## Title

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# Feedback Control of Nonlinear Dissipative Systems by Finite Determining Parameters - A Reaction-diffusion Paradigm

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**Abstract**

We introduce here a simple finite-dimensional feedback control scheme for stabilizing solutions of infinite-dimensional dissipative evolution equations, such as reaction-diffusion systems, the Navier-Stokes equations and the Kuramoto-Sivashinsky equation. The designed feedback control scheme takes advantage of the fact that such systems possess finite number of determining parameters (degrees of freedom), namely, finite number of determining Fourier modes, determining nodes, and determining interpolants and projections. In particular, the feedback control scheme uses finitely many of such observables and controllers. This observation is of a particular interest since it implies that our approach has far more reaching applications, in particular, in data assimilation. Moreover, we emphasize that our scheme treats all kinds of the determining projections, as well as, the various dissipative equations with one unified approach. However, for the sake of simplicity we demonstrate our approach in this paper to a one-dimensional reaction-diffusion equation paradigm.

**Keywords.** Reaction-diffusion, Navier-Stokes equations, feedback control, data assimilation, determining modes, determining nodes, determining volume elements.
**Mathematics Subject Classification (2000):** 35K57, 37L25, 37L30, 37N35, 93B52, 93C20, 93D15.

# 1 Introduction

Dissipative dynamical systems, such as the Navier-Stokes equations, the Kuramoto-Sivashinsky equation, the complex Ginzburg-Landau equation and various reaction-diffusion systems are known to have a finite-dimensional asymptotic (in time) behavior (see, e.g., [6, 7, 9, 16, 24, 31, 34, 36], and references therein). This is evident due to the fact that such systems possess finite-dimensional global attractors ([3, 9, 10, 31, 34, 36]), and finite number of determining modes ([18, 17, 16, 28]), determining nodes ([16, 21, 22, 23, 26, 28, 29]), determining volume elements ([23, 27]) and other finite number of determining parameters (degrees of freedom) such as finite elements and other interpolation polynomials ([6, 7, 22].) Moreover, some of these systems, which enjoy the property of separation of spatial scales, are also known to have a finite dimensional inertial manifolds (see, e.g., [9, 10, 19, 20, 36], and references therein). That is, in the presence of separation of spatial scales the long-term dynamics of such a system is equivalent to that of a finite system of ordinary differential equations.

There has been some interesting work on reduction methods, with applications focused on scientific computing and feedback control theory, taking advantage of the finite-dimensional asymptotic behavior of these dissipative dynamical systems (see, e.g., [1, 11, 12, 25, 35] and references therein) . However, there has been very little rigorous analytical work, in particular in the context of feedback control theory, justifying these applications. In the case of separation of spatial scales, and hence the existence of inertial manifolds, the authors of [32] and [33] provide an example of finite-dimensional feedback control (lumped feedback control) that drives the dynamics of one-dimensional reaction-diffusion system to an a priori specified finite-dimensional dynamics. It is worth stressing again that in the case of inertial manifold the dynamics of the underlying evolution equation is equivalent to that of an ordinary differential equations to begin with. However, the main challenge is in being able to provide a representation of this ODE system in the relevant parameters dictated by the applications. In [23] and [7] the authors have shown that if a certain dissipative system has separation of scales, and hence an inertial manifold, then such a manifold can be parameterized by any set of adequate parameters, e.g. Fourier modes, nodal values, local volume averages, etc... In the above mentioned work of [32] and [33] the authors employed such an equivalence in the parameterization of the inertial manifolds to show their results.

In this paper we propose a new feedback control for controlling general dissipative evolution equations using any of the determining systems of parameters (modes, nodes, volume elements, etc...) without requiring the presence of separation in spatial scales, i.e. without assuming the existence of an inertial manifold. To fix ideas we demonstrate our idea for a simple reaction diffusion equation, the Chafee-Infante equation, which is the real Ginzburg-Landau equation. It is worth mentioning, however, that this new idea has a far more reaching areas of applications, other than feedback control, such as in data assimilations for weather prediction [2, 4]. In addition, one can use this approach to show that the long time-time dynamics of the underlying dissipative evolution equation, such as the two-dimensional Navier-Stokes equations, can be imbedded in an infinite-dimensional dynamical system that is induced by an ordinary differential equations, named *determining form*, which is governed by a globally Lipschitz vector field, cf. [13, 14] and [15].

In this paper we will use the Chafee-Infante reaction-diffusion equation

$$\frac{\partial u}{\partial t} - \nu\, u_{xx} - \alpha u + u^3 = 0 \tag{1}$$

$$u_x(0) = u_x(L) = 0 \tag{2}$$

for $\alpha > 0$, large enough, as a paradigm to fix ideas and to use the notions of finite number of determining modes, nodes and volume elements to design feedback control to stabilize the $\mathbf{v}(x) \equiv 0$ unstable steady state solution of (1)-(2). Indeed, by linearizing equation (1) about $\mathbf{v} \equiv 0$ one obtains the linear equation

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} - \nu \mathbf{v}_{xx} - \alpha \mathbf{v} = 0 \\ \mathbf{v}_x(0) = \mathbf{v}_x(L) = 0 \end{aligned} \tag{3}$$

We solve equation (3) with initial condition $\mathbf{v}_0(x) = A_k \cos(\frac{kx}{L}\pi)$, where $A_k \in \mathrm{I\!R}$, by seeking a solution of the form $\mathbf{v}(x,t) = a_k(t) \cos(\frac{k\,x}{L}\pi)$. Therefore, we obtain

$$\dot{a_k} + \nu a_k\, (\frac{\pi k}{L})^2 - \alpha\, a_k = 0, \quad \text{with} \quad a_k(0) = A_k;$$

and whose solution is

$$a_k(t) = A_k\, e^{\left(-\nu\,(\frac{\pi k}{L})^2 + \alpha\right) t}.$$

Therefore, for $\alpha > 0$, large enough, all the low wave numbers $k^2 < \frac{\alpha\, L^2}{\pi^2 \nu}$ are unstable. Consequently, the dimension of the unstable manifold of $\mathbf{v} \equiv 0$ behaves like $\sqrt{\frac{\alpha\, L^2}{\nu}}$ (see, for instance, [3, 24] and [36] for a similar analysis).

The aim of this paper is to design a feedback control that stabilizes $\mathbf{v} \equiv 0$, for example, either by observing the values of the solutions at certain nodal points, local averages of the solutions in subintervals of $[0, L]$, or by observing finitely many of their Fourier modes. Based on the above discussion, a naive analysis would suggest that one would need about $\sqrt{\frac{L^2 \alpha}{\nu}}$ feedback controllers to stabilize $\mathbf{v} \equiv 0$.

In this paper we will give a rigorous justification to this assertion. First, we demonstrate our result for the case of local averages, which is the most straightforward approach. Later, we present a more general abstract result, that unifies our approach, utilizing all sorts of approximate interpolant operators, as observables and controllers, and show that this abstract approach applies to the Fourier modes, local volumes (i.e. local averages) and nodal values as particular examples. It is worth mentioning that the same feedback control scheme can be used to stabilize any other time-dependent solution, $v(x,t)$, of (1)-(2). The details of the proof are similar to the ones presented here for stabilizing the zero solution; thus, for the sake of simplicity they will not be provided. Furthermore, similar scheme can be also implemented for feedback control of other nonlinear dissipative dynamical systems, such as the two-dimensional Navier-Stokes equations, the Kuramoto-Sivashinsky equation and reaction-diffusion systems. A computational study concerning the implementation of this feedback control scheme for various nonlinear dissipative equations will be reported in a forthcoming work [30]. In addition, one can design similar feedback control algorithm with stochastically noisy observables and controllers to stabilize, in the average, given solutions; within errors that are determined by the standard deviation of the noise. This can be achieved by combining some of the ideas presented in [4] with those presented in the present paper, a subject of future work.

# 2 Finite volume elements feedback control

To fix ideas we propose the following feedback control system for (1)-(2) in order to stabilize the steady state solution $\mathbf{v} \equiv 0$,

$$\frac{\partial u}{\partial t} - \nu\, u_{xx} - \alpha u + u^3 = -\mu \sum_{k=1}^{N} \overline{u}_k\, \chi_{_{J_k}}(x) \tag{4}$$

$$u_x(0) = u_x(L) = 0, \tag{5}$$

where $J_k = \left[(k-1)\frac{L}{N}, k\frac{L}{N}\right)$, for $k = 1, \ldots, N-1$, and $J_N = \left[(N-1)\frac{L}{N}, L\right]$; moreover, $\chi_{_{J_k}}(x)$ is the characteristic function of the interval $J_k$, for $k = 1, \ldots, N$, and

$$\bar{\varphi}_k = \frac{1}{|J_k|} \int_{J_k} \varphi(x)\; dx = \frac{N}{L} \int_{J_k} \varphi(x)\; dx.$$

Here, the local averages of the solution, $\overline{u}_k$, for $k = 1, ..., N$, are the observables, and they are also used as the feedback controllers in (4). It is easy to observe $\mathbf{v} \equiv 0$ is also a steady state solution for (4)-(5).
For $\varphi \in H^1([0, L])$ we define

$$\|\varphi\|_{H^1}^2 := \frac{1}{L^2} \int_0^L \varphi^2(x)\, dx + \int_0^L \varphi_x^2(x)\, dx. \tag{6}$$

Before showing that (4)-(5) globally stabilizes the steady state $\mathbf{v} \equiv 0$, one has to prove first the global existence and uniqueness of the feedback system (4)-(5). In section 4, we will show in Theorem 4.1 a result concerning global existence, uniqueness and stabilization for a general family of finite-dimensional feedback control that includes system (4)-(5) as a particular case. Therefore, we will postpone this task of proving the global existence and uniqueness until section 4, and we only show here the global stability of $\mathbf{v} \equiv 0$. This is in order to fix ideas and to demonstrate our general approach.
Assuming the global existence and uniqueness of (4)-(5), we will show in this section that every solution $u$ of (4)-(5) tends to zero, as $t \to \infty$, under specific explicit assumptions on $N, \nu, \alpha, L$ and $\mu$ (see Theorem 2.1 for details). But first we need the following proposition to prove our result. We observe that similar propositions were introduced and proved in [8, 23, 26, 27] and [28] (see also [32] and [33]). We adapt here similar ideas from [8] for our proof.

**Proposition 2.1.** *Let* $\varphi \in H^1([0,L])$ *then*

$$\|\varphi(\cdot) - \sum_{k=1}^{N} \overline{\varphi}_k \, \chi_{_{J_k}}(\cdot)\|_{L^2} \, \le \, h \, \|\varphi_x\|_{L^2} \le \, h \, \|\varphi\|_{H^1}, \tag{7}$$

*where* $h = \frac{L}{N}$. *Moreover,*

$$\|\varphi\|_{L^2}^2 \le h \, \gamma^2(\varphi) + \left(\frac{h}{2\,\pi}\right)^2 \, \|\varphi_x\|_{L^2}^2, \tag{8}$$

*where*

$$\gamma^2(\varphi) = \sum_{k=1}^{N} \overline{\varphi}_k^2.$$

**Proof.**

$$\begin{aligned} \|\varphi(\cdot) - \sum_{k=1}^{N} \overline{\varphi}_k \, \chi_{_{J_k}}(\cdot)\|_{L^2}^2 &= \int_0^L \left(\varphi(x) - \sum_{k=1}^{N} \overline{\varphi}_k \, \chi_{_{J_k}}(x)\right)^2 dx \\ &= \int_0^L \left(\varphi(x) \sum_{k=1}^{N} \chi_{_{J_k}}(x) - \sum_{k=1}^{N} \overline{\varphi}_k \, \chi_{_{J_k}}(x)\right)^2 dx, \end{aligned}$$

where in the last equality we used the fact that $\sum_{k=1}^{N} \chi_{_{J_k}}(x) \equiv 1$. Therefore,

$$\begin{aligned} \|\varphi(\cdot) - \sum_{k=1}^{N} \overline{\varphi}_k \, \chi_{_{J_k}}(\cdot)\|_{L^2}^2 &= \int_0^L \left(\sum_{k=1}^{N} (\varphi(x) - \overline{\varphi}_k \,) \, \chi_{_{J_k}}(x))\right) \left(\sum_{l=1}^{N} (\varphi(x) - \overline{\varphi}_l \,) \, \chi_{_{J_l}}(x)\right) dx \\ &= \int_0^L \sum_{k,l=1}^{N} (\varphi(x) - \overline{\varphi}_k) \, (\varphi(x) - \overline{\varphi}_l) \, \chi_{_{J_k}}(x) \, \chi_{_{J_l}}(x) dx. \end{aligned}$$

Since $\chi_{_{J_l}}(x) \, \chi_{_{J_k}}(x) \equiv \chi_{_{J_k}}(x) \delta_{kl}$, it follows from the above that

$$\begin{aligned} \|\varphi(\cdot) - \sum_{k=1}^{N} \overline{\varphi}_k \chi_{_k}(\cdot)\|_{L^2}^2 &= \int_0^L \sum_{k=1}^{N} (\varphi(x) - \overline{\varphi}_k)^2 \, \chi_{_{J_k}}(x) \, dx \\ &= \sum_{k=1}^{N} \int_{J_k} (\varphi(x) - \overline{\varphi}_k)^2 \, dx. \end{aligned} \tag{9}$$

By virtue of Poincaré inequality we have

$$\int_{J_k} (\varphi(x) - \overline{\varphi}_k)^2 \, dx \le \left(\frac{h}{2\pi}\right)^2 \int_{J_k} (\varphi'(x))^2 \, dx. \tag{10}$$

Thus, (9) and (10) imply

$$\begin{aligned} \|\varphi(\cdot) - \sum_{k=1}^{N} \overline{\varphi}_k \chi_{_{J_k}}(\cdot)\|_{L^2}^2 &\le \left(\frac{h}{2\pi}\right)^2 \sum_{k=1}^{N} \int_{J_k} (\varphi'(x))^2 \, dx \\ &= \left(\frac{h}{2\pi}\right)^2 \int_0^L (\varphi'(x))^2 \, dx, \end{aligned} \tag{11}$$

which proves inequality (7) in the Proposition 2.1.
Next, we prove inequaliy (8). From the Poincaré inequality (10) we have

$$\int_{J_k} \varphi^2(x)\,dx - \overline{\varphi}_k^2\,h \le \left(\frac{h}{2\pi}\right)^2 \int_{J_k} \left(\varphi'(x)\right)^2\,dx. \tag{12}$$

Thus, by summing over $k = 1, \dots, N$, in the above inequality we conclude inequality (8) of the Proposition 2.1.

**Theorem 2.1.** *Let $N$ and $\mu$ be large enough such that $\mu \ge \nu \left(\frac{2\pi}{h}\right)^2 > \alpha$, where $\alpha > 0$ and $h = \frac{L}{N}$. Then $\|u(t)\|_{L^2}$ tends to zero, as $t \to \infty$, for every solution $u(t)$ of (4)-(5).*

**Proof.** We take the $L^2$ inner product of equation (4) with $u$, and integrate by parts to obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \nu\,\|u_x\|_{L^2}^2 - \alpha\|u\|_{L^2}^2 + \|u\|_{L^4}^4 = -\mu\sum_{j=1}^{N}\frac{L}{N}\,\overline{u}_j^2 = -\mu\frac{L}{N}\,\gamma^2(u),$$

which implies that

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \nu\,\|u_x\|_{L^2}^2 + \mu\,h\,\gamma^2(u) - \alpha\,\|u\|_{L^2}^2 \le 0. \tag{13}$$

Using (8), from Proposition 2.1, and the assumption that $\mu \ge \nu\left(\frac{2\pi}{h}\right)^2$ we have

$$\begin{aligned}\nu\,\|u_x\|_{L^2}^2 + \mu h \gamma^2(u) = \;& \nu\,\left(\tfrac{h}{2\pi}\right)^{-2}\left(\left(\tfrac{h}{2\pi}\right)^2\|u_x\|_{L^2}^2 + h\gamma^2(u)\right)\\ &+\left(\mu h\; - \nu\tfrac{4\pi^2}{h}\right)\gamma^2(u)\\ &\ge\; \tfrac{4\pi^2\,\nu}{h^2}\,\|u\|_{L^2}^2. \end{aligned}\tag{14}$$

Substituting (14) in (13) we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + (\frac{\nu\,4\,\pi^2}{h^2} - \alpha)\,\|u\|_{L^2}^2 \le 0.$$

Therefore, by virtue of Gronwall's inequality and the assumption that $\nu > \alpha\,\frac{h^2}{4\,\pi^2}$ one obtains

$$\|u(t)\|_{L^2}^2 \;\le\; e^{-(\nu(\frac{2\,\pi\,N}{L})^2-\alpha)\,t}\|u(0)\|_{L^2}^2;$$

and the Theorem follows.

## Remark 2.1

It is worth mentioning that the assumptions of Theorem 2.1, in particular, that $N > \sqrt{\frac{L^2\,\alpha}{4\pi^2\nu}}$, is consistent with the fact that the dimension of the unstable manifold about $\mathbf{v} \equiv 0$ is of order of $\sqrt{\frac{L^2\,\alpha}{\nu}}$. That is, one needs at least this number of parameters to stabilize $\mathbf{v} \equiv 0$. In Theorem 4.1 we give a different and more general proof, that illustrates this point further. As we have mentioned earlier, one can use the same idea to stabilize any other solution, $v(x,t)$, of (1)-(2) by using a slightly modified feedback control in the right-hand side of (4)-(5) of the form $-\mu\sum_{k=1}^{N}(\overline{u}_k - \overline{v}_k)\;\chi_{J_k}(x)$.

# 3 Interpolant operators as feedback controllers

In this section we will consider a general linear map $I_h : H^1([0,L]) \to L^2([0,L])$ which is an interpolant operator that approximates identity with error of order $h$. Specifically, it approximates the inclusion map $i : H^1 \hookrightarrow L^2$, such that the estimate

$$\|\varphi - I_h(\varphi)\|_{L^2} \leq c\, h\, \|\varphi\|_{H^1}, \tag{15}$$

holds, for every $\varphi \in H^1([0,L])$. The last inequality is a version of the well-known Bramble-Hilbert inequality, that usually appears in the context of finite elements [5]. We propose here to consider the following general feedback system, to stabilize the solution $v(x,t)$ of (1)-(2), of the form

$$\frac{\partial u}{\partial t} - \nu\, u_{xx} - \alpha\, u + u^3 = -\mu\,(I_h(u) - I_h(v)), \tag{16}$$

$$u_x(0) = u_x(L) = 0. \tag{17}$$

To fix ideas we focus on stabilizing the steady state solution $\mathbf{v} \equiv 0$ of (1)-(2). Here one can think of $I_h(u)$ as the observables and controllers that will be used to stabilize our system.
Before we state and prove our general theorems concerning system (16)-(17), we will give some examples of the approximate interpolant $I_h(\varphi)$ which satisfy the approximation property (15). In particular, we are interested in interpolant operators, $I_h$, of finite-rank, and whose rank is of the order $O(1/h)$.

## 3.1 Examples of finite-rank approximate identity interpolant operators

### 3.1.1 Finite volume elements

Using the notation of section 2 we consider the interpolant operator

$$I_h(\varphi) = \sum_{j=1}^{N} \bar{\varphi}_k\, \chi_{J_k}(x), \tag{18}$$

that uses local spatial averages (finite volume elements) for approximating the local values of the underlying function. We observe that the interpolant operator, $I_h(\varphi)$, that is introduced in (18) and implemented in (16), is exactly the same one discussed in details in section 2. In particular, one can easily see that approximating inequality (15) holds in this case, thanks to Proposition 2.1.

### 3.1.2 An interpolant operator based on nodal values

In this example we consider the interpolant operator

$$I_h(\varphi) = \sum_{k=1}^{N} \varphi(x_k)\, \chi_{J_k}(x), \tag{19}$$

where $J_k$ and $\chi_{J_k}$ are as in section 2, and the points $x_k \in J_k$, for $k = 1, 2, \cdots, N$ are arbitrary. Next, we show that the interpolant operator given in (19) satisfied the approximation property (15). Here again we adopt ideas from [8, 23, 26] to prove the next proposition.

**Proposition 3.1.** *For every* $\varphi \in H^1([0, L])$

$$\|\varphi(\cdot) - \sum_{k=1}^{N} \varphi(x_k)\chi_{J_k}(\cdot)\|_{L^2} \leq h\,\|\varphi_x\|_{L^2} \leq h\,\|\varphi\|_{H^1}.$$

**Proof.**

$$\|\varphi(.) - \sum_{k=1}^{N} \varphi(x_k)\chi_{J_k}(.)\|_{L^2}^2 = \int_0^L \left(\varphi(x) - \sum_{k=1}^{N} \varphi(x_k)\chi_{J_k}(x)\right)^2 dx,$$

and since $\sum_{k=1}^{N} \chi_k(x) \equiv 1$, it follows that

$$\|\varphi(\cdot) - \sum_{k=1}^{N} \varphi(x_k)\,\chi_{J_k}(\cdot)\|_{L^2}^2 = \int_0^L \left(\sum_{k=1}^{N} (\varphi(x) - \varphi(x_k))\,\chi_{J_k}(x)\right)^2 dx.$$

As in the proof of the Proposition 2.1, we observe that $\chi_{J_k}(x)\,\chi_{J_l}(x) \equiv \chi_{J_k}(x)\,\delta_{kl}$ and then we obtain

$$\begin{aligned}
\|\varphi(\cdot) \quad - \quad & \sum_{k=1}^{N} \varphi(x_k)\chi_{J_k}(.)\|_{L^2}^2 = \int_0^L \sum_{k=1}^{N} (\varphi(x) - \varphi(x_k))^2\,\chi_{J_k}(x)\,dx \\
= \quad & \sum_{k=1}^{N} \int_{J_k} (\varphi(x) - \varphi(x_k))^2\,dx = \sum_{k=1}^{N} \int_{J_k} \left(\int_{x_k}^{x} \varphi'(y)\,dy\right)^2 dx \\
\leq \quad & \sum_{k=1}^{N} \int_{J_k} \left(\int_{J_k} |\varphi'(y)|\,dy\right)^2 dx \leq h \sum_{k=1}^{N} \left(\int_{J_k} |\varphi'(y)|\,dy\right)^2 dx.
\end{aligned}$$

Using Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\|\varphi(.) - \sum_{k=1}^{N} \varphi(x_k)\chi_{J_k}(.)\|_{L^2}^2 \quad & \leq \quad \sum_{k=1}^{N} h^2 \int_{J_k} \left|\varphi'(y)\right|^2 dy, \\
& = \quad h^2\|\varphi_x\|_{L^2}^2;
\end{aligned}$$

which concludes the proof of Proposition 3.1.
In view of (1), (16) and (19) we propose the following feedback controller for stabilizing $\mathbf{v} \equiv 0$

$$\frac{\partial u}{\partial t} - \nu\,u_{xx} - \alpha u + u^3 = -\mu \sum_{k=1}^{N} u(x_k)\,\chi_{J_k}(x) \tag{20}$$

$$u_x(0) = u_x(L) = 0, \tag{21}$$

which is a special case of (16).

#### 3.1.3 Projection onto Fourier modes as an interpolant operator

Here, we consider the following projection onto the first $N$ Fourier modes as an example of an interpolant operator;

$$I_h(\varphi) = \sum_{k=1}^{N} \hat{\varphi}_k \cos(\frac{k\,\pi x}{L}),\ \ h = \frac{L}{N}, \tag{22}$$

where the Fourier coefficients are given by

$$\hat{\varphi}_k = \frac{2}{L}\int_0^L \varphi(x)\ \cos(\frac{\pi k x}{L})dx.$$

Next, we observe that inequality (15) holds for the interpolant operator given in (22).

**Proposition 3.2.** *Let $\varphi \in H^1([-L,L])$ be an even function, i.e. $\varphi(-x) = \varphi(x)$. Then*

$$\|\varphi(x) - \sum_{k=1}^{N} \widehat{\varphi}_k \cos\left(\frac{k\,x\pi}{L}\right)\|_{L^2([0,L])} \leq c\,h\,\|\varphi_x\|_{L^2([0,L])}. \tag{23}$$

**Proof.** The proof of this proposition is a simple exercise in Fourier series. Thus, it will be omitted.

## 4 Existence, uniqueness and stabilization using the $I_h$ feedback control

In this section we establish the global existence and uniqueness for the general feedback system introduced in (16)-(17); and that the $I_h$ feedback control is stabilizing the steady state solution $v \equiv 0$ of the (1)-(2). This will be accomplished under the assumptions (15) and that $\mu$ is large enough, and $h$ is small enough, satisfying:

$$\mu \geq (2\alpha + 3\nu L^{-2}) \qquad \text{and} \quad \nu \geq \mu\, c^2\, h^2. \tag{24}$$

To this end one uses the standard Galerkin approximation procedure based on the eigenfunctions of the Laplacian, subject to the Neumann boundary condition, i.e., $\cos(\frac{\pi\, k\, x}{L})$ for $k = 1, 2....$ We will omit the details of this standard procedure and provide only the formal *a-priori* estimates (see, e.g., [36]). These estimates can be obtained rigorously through the Galerkin procedure, by passing to the limit while using the relevant compactness theorems.
Let us now establish the aformentioned formal *a-priori* bounds for the solution which are essential for guaranteeing globlal existence and uniqueness.
System (16)-(17) can be rewritten as

$$\frac{\partial u}{\partial t} - \nu\, u_{xx} + \frac{\nu}{L^2}u - (\alpha + \frac{\nu}{L^2})\, u = -u^3 - \mu I_h(u) \tag{25}$$

$$u_x(0) = u_x(L) = 0. \tag{26}$$

Taking the $L^2$- inner product of (25) with $u$, integrating by parts and using the Neumann boundary conditions, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^L u^2\,dx \quad + \quad \nu\int_0^L u_x^2\,dx + \frac{\nu}{L^2}\int_0^L u^2\,dx = -\int_0^L u^4 dx$$
$$+(\alpha + \frac{\nu}{L^2})\int_0^L u^2\,dx - \mu\int_0^L I_h(u)\,u\,dx.$$

Writing

$$I_h(u)\,u = (I_h(u) - u)\,u + u^2$$

and applying the Cauchy-Schwarz inequality, we get

$$\frac{1}{2}\frac{d}{dt}\int_0^L u^2\,dx + \nu\int_0^L u_x^2\,dx + \frac{\nu}{L^2}\int_0^L u^2\,dx \le -\int_0^L u^4 dx + (\alpha + \frac{\nu}{L^2})\|u\|_{L^2}^2$$
$$-\,\mu\int_0^L u^2\,dx + \mu\left(\int_0^L u^2\,dx\right)^{\frac{1}{2}}\left(\int_0^L |u - I_h(u)|^2\,dx\right)^{\frac{1}{2}}.$$

Using Young's inequality we reach

$$\frac{1}{2}\frac{d}{dt}\int_0^L u^2\,dx + \nu\int_0^L u_x^2\,dx \quad + \quad \frac{\nu}{L^2}\int_0^L u^2\,dx \le -\int_0^L u^4 dx + (\alpha + \frac{\nu}{L^2})\|u\|_{L^2}^2$$
$$- \quad \frac{\mu}{2}\int_0^L |u|^2\,dx + \frac{\mu}{2}\|u - I_h(u)\|_{L^2}^2\,dx.$$

Using (15), and the definition of the $H^1$-norm given in (6) we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^L u^2\,dx + \nu\int_0^L u_x^2\,dx + \frac{\nu}{L^2}\int_0^L u^2\,dx \le -\int_0^L u^4 dx + (\alpha + \frac{\nu}{L^2})\|u\|_{L^2}^2$$
$$-\frac{\mu}{2}\int_0^L |u|^2\,dx + \mu\,\frac{c^2\,h^2}{2}\left(\frac{1}{L^2}\int_0^L u^2\,dx + \int_0^L u_x^2\,dx\right).$$

Thanks to the assumption (24) we conclude

$$\frac{d}{dt}\|u\|_{L^2}^2 + \nu\,(\|u_x\|_{L^2}^2 + \frac{1}{L^2}\,\|u\|_{L^2}^2)\ \le 0. \tag{27}$$

Therefore, by dropping the $\|u_x\|_{L^2}^2$ term from the left-hand side of (27) and applying Gronwall's inequality we have

$$\|u(t)\|_{L^2}^2 \le e^{\frac{-\nu\,t}{L^2}}\,\|u(0)\|_{L^2}^2 =: K_0(t). \tag{28}$$

Notice that from (27) one also concludes that for every $\tau > 0$

$$\nu\int_t^{t+\tau}(\|u_x(s)\|_{L^2}^2 + \frac{1}{L^2}\,\|u(s)\|_{L^2}^2)\,ds\ \le \|u(t)\|_{L^2}^2. \tag{29}$$

Next, we show the continuous dependence of the solutions of (16) on the initial data and the uniqueness, provided the assumptions (15) and (24) hold. Indeed, let $u_1, u_2$ be two solutions and $w = u_1 - u_2$ of (16). From (16) we find that

$$\frac{\partial w}{\partial t} - \nu w_{xx} - \alpha\,w = u_2^3 - u_1^3 - \mu\,I_h(w).$$

Multiplying by $w$ and integrating with respect to $x$ over $[0, L]$ we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^L w^2\,dx + \nu\int_0^L w_x^2\,dx &= \alpha\int_0^L w^2\,dx - \int_0^L w^2\,\frac{(u_1+u_2)^2+u_1^2+u_2^2}{2}\,dx\\ &\quad -\mu\int_0^L I_h(w)\,w\;dx\\ &\leq (\alpha-\mu)\int_0^L w^2\,dx + \mu\int_0^L |I_h(w)-w|\,|w|\,dx.\end{aligned}$$

A straightforward computation, using the Cauchy-Schwarz and Young inequalities and assumption (15), yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^L w^2\,dx + \nu\int_0^L w_x^2\,dx &\leq (\alpha-\mu)\int_0^L w^2\,dx + \mu\,\|I_h(w)-w\|_{L^2}\,\|w\|_{L^2}\\ &\leq (\alpha-\mu)\,\|w\|_{L^2}^2\,dx + \frac{\mu}{2}\,\|I_h(w)-w\|_{L^2}^2 + \frac{\mu}{2}\|w\|_{L^2}^2\\ &\leq (\alpha-\frac{\mu}{2})\,\|w\|_{L^2}^2 + \frac{\mu}{2}\,c^2\,h^2\|w\|_{H^1}^2.\end{aligned}$$

Using (6), the definition of the $H^1$-norm, we reach

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^L w^2\,dx + \nu\int_0^L w_x^2\,dx &\leq \left(\alpha-\frac{\mu}{2}\right)\,\|w\|_{L^2}^2 + \frac{\mu}{2}\,c^2\,h^2\left(\frac{\|w\|_{L^2}^2}{L^2} + \|w_x\|_{L^2}^2\right)\\ &\leq \left(\alpha-\frac{\mu}{2}+\frac{\mu}{2}\,\frac{c^2\,h^2}{L^2}\right)\,\|w\|_{L^2}^2 + \frac{\mu}{2}\,c^2\,h^2\,\|w_x\|_{L^2}^2.\end{aligned}$$

By assumption (24) the above implies

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 + \frac{\nu}{2}\,\|w_x\|_{L^2}^2 \leq \left(\alpha-\frac{\mu}{2}+\frac{\nu}{2L^2}\right)\,\|w\|_{L^2}^2.$$

Using assumption (24) one more time the above inequality simplifies to

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 + \frac{\nu}{2}\,\|w_x\|_{L^2}^2 \leq -\frac{\nu}{L^2}\|w\|_{L^2}^2.$$

Finally, by Gronwall's inequality we have

$$\|w(t)\|_{L^2}^2 \leq e^{-\frac{\nu t}{L^2}}\,\|w(0)\|_{L^2}^2. \tag{30}$$

Thus, if $w(0) = 0$ then $\|w(t)\|_{L^2} \equiv 0$. Moreover, inequaliy (30) implies the continuous dependence of the solutions of (16)-(17) on the initial data. In conclusion, from the above, and in particular thanks to (28) and (29), we have the following theorem:

**Theorem 4.1.** *Let $\mu, \nu$ and $h$ be positive parameters satisfying assumption (24); and that $I_h$ satisfies (15). Suppose $T > 0$ and $u_0 \in L^2([0, L])$, then system (16)-(17) has a unique solution $u \in C([0, T], L^2) \cap L^2([0, T], H^1)$ which also depends continuously on the initial data. Moreover,*

$$\lim_{t\to\infty}\|u(t)\|_{L^2}^2 = 0;$$

*and for every $\tau > 0$*

$$\lim_{t\to\infty} \int_t^{t+\tau} \|u_x(s)\|_{L^2}^2 \, ds = 0.$$

*In particular, we concluded the under that above assumption the feedback control interpolant operator $I_h$ is stabilizing the steady state solution $v \equiv 0$ of* (1)-(2).

### Remark 4.1

Let us observe that in order to satisfy assumption (24) one can choose, for small values of $\nu$, $\mu = O(\alpha)$. As a result, assumption (24) will hold if we choose $h$ small enough such that $N := \frac{L}{h} = O(\sqrt{\frac{\alpha L^2}{\nu}})$, that is the number of feedback controllers is comparable to the dimension of the unstable manifold about $\mathbf{v} \equiv 0$. This is consistent with our earlier observation in the introduction and in Remark 2.1.

## 5 Stabilizing in the $H^1$-norm

In the previous section we have shown that the feedback system (16)-(17) stabilzes the steady state solution $\mathbf{v} \equiv 0$ in the $L^2$-norm, i.e., $\|u\|_{L^2} \to 0$, as $t \to \infty$, provided assumptions (24) holds. Next, we show that we also have $\|u(t)\|_{H^1} \to 0$, as $t \to \infty$. To this end it is enough to show that $\|u_x\|_{L^2} \to 0$, as $t \to \infty$.
Let us rewrite (16)-(17) as

$$u_t + \frac{\nu}{L^2}\, u - \nu\, u_{xx} - (\alpha + \frac{\nu}{L^2})\, u + u^3 = -\mu\, I_h(u) \tag{31}$$

$$u_x(0) = u_x(L) = 0. \tag{32}$$

Thanks to the estimate (29) we realize that the solution instantaneously becomes in $H^1$. Therefore, without loss of generality we can assume that the initial data $u_0 \in H^1$. Below we provide the formal arguments and estimates, which, as we have already indicated earlier, can be established rigorously by using a Galerkin approximation procedure. We take the $L^2$ inner product of (31) with $-u_{xx}$, integrating by parts, and using the Neumann boundary conditions (17) we obtain:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_x\|_{L^2}^2 &+ \nu\,\|u_{xx}\|_{L^2}^2 + \frac{\nu}{L^2}\,\|u_x\|_{L^2}^2 - (\alpha + \frac{\nu}{L^2})\|u_x\|^2 \\
&= \int_0^L u^3\, u_{xx}\, dx + \mu \int_0^L I_h(u)\, u_{xx}\, dx \\
&= -3\int_0^L u^2\, u_x^2\, dx + \mu \int_0^L (I_h(u) - u)\, u_{xx}\, dx + \mu \int_0^L u\, u_{xx}\, dx \\
&= -3\int_0^L u^2\, u_x^2\, dx + \mu \int_0^L (I_h(u) - u)\, u_{xx}\, dx - \mu \int_0^L u_x^2\, dx.
\end{aligned}$$

By Cauchy-Schwarz inequality we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_x\|_{L^2}^2 + \nu\,\|u_{xx}\|_{L^2}^2 + \frac{\nu}{L^2}\|u_x\|_{L^2}^2 &\le (\alpha + \frac{\nu}{L^2})\,\|u_x\|_{L^2}^2 - \mu\,\|u_x\|_{L^2}^2 \\
&+ \mu\,\|I_h(u) - u)\|_{L^2}\,\|u_{xx}\|_{L^2}.
\end{aligned}$$

Applying Young's inequality we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_x\|_{L^2}^2+\nu\,\|u_{xx}\|_{L^2}^2+\frac{\nu}{L^2}\|u_x\|_{L^2}^2 \le (\alpha+\frac{\nu}{L^2}-\mu)\|u_x\|^2+\frac{\nu}{2}\,\|u_{xx}\|_{L^2}^2+\frac{\mu^2}{2\,\nu}\,\|I_h(u)-u)\|_{L^2}^2.$$

Using property (15) and the definition of the $H^1-$norm in (6), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_x\|_{L^2}^2\,dx+\frac{\nu}{2}\,\|u_{xx}\|_{L^2}^2+\frac{\nu}{L^2}\|u_x\|_{L^2}^2 &\le (\alpha+\frac{\nu}{L^2}-\mu)\|u_x\|_{L^2}^2\\ &+ \frac{h^2\,\mu^2\,c^2}{2\,\nu}\left(\|u_x\|_{L^2}^2+\frac{1}{L^2}\|u\|_{L^2}^2\right).\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_x\|_{L^2}^2\,dx+\frac{\nu}{2}\,\|u_{xx}\|_{L^2}^2+\frac{\nu}{L^2}\|u_x\|_{L^2}^2 &\le (\alpha+\frac{\nu}{L^2}+\frac{h^2\,\mu^2\,c^2}{2\,\nu}-\mu)\|u_x\|_{L^2}^2\\ &+ \frac{h^2\,\mu^2\,c^2}{2\,\nu\,L^2}\,\|u\|_{L^2}^2.\end{aligned}$$

Thanks to assumption (24) we observe that

$$\frac{h^2\,\mu^2\,c^2}{2\,\nu}\le\frac{\mu}{2}\quad\text{and}\quad(\alpha+\frac{\nu}{L^2}+\frac{h^2\,\mu^2\,c^2}{2\,\nu}-\mu)\le-\frac{\nu}{2\,L^2}.$$

Therefore, the above implies

$$\frac{1}{2}\frac{d}{dt}\|u_x\|_{L^2}^2+\frac{3\,\nu}{2\,L^2}\|u_x\|_{L^2}^2\le\frac{\mu}{2}\,\|u\|_{L^2}^2.$$

Since $\lim_{t\to\infty}\|u(t)\|^2=0$, then by Gronwall's inequality it is easy to show that $\|u_x\|_{L^2}^2\to 0$, as $t\to\infty$, (see also special Gronwall's type Lemma in [26]).

# 6 Nodal observables and feedback controllers

In this section we propose a different feedback control based on nodal value observables and feedback controllers. Assume that the observables are the values of the solutions $u(\overline{x}_k)$, at the points $\overline{x}_k\in J_k=[(k-1)\,\frac{L}{N},k\,\frac{L}{N}]$, $k=1,...,N$, and that the feedback is at some points $x_k\in J_k$, $x_k$ is not necessarily the same as $\overline{x}_k$. That is the measurements are made at $\overline{x}_k$, while the feedback controllers are at $x_k$, for $k=1,2...,N$. To avoid technical issues that are dealing with boundary conditions, we focus here on the periodic boundary condition case. In this case the feedback system will read

$$\frac{\partial u}{\partial t}-\nu\,u_{xx}-\alpha u+u^3=-\mu\,\sum_{k=1}^{N}h\,u(\overline{x}_k)\,\delta(x-x_k),\tag{33}$$

$$u(x,t)=u(x+L,t),\tag{34}$$

where $h=\frac{L}{N}$; and $\delta(x-a)\in H^{-1}_{per}([0,L])$, for $a\in[0,L]$, and is extended periodically such that

$$<\delta(\cdot-a),\varphi>=\varphi(a)\tag{35}$$

for every $\varphi \in H^1_{per}([0,L])$.
The feedback control proposed in (33) -(34) is different than that of (16)-(17), since the right-hand side in (33) is a distribution that belongs to $H^{-1}_{per}([0,L])$, while the right-hand side in (16) belongs to $L^2([0,L])$.
In this section we will show that, under similar assumptions to those in Theorem 4.1, the proposed feedback system (33) stabilizes the steady state $\mathbf{v} \equiv 0$ in the $L^2-$norm. One should not expect here a stronger statement, as the one stated in section 5, in which the stabilizing is also valid in the $H^1-$norm. This is because the solutions of (33)-(34) are weaker than those of (16)-(17), since the right-hand side in (33) is less regular than its counterpart in (16).
As before, we will show, below, the formal steps, which demonstrate simultaneously the global existence, uniqueness and stabilizing effect. These formal steps and estimates can be justified rigorously by implementing the Galerkin procedure based on the eigenfunction of the Laplacian, subject to periodic boundary conditions, with period $L$ (see, e.g., [36]). First, let us prove the following Lemma, which is basically the embedding of the Hölder space of $C^{\frac{1}{2}} \subset H^1$ (see, e.g., [8, 36]).

**Lemma 6.1.** *Let $x_k, \overline{x}_k \in J_k = [(k-1)\,h, k\,h], k = 1,..,N$, where $h = \frac{L}{N}$, $N \in I\!N$. Then for every $\varphi \in H^1([0,L])$ we have*

$$\sum_{k=1}^{N} |\varphi(x_k) - \varphi(\overline{x}_k)|^2 \le h\, \|\varphi_x\|^2_{L^2}, \tag{36}$$

*and*

$$\|\varphi\|^2_{L^2} \le 2\left[h \sum_{k=1}^{N} |\varphi(x_k)|^2 + h^2\, \|\varphi_x\|^2_{L^2}\right]. \tag{37}$$

**Proof.** We prove inequality (36) for $\varphi \in C^1([0,L])$, and by the density of $C^1 \subset H^1$ the result follows for every $\varphi \in H^1$.

$$\begin{aligned} |\varphi(x_k) - \varphi(\overline{x}_k)|^2 &\le \left|\int_{\overline{x}_k}^{x_k} \varphi'(s)\, ds\right|^2 \le \left(\int_{J_k} |\varphi'(s)|\, ds\right)^2 \\ &\le |J_k| \int_{J_k} |\varphi'(s)|^2\, ds = h \int_{J_k} |\varphi'(s)|^2\, ds. \end{aligned}$$

By summing the above inequality over $k = 1,..,N$ we conclude (36).
To prove (37) we observe that for every $x \in J_k$ we have

$$|\varphi(x)| \le |\varphi(x_k)| + \int_{J_k} |\varphi'(s)|\, ds.$$

Thus

$$|\varphi(x)|^2 \le 2\left[|\varphi(x_k)|^2 + \left(\int_{J_k} |\varphi'(s)|\, ds\right)^2\right], \tag{38}$$

and by integrating with respect to $x$ over $J_k$, and using the Cauchy-Schwarz inequality, we obtain

$$\int_{J_k} |\varphi(x)|^2\, dx \le 2\,h\left[|\varphi(x_k)|^2 + h \int_{J_k} |\varphi'(s)|^2\, ds\right]. \tag{39}$$

Now we conclude (37) by summing over $k = 1, \dots, N$.

**Theorem 6.1.** *Let $\mu > 4\,\alpha$ and $h$ is small enough such that $\nu \geq 2\,\mu\,h^2$. Then for every $T > 0$, and every $u_0 \in L^2_{per}[0, L]$ system (33) has a unique solution*

$$u \in C([0,T]; L^2_{per}[0,L]) \cap L^2\left([0,T]; H^1_{per}[\,0,L]\right) \cap L^4\left([0,T]; L^4_{per}[0,L]\right),$$

*and*

$$\frac{\partial u}{\partial t} \in L^2\left([0,T]; H^{-1}_{per}\right).$$

*Moreover,*

$$\lim_{t\to\infty} \|u(t)\|_{L^2} = 0, \quad \textit{and for every} \quad \tau > 0 \quad \lim_{t\to\infty} \int_t^{t+\tau} \|u_x(s)\|^2_{L^2}\, ds = 0 \quad (40)$$

**Proof.** We will show here only the relevant *a priori* estimates. The rest of the regularity results are standard for nonlinear parabolic equations (see, e.g., [36]).
We take the $H^{-1}$ action of (33) on $u \in H^1$, and use Lemma of Lions-Magenes (cf. Chap. III-p.169, [37]), to obtain:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2} \;&+\; \nu\,\|u_x\|^2_{L^2} = \alpha\,\|u\|^2_{L^2} - \int_0^L u^4\,dx - \mu\,h \sum_{k=1}^N u(\overline{x}_k)\,u(x_k) \\
&= \alpha\,\|u\|^2_{L^2} - \int_0^L u^4\,dx - \mu\,h \sum_{k=1}^N |u(x_k)|^2 + \mu\,h \sum_{k=1}^N \left(u(x_k) - u(\overline{x}_k)\right)\,u(x_k) \\
&\leq \alpha\,\|u\|^2_{L^2} - \int_0^L u^4\,dx - \frac{\mu}{2}\,h \sum_{k=1}^N |u(x_k)|^2 + \frac{\mu}{2}\,h \sum_{k=1}^N |u(x_k) - u(\overline{x}_k)|^2,
\end{aligned}$$

where in the last step we applied the Young's inequality. Next, we apply (36) and (37) to the right-hand side

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2} + \nu\,\|u_x\|^2_{L^2} \leq \alpha\,\|u\|^2_{L^2} - \int_0^L u^4\,dx - \frac{\mu}{4}\,\|u\|^2_{L^2} + \mu\,h^2\,\|u_x\|^2_{L^2}.$$

Hence

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2} + (\nu - \mu\,h^2)\,\|u_x\|^2_{L^2} \leq (\alpha - \frac{\mu}{4})\,\|u\|^2_{L^2} - \int_0^L u^4\,dx.$$

Since $\nu \geq 2\mu\,h^2$ and $4\alpha < \mu$ we conclude:

$$\frac{d}{dt}\|u\|^2_{L^2} + \nu\,\|u_x\|^2_{L^2} + (\frac{\mu}{2} - 2\alpha)\,\|u\|^2_{L^2} \leq 0.$$

Thanks to Gronwall's inequality we conclude from the above (40), and the regularity of the solutions as stated in the theorem.

Next, we prove the uniqueness of solutions. Let $u_1$ and $u_2$ be any two solutions. Denote by $w = u_1 - u_2$. Then $w$ satisfies

$$\frac{\partial w}{\partial t} - \nu\,w_{xx} - \alpha\,w + (u_1^2 + u_1\,u_2 + u_2^2)\,w = -\mu\,h \sum_{k=1}^N w(\overline{x}_k)\,\delta(x - x_k).$$

Taking the $H^{-1}$ action on $w \in H^1$, and using again Lemma of Lions-Magenes (cf. Chap. III-p.169, [37]), we obtain

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2+\nu\,\|w_x\|_{L^2}^2-\alpha\,\|w\|_{L^2}^2=-\int_0^L (u_1^2+u_1\,u_2+u_2^2)\,w^2\,dx-\mu\,h\sum_{k=1}^{N} w(\overline{x}_k)\,w(x_k).$$

Since $\int_0^L (u_1^2+u_1\,u_2+u_2^2)\,w^2\,dx = \int_0^L \left(\frac{u_1^2+u_2^2}{2}+\left(\frac{u_1+u_2}{2}\right)^2\right) w^2\,dx \geq 0$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2+\nu\,\|w_x\|_{L^2}^2-\alpha\,\|w\|_{L^2}^2 \leq -\mu\,h\sum_{k=1}^{N} w(\overline{x}_k)\,w(x_k).$$

Next, we follow the same steps as in the beginning of the proof to obtain

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2+(\nu-\mu\,h^2)\,\|w_x\|_{L^2}^2 \leq (\alpha-\frac{\mu}{4})\,\|w\|_{L^2}^2.$$

Since $\nu \geq 2\mu\,h^2$ and $\mu > 4\,\alpha$ we conclude, thanks to Gronwall's inequality,

$$\|w(t)\|_{L^2}^2 \leq e^{(\alpha-\mu/4)\,t}\|w(0)\|_{L^2}^2. \tag{41}$$

Notice that (41) implies the uniqueness of the solutions and their continuous dependence on the initia data.

## Remark 6.1

Here again we observe that for small values of $\nu$ by choosing $\mu = O(\alpha)$ then the condition of the theorem imply that $N := \frac{L}{h} = O(\sqrt{\frac{\alpha}{\nu}})$; which is comparable to the dimension of the unstable manifold about the steady state $\mathbf{v} \equiv 0$.

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# References

[1] A. Armaou and P. D. Christofides, *Feedback control of the Kuramoto-Sivashinsky equation*, Physica D 137(1-2), pp. 49–61 (2000).

[2] A. Azouani, E. Olson and E.S. Titi, *Continuous data assimilation using general interpolant observables*, 24(2), pp. 277–304 (2014).

[3] A. V. Babin and M. Vishik, *Attractors of evolutionary partial differential equations*, North-Holland, Amsterdam, London, NewYork, Tokyo, (1992).

[4] H. Bessaih, E. Olson and E.S. Titi, *Continuous assimilation of data with stochastic noise*, Preprint.

[5] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, Classics in Applied Mathematics, 40, SIAM, (2002).

[6] B. Cockburn, D.A. Jones, E.S. Titi, *Degrés de liberté déterminants pour équations non linéaires dissipatives*, C.R. Acad. Sci.-Paris, Sér. I 321, pp. 563–568 (1995).

[7] B. Cockburn, D.A. Jones and E.S., Titi, *Estimating the number of asymptotic degrees of freedom for nonlinear dissipative systems*, Math. Comput. 97, pp. 1073–1087 (1997).

[8] P. Constantin, Ch. Doering and E.S. Titi, *Rigorous estimates of small scales in turbulent flows*, Journal of Mathematical Physics 37, pp. 6152-6156 (1996).

[9] P. Constantin and C. Foias, *Navier-Stokes Equations*, University of Chicago Press, Chicago, (1988).

[10] P. Constantin, C. Foias, B. Nicolaenko and R. Temam, *Integral Manifolds and Inertial Manifolds for Dissipative Partial Differential Equations*, Springer-Verlag, Applies Mathematical Sciences Series, Vol. 70 (1988).

[11] N. H. El-Farra, A. Armaou and P.D. Christofides, *Analysis and control of parabolic PDE systems with input constraints*, Automatica 39(4), pp. 715–725 (2003).

[12] C. Foias, M.S. Jolly, I.G. Kevrekidis, G.R. Sell and E.S. Titi, *On the computation of inertial manifolds*, Physics Letters A 131, pp. 433–436 (1988).

[13] C. Foias, M. Jolly and R. Karavchenko, *Determining forms for the Kuramoto-Sivashinsky and Lorenz equations: analysis and computations*, (in preparation).

[14] C. Foias, M. Jolly, R. Kravchenko and E.S. Titi, *A determining form for the 2D Navier-Stokes equations - the Fourier modes case*, Journal of Mathematical Physics 53, pp. 115623-1–115623-30 (2012).

[15] C. Foias, M. Jolly, R. Karavchenko and E.S. Titi, *A unified approach to determining forms for the 2D Navier-Stokes equations – the general interpolants case*, Russian Mathematical Surveys, (2013), (to appear). arXiv:1309.0247

[16] C. Foias, O.P. Manley, R. Rosa, R. Temam, *Navier-Stokes Equations and Turbulence*, Cambridge University Press, 2001.

[17] C. Foias, O.P. Manley, R. Temam and Y. Treve, *Asymptotic analysis of the Navier-Stokes equations*, Physica D 9, pp. 157–188 (1983).

[18] C. Foias, G. Prodi, *Sur le comportement global des solutions non stationnaires des équations de Navier-Stokes en dimension deux*, Rend. Sem. Mat. Univ. Padova 39, pp. 1–34 (1967).

[19] C. Foias, G.R. Sell and R. Temam, *Inertial manifolds for nonlinear evolutionary equations*, Journal of Differential Equations 73, pp. 309–353 (1988).

[20] C. Foias, G.R. Sell and E.S. Titi, *Exponential tracking and approximation of inertial manifolds for dissipative nonlinear equations*, Journal of Dynamics and Differential Equations 1, pp. 199–244 (1989).

[21] C. Foias and R. Temam, *Determination of the solutions of the Navier-Stokes equations by a set of nodal values*, Math. Comput. 43, pp. 117–133 (1984).

[22] C. Foias, R. Temam, *Asymptotic numerical analysis for the Navier-Stokes equations*, Nonlinear Dynamics and Turbulence, Edit. by Barenblatt, Iooss, Joseph, Boston: Pitman Advanced Pub. Prog., (1983).

[23] C. Foias and E.S. Titi, *Determining nodes, finite difference schemes and inertial manifolds*, Nonlinearity 4, pp. 135–153 (1991).

[24] J. K. Hale, *Asymptotic behavior of dissipative Systems, Math. Survey and Monographs*, 25, AMS, Providence, R. I. (1988).

[25] M.S. Jolly, and I.G. Kevrekidis and E.S. Titi, *Approximate inertial manifolds for the Kuramoto–Sivashinsky equation: Analysis and Computations*, Physica D 44, pp. 38–60 (1990).

[26] D. Jones and E.S. Titi, *On the number of determining nodes for the 2-D Navier-Stokes equations*, J. Math. Anal. Appl. 168(1), pp. 72–88 (1992).

[27] D. Jones and E.S. Titi, *Determining finite volume elements for the 2-D Navier–Stokes equations*, Physica D 60, pp. 165–174 (1992).

[28] D. Jones and E.S. Titi, *Upper bounds on the number of determining modes, nodes, and volume elements for the Navier-Stokes equations*, Indiana University Mathematics Journal 42, pp. 875-887 (1993).

[29] I. Kukavica, *On the number of determining nodes for the Ginzburg-Landau equation*, Nonlinearity 5, pp. 997–1006 (1992).

[30] E. Lunasin and E.S. Titi, *Finite determining parameters feedback control for distributed nonlinear dissipative systems - a computational study*, (in preparation).

[31] J. Robinson, *Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*, Cambridge Texts in Applied Mathematics, (2001).

[32] R. Rosa *Exact finite-dimensional feedback control via inertial manifold theory with application to the Chafee-Infante equation*, J. Dynamics and Diff. Eqs. 15(1), pp. 61–86 (2003).

[33] R. Rosa and R. Temam, *Finite-dimensional feedback control of a scalar reaction-diffusion equation via inertial manifold theory*, in Foundations of Computational Mathematics, Selected papers of a conference held at IMPA, Rio de Janeiro, RJ, Brazil, January 1997 (Eds. F. Cucker and M. Shub), pp. 382–391, Springer-Verlag, Berlin (1997).

[34] G.R. Sell, Y. You, *Dynamics of Evolutionary Equations*, Springer, 2002.

[35] S. Shvartsman, C. Theodoropoulos, R. Rico-Martinez, I.G. Kevrekidis, E.S. Titi, and T.J. Mountziares, *Order reduction of nonlinear dynamic models for distributed reacting systems*, Journal of Process Control 10, pp. 177–184 (2000).

[36] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, New York: Springer (2nd augmented edition, 1997).

[37] R. Temam, *Navier-Stokes Equations: Theory and Numerical analysis*, AMS Chelsea Publishing, Providence, RI, Theory and numerical analysis, Reprint of the 1984 edition. MR 1846644 (2001).