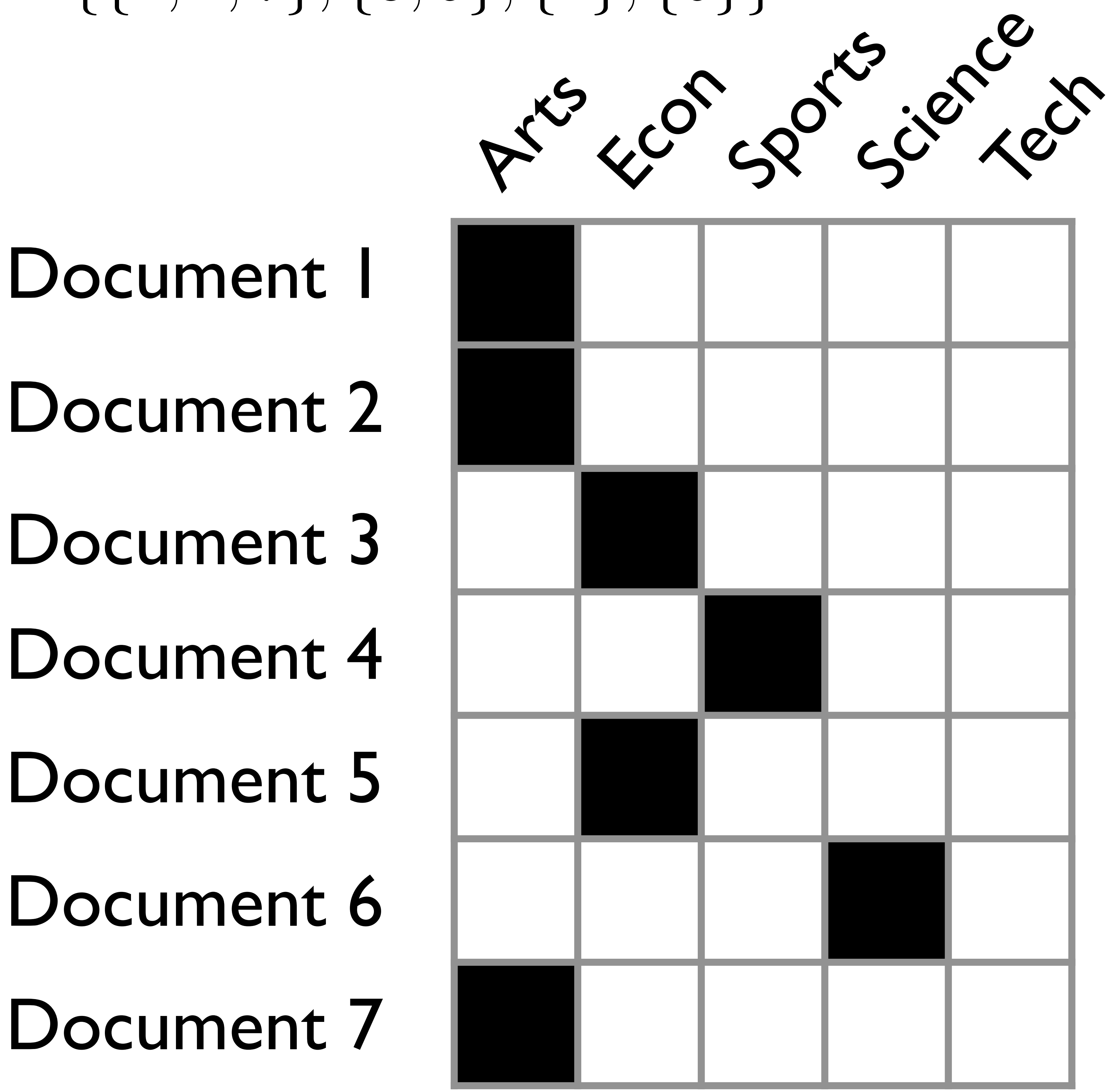
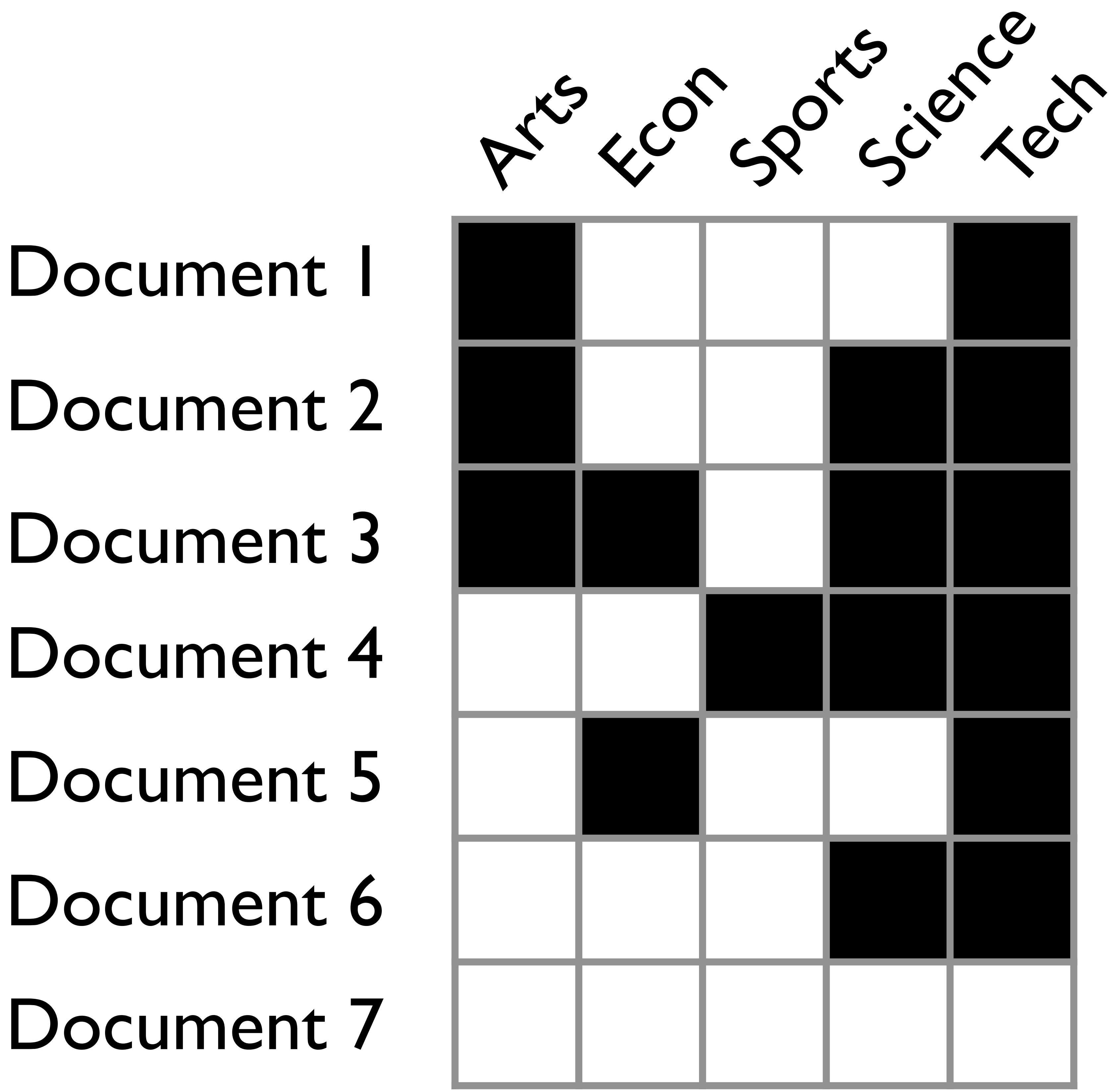



"clusters", "blocks (of a partition)"

# Latent feature allocation



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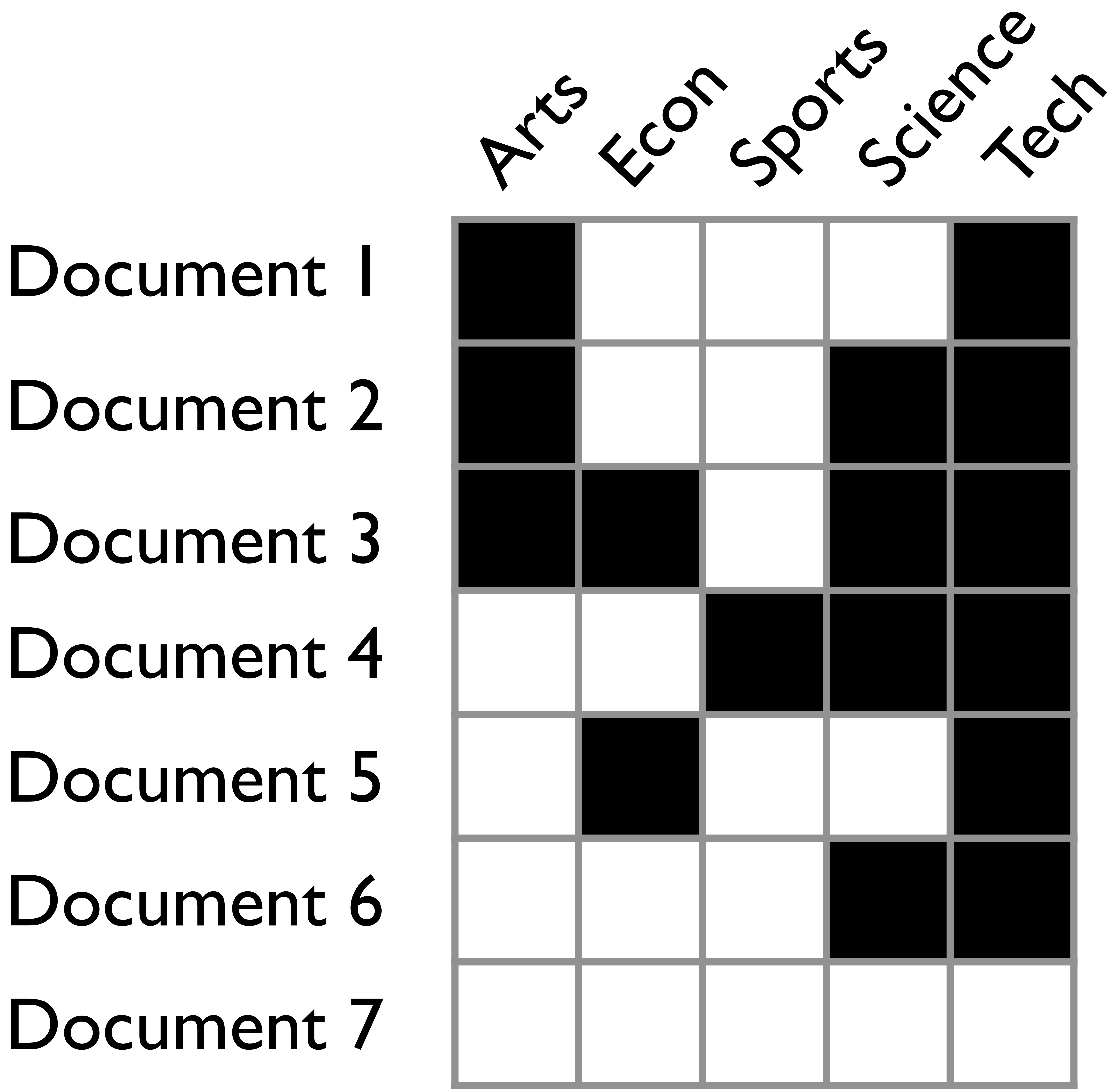
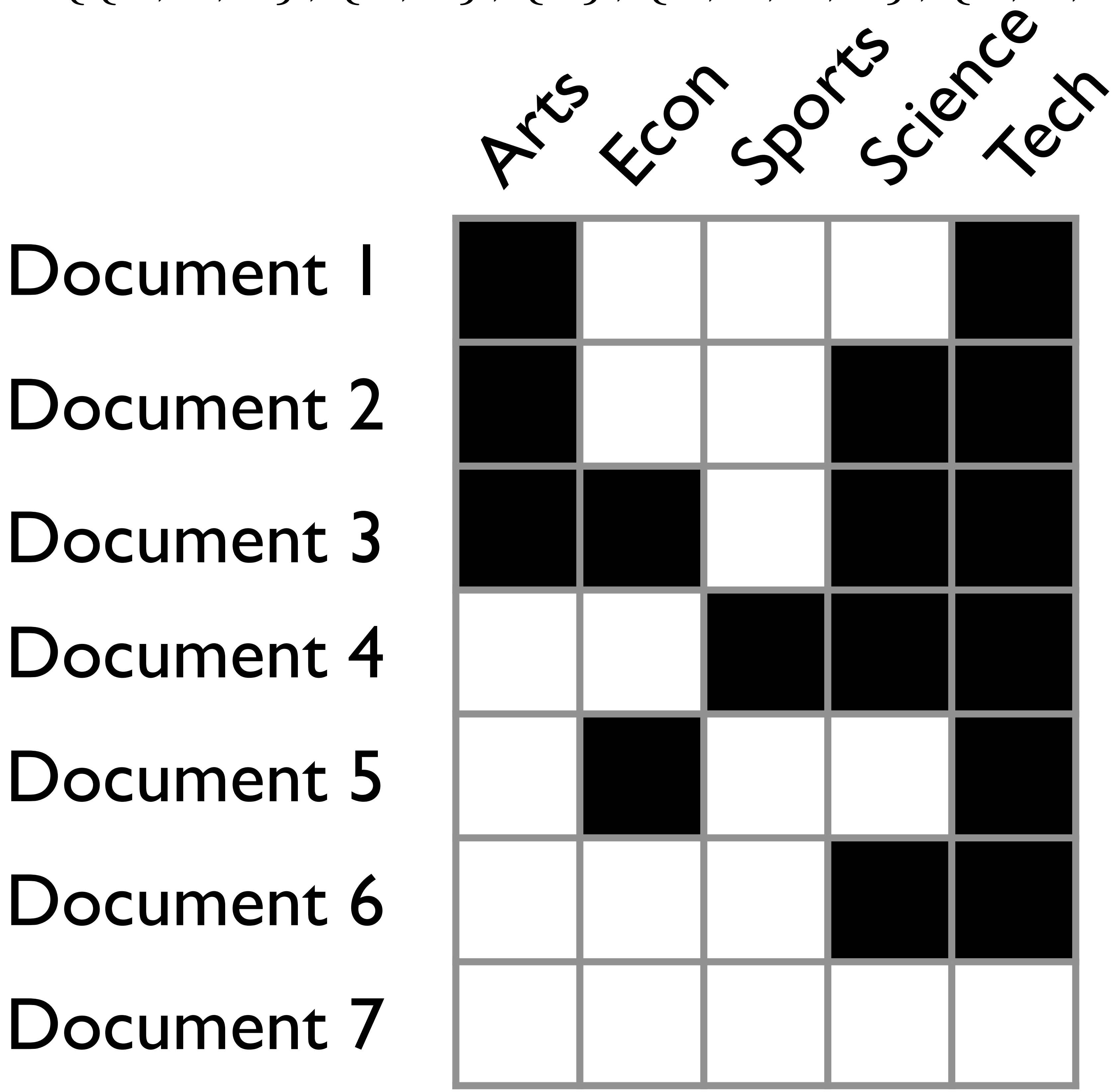



“features”,
“topics”

# Latent feature allocation

$$f_7 = \{\{1,2,3\},\{3,5\},\{4\},\{2,3,4,6\},\{1,2,3,4,5,6\}\}$$




“features”,
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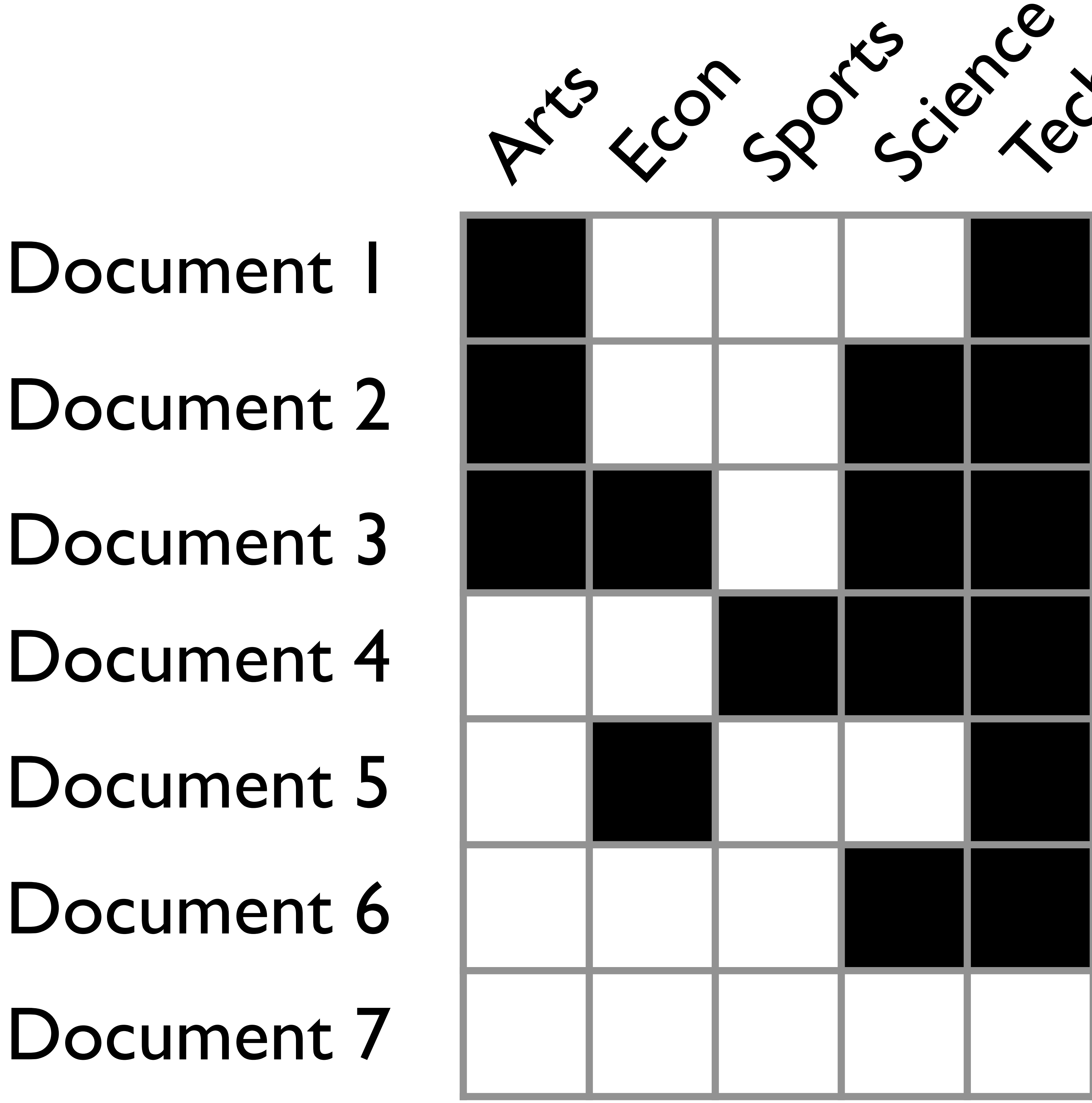
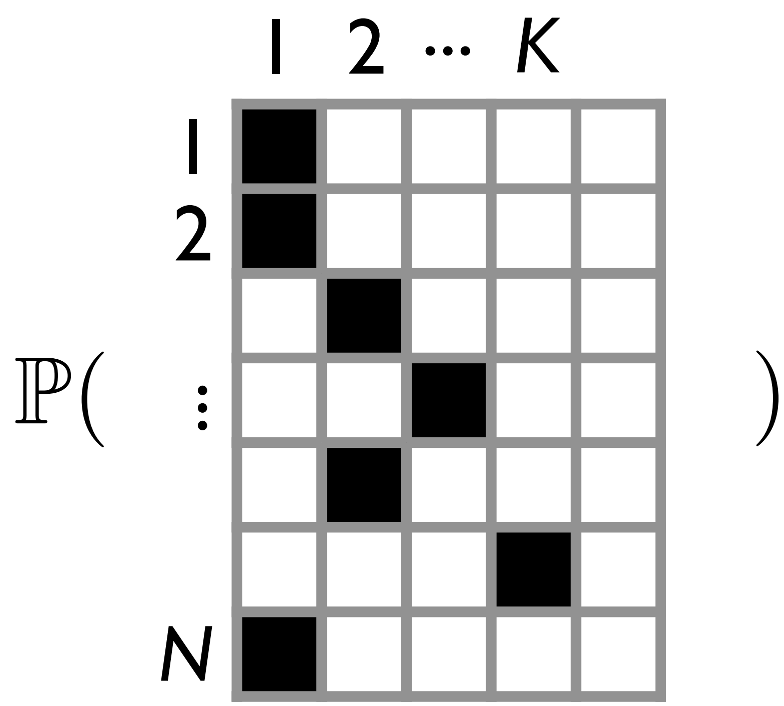
# Latent feature allocation

$$f_7 = \{\{1,2,3\},\{3,5\},\{4\},\{2,3,4,6\},\{1,2,3,4,5,6\}\}$$

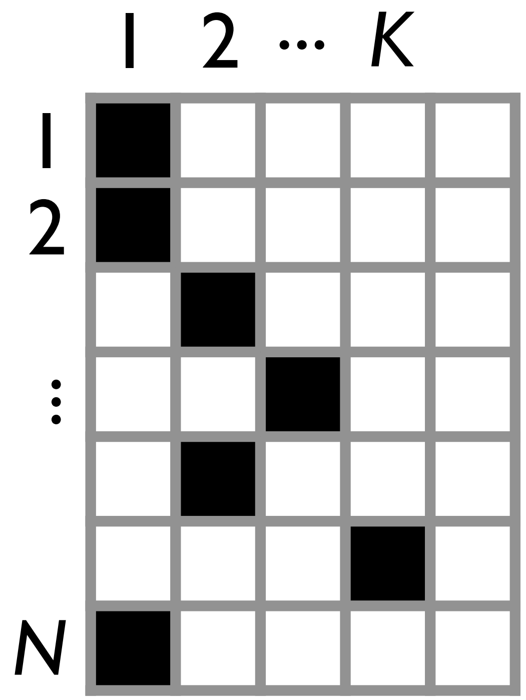



<u>Question</u>:
Can we characterize exchangeable feature distributions?

# Exchangeable probability functions



# Exchangeable probability functions

$\mathbb{P}($  $) = p\left(S_{N,1}, \ldots, S_{N,K}\right)$

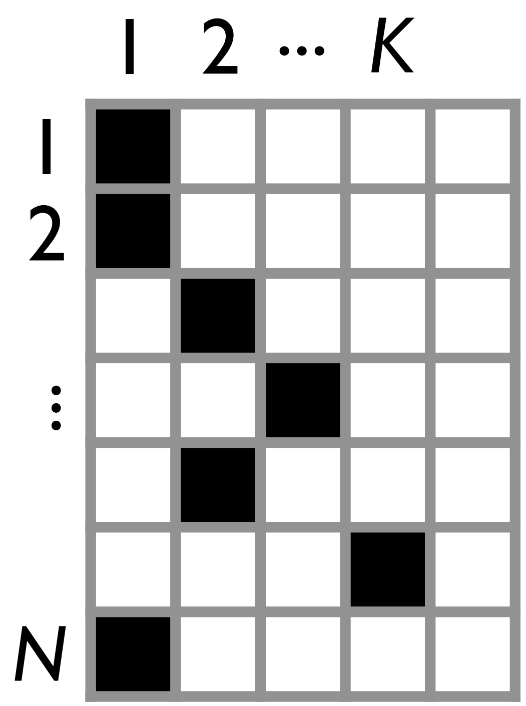




[Pitman 1995]

# Exchangeable probability functions

Size of *K*th cluster

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[Pitman 1995]

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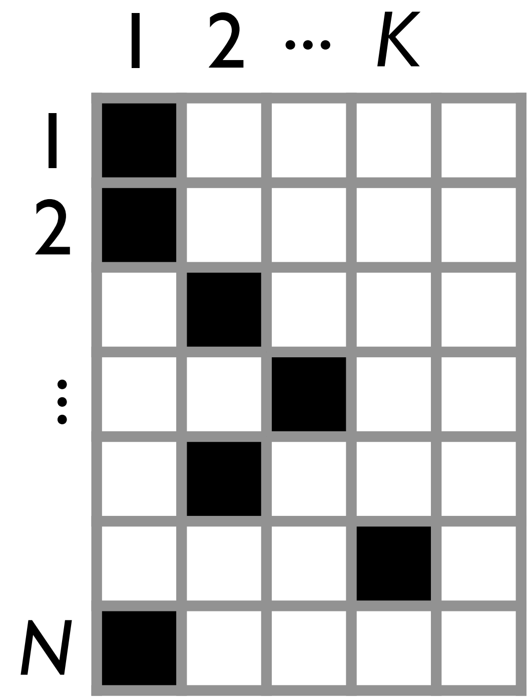
Exchangeable partition probability function (EPPF)




$$\mathbb{P}(\quad \text{[grid]} \quad) = p\left(S_{N,1}, \ldots, S_{N,K}\right)$$



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# Exchangeable probability functions

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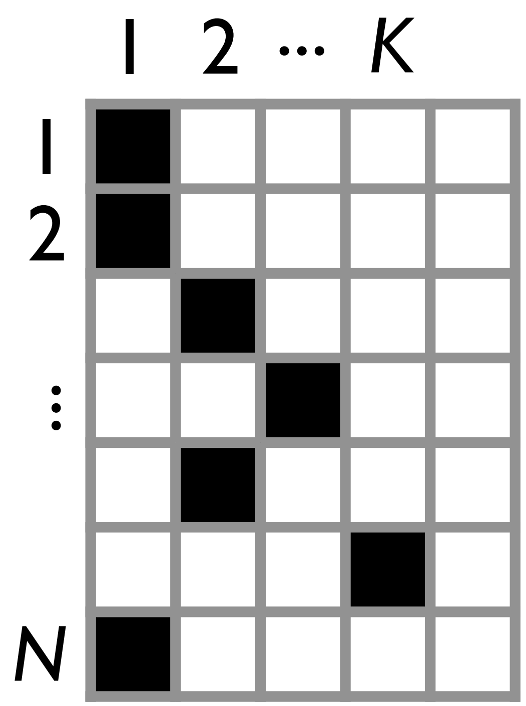


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"Exchangeable **feature** probability function" (EFPF)?

# Example: Indian buffet process



[Griffiths, Ghahramani 2006]

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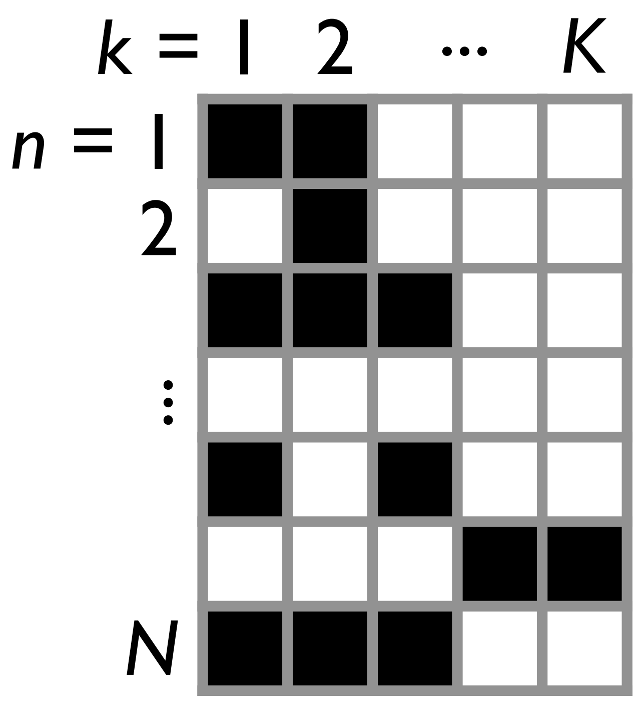




[Griffiths, Ghahramani 2006]

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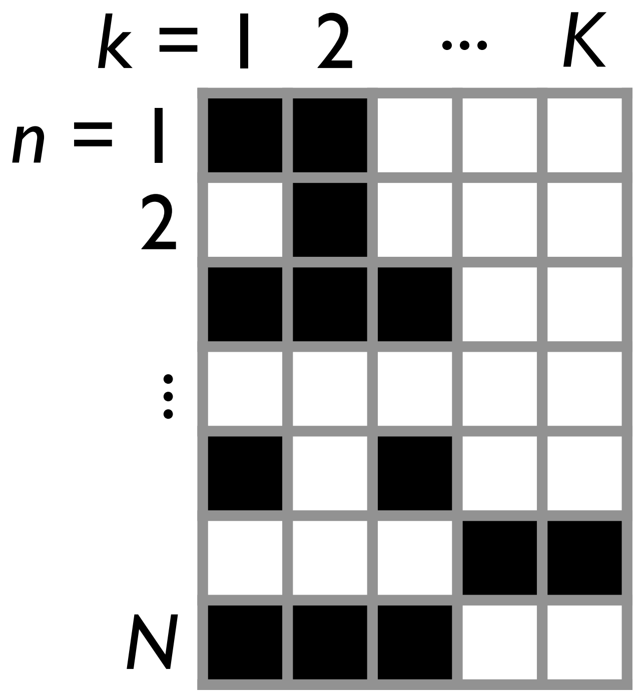


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[Griffiths, Ghahramani 2006]

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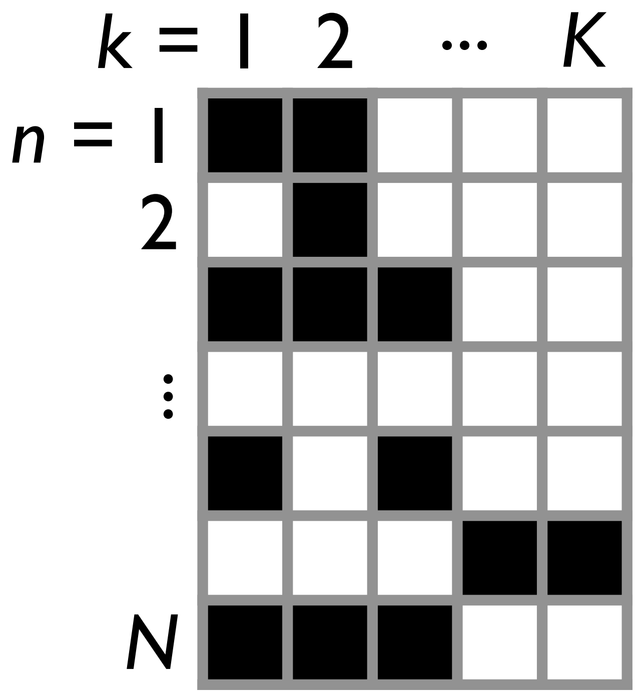


For $n$ = 1, 2, ..., N

1. Data point $n$ has an existing feature $k$ that has already occurred $S_{n-1,k}$ times with probability $\dfrac{S_{n-1,k}}{\theta + n - 1}$



[Griffiths, Ghahramani 2006]

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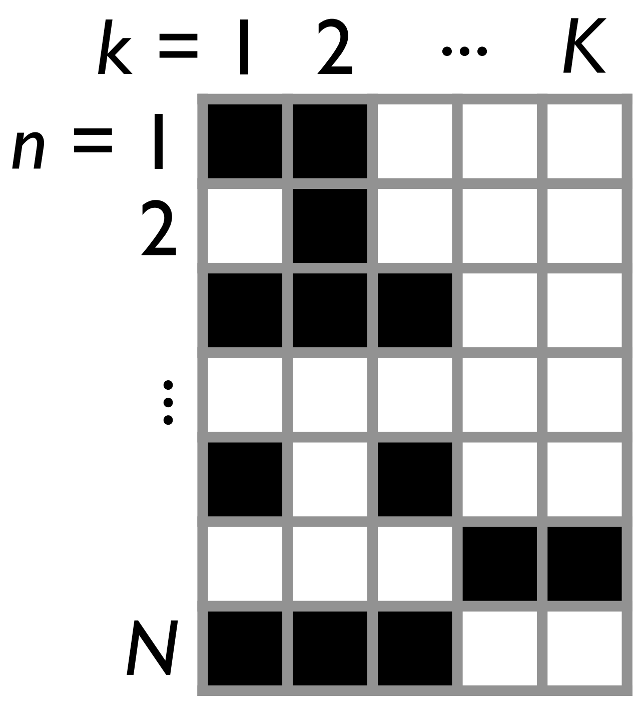


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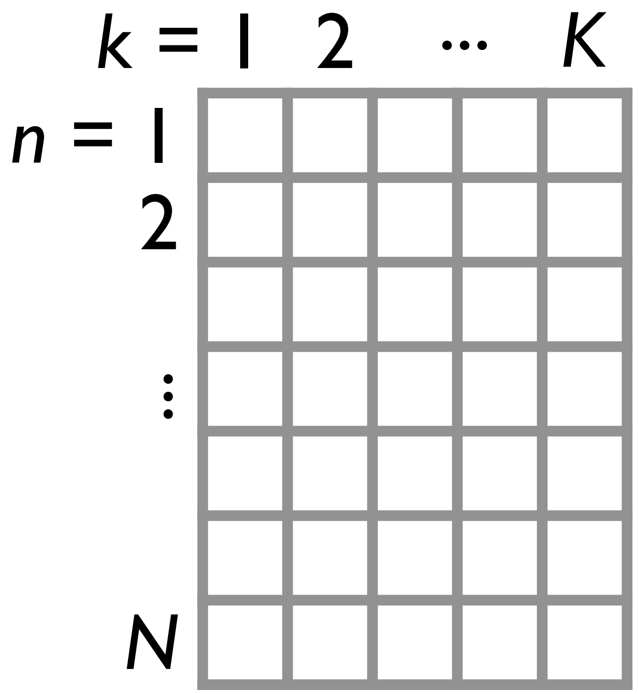


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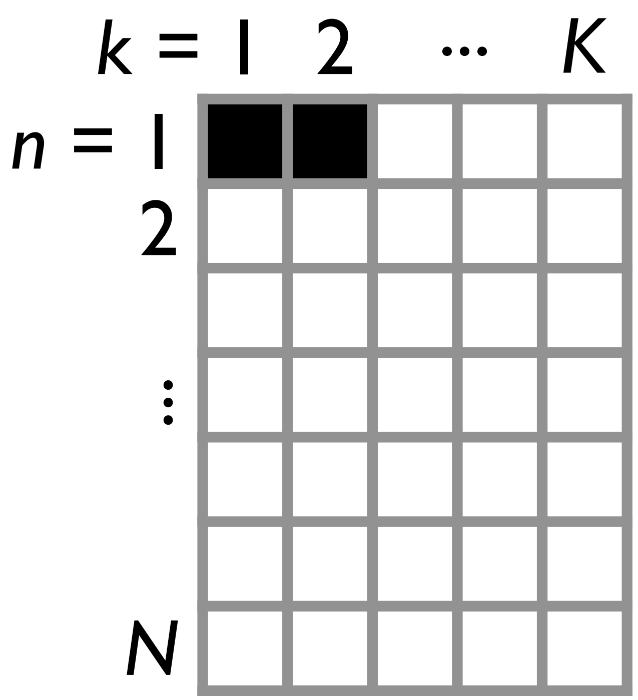


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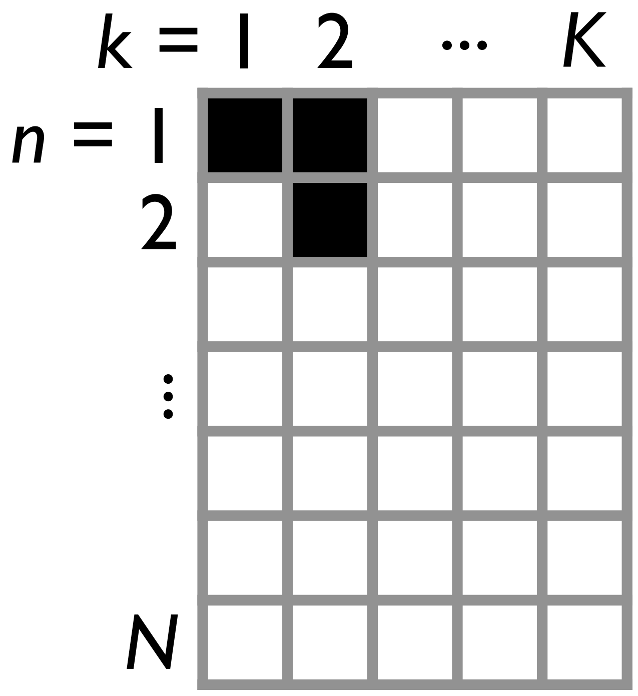


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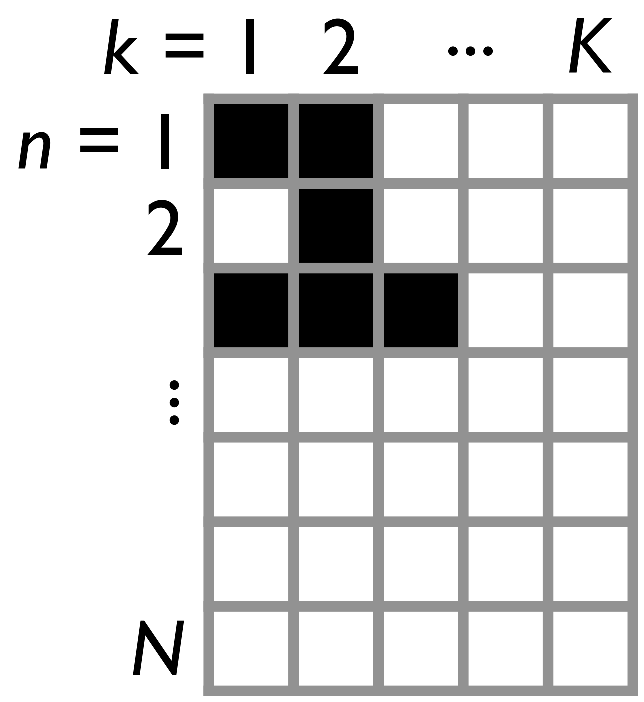


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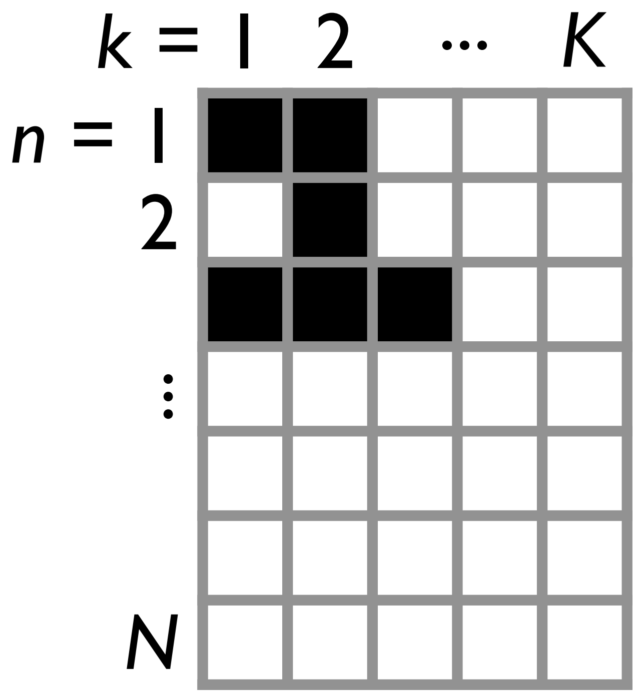


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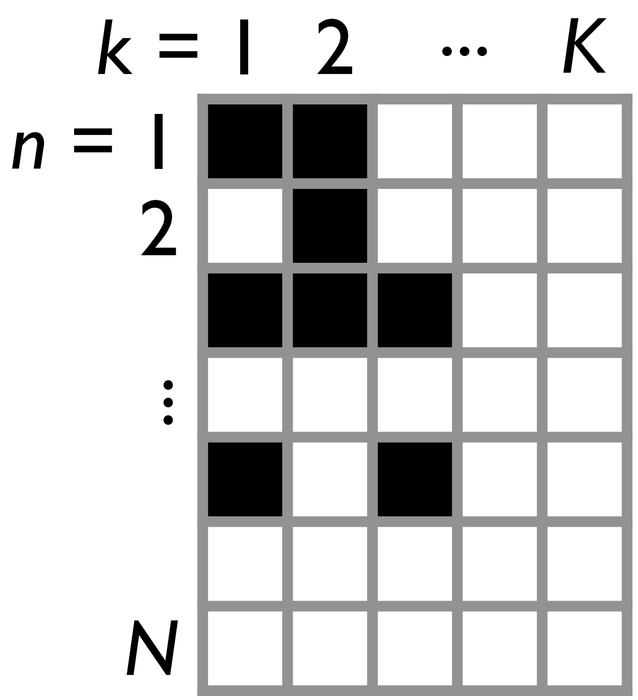


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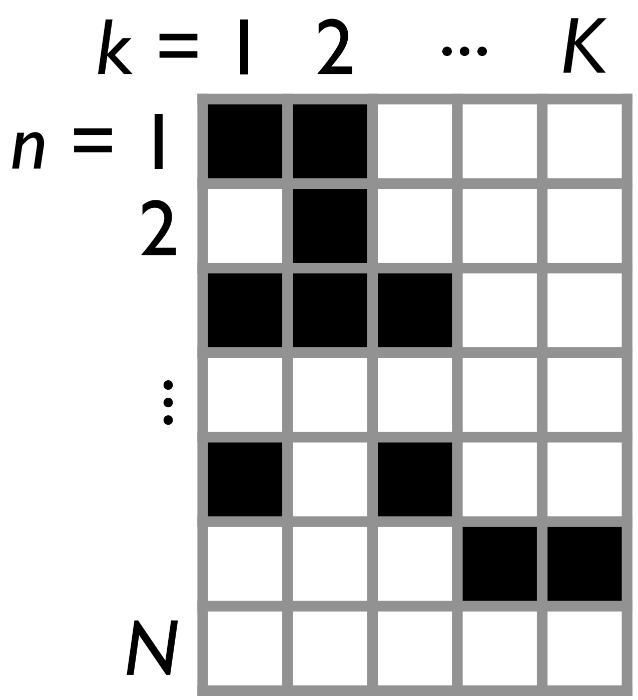


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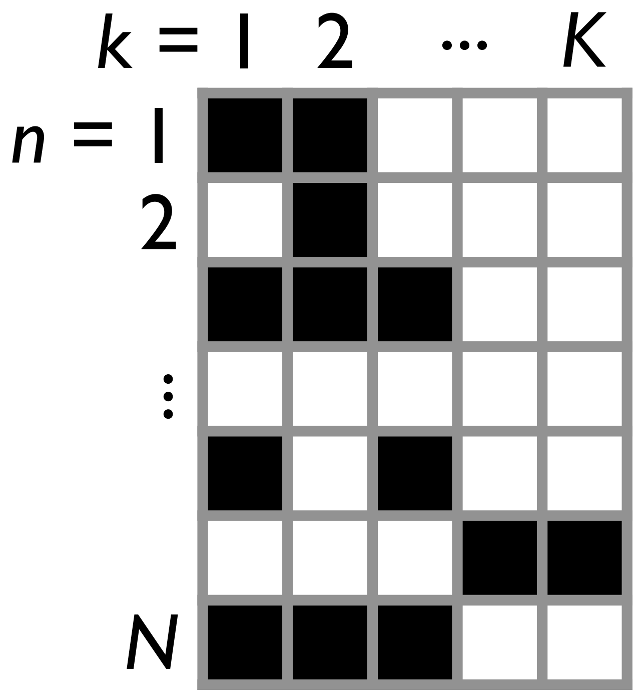


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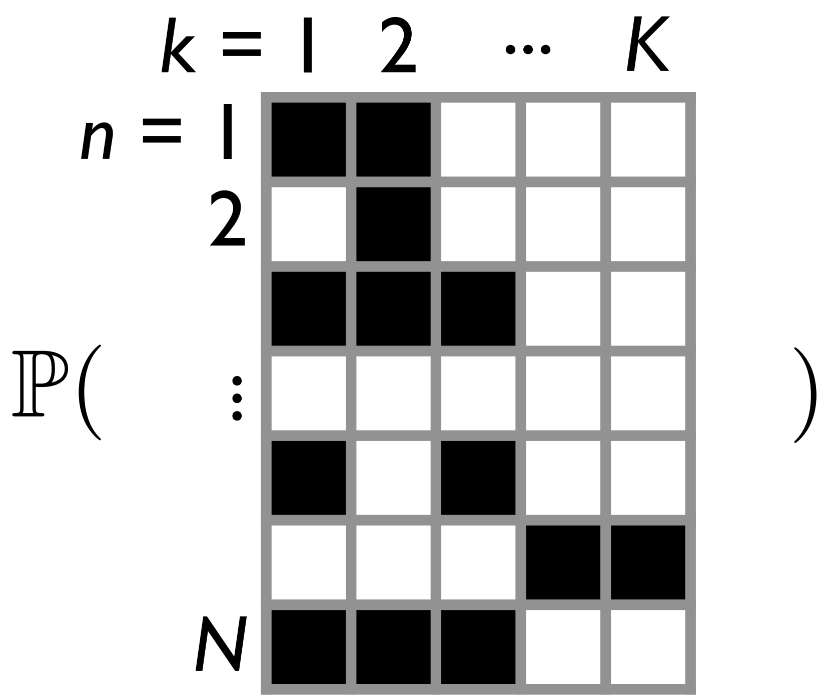

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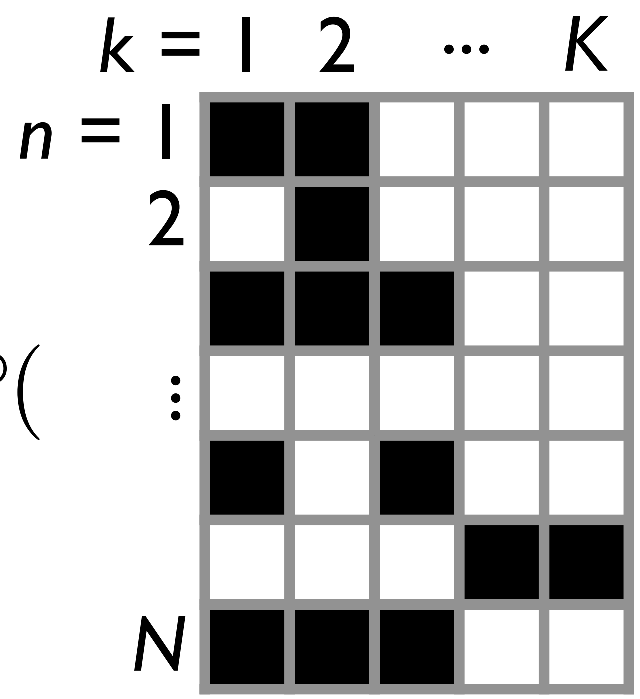
$\mathbb{P}($  $)$



$$= \frac{1}{K_N!}(\theta\gamma)^{K_N} \exp\left(-\theta\gamma \sum_{n=1}^{N} (\theta + n - 1)^{-1}\right) \prod_{k=1}^{K_N} \frac{\Gamma(S_{N,k})\Gamma(N - S_{N,k} + \theta)}{\Gamma(N + \theta)}$$

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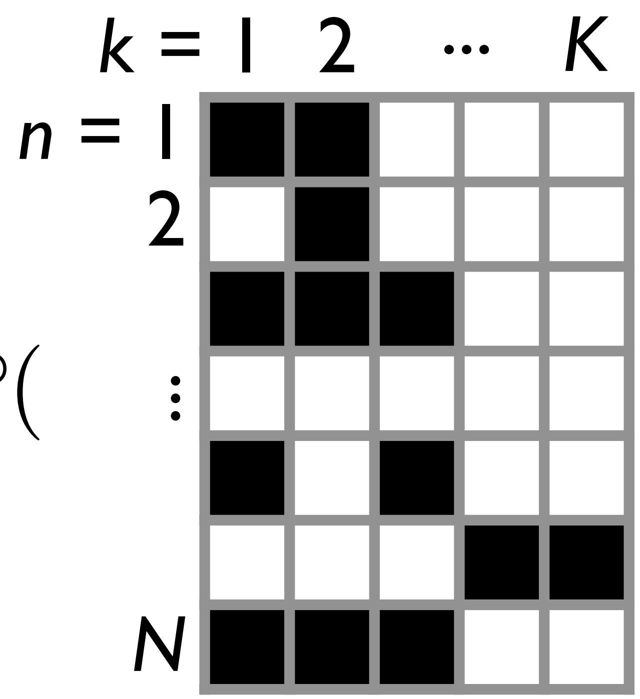
$\mathbb{P}($ 

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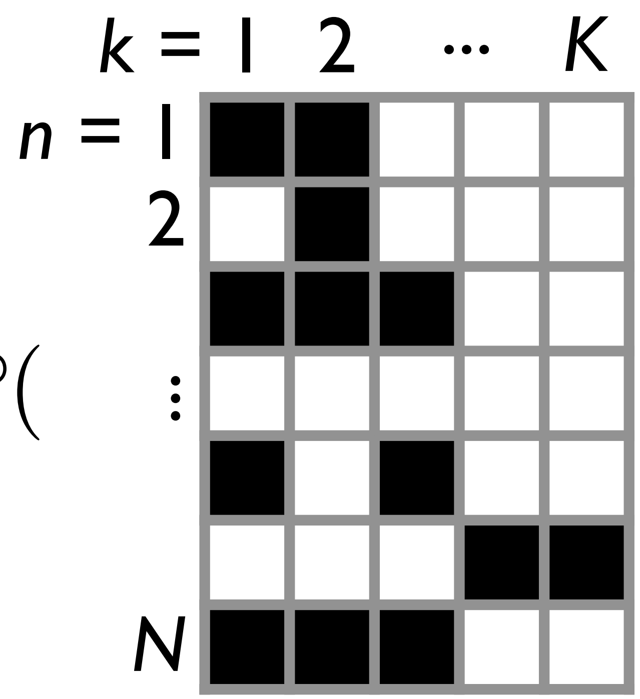
$\mathbb{P}($ 

$)$

Number of features

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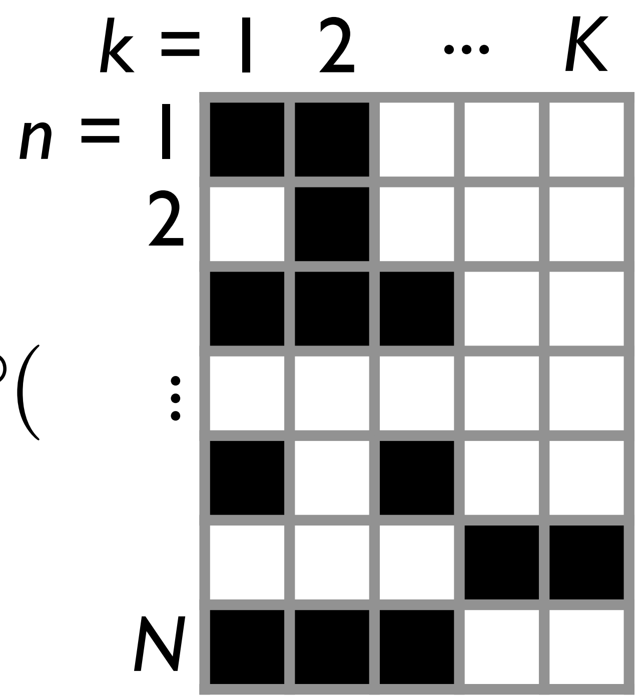


$)$

Number of data points

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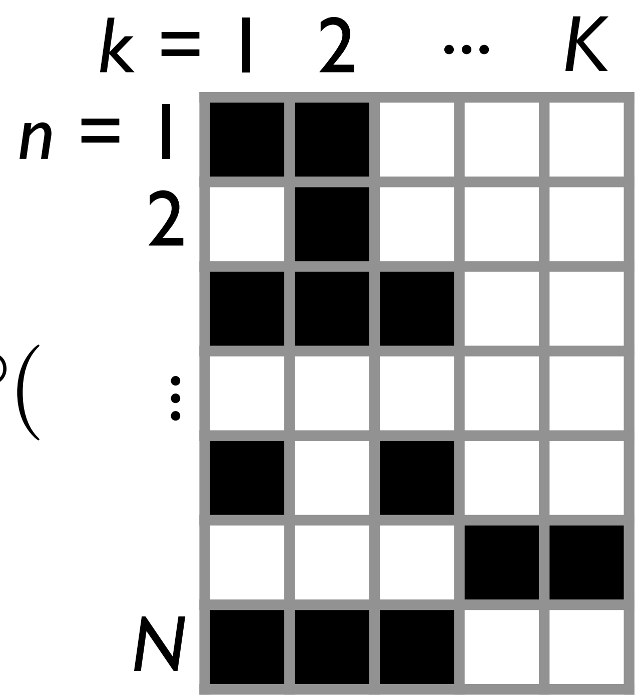


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$$= p(N; S_{N,1}, S_{N,2}, \ldots, S_{N,K})$$



[Broderick, Jordan, Pitman 2013]

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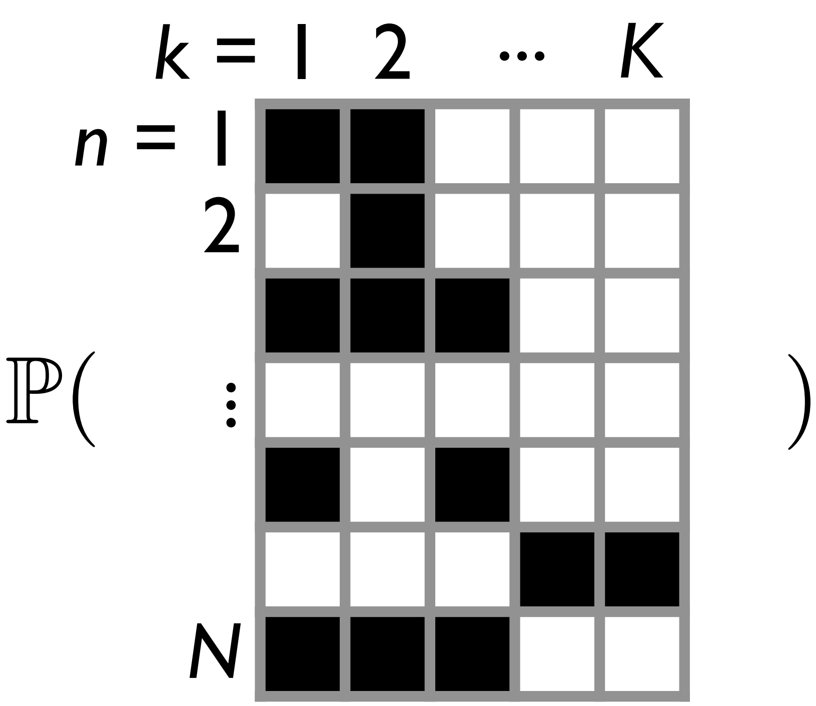


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"EFPF"



[Broderick, Jordan, Pitman 2013]

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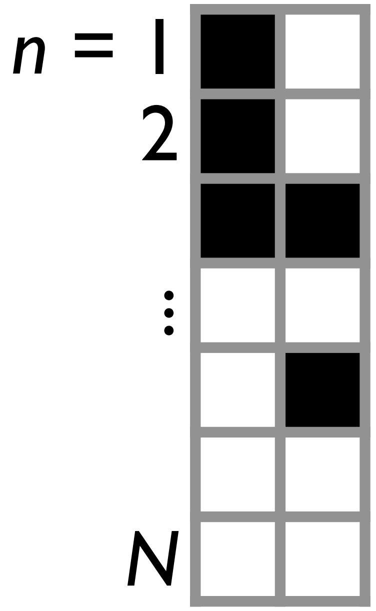
"Exchangeable feature probability function" (EFPF)?

Counterexample



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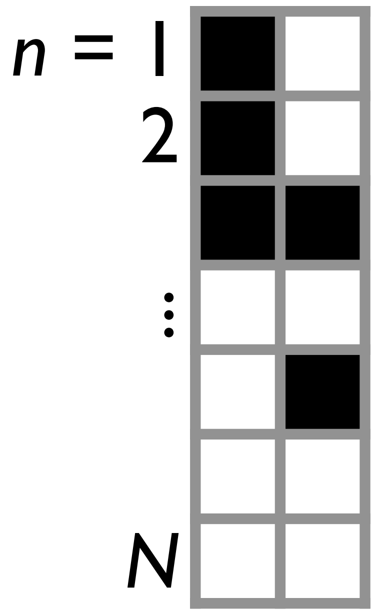
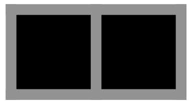
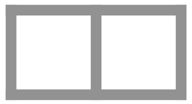



$\mathbb{P}(\text{row} =$  $) = p_1$

$\mathbb{P}(\text{row} =$  $) = p_2$

$\mathbb{P}(\text{row} =$  $) = p_3$

$\mathbb{P}(\text{row} =$  $) = p_4$



[Broderick, Jordan, Pitman 2013]

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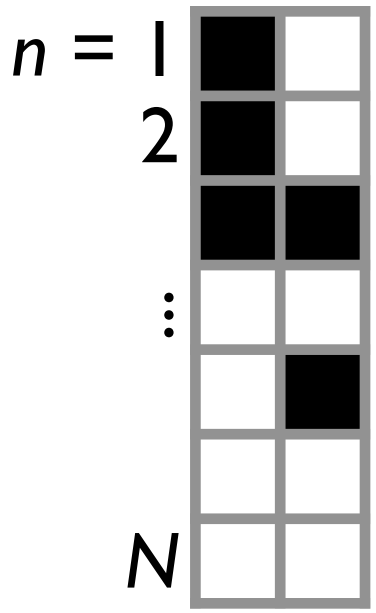


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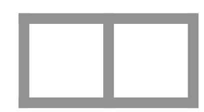
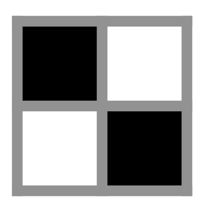
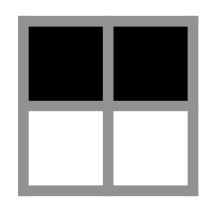
$\mathbb{P}(\text{row} =$  $) = p_4$







[Broderick, Jordan, Pitman 2013]

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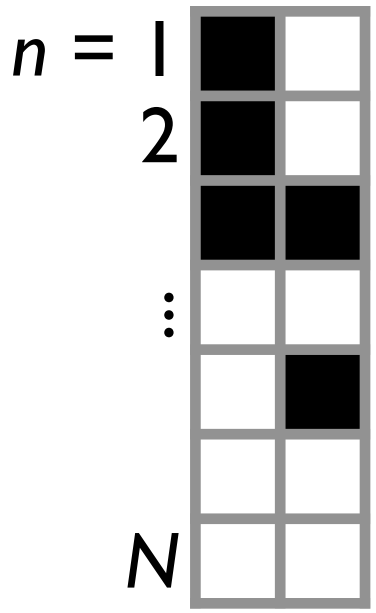


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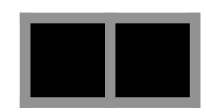
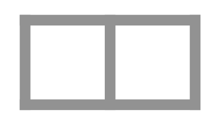
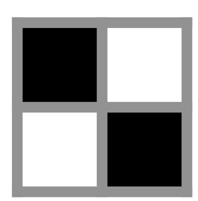
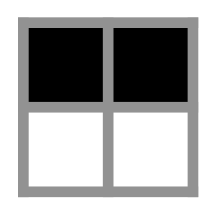
$\mathbb{P}(\text{row} =$  $) = p_3$

$\mathbb{P}(\text{row} =$  $) = p_4$

$\mathbb{P}($  $)$ $\mathbb{P}($  $)$



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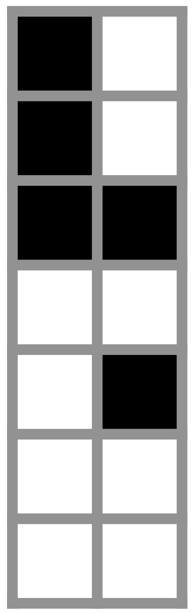
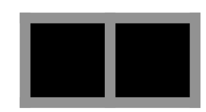
Counterexample

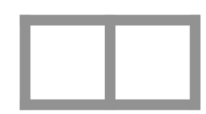
$n$ = 1
2
⋮
$N$

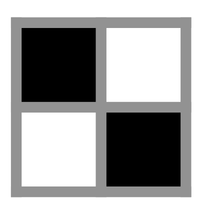
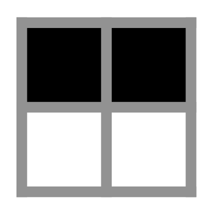


$\mathbb{P}(\text{row} =$  $) = p_1$

$\mathbb{P}(\text{row} =$  $) = p_2$

$\mathbb{P}(\text{row} =$  $) = p_3$

$\mathbb{P}(\text{row} =$  $) = p_4$

$\mathbb{P}($  $)$ $\quad$ $\mathbb{P}($  $)$

$p_1 p_2 \quad p_3 p_4$

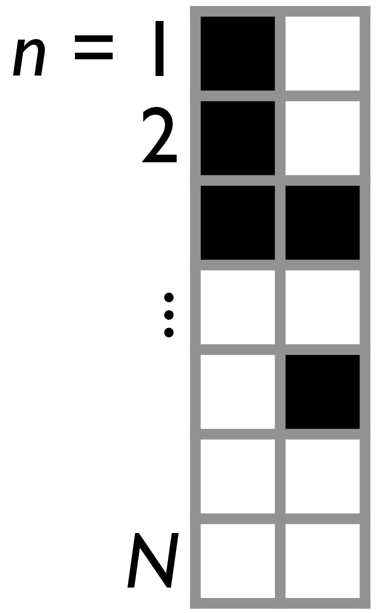


[Broderick, Jordan, Pitman 2013]

# Exchangeable probability functions

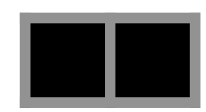
"Exchangeable feature probability function" (EFPF)?

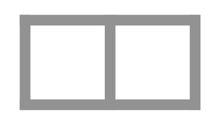
Counterexample

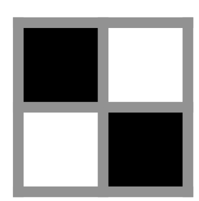
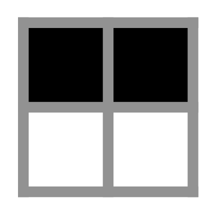



$\mathbb{P}(\text{row} =$  $) = p_1$

$\mathbb{P}(\text{row} =$  $) = p_2$

$\mathbb{P}(\text{row} =$  $) = p_3$

$\mathbb{P}(\text{row} =$  $) = p_4$

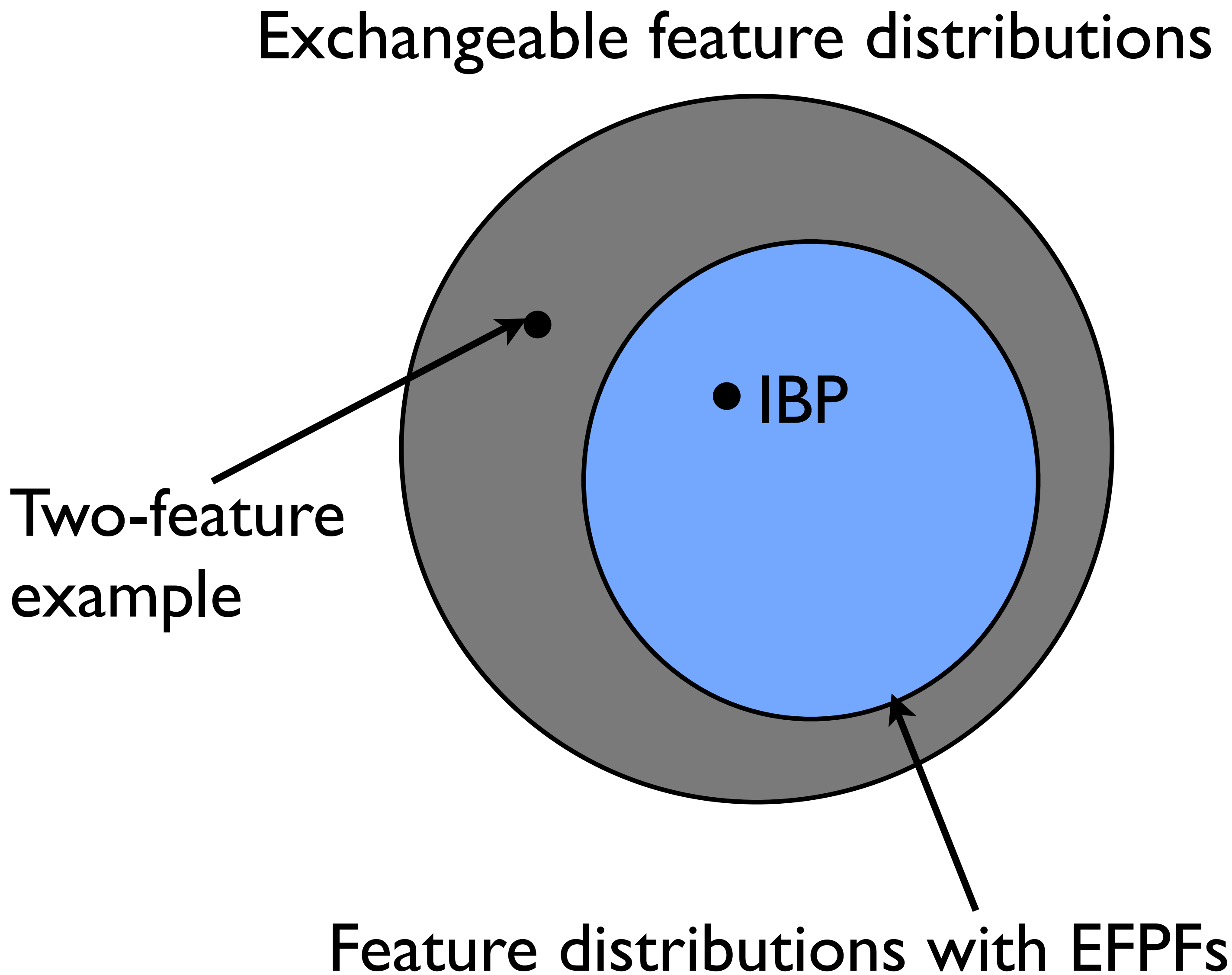
$\mathbb{P}($  $) \neq \mathbb{P}($  $)$

$$p_1 p_2 \neq p_3 p_4$$



[Broderick, Jordan, Pitman 2013]

# Exchangeable probability functions






[Broderick, Jordan, Pitman 2013]

# Paintboxes

Exchangeable partition: Kingman paintbox



[Kingman 1978]

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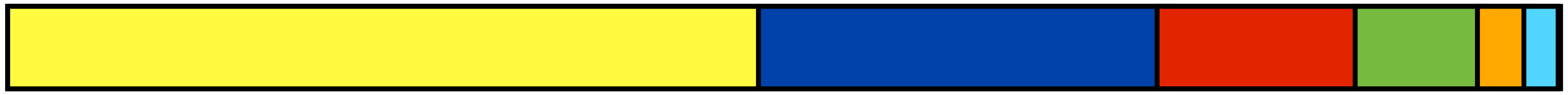
Exchangeable partition: Kingman paintbox





[Kingman 1978]

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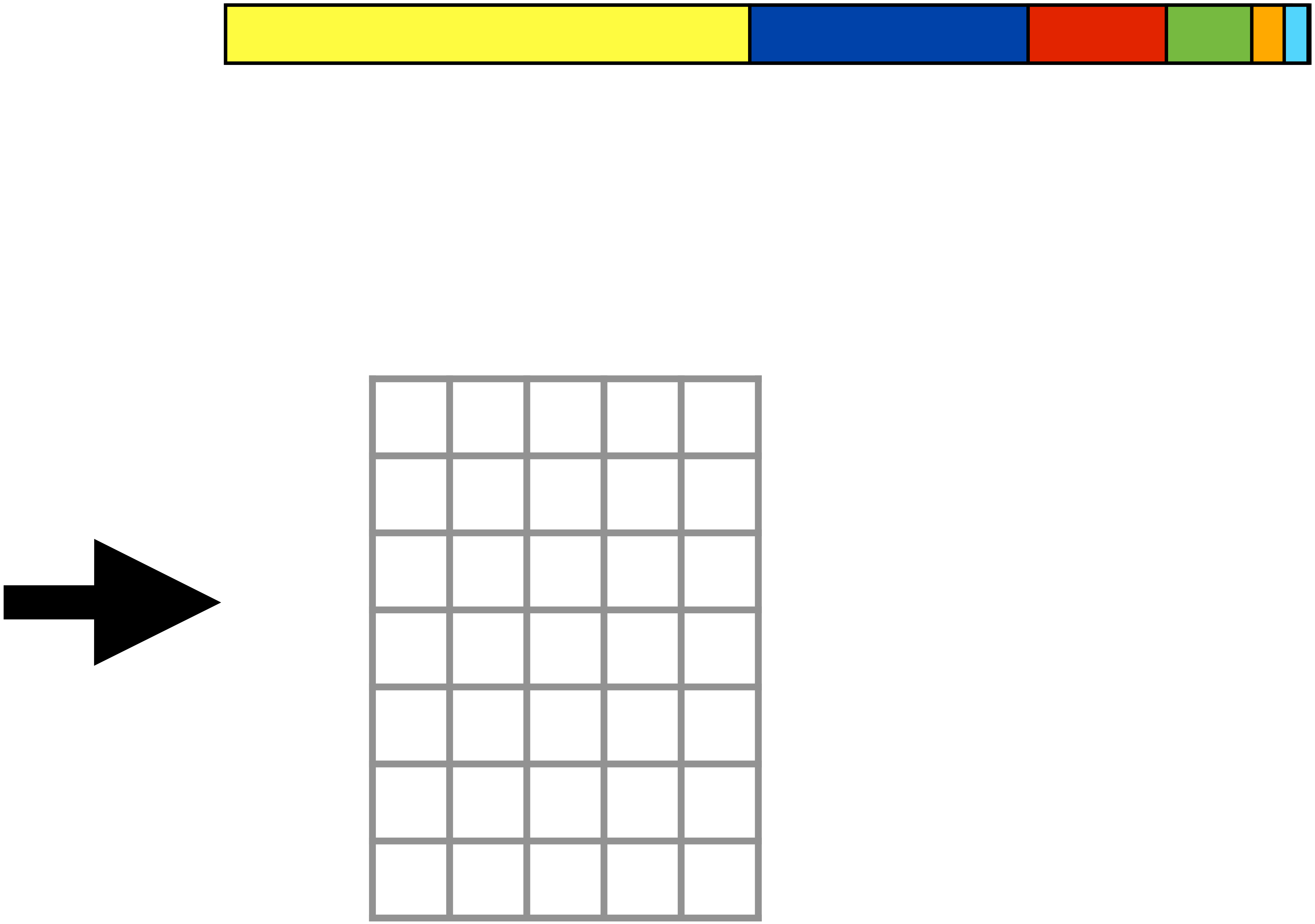
Exchangeable partition: Kingman paintbox





[Kingman 1978]

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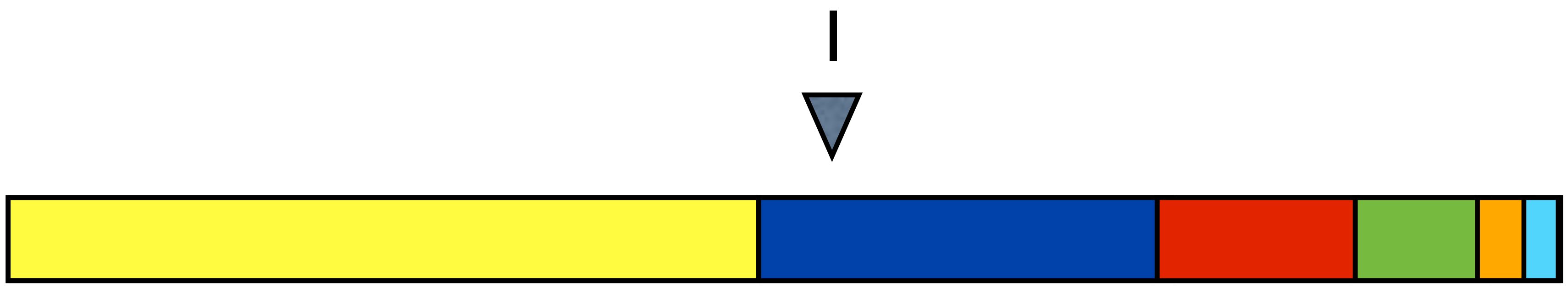
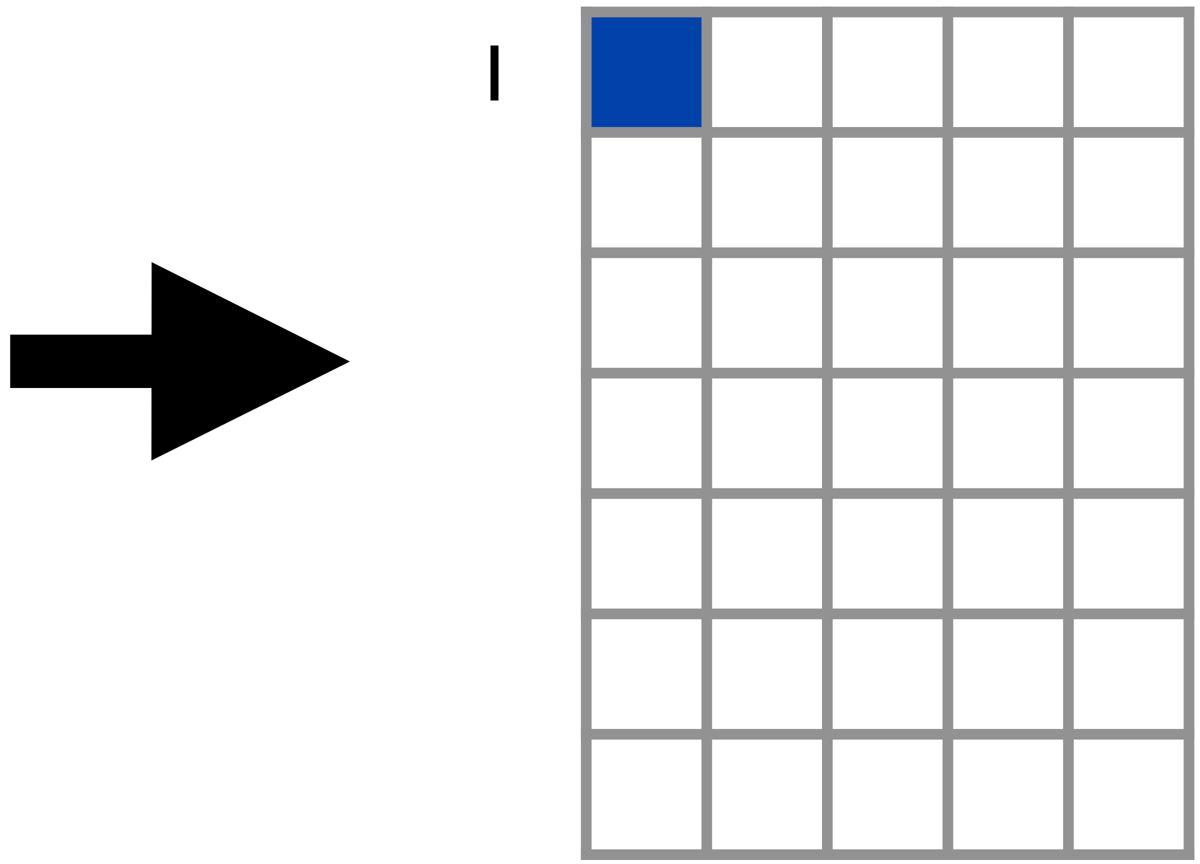
Exchangeable partition: Kingman paintbox






[Kingman 1978]

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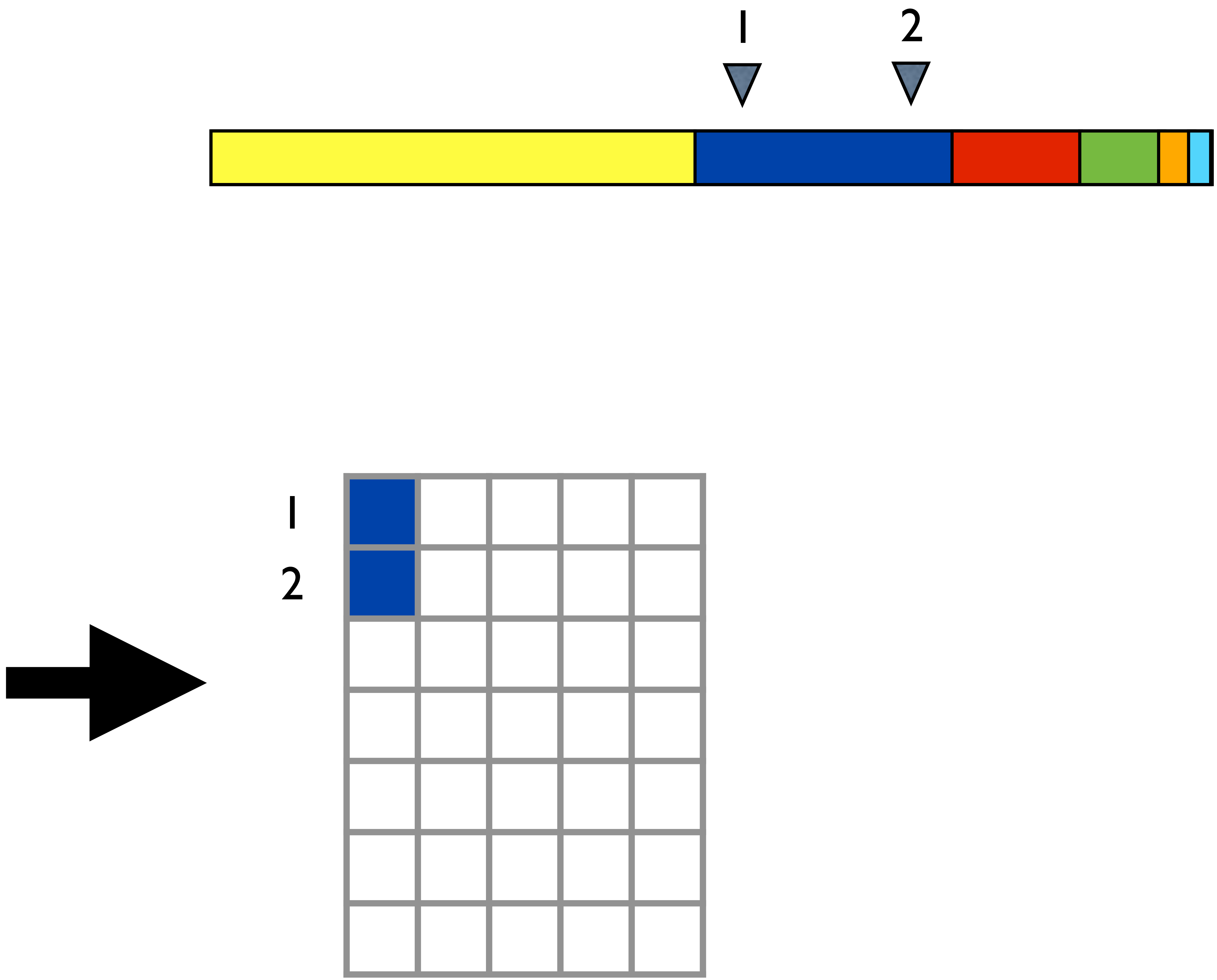
## Exchangeable partition: Kingman paintbox






[Kingman 1978]

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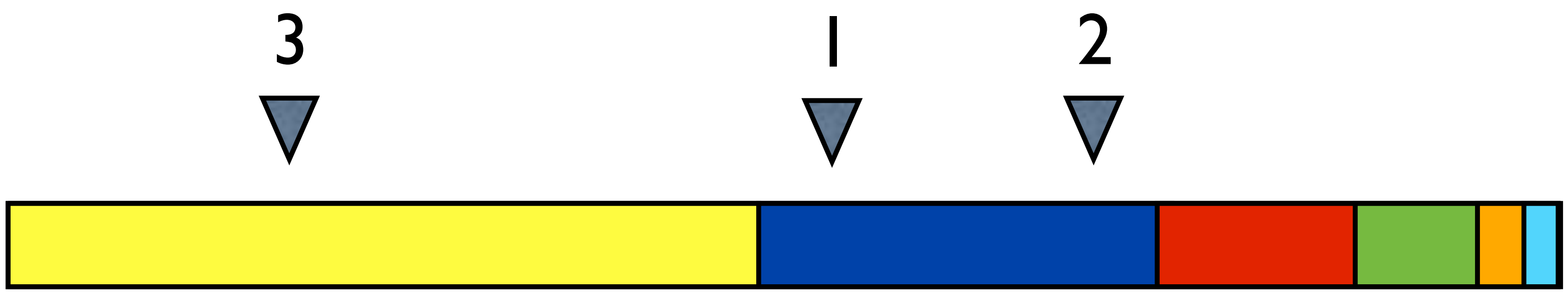
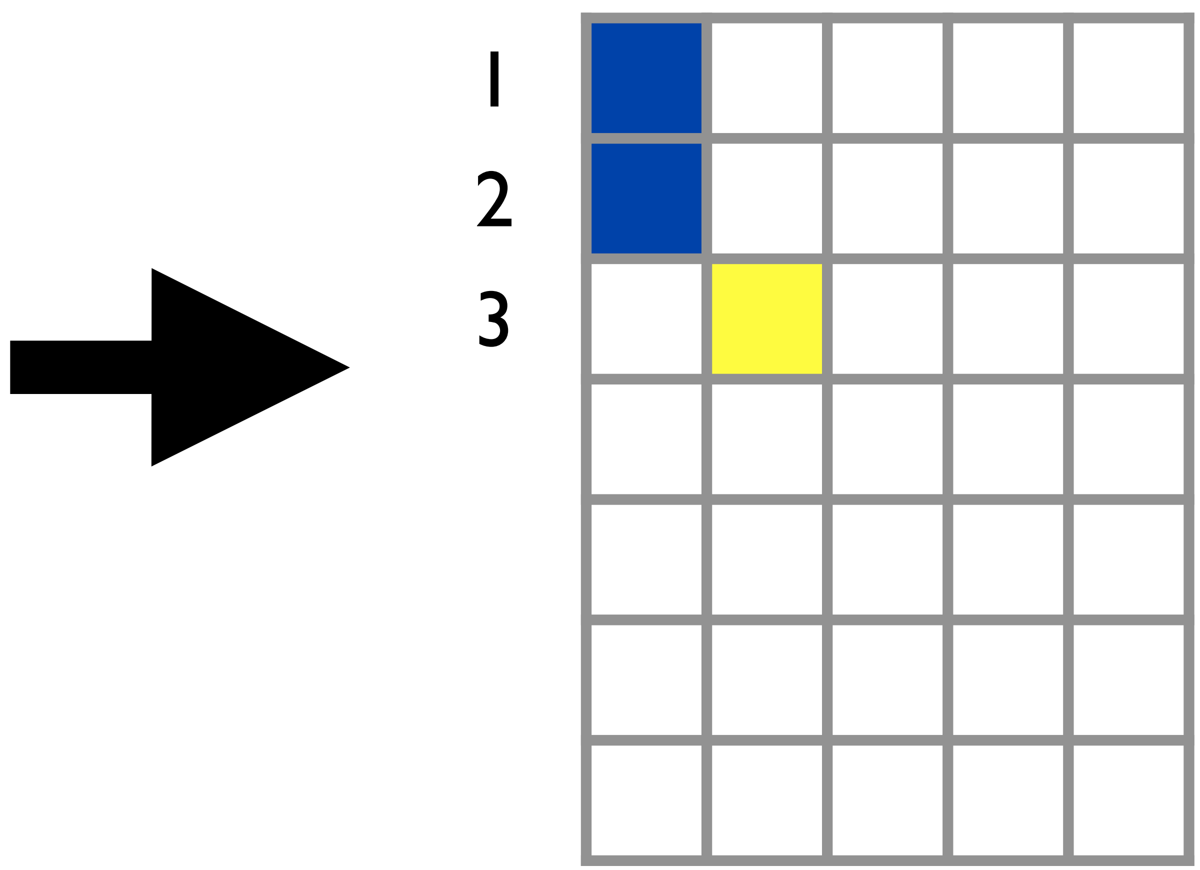
Exchangeable partition: Kingman paintbox




[Kingman 1978]

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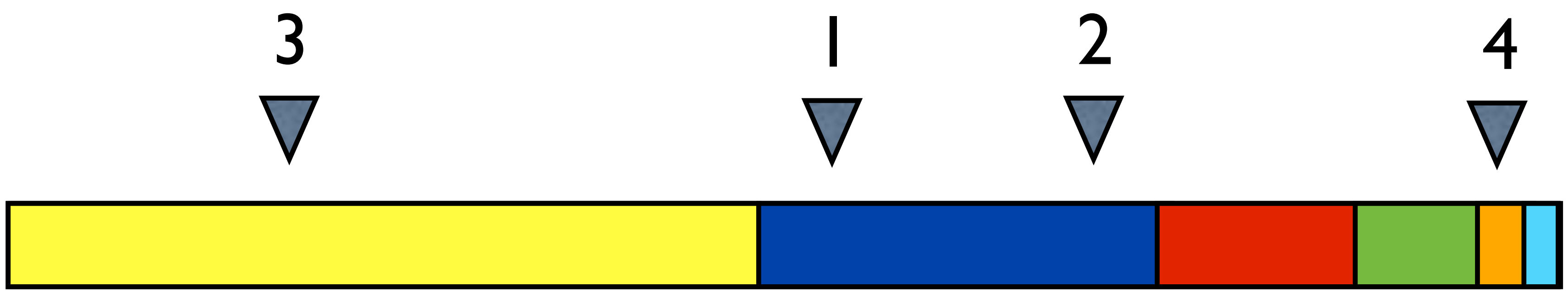
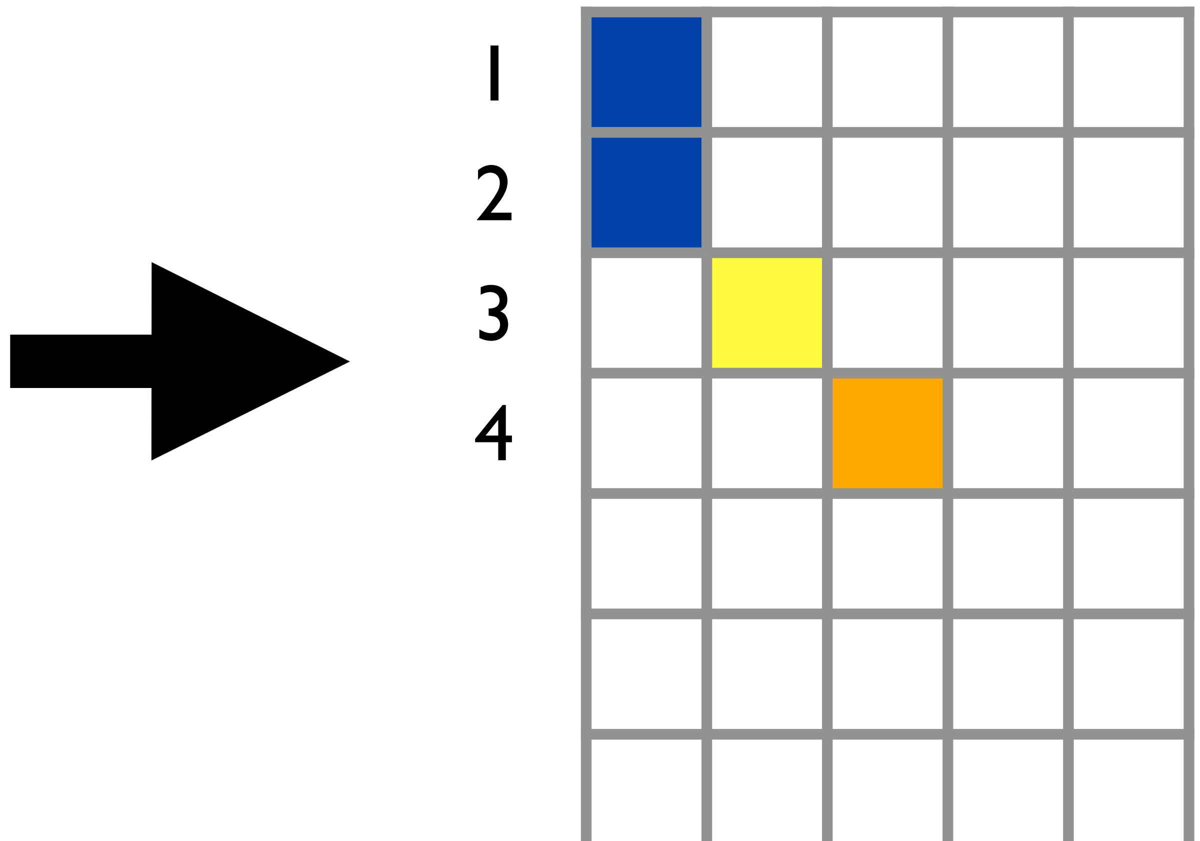
Exchangeable partition: Kingman paintbox






[Kingman 1978]

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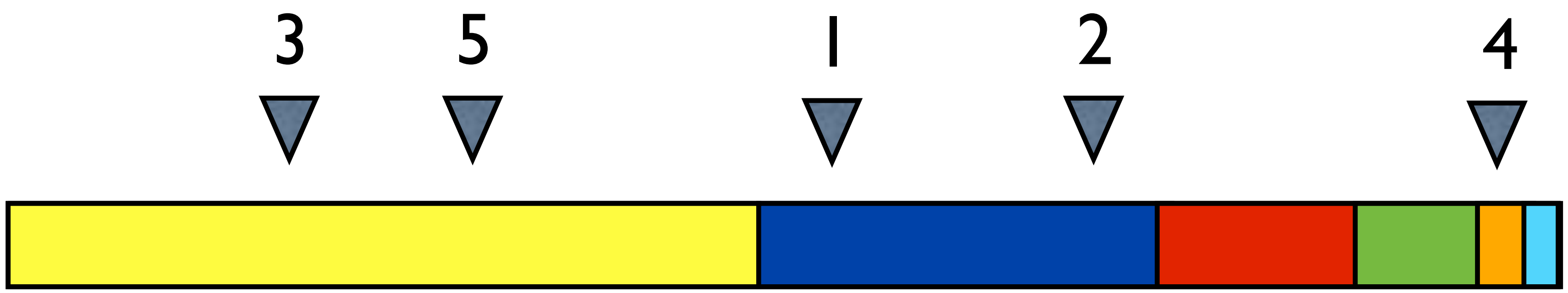
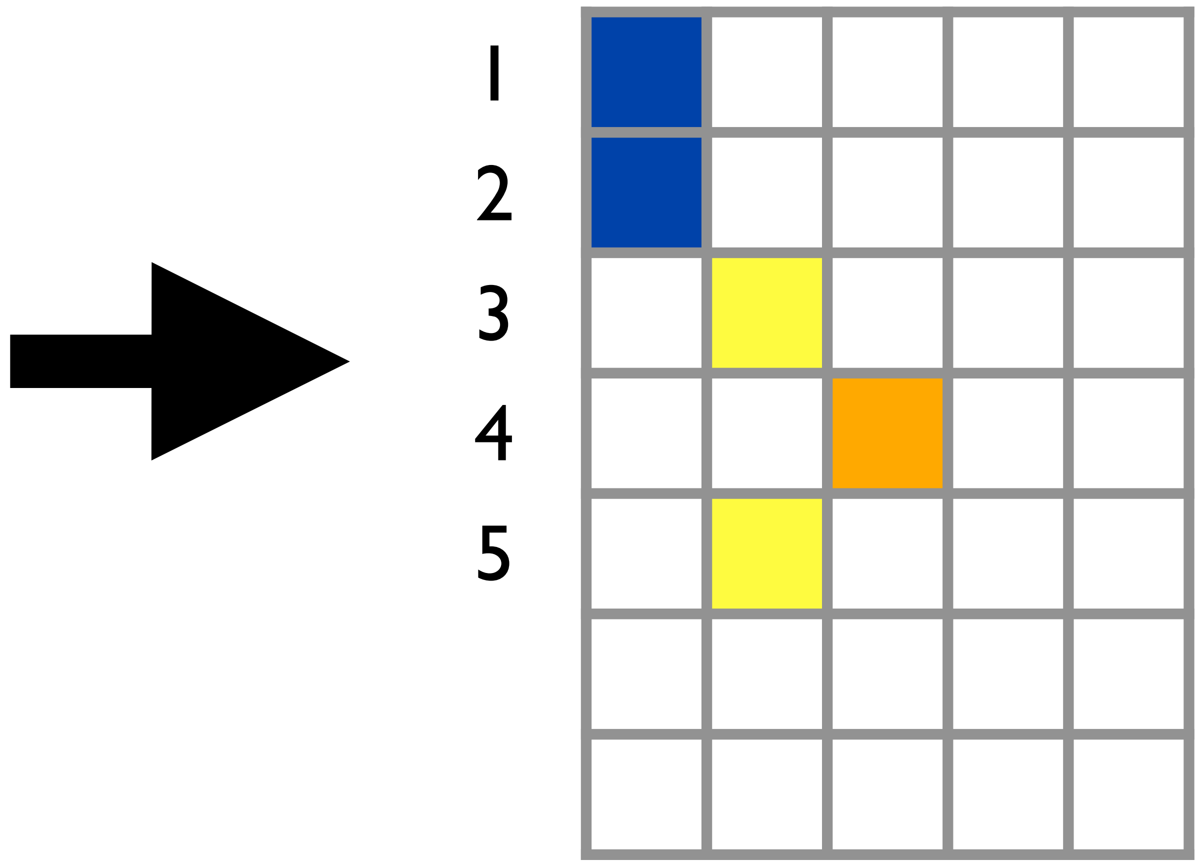
Exchangeable partition: Kingman paintbox






[Kingman 1978]

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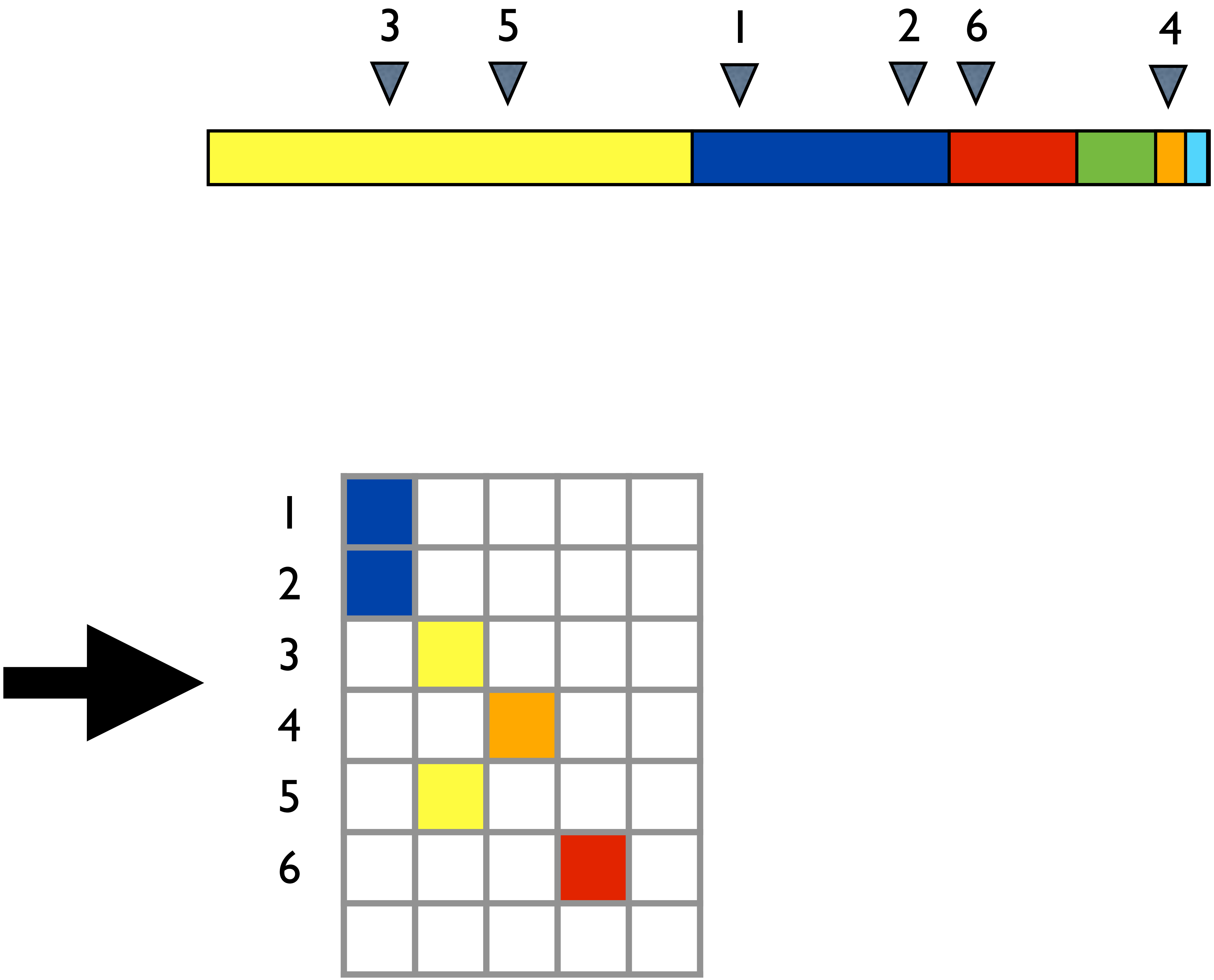


[Kingman 1978]

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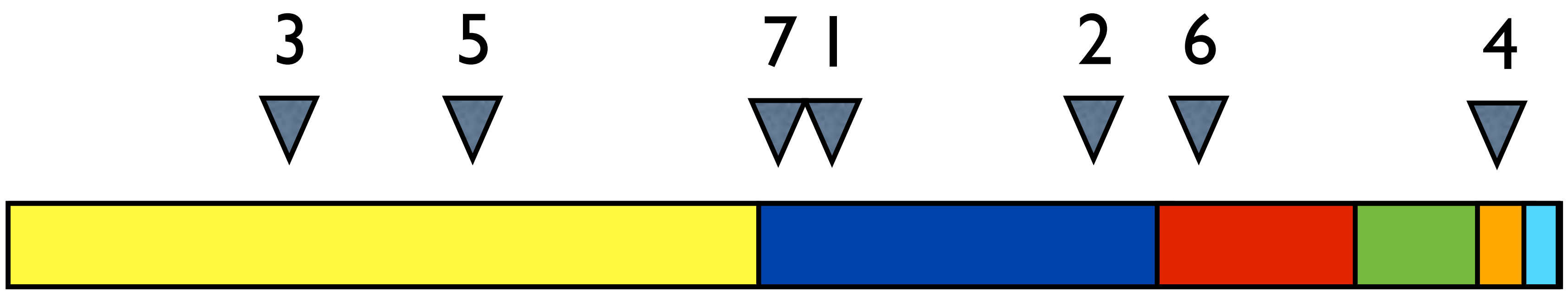


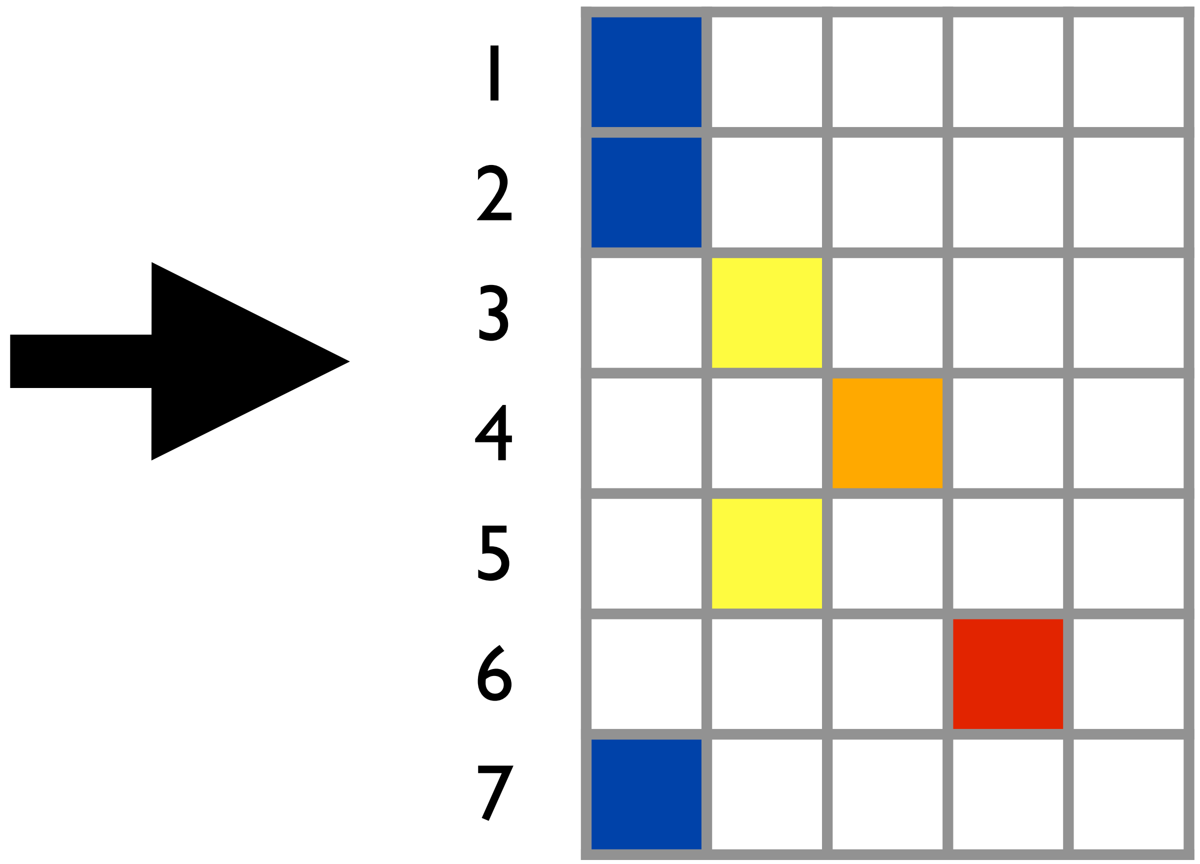




[Kingman 1978]

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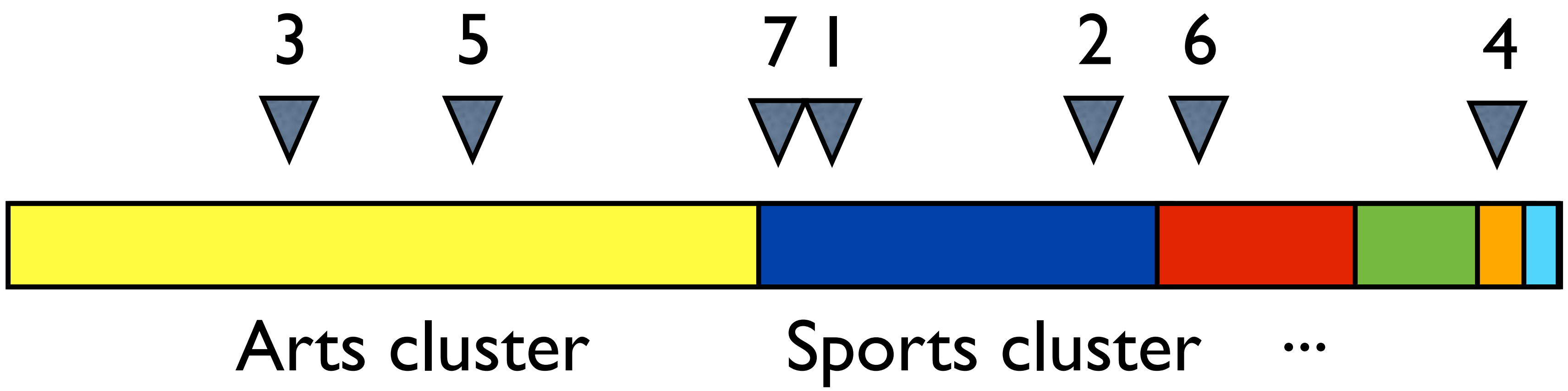


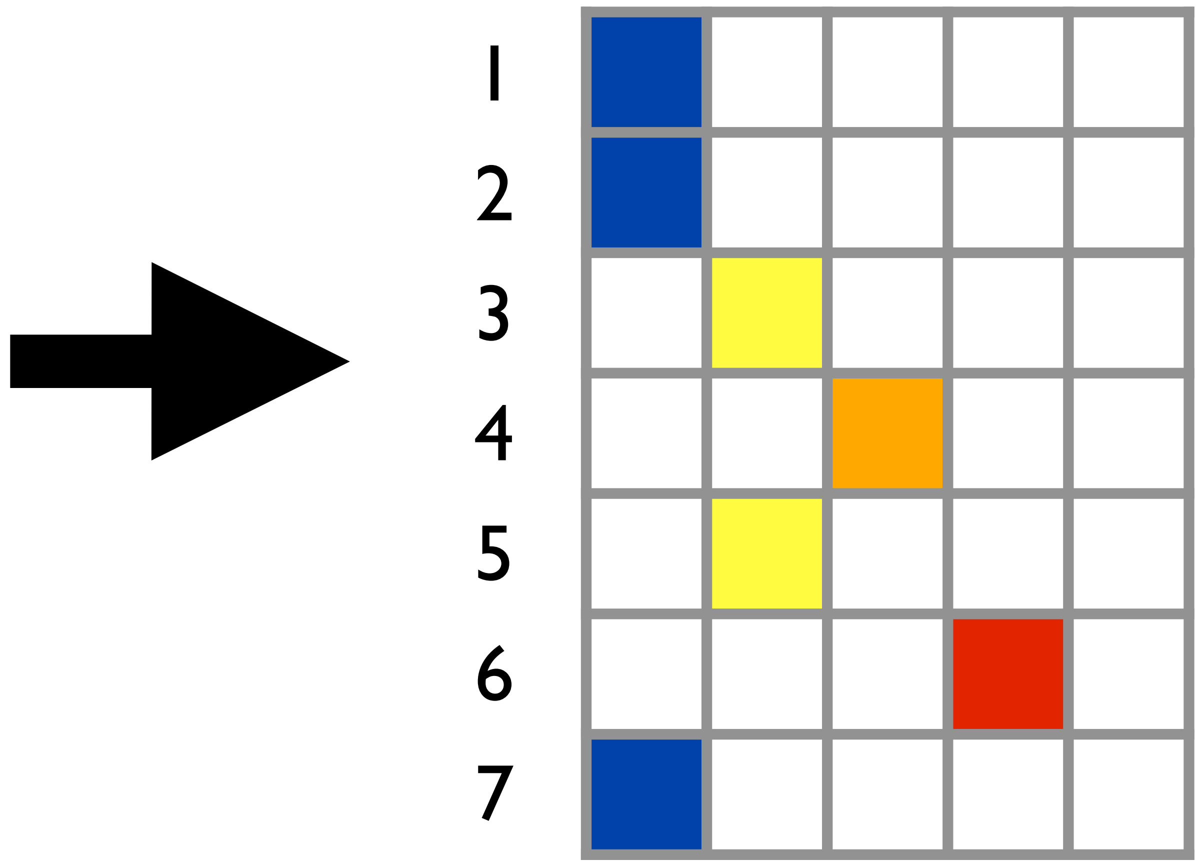




[Kingman 1978]

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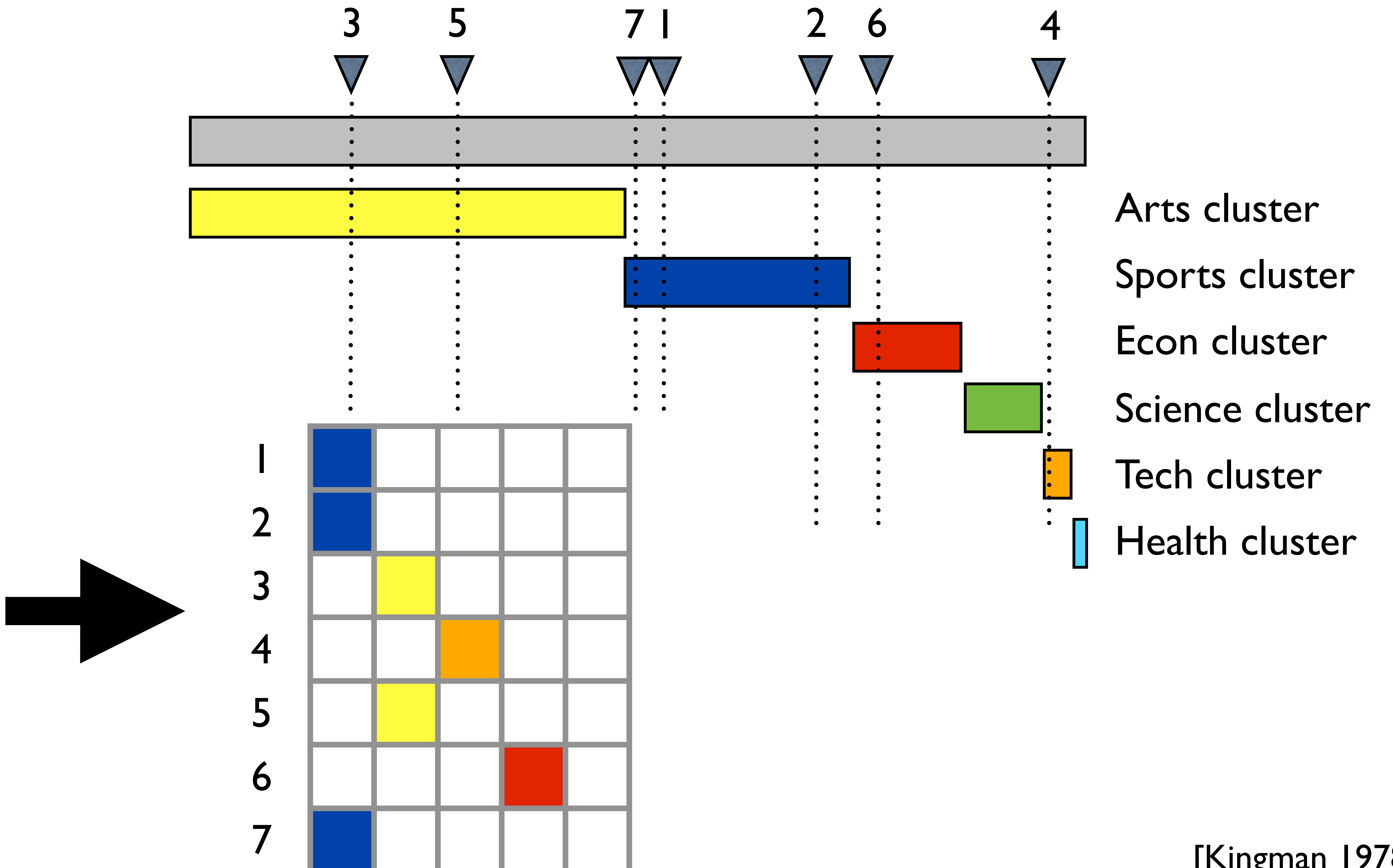




[Kingman 1978]

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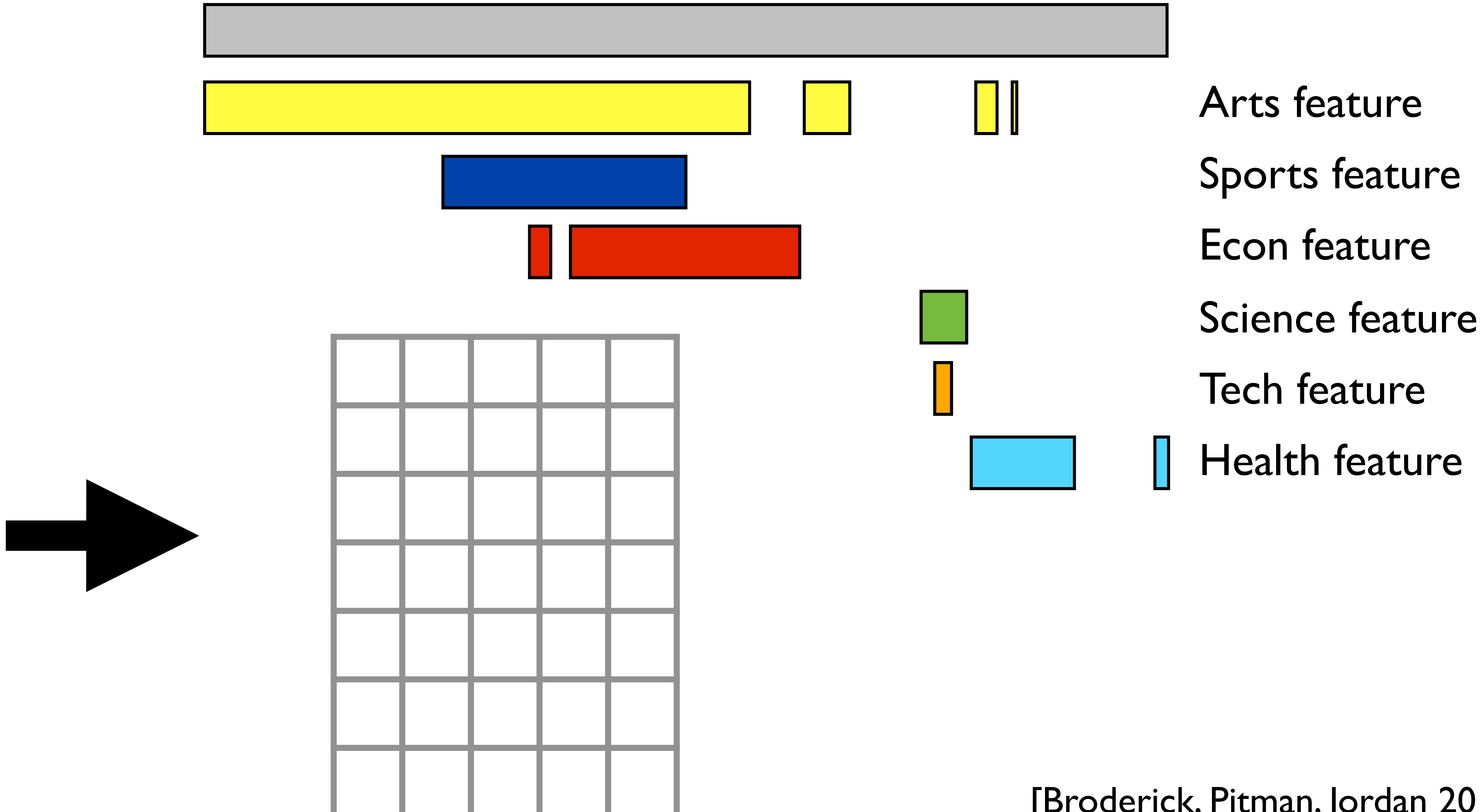
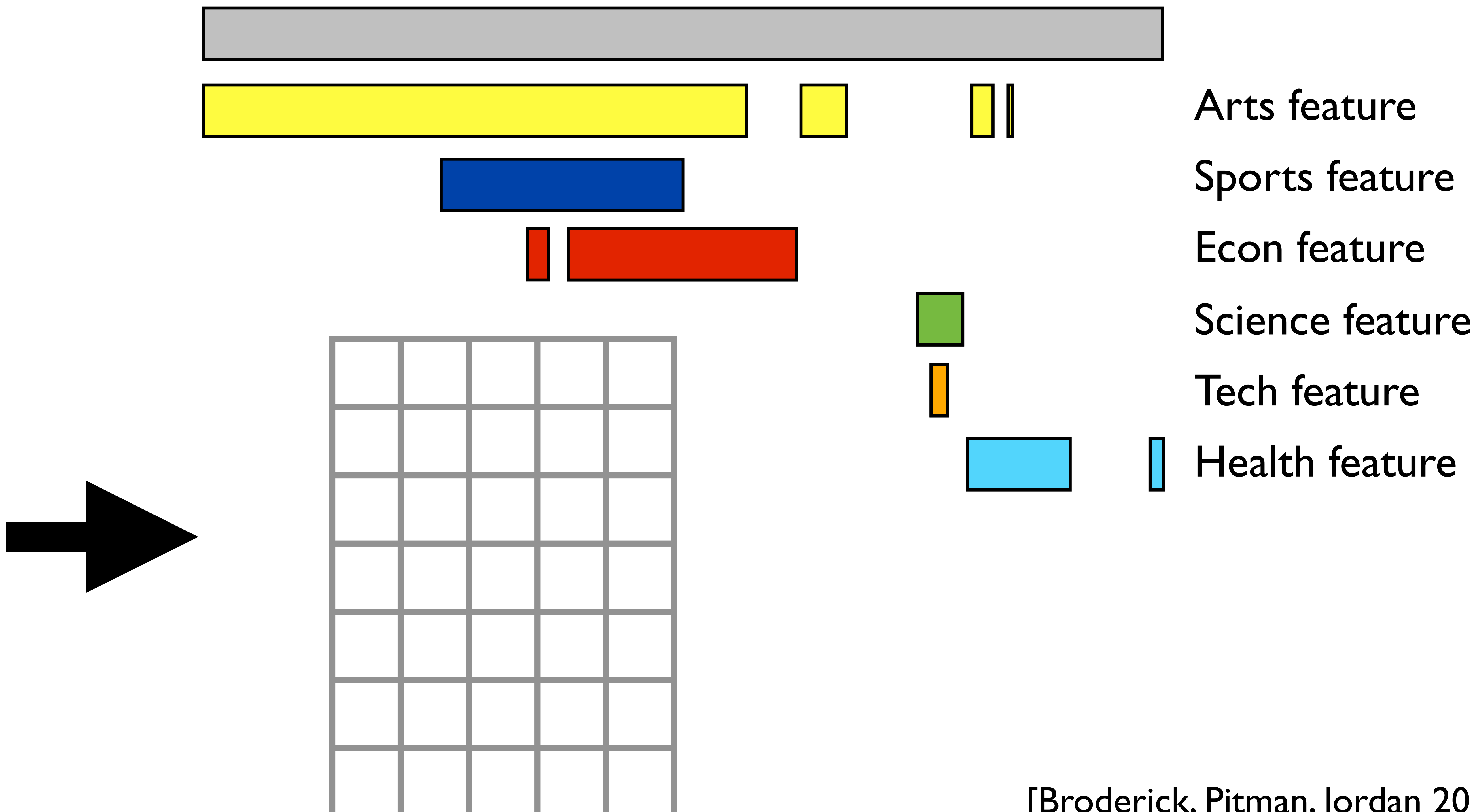





[Broderick, Pitman, Jordan 2013]

# Paintboxes

Exchangeable feature allocation: feature paintbox






[Broderick, Pitman, Jordan 2013]

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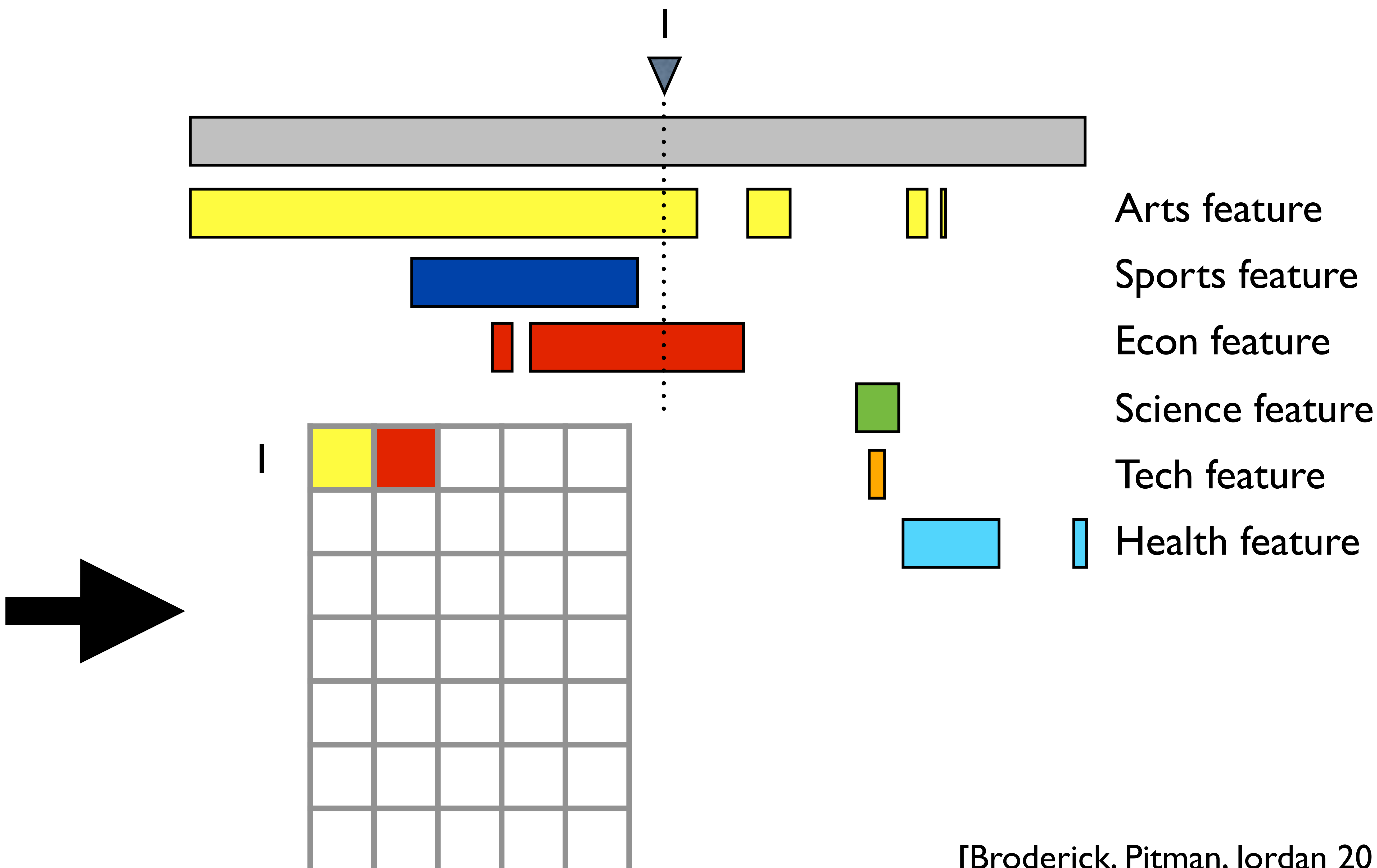
Exchangeable feature allocation: feature paintbox






[Broderick, Pitman, Jordan 2013]

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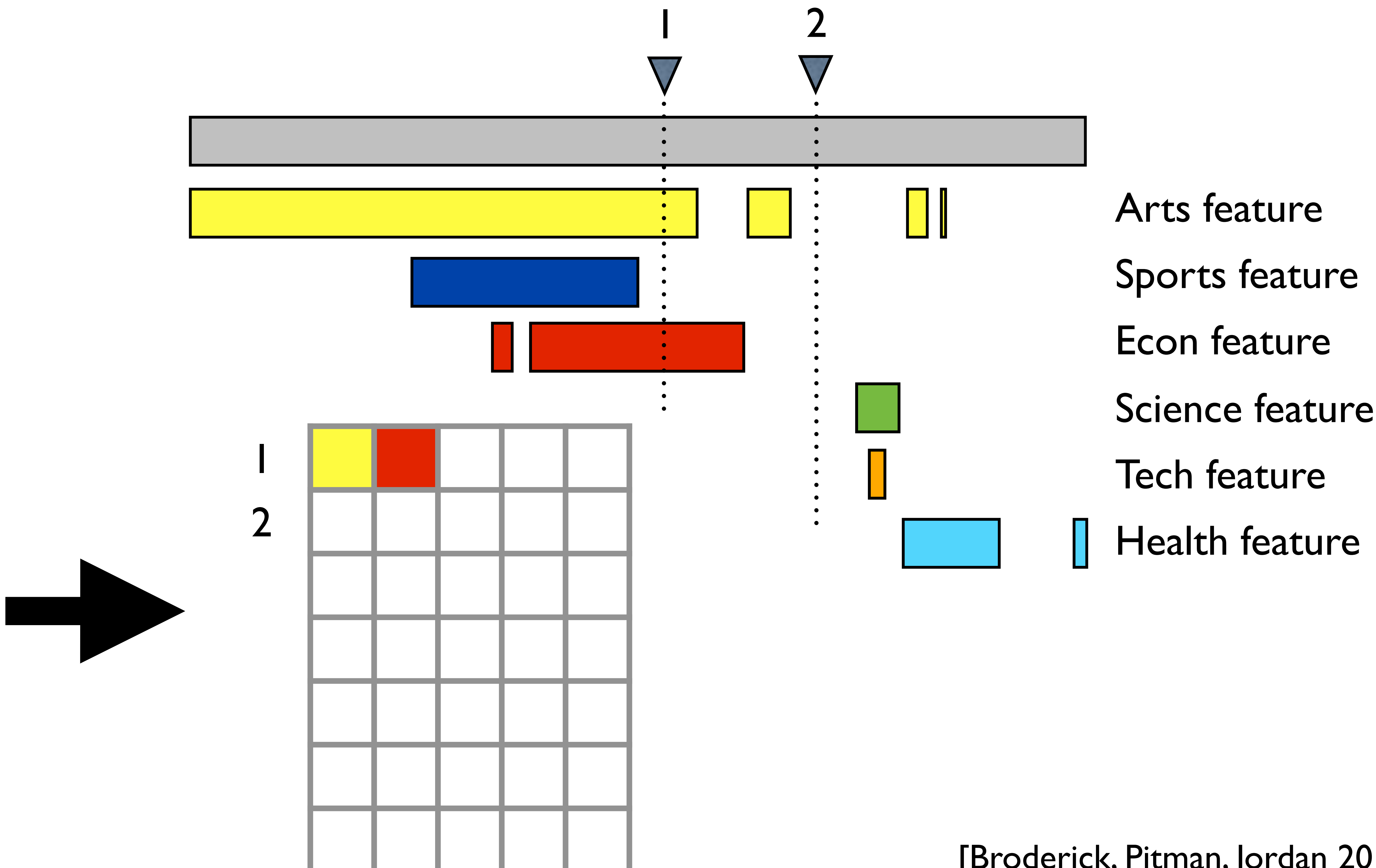
Exchangeable feature allocation: feature paintbox






[Broderick, Pitman, Jordan 2013]

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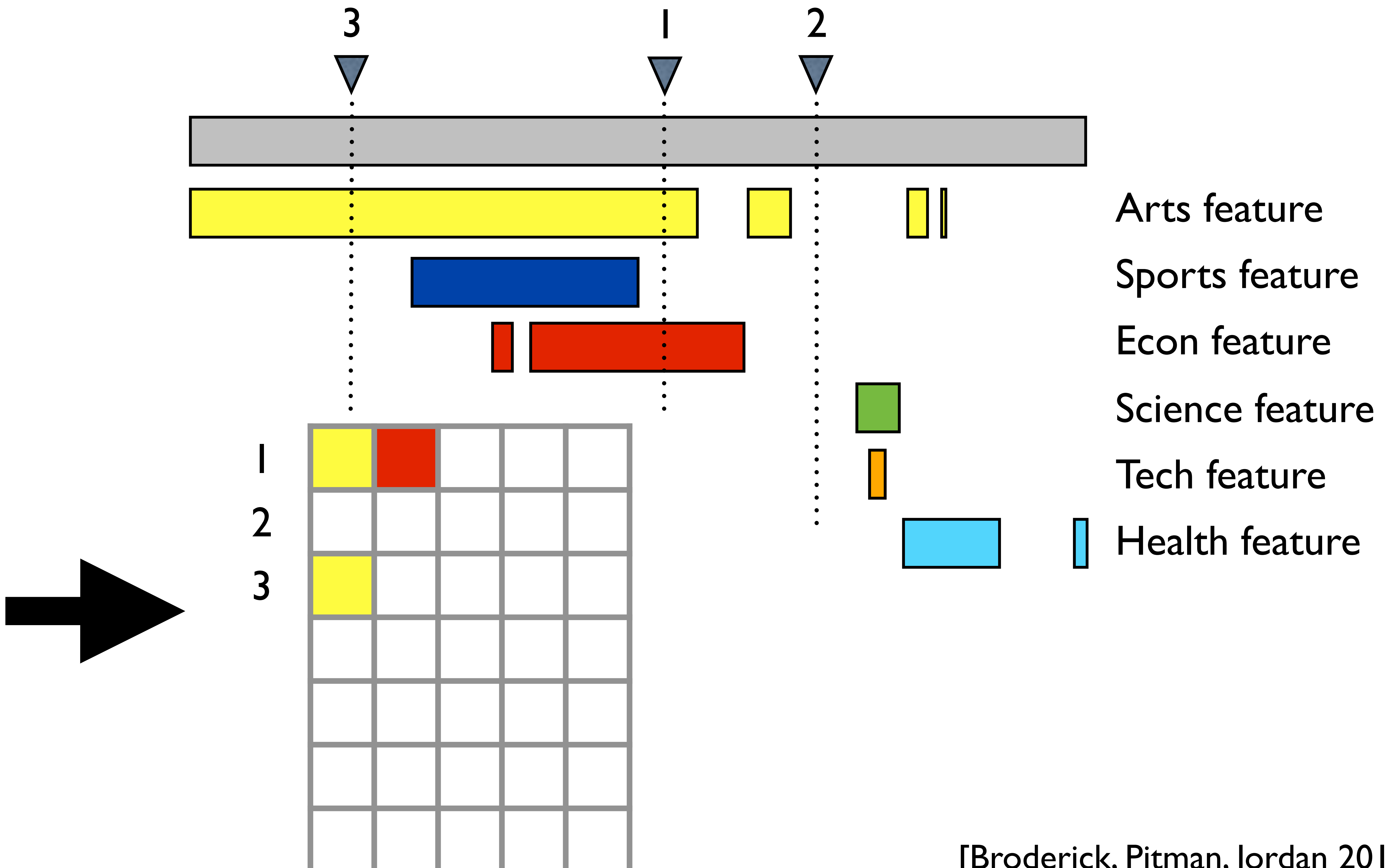




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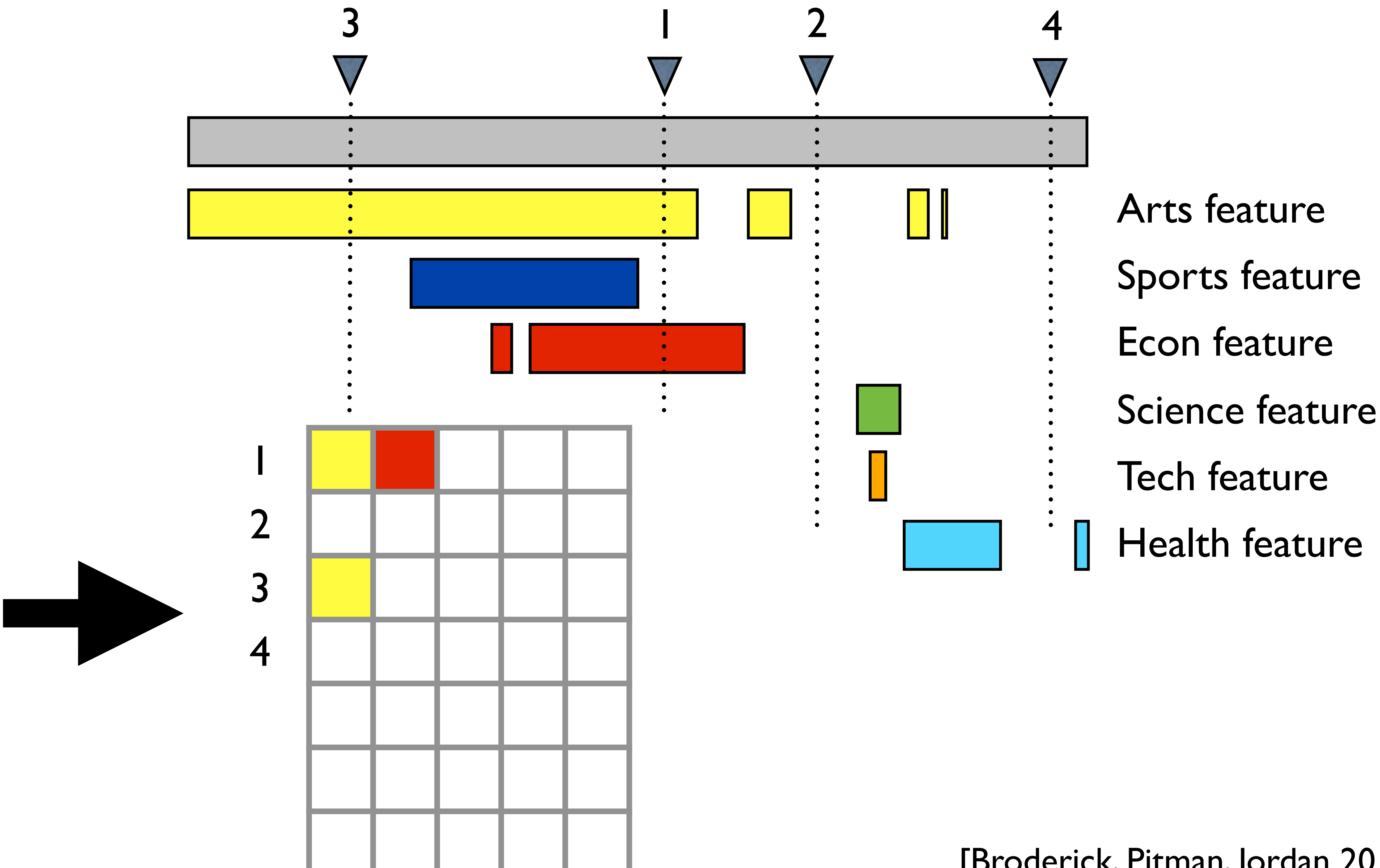




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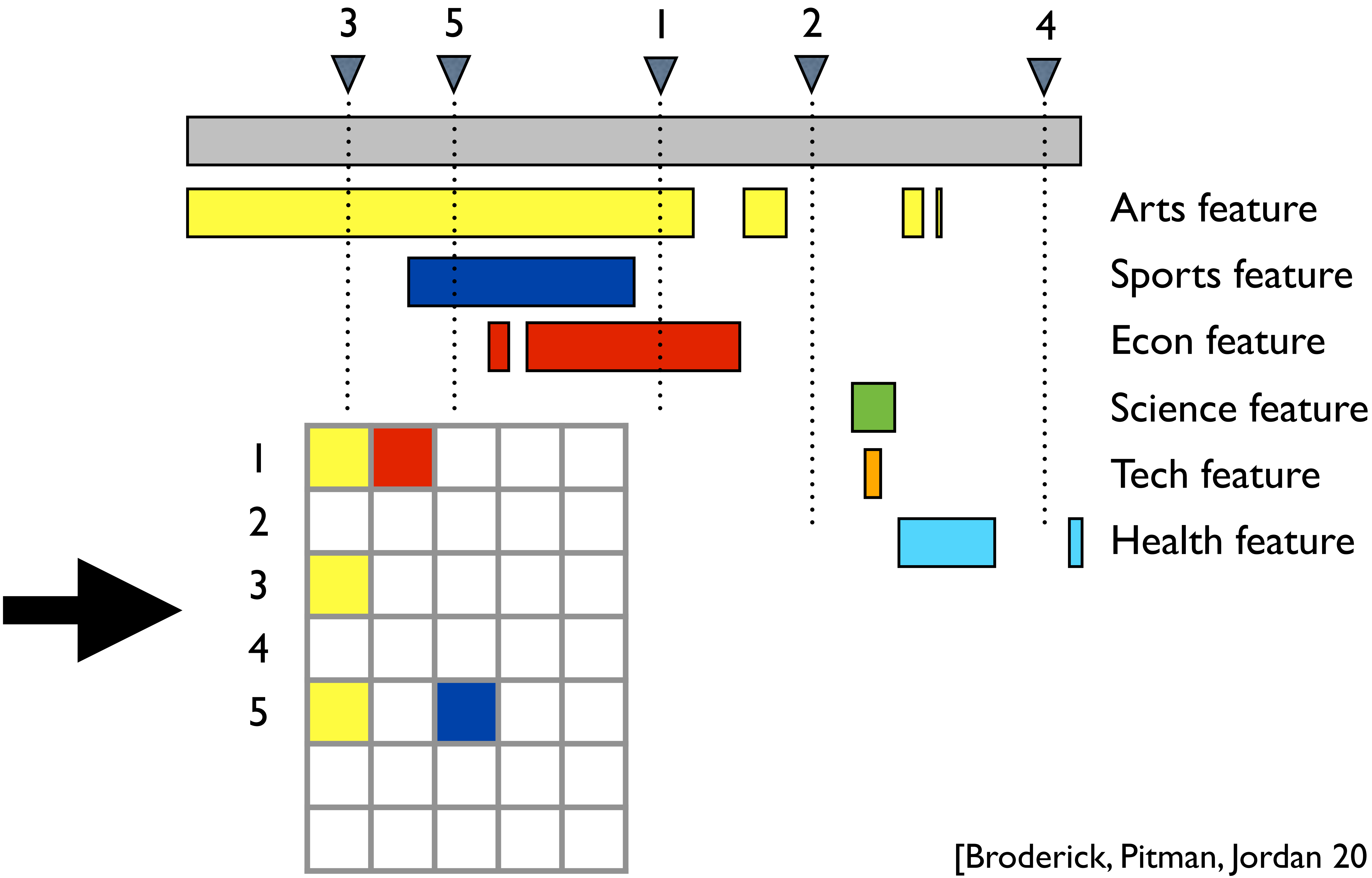




[Broderick, Pitman, Jordan 2013]

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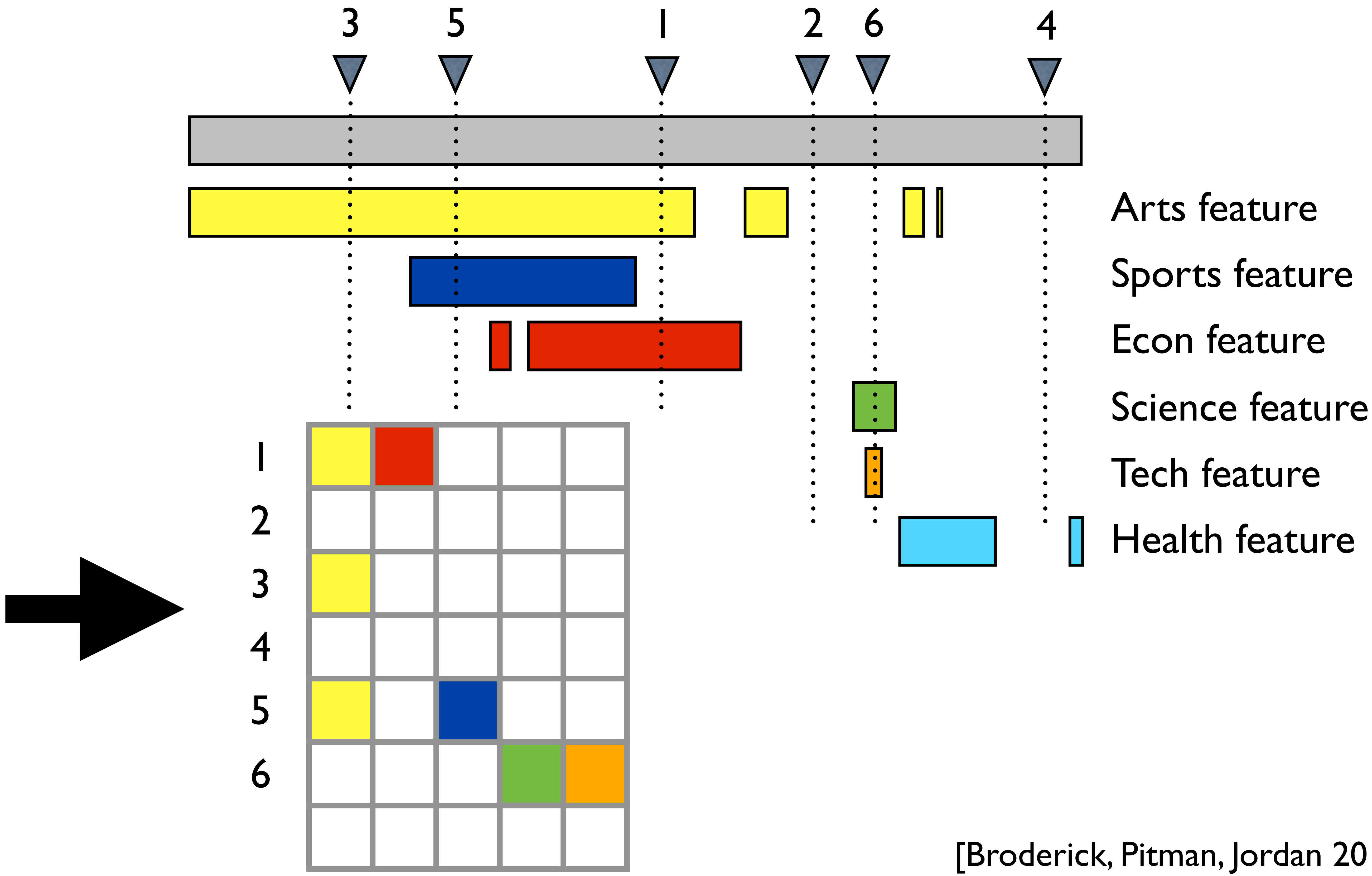




[Broderick, Pitman, Jordan 2013]

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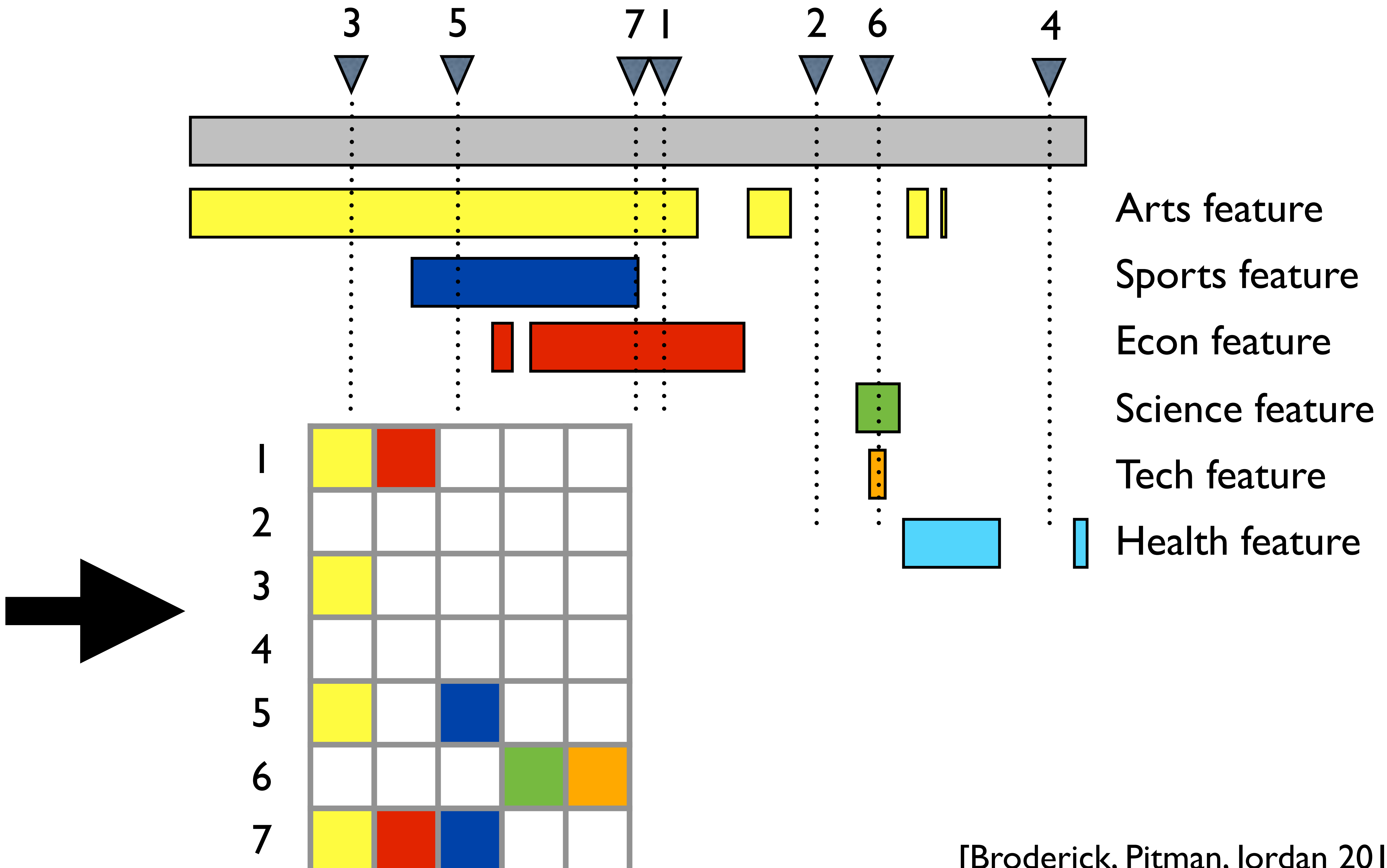
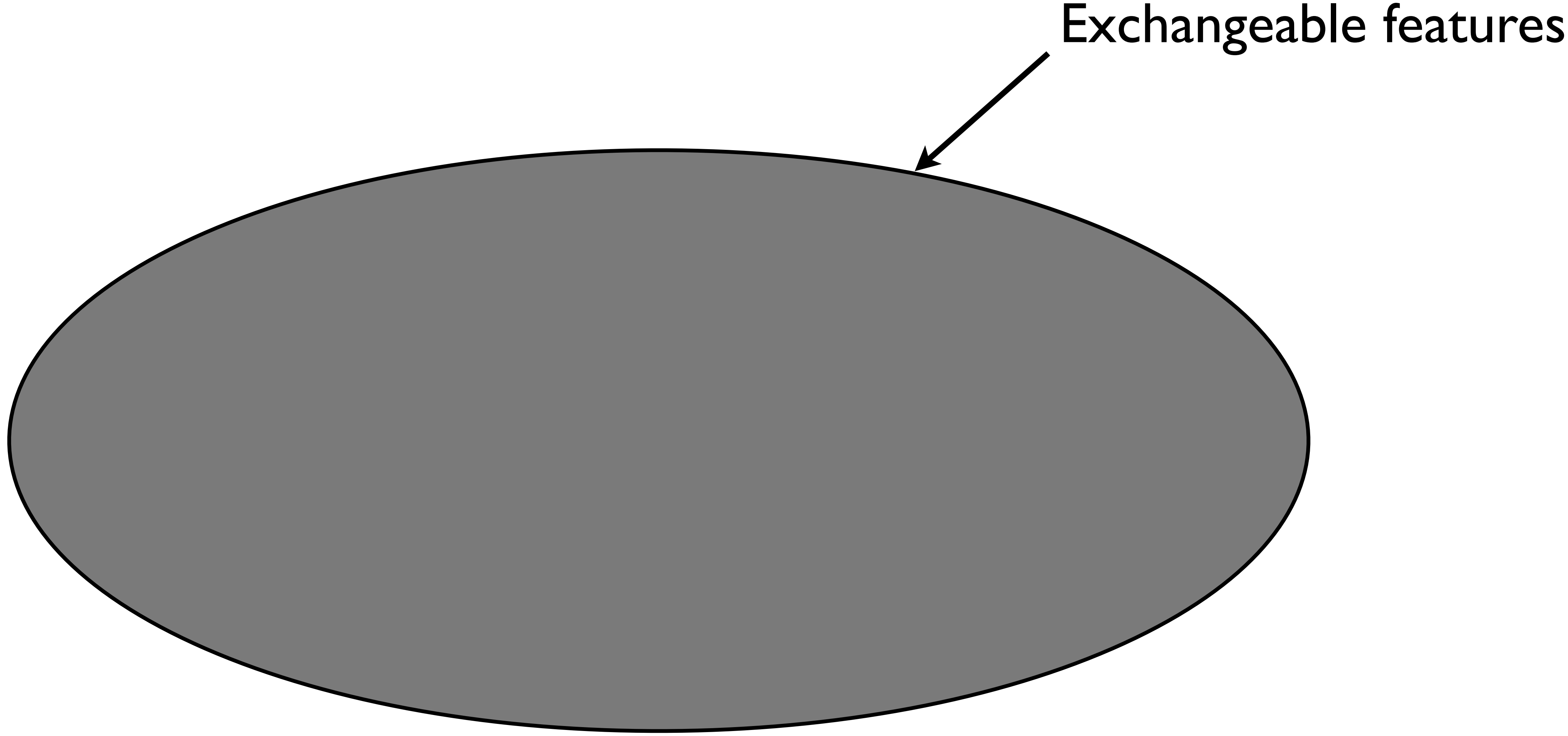





[Broderick, Pitman, Jordan 2013]

# Conclusions



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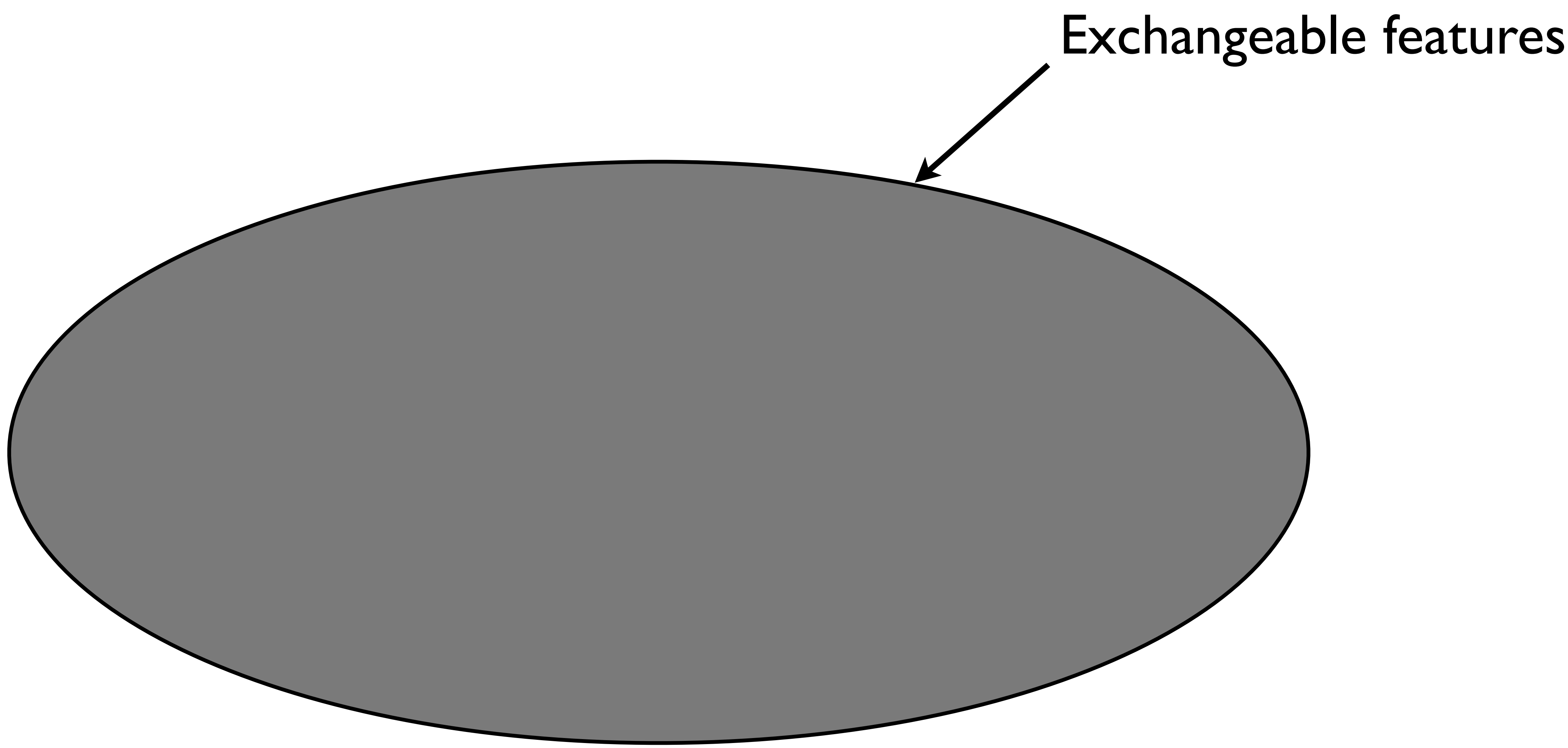
- Feature paintbox: characterization of exchangeable feature models



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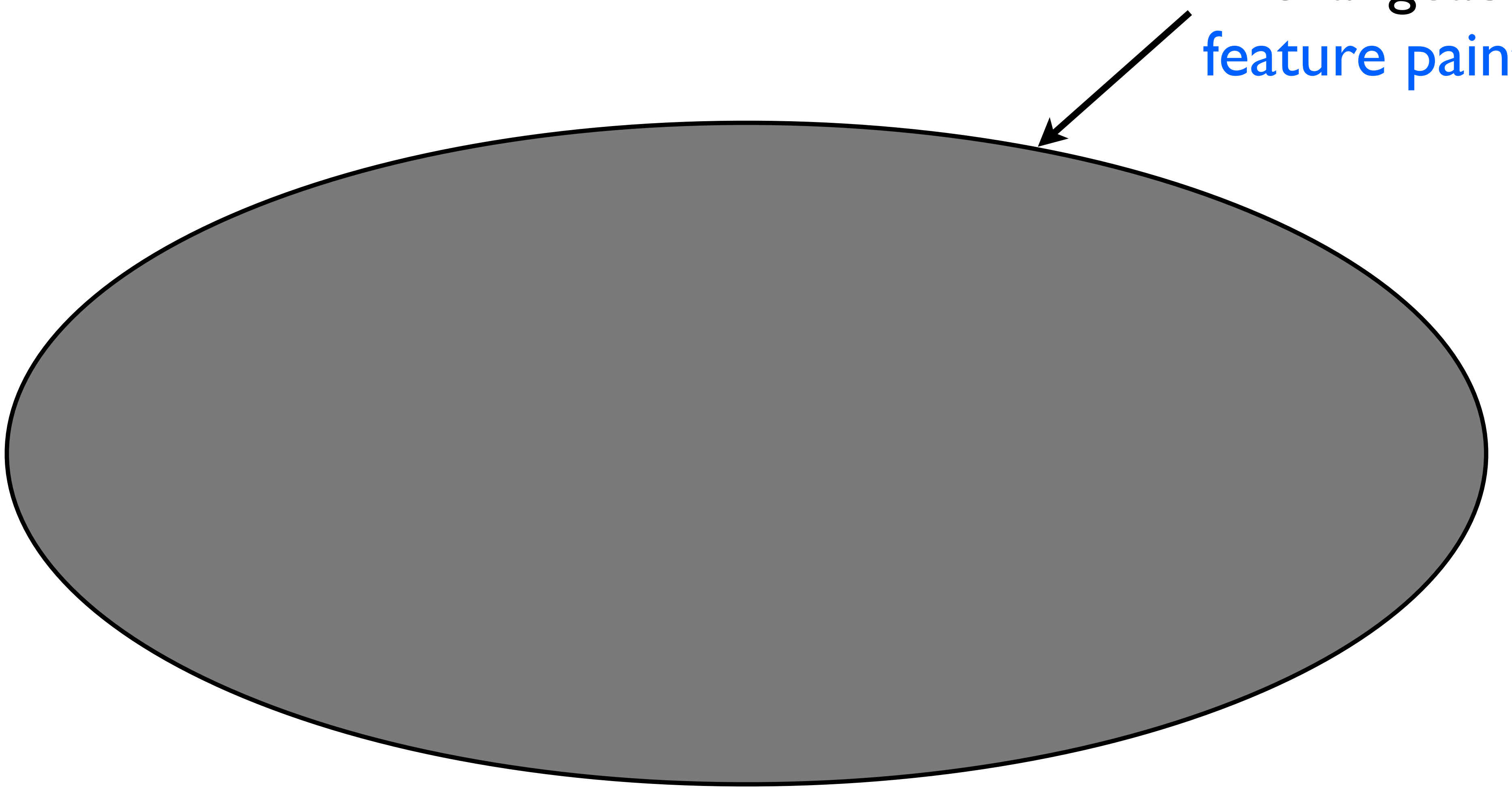

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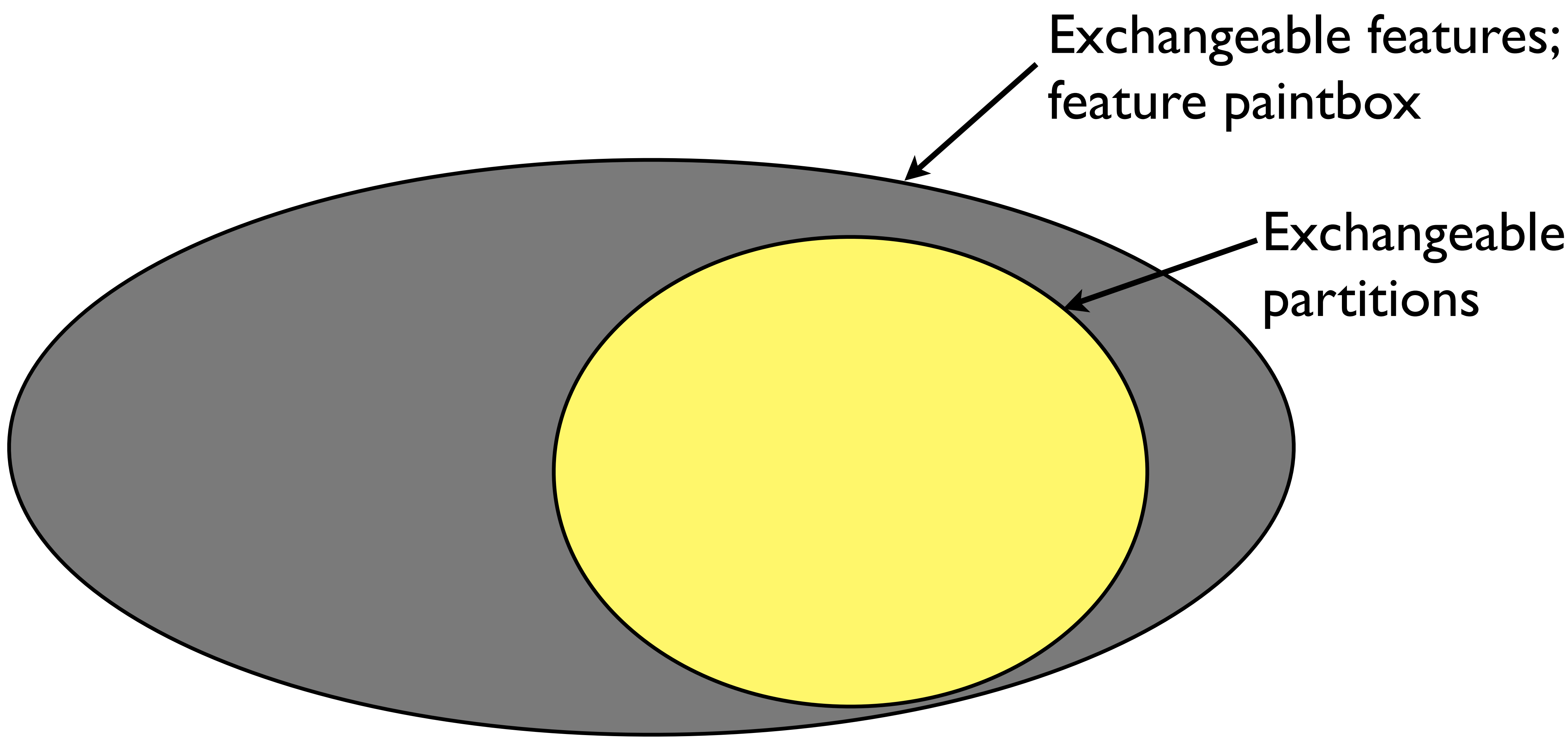

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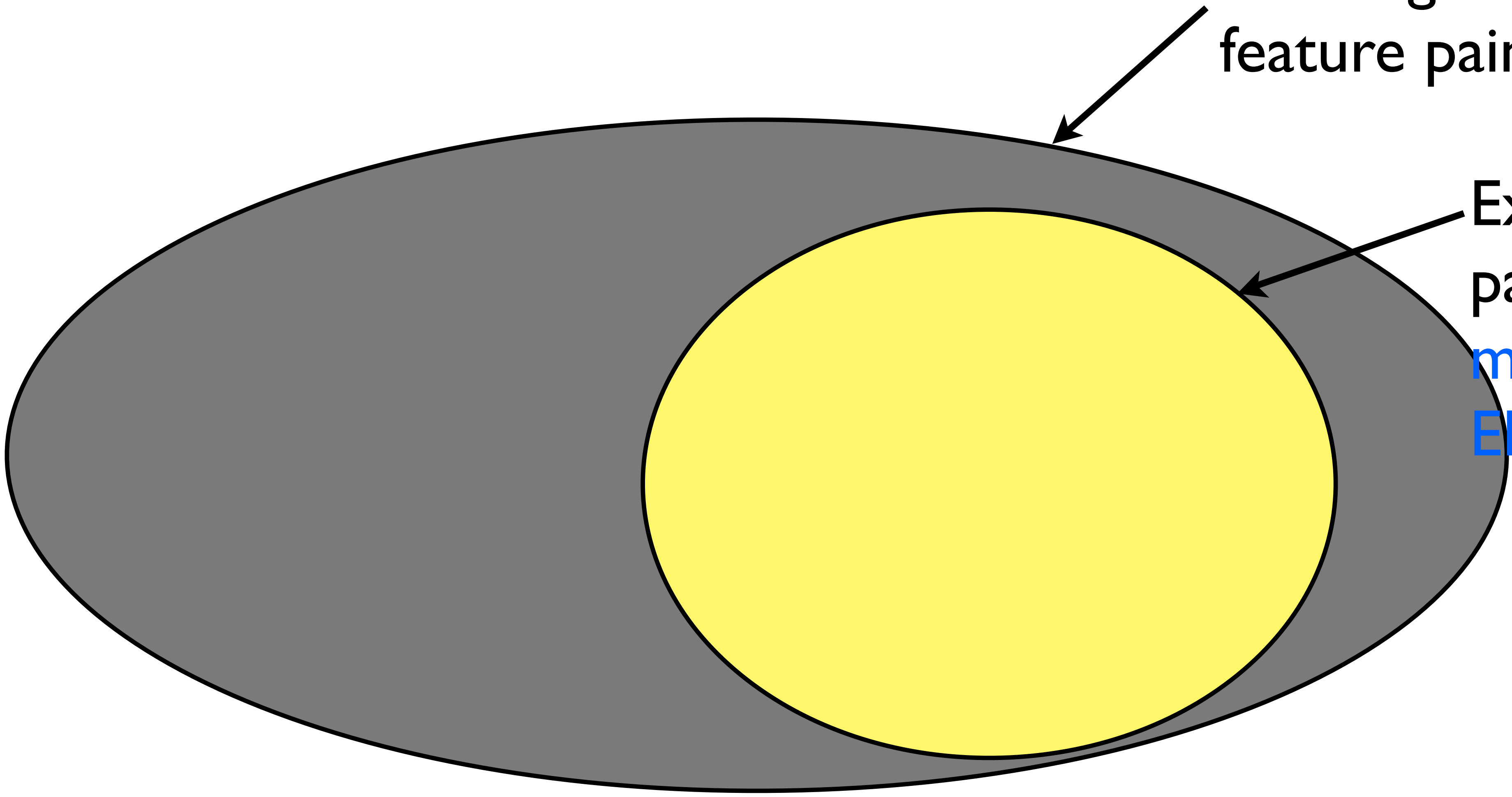

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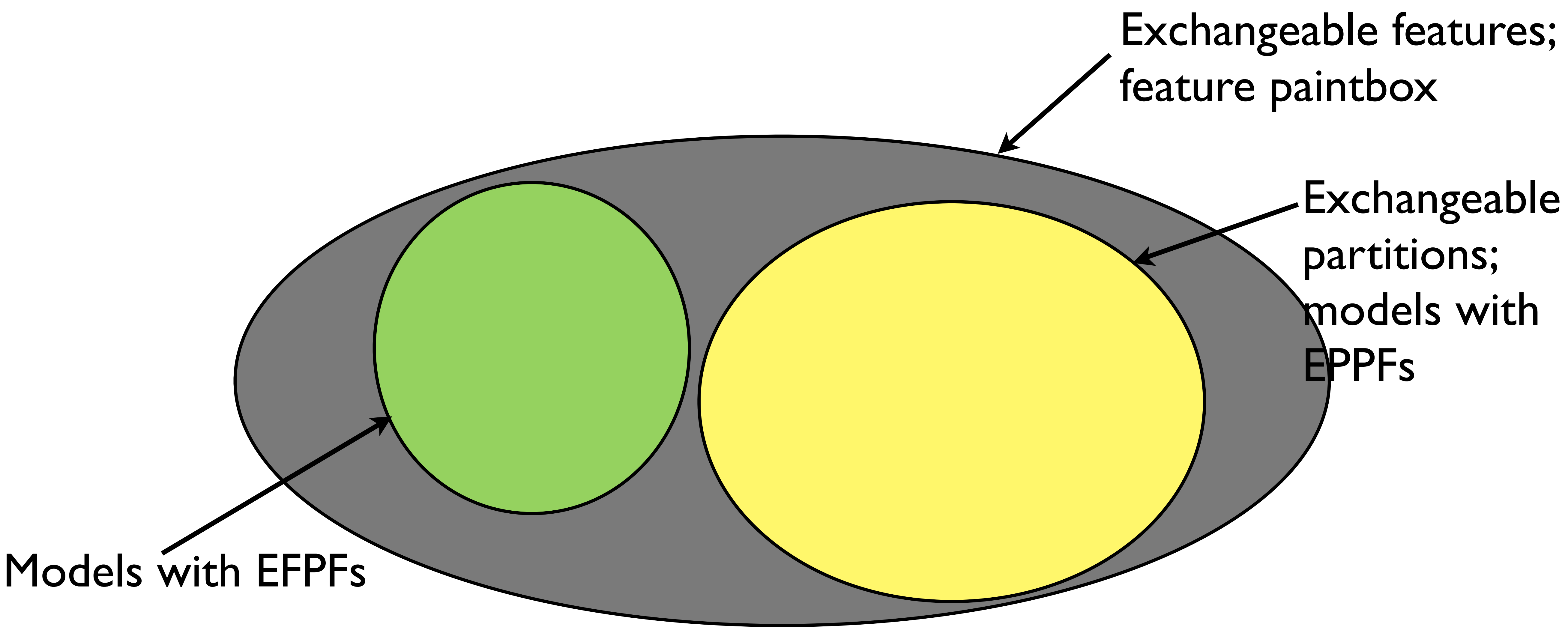

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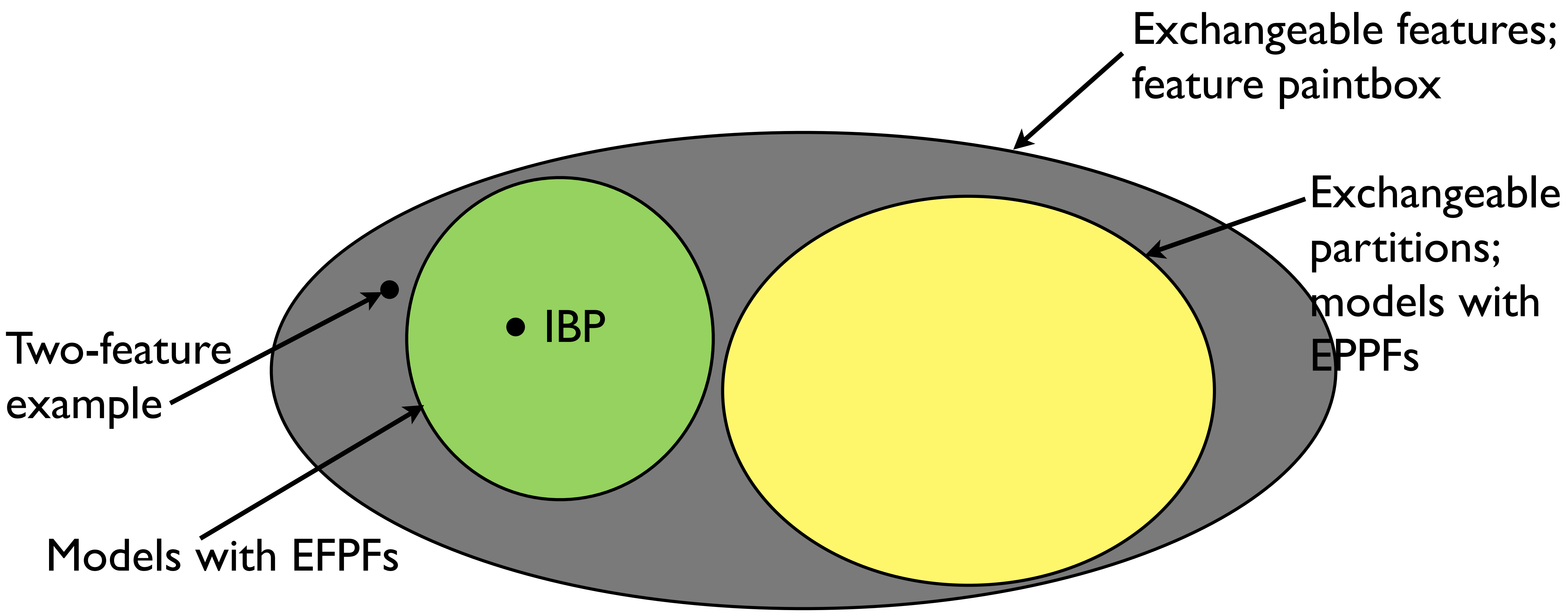

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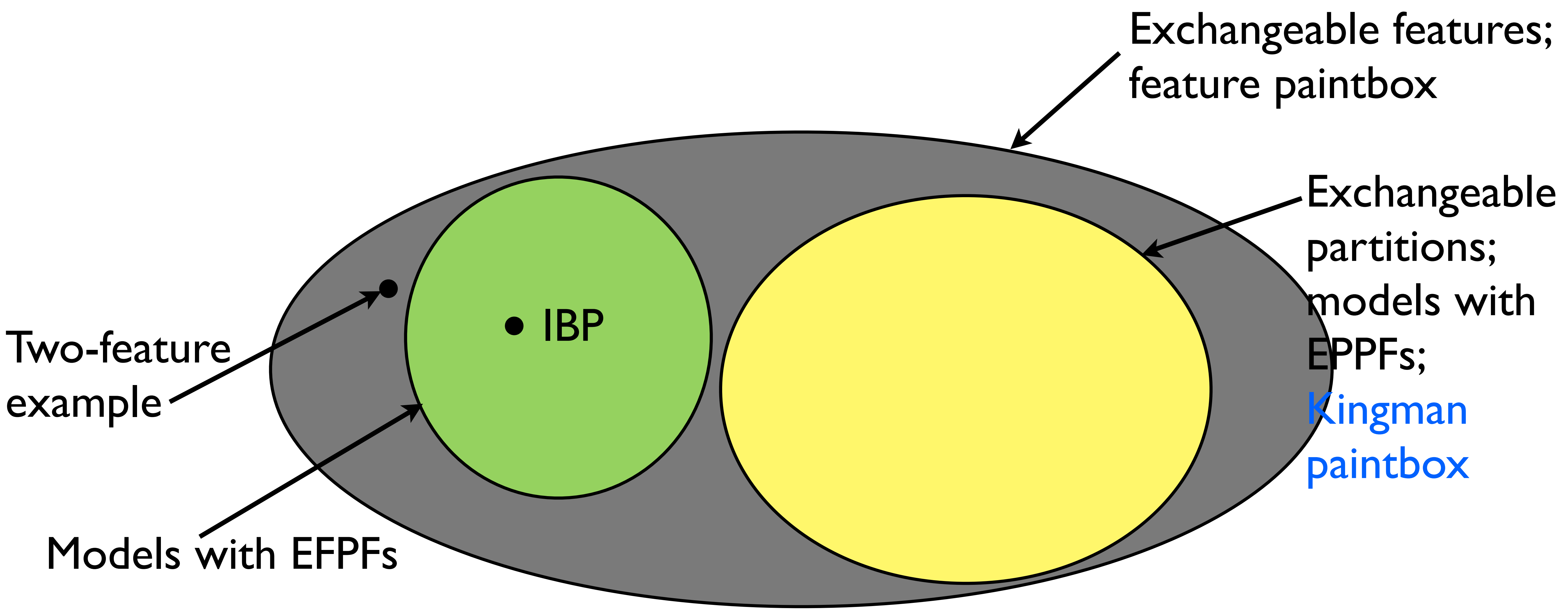

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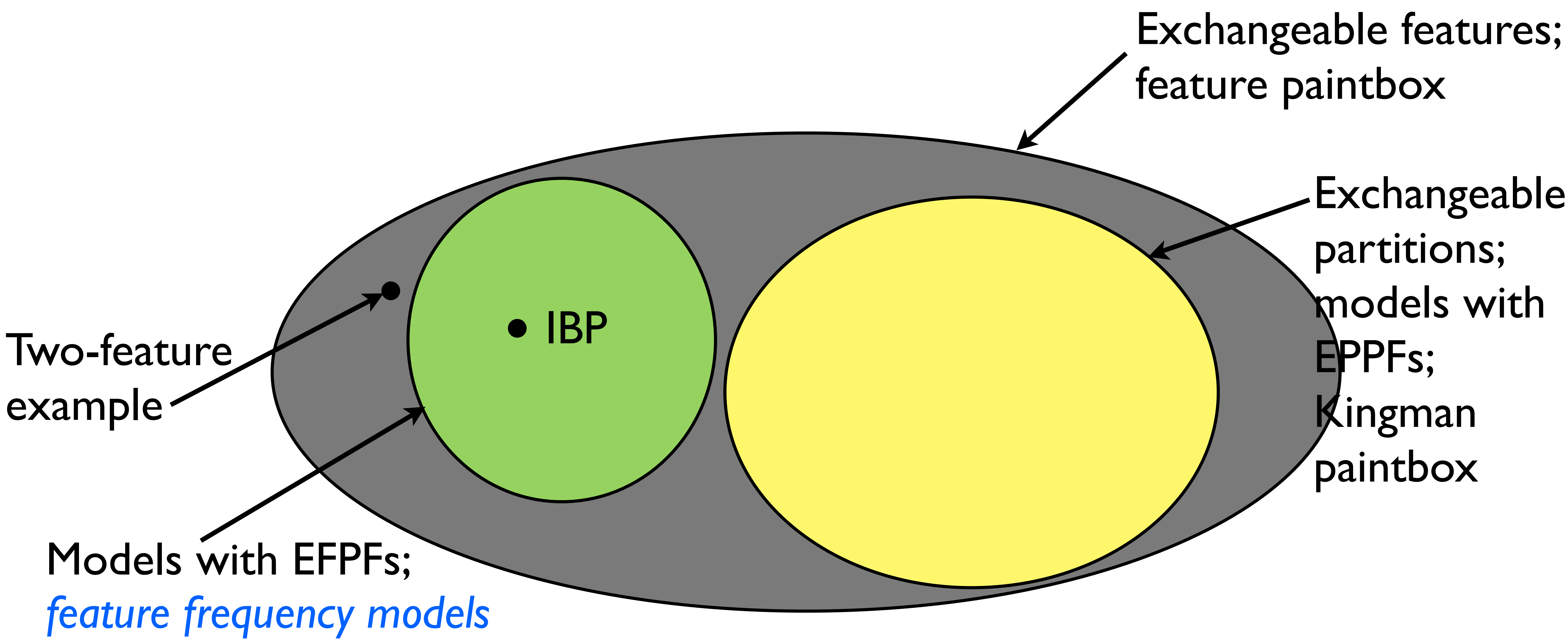

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# Further References


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