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## Feedback Limitations in Nonlinear Systems: From Bode Integrals to Cheap Control

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**Abstract**—Feedback limitations of nonlinear systems are investigated using the cheap control approach. The main result is that in the limit, when the control effort is free, the smallest achievable  $L_2$  norm of the output is equal to the least amount of control energy ( $L_2$  norm) needed to stabilize the unstable zero dynamics. This nonlinear result is structurally similar to an earlier linear result by Qiu and Davison (1993), which, in turn, is connected with a Bode-type integral derived by Middleton (1991).

**Index Terms**—Cheap optimal control, nonlinear systems, nonminimum phase systems, performance limitations, zero dynamics.

### I. INTRODUCTION

In this paper we investigate and quantify obstacles limiting the performance of nonlinear systems. We clarify which structural properties of the nonlinear system prevent any feedback controller from reducing to zero the  $L_2$  norm of the output.

For linear systems such questions have been answered by classical and modern results, extensively reviewed in a recent monograph [1].

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However, most of these linear results are formulated in the frequency domain and are not directly applicable to nonlinear systems.

Feedback limitations of linear systems have also been approached via "cheap" optimal control, which we now extend to nonlinear systems. The key idea is that an optimal controller will be close to its *ideal performance* if the control effort is cheap and will attain it if the control effort is free. The penalty on the control effort is thus scaled by a small parameter  $\varepsilon > 0$  and the ideal performance is evaluated in the limit as  $\varepsilon \rightarrow 0$ .

Although the linear cheap control problem is well understood [2], [3], we revisit it in Section II for two reasons. First, we use it to introduce a singular perturbation technique which we subsequently apply to the nonlinear problem. This parallel development makes the structural similarities of the two problems more apparent. Second, we point out to a heretofore unnoticed connection between Bode-type integrals and linear cheap control. Specifically, we show that a  $T$ -integral formula derived by Middleton [4] reduces to a limiting cheap control formula derived by Qiu and Davison [3]. The new insight gained is that the value of the  $T$ -integral is, in fact, the least amount of energy ( $L_2$  norm) required to stabilize the unstable zeros of the linear system.

Because of this insight, the nonlinear cheap control analysis in Section III focuses on the unstable zero dynamics—the nonlinear counterpart of nonminimum-phase (NMP) zeros. For simplicity, we assume that the zero dynamics are *antistable* (asymptotically stable in reverse time) and affinely driven by the output and that the relative degree of the nonlinear system is one. In this way we avoid complicated derivations and gain in clarity. Our main result is that in the limit, as  $\varepsilon \rightarrow 0$ , the cheap control problem reduces to the minimum energy ( $L_2$  norm) problem in which the system output is used to stabilize the nonlinear zero dynamics. This result is a nonlinear counterpart of the Qiu–Davison formula, through which it is also connected with Middleton's integral formula.

### II. FEEDBACK LIMITATIONS IN LINEAR SYSTEMS

#### A. Cheap Optimal Control

The cheap control problem for a stabilizable and detectable linear time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), & x \in R^n, & \quad u \in R^m \\ y(t) &= Cx(t), & y \in R^m \end{aligned} \quad (1)$$

consists of finding a stabilizing state feedback control which minimizes the functional

$$J_\varepsilon = \frac{1}{2} \int_0^\infty [y^T(t)y(t) + \varepsilon^2 u^T(t)u(t)] dt \quad (2)$$

when  $\varepsilon > 0$  is small.

As  $\varepsilon \rightarrow 0$ , the optimal value  $J_\varepsilon^*$  tends to  $J_0^*$ , the *ideal performance*. Kwakernaak and Sivan proved that  $J_0^* = 0$  if and only if the system (1) is minimum phase and right invertible [2, Th. 3.14]. Qiu and Davison derived an explicit formula as a function of the NMP zeros [3]. We will rederive the Qiu–Davison formula under the assumption that the system (1) has relative degree one ( $\text{rank } CB = m$ ). Then, a change of coordinates  $x \mapsto (y, z)$  exists which transforms (1) into the form

$$\begin{aligned} \dot{y} &= A_1 y + A_2 z + B_1 u, & B_1 &= CB \\ \dot{z} &= B_0 y + A_0 z, & y \in R^m, & \quad z \in R^{n-m}. \end{aligned} \quad (3)$$

The zeros of  $C(sI - A)^{-1}B$  are the eigenvalues of  $A_0$ . We assume that all the zeros are NMP, that is,  $-A_0$  is Hurwitz.

We now minimize the functional (2) for the system (3). The associated algebraic Riccati equation is

$$\begin{bmatrix} A_1 & A_2 \\ B_0 & A_0 \end{bmatrix}^T P(\varepsilon) + P(\varepsilon) \begin{bmatrix} A_1 & A_2 \\ B_0 & A_0 \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \\ = \frac{1}{\varepsilon^2} P(\varepsilon) \begin{bmatrix} B_1 B_1^T & 0 \\ 0 & 0 \end{bmatrix} P(\varepsilon). \quad (4)$$

To find the limit of the positive definite solution  $P(\varepsilon)$  as  $\varepsilon \rightarrow 0$ , we must resolve the singularity at  $\varepsilon = 0$ . Following Jameson and O'Malley [5] we seek  $P(\varepsilon)$  in the form of a series in  $\varepsilon$

$$P(\varepsilon) = \begin{bmatrix} \varepsilon P_1 & \varepsilon P_2 \\ \varepsilon P_2^T & P_0 + \varepsilon P_3 \end{bmatrix} + O(\varepsilon^2) \quad (5)$$

where  $P_0, P_1, P_2$ , and  $P_3$  are independent of  $\varepsilon$ . The substitution of (5) into (4) yields

$$I - P_1 B_1 B_1^T P_1 + O(\varepsilon) = 0 \quad (6)$$

$$B_0^T P_0 - P_1 B_1 B_1^T P_2 + O(\varepsilon) = 0 \quad (7)$$

$$A_0^T P_0 + P_0 A_0 - P_2^T B_1 B_1^T P_2 + O(\varepsilon) = 0. \quad (8)$$

Setting  $\varepsilon = 0$ , we find  $P_1 = (B_1 B_1^T)^{-1/2}$  from (6), and  $P_2 = (B_1 B_1^T)^{-1/2} B_0^T P_0$  from (7), and, substituting in (8), we obtain

$$A_0^T P_0 + P_0 A_0 = P_0 B_0 B_0^T P_0. \quad (9)$$

Because system (3) is stabilizable and  $-A_0$  is Hurwitz, the unique positive definite  $P_0$  exists. By equating to zero the  $O(\varepsilon)$  terms (not shown in (6)–(8)) we find  $P_3 = P_0 B_0 (B_1 B_1^T)^{-1/2} B_0^T P_0$ .

Finally, the optimal value  $J_\varepsilon^* = V(y, z; \varepsilon)$  is calculated to be

$$J_\varepsilon^* = \frac{1}{2} z^T P_0 z + \frac{\varepsilon}{2} [y + B_0^T P_0 z]^T \\ \times (B_1 B_1^T)^{-1} [y + B_0^T P_0 z] + O(\varepsilon^2) \\ = V_0(z) + \varepsilon V_1(y, z) + O(\varepsilon^2). \quad (10)$$

Its limit as  $\varepsilon \rightarrow 0$

$$V_0(z) = \frac{1}{2} z^T P_0 z \quad (11)$$

defines the *ideal performance* of system (3).

We interpret this result via the *minimum energy problem*: Find  $y$  to stabilize  $\dot{z} = A_0 z + B_0 y$  and to minimize

$$J = \frac{1}{2} \int_0^\infty y^T(t) y(t) dt. \quad (12)$$

Here the system output  $y$  acts as the control variable for the zero dynamics. The optimal value is  $V_0(z)$  given in (11). This means that *the limiting optimal value for the full system is equal to the least amount of energy required to stabilize the zero-dynamics subsystem*.

### B. Singular Perturbation Analysis

When the optimal feedback control

$$u^* = -\frac{1}{\varepsilon} B_1^T (B_1 B_1^T)^{-1/2} [y + B_0^T P_0 z + O(\varepsilon)]$$

is applied to (3), the resulting closed-loop system is in the standard singular perturbation form

$$\varepsilon \dot{y} = -(B_1 B_1^T)^{1/2} [y + B_0^T P_0 z] + O(\varepsilon) \\ \dot{z} = B_0 y + A_0 z. \quad (13)$$

For a slow-fast analysis [6] we set  $\varepsilon = 0$  and get  $y = -B_0^T P_0 z$ . Then, substituting in the  $z$ -equation, we find that the slow subsystem of (13) is the optimal zero-dynamics subsystem

$$\dot{z} = -P_0^{-1} A_0^T P_0 z. \quad (14)$$

These slow dynamics evolve in the *slow invariant subspace*  $y + B_0^T P_0 z = 0$ , while the fast dynamics represent the convergence of  $y + B_0^T P_0 z$  to zero. The two parts of the optimal value are  $V_0(z)$  *slow* and  $\varepsilon V_1(y, z)$  *fast*. We note that the eigenvalues of (14) are the mirror image of the NMP zeros of (3), a fact known from [2, Th. 3.12].

### C. Cheap Control and the Bode T-Integral

We now apply the above results to the problem of regulating the output  $y$  of (3) to a constant setpoint  $r$ . With the feedforward term  $\bar{u} = -B_1^{-1}(A_1 - A_2 A_0^{-1} B_0) r$  we place the equilibrium of (3) at  $y = r$ . Introducing the error variables  $e = y - r$ ,  $\tilde{z} = z + A_0^{-1} B_0 r$ , and  $\tilde{u} = u - \bar{u}$ , we rewrite (3) as

$$\dot{e} = A_1 e + A_2 \tilde{z} + B_1 \tilde{u} \\ \dot{\tilde{z}} = A_0 \tilde{z} + B_0 e. \quad (15)$$

Then the cheap control problem is the same as for (3), but with  $(e, \tilde{z})$  replacing  $(y, z)$ . The ideal performance is  $V_0(\tilde{z})$ . We are interested in  $V_0(\tilde{z}(0))$  for the initial condition  $e(0) = r$ ,  $\tilde{z}(0) = A_0^{-1} B_0 r$ , which corresponds to the state  $y$  being transferred from zero to  $r$ .

*Qiu-Davison Formula*: For the initial condition  $e(0) = r$ ,  $\tilde{z}(0) = A_0^{-1} B_0 r$ , the ideal performance of (15) is

$$V_0(\tilde{z}(0)) = \frac{1}{2} r^T H r, \quad \text{where } \text{trace } H = 2 \sum_{i=1}^k \frac{1}{\alpha_i} \quad (16)$$

and  $\alpha_1, \dots, \alpha_k$  are the unstable eigenvalues of  $A_0$ , that is the NMP zeros of system (15). This formula is obtained from (11) and (9)

$$V_0(\tilde{z}(0)) = \frac{1}{2} \tilde{z}^T(0) P_0 \tilde{z}(0) \\ = \frac{1}{2} r^T B_0^T (A_0^T)^{-1} P_0 A_0^{-1} B_0 r \triangleq \frac{1}{2} r^T H r \\ \text{trace } H = \text{trace} [P_0^{1/2} A_0^{-1} B_0 B_0^T (A_0^T)^{-1} P_0^{1/2}] \\ = \text{trace} [P_0^{1/2} A_0^{-1} P_0^{-1/2} + P_0^{-1/2} (A_0^T)^{-1} P_0^{1/2}] \\ = 2 \text{trace } A_0^{-1} = 2 \sum_{i=1}^k \frac{1}{\alpha_i}.$$

How is this result related to the performance limitations defined by well-known Bode integrals? An integral formula for the complementary sensitivity function  $T$  derived by Middleton [4] is

$$\frac{1}{\pi} \int_0^\infty \log |T(j\omega)| \frac{d\omega}{\omega^2} + \frac{1}{2K_v} = \sum_{i=1}^k \frac{1}{\alpha_i} \quad (17)$$

where  $K_v$  is the *velocity constant*. The glaring similarity of the Middleton formula (17) and the Qiu-Davison formula (16) has, apparently, remained unnoticed.

*Proposition (Cheap Control and Bode T-Integral)*: Let  $G$  and  $K$  be the single-input/single-output transfer functions of a plant and a minimum phase controller, respectively. Suppose that  $T = GK(1 + GK)^{-1}$  is stable and  $T(0) = 1$ . Then

$$\frac{1}{\pi} \int_0^\infty \log |T(j\omega)| \frac{d\omega}{\omega^2} + \frac{1}{2K_v} = \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \int_0^\infty e^2(t) dt$$

where  $e(t) = y(t) - r$  is the error in transferring the system from rest to the setpoint  $r = 1$  using the cheap optimal control.

A dual result relates the classical Bode integral for the sensitivity function  $S$  [7] with the minimum energy ( $L_2$  norm) required to transfer the system to rest from the initial condition originating from a unit impulse at the input [8]. Similar relations have been derived in linear filtering [9].

## III. FEEDBACK LIMITATIONS IN NONLINEAR SYSTEMS

## A. Cheap Optimal Control

We now proceed to the main subject of this paper, i.e., the lowest attainable  $L_2$  norm of the output  $y$  of the nonlinear system

$$\begin{aligned} \dot{y} &= f(y, z) + g(y, z)u, & y, u &\in R^m \\ \dot{z} &= f_0(z) + g_0(z)y, & z &\in R^{n-m} \end{aligned} \quad (18)$$

where  $f(0, 0) = 0$ ,  $f_0(0) = 0$ . The following assumption implies that the relative degree of (18) is one and that its zero dynamics are governed by  $\dot{z} = f_0(z)$  [10].

*Assumption 1:* There exists  $\gamma > 0$  such that the smallest singular value of  $g(y, z)$  is greater than or equal to  $\gamma$  for all  $y$  and  $z$ .

For the nonlinear system (18), the cheap optimal control problem consists of finding a feedback control  $u$  which guarantees asymptotic stability and minimizes the functional (2) when  $\varepsilon > 0$  is small. It is well known (e.g., [11, p. 91]) that this problem has a solution if there exists a positive semidefinite optimal value  $V(y, z; \varepsilon)$  which satisfies the Hamilton–Jacobi–Bellman equation (HJBE)

$$\begin{aligned} \frac{\partial V}{\partial y} f(y, z) + \frac{\partial V}{\partial z} [f_0(z) + g_0(z)y] + \frac{1}{2} y^T y \\ - \frac{1}{2\varepsilon^2} \frac{\partial V}{\partial y} g(y, z) g^T(y, z) \frac{\partial^T V}{\partial y} = 0, \quad V(0, 0; \varepsilon) = 0 \end{aligned} \quad (19)$$

and such that the feedback control

$$u = -\frac{1}{\varepsilon^2} g^T(y, z) \frac{\partial^T V}{\partial y} \quad (20)$$

asymptotically stabilizes (18). We are interested in the limit of  $V(y, z; \varepsilon)$  as  $\varepsilon \rightarrow 0$ . At  $\varepsilon = 0$  the HJBE (19) has a singularity, which we try to resolve by seeking a solution in the form

$$V(y, z; \varepsilon) = V_0(z) + \varepsilon V_1(y, z) + O(\varepsilon^2). \quad (21)$$

Its substitution in (19) yields

$$\begin{aligned} \frac{\partial V_0}{\partial z} [f_0(z) + g_0(z)y] + \frac{1}{2} y^T y - \frac{1}{2} \frac{\partial V_1}{\partial y} g(y, z) g^T(y, z) \frac{\partial^T V_1}{\partial y} \\ + O(\varepsilon) = 0. \end{aligned} \quad (22)$$

A closer examination of (22) shows that the first two terms represent the Hamiltonian for the minimum energy problem: Find  $y$  to asymptotically stabilize the zero-dynamics subsystem

$$\dot{z} = f_0(z) + g_0(z)y \quad (23)$$

and to minimize the cost (12). We therefore make the following assumption.

*Assumption 2:* The zero dynamics of (18) are *antistable* ( $\dot{z} = -f_0(z)$  is asymptotically stable), and there exists a positive definite function  $V_0(z)$  that satisfies the HJBE

$$\frac{\partial V_0}{\partial z} f_0(z) - \frac{1}{2} \frac{\partial V_0}{\partial z} g_0(z) g_0^T(z) \frac{\partial^T V_0}{\partial z} = 0, \quad V_0(0) = 0 \quad (24)$$

and such that the feedback control

$$y = \alpha_0(z) \triangleq -g_0^T(z) \frac{\partial^T V_0}{\partial z} \quad (25)$$

achieves global asymptotic stability (GAS) of (23).

With  $V_0(z)$  so defined, we return to (22), add and subtract  $\frac{1}{2} \|\alpha_0(z)\|^2 = \frac{1}{2} \|g_0^T \frac{\partial^T V_0}{\partial z}\|^2$ , and, using (24), obtain

$$\frac{1}{2} [y - \alpha_0(z)]^T [y - \alpha_0(z)] - \frac{1}{2} \frac{\partial V_1}{\partial y} g(y, z) g^T(y, z) \frac{\partial^T V_1}{\partial y} = O(\varepsilon). \quad (26)$$

Defining  $\eta = y - \alpha_0(z)$ ,  $\tilde{g}(\eta, z) = g(\eta + \alpha_0(z), z)$ , and  $\tilde{V}_1(\eta, z) = V_1(\eta + \alpha_0(z), z)$ , and letting  $\varepsilon = 0$ , we rewrite (26) as

$$\frac{1}{2} \eta^T \eta - \frac{1}{2} \frac{\partial \tilde{V}_1}{\partial \eta} \tilde{g}(\eta, z) \tilde{g}^T(\eta, z) \frac{\partial^T \tilde{V}_1}{\partial \eta} = 0. \quad (27)$$

This is the HJBE for the cheap control problem: Find  $u$  to stabilize the system  $\dot{\eta} = \tilde{g}(\eta, z)u$ , where  $z$  is treated as a constant vector, and to minimize the functional  $\frac{1}{2} \int_0^\infty [\eta^T \eta + \varepsilon^2 u^T u] dt$ . The optimal value for this problem is  $\varepsilon \tilde{V}_1(\eta, z)$  and the optimal control is

$$u = -\frac{1}{\varepsilon} \tilde{g}^T \frac{\partial^T \tilde{V}_1}{\partial \eta} = -\frac{1}{\varepsilon} \tilde{g}^T (\tilde{g} \tilde{g}^T)^{-1/2} \eta. \quad (28)$$

Thus, the original cheap control problem (18), (2) has decomposed in two subproblems: a minimum energy problem for the zero-dynamics subsystem (23) and a cheap control problem to rapidly regulate  $\eta = y - \alpha_0(z)$  to zero.

## B. Singular Perturbation Analysis

We treat (28) as an approximation of the optimal control (20) and apply it to (18). In the  $(\eta, z)$ -coordinates, the resulting feedback system is

$$\begin{aligned} \varepsilon \dot{\eta} &= -[\tilde{g}(\eta, z) \tilde{g}^T(\eta, z)]^{1/2} \eta + \varepsilon \phi(\eta, z) \\ \dot{z} &= f_0(z) + g_0(z) \alpha_0(z) + g_0(z) \eta \end{aligned} \quad (29)$$

where  $\phi(\eta, z) = f(\eta + \alpha_0(z), z) - \dot{\alpha}_0(z)$ .

This system is in standard singular perturbation form [6]. We investigate its stability properties. In the slow manifold  $\eta = 0$  the system reduces to the slow subsystem

$$\dot{z} = f_0(z) + g_0(z) \alpha_0(z) \quad (30)$$

which, by Assumption 2, is GAS. Convergence to the slow manifold is governed by the fast (boundary-layer) subsystem

$$\frac{d\eta}{d\tau} = -[\tilde{g}(\eta, z) \tilde{g}^T(\eta, z)]^{1/2} \eta, \quad \tau = \frac{t}{\varepsilon} \quad (31)$$

with  $z$  treated as a constant vector. The fast subsystem (31) is globally exponentially stable by Assumption 1.

As shown in [12], these stability properties of the slow and fast subsystems (30) and (31) guarantee semiglobal (in  $\varepsilon$ ) boundedness of the solutions of (29). For each pair of positive numbers  $(R, \rho)$ , there exists  $\varepsilon_R > 0$  such that for all  $\varepsilon \in (0, \varepsilon_R]$  the solutions of (29) starting in the ball  $B_R$  of radius  $R$  enter in finite time the ball  $B_\rho$ .

However, this is not sufficient for asymptotic stability of the equilibrium  $(\eta, z) = (0, 0)$ : the system  $\varepsilon \dot{\eta} = -\eta + \varepsilon z$ ,  $\dot{z} = -z^3 + \eta$  has slow and fast subsystems as above, but for  $\varepsilon > 0$  its equilibrium is unstable [12]. As suggested in [13], we need to restrict the interconnection  $\phi(\eta, z)$ .

*Lemma (Asymptotic Stability):* Suppose  $\|\phi(\eta, z)\| \leq k_1 \|\eta\| + k_2 \|\alpha_0(z)\|$  in  $B_\delta$  for some  $\delta > 0$ . Then, under Assumptions 1 and 2, to each  $R > 0$  there corresponds an  $\varepsilon_R > 0$  such that for all  $\varepsilon \in (0, \varepsilon_R]$  the equilibrium  $(\eta, z) = (0, 0)$  of (29) is asymptotically stable and its basin of attraction contains  $B_R$ .

*Proof:* Using (24) and (25), the derivative of the Lyapunov function  $W(\eta, z) = \frac{1}{2} \eta^T \eta + V_0(z)$  along the solutions of (29) in  $B_\delta$  satisfies

$$\dot{W} \leq -\frac{1}{2} \begin{bmatrix} \|\eta\| \\ \|\alpha_0(z)\| \end{bmatrix}^T \begin{bmatrix} 2(\frac{\gamma}{\varepsilon} - k_1) & 1 + k_2 \\ 1 + k_2 & 1 \end{bmatrix} \begin{bmatrix} \|\eta\| \\ \|\alpha_0(z)\| \end{bmatrix} \quad (32)$$

where  $\gamma$  is given by Assumption 1.

Thus,  $\dot{W} \leq 0$  for all  $\varepsilon \in (0, \varepsilon_a]$ , where  $\varepsilon_a = \gamma / [(1 + k_2)^2 + 2k_1]$ . Pick  $c$  such that  $\Omega_c = \{(\eta, z) : W(\eta, z) \leq c\} \subset B_\delta$ . Then, by LaSalle's invariance theorem, every solution of (29) starting in  $\Omega_c$

converges to the largest invariant set contained in  $E = \{(\eta, z) : \dot{W} = 0\} \subset \{(\eta, z) : \eta = 0, \alpha_0(z) = 0\}$ . Because, by Assumption 2,  $y = \alpha_0(z)$  asymptotically stabilizes (23), this invariant set is just  $(\eta, z) = (0, 0)$ , which is then asymptotically stable with a basin of attraction that contains  $\Omega_c$ .

Next, choose  $\rho > 0$  such that  $B_\rho \subset \Omega_c$ . Then, [12, Th. 1] guarantees the existence of  $\varepsilon_R \in (0, \varepsilon_a]$  such that for all  $\varepsilon \in (0, \varepsilon_R]$  the solutions of (29) starting in  $B_R$  enter  $B_\rho$  in finite time and thereafter converge to  $(\eta, z) = (0, 0)$ .  $\square$

### C. Limitations to Nonlinear Ideal Performance

Thus far we have established that  $V_0 + \varepsilon V_1$  satisfies the HJBE (19) with an  $O(\varepsilon)$  error and that the corresponding control, given by (28)

$$u = -\frac{1}{\varepsilon^2} g^T \frac{\partial^T (V_0 + \varepsilon V_1)}{\partial y} = -\frac{1}{\varepsilon} \tilde{g}^T \frac{\partial^T \tilde{V}_1}{\partial \eta} = -\frac{1}{\varepsilon} \tilde{g}^T (\tilde{g} \tilde{g}^T)^{-1} V^{1/2} \eta$$

asymptotically stabilizes (18), written in closed-loop form as (29).

Our analysis has decomposed the optimal cheap control problem in two separate subproblems: a minimum energy problem for asymptotic stabilization of the zero-dynamics subsystem (23) and a cheap control problem for asymptotic stabilization of the boundary-layer subsystem (31). The solutions of (29), as well as those of (18) and (20), begin by rapidly ("instantaneously" as  $\varepsilon \rightarrow 0$ ) converging to the zero-dynamics manifold  $\eta = y - \alpha_0(z) = 0$ . After that, they settle in a slow motion during which the zero dynamics are controlled by  $y = \alpha_0(z)$ . The cost of the rapid transient vanishes as  $\varepsilon \rightarrow 0$ . What remains is the cost  $V_0(z)$  of the slow motion needed to stabilize the unstable zero dynamics. The overall cost cannot be reduced below  $V_0(z)$ . We are now ready to state and prove our main result.

*Theorem (Ideal Performance):* Under the conditions of the lemma, for every initial state  $(y(0), z(0))$  for which the cheap control problem (18), (2) has a solution, the optimal value satisfies

$$J_\varepsilon^* = V(y(0), z(0); \varepsilon) = V_0(z(0)) + O(\varepsilon) \quad (33)$$

and thus the ideal performance is  $V_0(z(0))$ , the optimal value of the minimum energy problem for the zero-dynamics subsystem (23) controlled by the output  $y$ .

*Proof:* We prove that the cost produced by the control (28) satisfies (33). Since the optimal control cannot produce a larger cost, (33) will therefore hold for the optimal value. Let  $t_\delta$  be the finite time such that  $(\eta(t), z(t)) \in B_\delta$  for all  $t \geq t_\delta$ . By integrating (32) over  $[t_\delta, \infty)$  we obtain

$$\int_{t_\delta}^{\infty} \left[ \frac{1}{2} \|\alpha_0(z)\|^2 + (1 + k_2) \|\alpha_0(z)\| \|\eta\| + \left( \frac{\gamma}{\varepsilon_R} - k_1 \right) \|\eta\|^2 \right] dt \leq V_0(z(t_\delta)) + \frac{1}{2} \|\eta(t_\delta)\|^2 < \infty. \quad (34)$$

This, along with the boundedness of the solutions of (29) on  $[0, t_\delta]$ , guarantees that the cost

$$J_\varepsilon = \int_0^{\infty} \left[ \frac{1}{2} \|\alpha_0(z)\|^2 + \alpha_0^T(z) \eta + \|\eta\|^2 \right] dt \quad (35)$$

is finite. Observing that (24) and (29) yield  $\frac{1}{2} \|\alpha_0(z)\|^2 + \alpha_0^T(z) \eta = -\dot{V}_0$ , we express (35) as

$$J_\varepsilon = V_0(z(0)) + \int_0^{\infty} \|\eta\|^2 dt. \quad (36)$$

Multiplying the  $\eta$  equation in (29) by  $\eta^T$

$$\eta^T \dot{\eta} = -\frac{1}{\varepsilon} \eta^T (\tilde{g} \tilde{g}^T)^{-1/2} \eta + \eta^T \phi \leq -\frac{\gamma}{\varepsilon} \|\eta\|^2 + \eta^T \phi$$

and integrating we get

$$\frac{\gamma}{\varepsilon} \int_0^{\infty} \|\eta\|^2 dt \leq \frac{1}{2} \|\eta(0)\|^2 + \int_0^{t_\delta} \eta^T \phi dt + \int_{t_\delta}^{\infty} \eta^T \phi dt. \quad (37)$$

Since, by assumption, the bound  $\|\phi\| \leq k_1 \|\eta\| + k_2 \|\alpha_0\|$  holds on  $[t_\delta, \infty)$ , it follows from (34) and (37) that  $\int_0^{\infty} \|\eta\|^2 dt$  in (36) is  $O(\varepsilon)$ , which proves (33).  $\square$

Thus, the lowest attainable  $L_2$  norm of the output of (18) is the least amount of energy required to stabilize the unstable zero dynamics. Although essentially nonlinear, this result is structurally similar to its linear counterpart.

It is also of interest to find a nonlinear analog of the property that, as  $\varepsilon \rightarrow 0$ , the finite poles of the optimal linear system converge to the mirror image of the NMP zeros. An expression of this property is that the zero dynamics are "as stable in the closed loop as they are unstable in the open loop." Using  $V_0(z)$  as a Lyapunov function we reveal an analogous property of the nonlinear zero dynamics.

*Corollary (Mirroring Property):* For the open- and closed-loop zero dynamics,  $V_0(z)$  satisfies

$$\frac{\partial V_0}{\partial z} f_0(z) = -\frac{\partial V_0}{\partial z} [f_0(z) + g_0(z) \alpha_0(z)]. \quad (38)$$

This identity is immediate from (24) and (30).

### D. Example

The essentially nonlinear character of the obtained results will be illustrated on the system

$$\begin{aligned} \dot{y} &= 6z^5 + u \\ \dot{z} &= z^3 + y. \end{aligned} \quad (39)$$

Its linearization  $\dot{y} = u, \dot{z} = y$  is not detectable so that the linear analysis does not apply.

Upon the substitution of  $V = V_0 + \varepsilon V_1 + O(\varepsilon^2)$  in the HJBE (19), we obtain

$$\frac{\partial V_0}{\partial z} [z^3 + y] + \frac{1}{2} y^2 - \frac{1}{2} \left[ \frac{\partial V_1}{\partial y} \right]^2 + O(\varepsilon) = 0. \quad (40)$$

The HJBE for the minimum energy stabilization of the zero-dynamics subsystem  $\dot{z} = z^3 + y$  is

$$\frac{\partial V_0}{\partial z} z^3 - \frac{1}{2} \left[ \frac{\partial V_0}{\partial z} \right]^2 = 0, \quad V_0(0) = 0. \quad (41)$$

The optimal value  $V_0(z)$  and the corresponding control law are

$$V_0(z) = \frac{z^4}{2}, \quad y = \alpha_0(z) = -\frac{\partial V_0}{\partial z} = -2z^3.$$

The substitution of  $V_0(z) = z^4/2$  and  $\varepsilon = 0$  in (40) yields  $\partial V_1 / \partial y = y + 2z^3$ , and, hence

$$u = -\frac{1}{\varepsilon^2} \frac{\partial (V_0 + \varepsilon V_1)}{\partial y} = -\frac{1}{\varepsilon} \frac{\partial V_1}{\partial y} = -\frac{1}{\varepsilon} (y + 2z^3) = -\frac{1}{\varepsilon} \eta.$$

When this control is applied to (39), the closed-loop system is

$$\dot{\eta} = -\frac{1}{\varepsilon} \eta + 6z^2 \eta, \quad \dot{z} = -z^3 + \eta. \quad (42)$$

Our proof of asymptotic stability of (29) was in two steps: we first showed semiglobal boundedness and then local asymptotic stability. However, for (42) we give a direct proof using  $W = \eta^2/2 + z^4/2$  as a Lyapunov function. Its derivative along the solutions of (42)

$$\dot{W} = -\begin{bmatrix} \eta \\ z^3 \end{bmatrix}^T \begin{bmatrix} \frac{1}{\varepsilon} - 6z^2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} \eta \\ z^3 \end{bmatrix}$$

is negative definite if  $12z^2 < (2 - \varepsilon)/\varepsilon$ . For  $\varepsilon < 1$  this inequality is satisfied if  $z^2 \leq (12\varepsilon)^{-1}$ . Therefore, the invariant set  $\{(\eta, z) : \eta^2 + z^4 \leq (12\varepsilon)^{-2}\}$  belongs to the basin of attraction of the equilibrium  $(\eta, z) = (0, 0)$  of (42). We conclude that the lowest achievable  $L_2$ -norm of  $y$  in (39) is  $J_0^* = V_0(z) = z^4/2$ .

## IV. CONCLUSION

The main result in this paper establishes that the ideal performance of a nonlinear system, that is the lowest attainable  $L_2$  norm of its output, is the least amount of energy required to stabilize the zero dynamics. Thus, unstable zero dynamics represent a structural obstacle to attaining zero ideal performance. Although our results are for systems of relative degree one with a local growth condition, a similar structural obstacle is present in a wider class of nonlinear systems for which its quantification would require a more elaborate analysis.

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## A New Representation of the Parameters of Lifted Systems

Leonid Mirkin and Zalman J. Palmor

**Abstract**—Lifting, i.e., discretization with built-in intersample behavior, is an emerging technique for the analysis and design of sampled-data systems. The applicability of the lifting technique, however, is severely limited owing to difficulties in dealing with the parameters of the lifted systems, which are operators over infinite-dimensional spaces rather than finite-dimensional matrices. In this paper, a new representation for the parameters of the lifted systems is proposed. The technical machinery developed in the paper based on this representation simplifies considerably algebraic manipulation over parameters of the lifted systems, thus extending the scope of applicability of the lifting technique. To illustrate the advantages of the proposed approach, the computational issues in the sampled-data  $H^\infty$  problem are considered.

**Index Terms**—Generalized sampling and hold circuits, lifting technique, sampled-data systems.

## I. INTRODUCTION

The main motivation for this paper comes from the consideration of the intersample behavior of sampled-data systems, that is systems consisting of a continuous-time plant and a discrete-time controller, which are connected by sampling and hold devices. Dealing with the continuous-time behavior of sampled-data systems is complicated by 1) the simultaneous presence of continuous- and discrete-time signals and 2) inherent periodicity of the interconnection of a sampler, a discrete-time controller, and a hold. An elegant way to circumvent these difficulties is offered by the so-called *lifting technique* [1]–[3] (see also [4] and [5]), which allows one to convert periodic continuous/discrete systems to equivalent, in some sense, *pure discrete-time shift-invariant* ones.

Although the reduction of hybrid periodic problems to discrete-time time-invariant ones is clearly advantageous, the lifting method gives rise to another problem: input and output spaces of the lifted systems become infinite-dimensional. Conceptually, this fact does not lead to considerable difficulties since many LTI system notions have almost one-to-one counterparts in the lifted domain (see [3] and [6]). Moreover, the preservation of the state-space dimensions under lifting guarantees that any observer-based (including  $H^2$  and  $H^\infty$ ) controller in the lifted domain has a finite state dimension. Thus, analytic solutions to various sampled-data analysis and design problems can be obtained in terms of the parameters of the lifted models in quite a straightforward manner. Yet the latter parameters are no longer finite-dimensional matrices but rather operators over infinite-dimensional input and/or output spaces, such as  $L^2[0, h]$ . For this reason, the usefulness of a lifted solution relies essentially upon the ability to treat such infinite-dimensional operators.

Unfortunately, lifted parameters are not readily treatable. Consequently, sampled-data problems solved so far by means of the lifting technique share one key property: the possibility to separate beforehand the infinite dimensionality from the design by converting the problem to an equivalent finite-dimensional one. If a problem admits such a separation, the complexity of the operations over the

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