FEEDBACK STABILIZATION OF A FLUID–STRUCTURE MODEL

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Abstract. We study a system coupling the incompressible Navier-Stokes equations in a 2D rectangular type domain with a damped Euler-Bernoulli beam equation, where the beam is a part of the upper boundary of the domain occupied by the fluid. Due to the deformation of the beam the fluid domain depends on time. We prove that this system is exponentially stabilizable, locally about the null solution, with any prescribed decay rate, by a feedback control corresponding to a force term in the beam equation. The feedback is determined, via a Riccati equation, by solving an infinite time horizon control problem for the linearized model. A crucial step in this analysis consists in showing that this linearized system can be rewritten thanks to an analytic semigroup of which the infinitesimal generator has a compact resolvent.

Key words. Fluid-structure interaction, feedback control, stabilization, Navier-Stokes equations, Beam equation

AMS subject classifications. 93B52, 93C20, 93D15, 35Q30, 76D55, 76D05, 74F10

1. Setting of the problem. Let Ω be the rectangular domain $(0,L)\times(0,1)\subset\mathbb{R}^2$, with boundary Γ. Let us set $\Gamma_s=(0,L)\times\{1\}$, the upper part of the boundary of Ω, and $\Gamma_0=\Gamma\setminus\Gamma_s$. For a given function η from $\Gamma_s\times(0,\infty)$ into $(-1,\infty)$, we denote by $\Omega_{\eta(t)}$ and $\Gamma_{s,\eta(t)}$ the sets

$$\begin{aligned}\Omega_{\eta(t)} &= \Big\{(x,y)\mid x\in(0,L),\ 0<y<1+\eta(x,t)\Big\},\\ \Gamma_{s,\eta(t)} &= \Big\{(x,y)\mid x\in(0,L),\ y=1+\eta(x,t)\Big\}.\end{aligned}$$

For $0<T<\infty$ or $T=\infty$, we also use the notation

$$\begin{aligned}&\Sigma_T^0=\Gamma_0\times(0,T), &&\Sigma_T=\Gamma\times(0,T),\\ &Q_T=\Omega\times(0,T), &&\widetilde{Q}_T=\textstyle\bigcup_{t\in(0,T)}\Omega_{\eta(t)}\times\{t\},\\ &\Sigma_T^s=\Gamma_s\times(0,T), &&\widetilde{\Sigma}_T^s=\textstyle\bigcup_{t\in(0,T)}\Gamma_{s,\eta(t)}\times\{t\}.\end{aligned}$$

We consider the following fluid-structure model coupling the Navier-Stokes equations with a damped Euler-Bernoulli beam equation:

$$\begin{aligned}&\mathbf{u}_t+(\mathbf{u}\cdot\nabla)\mathbf{u}-\operatorname{div}\sigma(\mathbf{u},p)=0,\quad \operatorname{div}\mathbf{u}=0\quad \text{in } \widetilde{Q}_\infty,\\ &\mathbf{u}=\eta_t\vec{e}_2\ \text{ on } \widetilde{\Sigma}_\infty^s,\quad \mathbf{u}=0\ \text{ on } \Sigma_\infty^0,\quad \mathbf{u}(0)=\mathbf{u}^0 \text{ in } \Omega_{\eta(0)}=\Omega_{\eta_1^0},\\ &\eta_{tt}-\beta\eta_{xx}-\delta\eta_{txx}+\alpha\eta_{xxxx}=\rho_1 p+H(\mathbf{u},\eta)+f\ \text{ on } \Sigma_\infty^s,\\ &\eta=0\quad\text{and}\quad \eta_x=0\ \text{ on } \big\{0,L\big\}\times(0,\infty),\\ &\eta(0)=\eta_1^0\quad\text{and}\quad \eta_t(0)=\eta_2^0\ \text{ in } \Gamma_s,\end{aligned}\tag{1.1}$$

with

$$\begin{aligned}&H(\mathbf{u},\eta)=-\rho_2\nu(\nabla\mathbf{u}+\nabla\mathbf{u}^T)(-\eta_x\vec{e}_1+\vec{e}_2)\cdot\vec{e}_2,\\ &\sigma(\mathbf{u},p)=\nu(\nabla\mathbf{u}+\nabla\mathbf{u}^T)-p\,I,\quad \vec{e}_1=(1,0),\quad \vec{e}_2=(0,1).\end{aligned}$$

In this setting $\nu>0$ is the fluid viscosity, $\alpha>0$, $\beta\geq0$, and $\delta>0$ are the adimensional rigidity, stretching, and friction coefficients of the beam, ρ_1 and ρ_2 are positive constants related to the density of the fluid and the density of the structure (see [4]), f is a control function. Our objective is to determine f in feedback form, able to stabilizes the system (1.1) (in an appropriate space) with a prescribed exponential decay rate $-\omega<0$, locally about $(\mathbf{0},0,0,0)$. Existence of a local strong solution for system (1.1) with $f=0$ has been proved in [4] (with periodic boundary conditions on the lateral boundary of Ω), under smallness conditions on the data, while existence of Hopf solutions for a slightly different model is proved

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in [13] (see also [8] and [12] for other models for the beam equation and for existence results in the three dimensional case). To the author knowledge nothing is known about control and stabilization of such a system. To study the control system (1.1), as in [4], we make a change of variable in order to rewrite system (1.1) in the cylindrical domain $\Omega \times (0, \infty)$ and we denote by $(\hat{\mathbf{u}}, \hat{p})$ the image of $(\mathbf{u}, p)$ by this transformation. Since we are looking for solutions satisfying a prescribed exponential decay rate $-\omega$, we rewrite the system as a first order system by setting $\eta = \eta_1$ and $\eta_t = \eta_2$ and we study the control system satisfied by $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2) = e^{\omega t}(\hat{\mathbf{u}}, \hat{p}, \eta_1, \eta_2)$. We linearize the system satisfied by $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2)$ about $(\mathbf{0}, 0, 0, 0)$ and we determine a feedback control, able to stabilize the linearized system satisfied by $(\mathbf{v}, p, \eta_1, \eta_2)$, by solving an infinite time horizon control problem. Next we prove that this linear feedback law, applied in the nonlinear system satisfied by $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2)$, is able to stabilize the nonlinear system provided that the intial condition is small enough in an appropriate norm.

The analysis that we do for the linearized system is completely new for this type of fluid–structure system. Indeed we show that the linearized system satisfied by $(\mathbf{v}, p, \eta_1, \eta_2)$ is equivalent to a system of the form

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \mathcal{A}_\omega \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} + \mathcal{B}\begin{pmatrix} \underline{0} \\ 0 \\ f \end{pmatrix}, \qquad \begin{pmatrix} P\mathbf{v}(0) \\ \eta_1(0) \\ \eta_2(0) \end{pmatrix} = \begin{pmatrix} P\mathbf{v}^0 \\ \eta_1^0 \\ \eta_2^0 \end{pmatrix}, \\
&(I-P)\mathbf{v}(t) = (I-P)D(\eta_2(t)\,\vec{e}_2\,\chi_{\Gamma_s}),
\end{aligned} \tag{1.2}$$

where P is the so-called Leray projector and D is the Dirichlet operator associated with the stationary Stokes equation (P and D are defined in section 3, while $\mathcal{A}_\omega$ and $\mathcal{B}$ are defined in section 4). This type of decomposition of velocity fields, into $P\mathbf{v}$ and $(I-P)\mathbf{v}$, has already been introduced for the Navier-Stokes equations with nonhomogeneous boundary conditions in [22]. Finding again this decomposition for system (1.2) is not totally obvious because the pressure, which is eliminated in the Navier-Stokes equations thanks to the projector P, also appears in the beam equation. Rewriting the system satisfied by $(\mathbf{v}, p, \eta_1, \eta_2)$ in the form (1.2) is crucial to prove the stabilizability of this system. Indeed, we show that the operator $(\mathcal{A}_\omega, D(\mathcal{A}_\omega))$ is the infinitesimal generator of an analytic semigroup on the space $\mathbf{H} = \mathbf{V}_n^0(\Omega) \times (H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)) \times L_0^2(\Gamma_s)$ and has a compact resolvent in this space (for the precise definitions of these spaces we refer to section 3). We show that the stabilizability of system (1.2) reduces to proving an approximate controllability result for a projected system. Such an approximate controllability result can be deduced from [18] in the case of a rectangular domain (see also [19, 20] for supplementary approximate controllability results).

The plan of the paper is as follows. Section 2 is devoted to rewriting system (1.1) in a fixed domain and to the obtention of a linearized system. We study the semigroup of the linearized system and properties of its infinitesimal generator in section 3. Existence and regularity results for the linearized system are stated in section 4. We study the stabilizability of the linearized system in section 5. Three feedback control laws for the linearized system (1.2) are introduced in section 6. The first one is a feedback law for system (1.2) written as a system of partial differential equations, involving the pressure (see system (2.6)). The second one is a feedback law obtained by the classical approach introduced in [2] or in [15]. In that case the pressure is eliminated since it does not appear in (1.2). The corresponding feedback law is defined via the solution to a Riccati equation of the form

$$\widetilde{\Pi} \in \mathcal{L}(\mathbf{H}), \quad \widetilde{\Pi} = \widetilde{\Pi}^* \geq 0, \quad \widetilde{\Pi}\mathcal{A}_\omega + \mathcal{A}_\omega^*\widetilde{\Pi} - \widetilde{\Pi}\mathcal{B}\mathcal{B}^*\widetilde{\Pi} + C^*C = 0.$$

(See equation (6.2) for the definition of C.) Since $\mathcal{A}_\omega^*$, which is determined in section 3.5, cannot be interpreted only in terms of partial operators (contrarily to $\mathcal{A}_\omega$), we introduce a third feedback law obtained by solving a Riccati equation of the form

$$\widehat{\Pi} \in \mathcal{L}(\widehat{\mathbf{H}}), \quad \widehat{\Pi} = \widehat{\Pi}^* \geq 0, \quad \widehat{\Pi}\mathcal{A}_\omega + \mathcal{A}_\omega^\sharp\widehat{\Pi} - \widehat{\Pi}\mathcal{B}\mathcal{B}^\sharp\widehat{\Pi} + I = 0,$$

where $\widehat{\mathbf{H}}$ is the space $\mathbf{H}$ equipped with another inner product (see section 3.5), $\mathcal{A}_\omega^\sharp \in \mathcal{L}(\widehat{\mathbf{H}})$ is the adjoint of $\mathcal{A}_\omega \in \mathcal{L}(\widehat{\mathbf{H}})$ and $\mathcal{B}^\sharp \in \mathcal{L}(\widehat{\mathbf{H}}, L_0^2(\Gamma_s))$ is the adjoint of $\mathcal{B} \in \mathcal{L}(L_0^2(\Gamma_s), \widehat{\mathbf{H}})$. The main interest of this approach is that $\mathcal{A}_\omega^\sharp$ can be interpreted in terms of partial differential operators (which can be helpful for numerical calculations). Moreover, we are able to establish the precise relationship between the feedback operators obtained by the first approach and the third one.

The optimal control problems corresponding to the first approach are studied in details in sections 7 and 8.1. In these sections, all the calculations are made in a very simple way via integrations by parts. Therefore they can be easily checked and do not need a sophisticated functional analysis framework. However the feedback law corresponding to the first approach is expressed in terms of an operator Π which is not, at that stage, characterized by a Riccati equation. This is why the third approach is helpful even if in that case the representation of the state and adjoint systems via $\mathcal{A}_\omega$ and $\mathcal{A}_\omega^\sharp$ cannot be avoided.

To deal with the nonlinear closed loop system, we first study the nonhomogeneous linearized closed loop system in section 9. The main results of the paper are stated in section 10 (Theorems 10.2 and 10.3). Some Lipschitz properties of the nonlinear terms in the nonlinear system are established in section 11. These properties are next used in section 12 in the proof of the main results.

Let us finally give some references which are connected to the present work. The control of a channel flow with periodic boundary conditions have been studied in [5, 31, 32, 33]. We think that the results in those papers may be very useful to study the control of a channel flow coupled with a beam equation, with periodic boundary conditions at the lateral boundary $\{0\}\times[0,L]\cup\{L\}\times[0,L]$. This will be investigated in a future work. Let us also mention some controllability results obtained for systems coupling the Navier-Stokes equations with finite dimensional solid-structure models [6, 21, 26] (and see also [25] for a simplified model). These controllability resuls are mainly based on results first obtained for the Navier-Stokes equations in [3]. In those models the controls act in the fluid equation and not in the structure equation as in (1.1). Thus the problems are quite different. The feedback stabilization of the Navier-Stokes equations in the three dimensional case is studied in [24]. It can be a starting point to study the stabilization of systems similar to (1.1) in the 3D case.

2. The linearized system. The solutions to system (1.1) obey

$$0=\int_{\Omega_{\eta(t)}}\operatorname{div}\mathbf{u}(t)=\int_{\Gamma_{s,\eta(t)}}\mathbf{u}(t)\cdot\mathbf{n}(t)=\int_{\Gamma_s}\eta_t(t)=\int_0^L\eta_t(x,t)dt,$$

since the unit normal to $\Gamma_{s,\eta(t)}$ outward $\Omega_{\eta(t)}$ is

$$\mathbf{n}(t)=\left(\frac{-\eta_x(t)}{\sqrt{1+\eta_x^2(t)}},\frac{1}{\sqrt{1+\eta_x^2(t)}}\right)^T.$$

Thus we must choose η_2^0 in the space

$$L_0^2(\Gamma_s)=\Big\{\eta\in L^2(\Gamma_s)\mid\int_{\Gamma_s}\eta=0\Big\}.$$

If η_1^0 also belongs to $L_0^2(\Gamma_s)$, then we have

$$\int_{\Gamma_s}\eta(t)=0\quad\text{and}\quad\int_{\Gamma_s}\eta_t(t)=0\quad\text{for all } t\geq 0.$$

Everywhere throughout the paper we shall choose η_1^0 and η_2^0 with zero mean value over Γ_s. If we denote by M_s the orthogonal projection in $L^2(\Gamma_s)$ onto $L_0^2(\Gamma_s)$, the equation satisfied by η in system (1.1) must be written in the form

$$\eta_{tt}-\beta\eta_{xx}-\delta\eta_{txx}+\alpha M_s(\eta_{xxxx})=M_s(\rho_1 p+H(\mathbf{u},\eta)+f)\quad\text{on } \Sigma_\infty^s.$$

Observe that due to the boundary conditions

$$\eta=0\quad\text{and}\quad\eta_x=0\;\;\text{on } \big\{0,L\big\}\times(0,\infty),$$

we have (for solutions regular enough and when η_1^0 and η_2^0 belong to $L_0^2(\Gamma_s)$)

$$\int_{\Gamma_s}\eta_{tt}=0,\quad\int_{\Gamma_s}\eta_{xx}=0,\quad\text{and}\quad\int_{\Gamma_s}\eta_{txx}=0,$$

but we do not necessarily have

$$\int_{\Gamma_s} \eta_{xxxx} = 0.$$

This is why, in the equation satisfied by η, we have to write $M_s(\eta_{xxxx})$ in place of η_{xxxx}. But for simplicity we shall skip the writing of M_s in the different equations, except if we want to stress on the role of the operator M_s (which is for example the case when we shall define the operator $(\mathcal{A}_\omega, D(\mathcal{A}_\omega))$).

We consider system (1.1) for initial conditions $\mathbf{u}^0$ such that $\operatorname{div} \mathbf{u}^0 = \mathbf{u}^0_{1,x} + \mathbf{u}^0_{2,y} = 0$ and obeying the compatibility condition

$$\mathbf{u}^0 = 0 \ \text{ on } \Gamma_0, \quad \mathbf{u}^0(x, 1+\eta(x,0)) = \mathbf{u}^0(x, 1+\eta_1^0(x)) = \eta_2^0(x)\vec{e}_2 \ \text{ for } x \in (0,L). \tag{2.1}$$

As in [4], for a given function $\eta : (0,L) \times (0,T) \mapsto \mathbb{R}$ satisfying $\eta > -1$, we consider the changes of variables

$$\mathcal{T}_\eta : (x,y,t) \longmapsto (x,z,t) = \left(x, \frac{y}{1+\eta(x,t)}, t\right) \quad \text{and} \quad \mathcal{T}_{\eta(t)} : (x,y) \longmapsto (x,z) = \left(x, \frac{y}{1+\eta(x,t)}\right). \tag{2.2}$$

The mapping $\mathcal{T}_{\eta_1^0}$ is defined in a similar way. The mapping $\mathcal{T}_{\eta(t)}$ transforms $\Omega_{\eta(t)}$ into $\Omega = (0,L) \times (0,1)$. Setting

$$\hat{\mathbf{u}}(x,z,t) = \mathbf{u}(x,y,t), \quad \hat{p}(x,z,t) = p(x,y,t),$$

the nonlinear system (1.1) is rewritten in the form

$$\begin{aligned}
&\hat{\mathbf{u}}_t + (\hat{\mathbf{u}} \cdot \nabla)\hat{\mathbf{u}} - \nu\Delta\hat{\mathbf{u}} - \nabla\hat{p} = \hat{F}(\hat{\mathbf{u}}, \hat{p}, \eta), \quad \operatorname{div} \hat{\mathbf{u}} = \hat{G}(\hat{\mathbf{u}}, \eta) \quad \text{ in } Q_\infty,\\
&\hat{\mathbf{u}} = \eta_t \vec{e}_2 \ \text{ on } \Sigma^s_\infty, \quad \hat{\mathbf{u}} = 0 \ \text{ on } \Sigma^0_\infty, \quad \hat{\mathbf{u}}(0) = \hat{\mathbf{u}}^0 \text{ in } \Omega,\\
&\eta_{tt} - \beta\eta_{xx} - \delta\eta_{txx} + \alpha\eta_{xxxx} = \rho_1\hat{p} + \hat{H}(\hat{\mathbf{u}}, \eta) + f \ \text{ on } \Sigma^s_\infty,\\
&\eta = 0 \quad \text{and} \quad \eta_x = 0 \ \text{ on } \big\{0, L\big\} \times (0,\infty),\\
&\eta(0) = \eta_1^0 \quad \text{and} \quad \eta_t(0) = \eta_2^0 \ \text{ in } \Gamma_s,
\end{aligned} \tag{2.3}$$

where $\hat{\mathbf{u}}^0(x,z) = \mathbf{u}^0(x,y) = \mathbf{u}^0(x, z(1+\eta(x,0))) = \mathbf{u}^0(x, z(1+\eta_1^0(x))) = \mathbf{u}^0 \circ \mathcal{T}^{-1}_{\eta_1^0}(x,z)$,

$$\begin{aligned}
&\hat{F}(\hat{\mathbf{u}}, \hat{p}, \eta)\\
&= -\eta\hat{\mathbf{u}}_t + \left(z\eta_t + \nu z\left(\tfrac{\eta_x^2}{1+\eta} - \eta_{xx}\right)\right)\hat{\mathbf{u}}_z\\
&\quad + \nu\left(-2z\eta_x\hat{\mathbf{u}}_{xz} + \eta\hat{\mathbf{u}}_{xx} + \left(\tfrac{z^2\eta_x^2 - \eta}{1+\eta}\right)\hat{\mathbf{u}}_{zz}\right)\\
&\quad + z(\eta_x\hat{p}_z - \eta\hat{p}_x)\vec{e}_1 - (1+\eta)\hat{\mathbf{u}}_1\hat{\mathbf{u}}_x + (z\eta_x\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2)\hat{\mathbf{u}}_z,
\end{aligned}$$

$$\hat{G}(\hat{\mathbf{u}}, \eta) = -\eta\hat{\mathbf{u}}_{1,x} + z\eta_x\hat{\mathbf{u}}_{1,z} = \operatorname{div}(\hat{\mathbf{w}}) \quad \text{with } \hat{\mathbf{w}} = -\eta\hat{\mathbf{u}}_1\vec{e}_1 + z\eta_x\hat{\mathbf{u}}_1\vec{e}_2,$$

and

$$\hat{H}(\hat{\mathbf{u}}, \eta) = \rho_2\nu\left(\tfrac{\eta_x}{1+\eta}\hat{\mathbf{u}}_{1,z} + \eta_x\hat{\mathbf{u}}_{2,x} - \tfrac{2+\eta_x^2}{1+\eta}\hat{\mathbf{u}}_{2,z}\right) = -2\rho_2\nu\hat{\mathbf{u}}_{2,z} + \rho_2\nu\left(\tfrac{\eta_x}{1+\eta}\hat{\mathbf{u}}_{1,z} + \eta_x\hat{\mathbf{u}}_{2,x} - \tfrac{\eta_x^2 - 2\eta}{1+\eta}\hat{\mathbf{u}}_{2,z}\right).$$

Due to (2.1), we can see that

$$\operatorname{div}(\hat{\mathbf{u}}^0 - \hat{\mathbf{w}}(0)) = 0 \ \text{ in } \Omega, \quad \hat{\mathbf{u}}^0 - \hat{\mathbf{w}}(0) = 0 \ \text{ on } \Gamma_0, \quad \hat{\mathbf{u}}^0 - \hat{\mathbf{w}}(0) = \eta_2^0\vec{e}_2 \ \text{ on } \Gamma_s. \tag{2.4}$$

For $-\omega < 0$, we make the following change of variables:

$$\tilde{\mathbf{u}} = e^{\omega t}\hat{\mathbf{u}}, \quad \tilde{p} = e^{\omega t}\hat{p}, \quad \tilde{\eta}_1 = e^{\omega t}\eta, \quad \tilde{\eta}_2 = e^{\omega t}\eta_t.$$

The system (2.3) is transformed into

$$
\begin{aligned}
&\tilde{\mathbf{u}}_t + e^{-\omega t}(\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}} - \nu\Delta\tilde{\mathbf{u}} - \nabla\tilde{p} - \omega\tilde{\mathbf{u}} = e^{-\omega t}\tilde{F}(\tilde{\mathbf{u}},\tilde{p},\tilde{\eta}_1,\tilde{\eta}_2), \quad \operatorname{div}\tilde{\mathbf{u}} = e^{-\omega t}\tilde{G}(\tilde{\eta}_1,\tilde{\mathbf{u}}) \quad \text{in } Q_\infty,\\
&\tilde{\mathbf{u}} = \tilde{\eta}_2\vec{e}_2 \ \text{ on } \Sigma^s_\infty, \quad \tilde{\mathbf{u}} = 0 \ \text{ on } \Sigma^0_\infty, \quad \tilde{\mathbf{u}}(0) = \hat{\mathbf{u}}^0 \text{ in } \Omega,\\
&\tilde{\eta}_{1,t} = \tilde{\eta}_2 + \omega\tilde{\eta}_1 \quad \text{on } \Sigma^s_\infty,\\
&\tilde{\eta}_{2,t} - \omega\tilde{\eta}_2 - \beta\tilde{\eta}_{1,xx} - \delta\tilde{\eta}_{2,xx} + \alpha\tilde{\eta}_{1,xxxx} = \rho_1\tilde{p} - 2\nu\rho_2\tilde{\mathbf{u}}_{2,z} + e^{-\omega t}\tilde{H}(\tilde{\mathbf{u}},\tilde{\eta}_1) + \tilde{f} \ \text{ on } \Sigma^s_\infty,\\
&\tilde{\eta}_1 = 0 \quad \text{and} \quad \tilde{\eta}_{1,x} = 0 \ \text{ on } \big\{0,L\big\}\times(0,\infty),\\
&\tilde{\eta}_1(0) = \eta^0_1 \quad \text{and} \quad \tilde{\eta}_2(0) = \eta^0_2 \ \text{ in } \Gamma_s,
\end{aligned}
\tag{2.5}
$$

with

$$
\begin{aligned}
&\tilde{f} = e^{\omega t}f,\\
&\tilde{F}(\tilde{\mathbf{u}},\tilde{p},\tilde{\eta}_1,\tilde{\eta}_2) = -\tilde{\eta}_1(\tilde{\mathbf{u}}_t - \omega\tilde{\mathbf{u}}) + \left(z\tilde{\eta}_2 + \nu z\left(\tfrac{\tilde{\eta}^2_{1,x}}{e^{\omega t}+\tilde{\eta}_1} - \tilde{\eta}_{1,xx}\right)\right)\tilde{\mathbf{u}}_z\\
&\quad +\nu\left(-2z\tilde{\eta}_{1,x}\tilde{\mathbf{u}}_{xz} + \tilde{\eta}_1\tilde{\mathbf{u}}_{xx} + \left(\tfrac{z^2\tilde{\eta}^2_{1,x} - e^{-\omega t}\tilde{\eta}_1}{e^{\omega t}+\tilde{\eta}_1}\right)\tilde{\mathbf{u}}_{zz}\right)\\
&\quad +z(\tilde{\eta}_{1,x}\tilde{p}_z - \tilde{\eta}_1\tilde{p}_x)\vec{e}_1 - (1+e^{-\omega t}\tilde{\eta}_1)\tilde{\mathbf{u}}_1\tilde{\mathbf{u}}_x + (ze^{-\omega t}\tilde{\eta}_{1,x}\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2)\tilde{\mathbf{u}}_z,\\
&\tilde{G}(\tilde{\eta}_1,\tilde{\mathbf{u}}) = -\tilde{\eta}_1\tilde{\mathbf{u}}_{1,x} + z\tilde{\eta}_{1,x}\tilde{\mathbf{u}}_{1,z} = \operatorname{div}\left(-\tilde{\eta}_1\tilde{\mathbf{u}}_1\vec{e}_1 + z\tilde{\eta}_{1,x}\tilde{\mathbf{u}}_1\vec{e}_2\right),\\
&\tilde{H}(\tilde{\mathbf{u}},\tilde{\eta}_1) = \nu\left(\tfrac{e^{-\omega t}\tilde{\eta}_{1,x}}{e^{\omega t}+\tilde{\eta}_1}\tilde{\mathbf{u}}_{1,z} + \tilde{\eta}_{1,x}\tilde{\mathbf{u}}_{2,x} - \tfrac{\tilde{\eta}^2_{1,x}}{e^{\omega t}+\tilde{\eta}_1}\tilde{\mathbf{u}}_{2,z} + \tfrac{2e^{-\omega t}\tilde{\eta}_1}{e^{\omega t}+\tilde{\eta}_1}\tilde{\mathbf{u}}_{2,z}\right).
\end{aligned}
$$

If we linearize (2.5) about $(\mathbf{0},0,0,0)$, we obtain the system

$$
\begin{aligned}
&\mathbf{v}_t - \operatorname{div}\sigma(\mathbf{v},p) - \omega\mathbf{v} = 0,\\
&\operatorname{div}\mathbf{v} = 0 \quad \text{in } Q_\infty,\\
&\mathbf{v} = \eta_2\vec{e}_2 \ \text{ on } \Sigma^s_\infty, \quad \mathbf{v} = 0 \ \text{ on } \Sigma^0_\infty, \quad \mathbf{v}(0) = \mathbf{v}^0 \text{ in } \Omega,\\
&\eta_{1,t} = \eta_2 + \omega\eta_1 \quad \text{on } \Sigma^s_\infty,\\
&\eta_{2,t} - \omega\eta_2 - \beta\eta_{1,xx} - \delta\eta_{2,xx} + \alpha M_s\eta_{1,xxxx} = M_s(\rho_1 p - 2\nu\mathbf{v}_{2,z} + f) \ \text{ on } \Sigma^s_\infty,\\
&\eta_1 = 0 \quad \text{and} \quad \eta_{1,x} = 0 \ \text{ on } \big\{0,L\big\}\times(0,\infty),\\
&\eta_1(0) = \eta^0_1 \quad \text{and} \quad \eta_2(0) = \eta^0_2 \ \text{ in } \Gamma_s.
\end{aligned}
\tag{2.6}
$$

Observe that

$$\mathbf{v}_{1,x} + \mathbf{v}_{2,z} = 0 \quad \text{implies} \quad \mathbf{v}_{2,z}|_{\Gamma_s} = 0,$$

if for example $\mathbf{v}$ belongs to $L^2(0,\infty;\mathbf{H}^2(\Omega))$. This is why the term $-2\nu\mathbf{v}_{2,z}$ will be dropped out from the equation satisfied by η_2. Let us notice that $\tilde{\mathbf{u}}_{2,z}$ cannot be dropped out in system (2.5).

3. Definition of an analytic semigroup.

3.1. Transformation of system (2.6). Let us recall that $\mathbf{L}^2(\Omega) = L^2(\Omega;\mathbb{R}^2)$ admits the following orthogonal decomposition

$$\mathbf{L}^2(\Omega) = \mathbf{V}^0_n(\Omega) \oplus \operatorname{grad} H^1(\Omega),$$

with

$$\mathbf{V}^0_n(\Omega) = \Big\{\mathbf{y} \in \mathbf{L}^2(\Omega) \mid \ \operatorname{div}\mathbf{y} = 0, \ \mathbf{y}\cdot\mathbf{n} = \mathbf{0} \text{ on } \Gamma\Big\},$$

and let us denote by $P\ :\ \mathbf{L}^2(\Omega)\longmapsto \mathbf{V}^0_n(\Omega)$ the so-called Leray or Helmholtz projector. We also introduce the notations

$$\mathbf{V}^0(\Omega)=\Big\{\mathbf{y}\in\mathbf{L}^2(\Omega)\mid\ \mathrm{div}\,\mathbf{y}=0\Big\},\quad \mathbf{H}^1_0(\Omega)=H^1_0(\Omega;\mathbb{R}^2),\quad \mathbf{H}^2(\Omega)=H^2(\Omega;\mathbb{R}^2),$$
$$\mathbf{V}^2(\Omega)=\mathbf{H}^2(\Omega)\cap\mathbf{V}^0(\Omega),\quad \mathbf{V}^1_0(\Omega)=\mathbf{H}^1_0(\Omega)\cap\mathbf{V}^0_n(\Omega),\quad \mathbf{V}^{-1}(\Omega)=(\mathbf{V}^1_0(\Omega))',$$
$$L^2_0(\Omega)=\Big\{p\in L^2(\Omega)\mid \textstyle\int_\Omega p=0\Big\},\quad \mathcal{H}^\sigma(\Omega)=H^\sigma(\Omega)\cap L^2_0(\Omega),\quad \mathbf{V}^\sigma_n(\Omega)=\mathbf{H}^\sigma(\Omega)\cap\mathbf{V}^0_n(\Omega)\quad\text{for }\sigma\ge 0,$$
$$\text{for }\sigma<0,\quad \mathcal{H}^\sigma(\Omega)=(\mathcal{H}^{-\sigma}(\Omega))',\quad (\mathcal{H}^{-\sigma}(\Omega))'\text{ is the dual of }\mathcal{H}^{-\sigma}(\Omega)\text{ with }L^2_0(\Omega)\text{ as pivot space},$$
$$L^2_0(\Gamma_s)=\Big\{\eta\in L^2(\Gamma_s)\mid \textstyle\int_{\Gamma_s}\eta=0\Big\},\quad L^2_0(\Gamma)=\Big\{\pi\in L^2(\Gamma)\mid \textstyle\int_\Gamma \pi=0\Big\},$$
$$\mathcal{H}^\sigma(\Gamma_s)=H^\sigma(\Gamma_s)\cap L^2_0(\Gamma_s)\quad\text{and}\qquad \mathcal{H}^\sigma(\Gamma)=H^\sigma(\Gamma)\cap L^2_0(\Gamma)\quad\text{for }\sigma\ge 0,$$
$$\text{for }\sigma<0,\quad \mathcal{H}^\sigma(\Gamma)=(\mathcal{H}^{-\sigma}(\Gamma))'\quad\text{where }(\mathcal{H}^{-\sigma}(\Gamma))'\text{ is the dual of }\mathcal{H}^{-\sigma}(\Gamma)\text{ with }L^2_0(\Gamma)\text{ as pivot space},$$
$$\text{for }\sigma<0,\quad \mathcal{H}^\sigma(\Gamma_s)=(\mathcal{H}^{-\sigma}(\Gamma_s))',\quad (\mathcal{H}^{-\sigma}(\Gamma_s))'\text{ is the dual of }\mathcal{H}^{-\sigma}(\Gamma_s)\text{ with }L^2_0(\Gamma_s)\text{ as pivot space.}$$

We denote by $A_0=\nu P\Delta$ the Stokes operator in $\mathbf{V}^0_n(\Omega)$ with domain

$$D(A_0)=\mathbf{V}^2(\Omega)\cap\mathbf{V}^1_0(\Omega).$$

It is well known that, by the extrapolation method, the Stokes operator can be extended as an unbounded operator in $(\mathbf{V}^2(\Omega)\cap\mathbf{V}^1_0(\Omega))'$ with domain $\mathbf{V}^0_n(\Omega)$. This extension will be still denoted by A_0, and we shall see that it does not lead to confusion. The operator P may also be extended to a bounded operator from $\mathbf{H}^{-1}(\Omega)$ (the dual of $\mathbf{H}^1_0(\Omega)$ with $\mathbf{L}^2(\Omega)$ as pivot space) to $\mathbf{V}^{-1}(\Omega)$ (the dual of $\mathbf{V}^1_0(\Omega)$ with $\mathbf{V}^0_n(\Omega)$ as pivot space) by the formula

$$\langle P\mathbf{u},\mathbf{\Phi}\rangle_{\mathbf{V}^{-1}(\Omega),\mathbf{V}^1_0(\Omega)}=\langle \mathbf{u},\mathbf{\Phi}\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}^1_0(\Omega)}\quad\text{for all}\ \ \mathbf{\Phi}\in\mathbf{V}^1_0(\Omega).$$

In that case P is a projector in $\mathbf{H}^{-1}(\Omega)$ but no longer an orthogonal projector.

We only need to consider system (2.6) in the case when $\omega=0$. Following [22], it is convenient to rewrite the equation satisfied by $\mathbf{v}$ in system (2.6) (for $\omega=0$) into two equations, one satisfied by $P\mathbf{v}$ and the other one by $(I-P)\mathbf{v}$. More precisely we have

$$\begin{aligned}&P\mathbf{v}'=A_0P\mathbf{v}+(-A_0)PD(\eta_2\vec{e}_2\chi_{\Gamma_s}),\quad \mathbf{v}(0)=\mathbf{v}^0\quad\text{in}\quad\Omega,\\ &(I-P)\mathbf{v}(t)=(I-P)D(\eta_2(t)\vec{e}_2\chi_{\Gamma_s}).\end{aligned}$$

In this setting A_0 is the Stokes operator in $(\mathbf{V}^2(\Omega)\cap\mathbf{V}^1_0(\Omega))'$ with domain $\mathbf{V}^0_n(\Omega)$, χ_{Γ_s} denotes the characteristic function of Γ_s, and D is defined by $D\mathbf{g}=\mathbf{w}$, where $(\mathbf{w},q)$ is the solution to the Dirichlet problem

$$-\nu\Delta\mathbf{w}+\nabla q=0\quad\text{and}\qquad \mathrm{div}\,\mathbf{w}=0\quad\text{in}\quad\Omega,\quad \mathbf{w}=\mathbf{g}\quad\text{on}\quad\Gamma.$$

We shall also set

$$D_s\eta_2=D(\eta_2\,\vec{e}_2\,\chi_{\Gamma_s}).$$

This rewriting is a way to eliminate the pressure in the equation satisfied by $\mathbf{v}$. However, since the pressure p also appears in the equation satisfied by η_2, we have to express p in terms of $P\mathbf{v}$ and $(I-P)\mathbf{v}$. For that we can notice that $(I-P)\mathbf{v}$ is the gradient of the function $q\in\mathcal{H}^1(\Omega)$ solution to the Neumann problem

$$\Delta q(t)=0\quad\text{in}\quad\Omega,\quad \frac{\partial q(t)}{\partial\mathbf{n}}=\eta_2(t)\quad\text{on}\quad\Gamma_s,\quad \frac{\partial q(t)}{\partial\mathbf{n}}=0\quad\text{on}\quad\Gamma_0. \tag{3.1}$$

We denote by $N_s\in\mathcal{L}(L^2_0(\Gamma_s),\mathcal{H}^{3/2}(\Omega))$ the operator defined by $N_s\eta_2(t)=q(t)$. In [22] it is shown that the pressure p appearing in the first equation in (2.6) satisfies

$$p=\pi-q_t,$$

where q_t is the time derivative of q and $\pi(t)$ is the solution of the other Neumann problem

$$\Delta\pi(t) = 0 \quad \text{in} \quad \Omega, \qquad \frac{\partial \pi(t)}{\partial \mathbf{n}} = \nu \Delta P\mathbf{v}(t)\cdot\mathbf{n} \quad \text{on} \quad \Gamma. \tag{3.2}$$

Let us notice that $\Delta P\mathbf{v}(t)\cdot\mathbf{n}$ is well defined in $H^{-1/2}(\Gamma)$ if $\Delta P\mathbf{v}(t)$ belongs to $\mathbf{L}^2(\Omega)$. Indeed in that case $\Delta P\mathbf{v}(t) \in L^2(\Omega)$ and $\operatorname{div}(\Delta P\mathbf{v}(t)) = 0 \in L^2(\Omega)$. Moreover $\langle \Delta P\mathbf{v}(t)\cdot\mathbf{n}, 1\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}$ (see [28, Chapter 1, Theorem 1.2]). Therefore if the solution to system (2.6) is such that $P\mathbf{v} \in L^2(0,\infty;\mathbf{V}^2(\Omega))$, the solution π to the above Neumann problem belongs to $L^2(0,\infty;\mathcal{H}^1(\Omega))$. We denote by $N_0 \in \mathcal{L}(\mathcal{H}^{-1/2}(\Gamma),\mathcal{H}^1(\Omega))$ the operator defined by $N_0(\nu\Delta P\mathbf{v}(t)\cdot\mathbf{n}) = \pi(t)$, when $\Delta P\mathbf{v}(t)\cdot\mathbf{n} \in \mathcal{H}^{-1/2}(\Gamma)$. We denote by γ_s the modified trace operator on Γ_s defined by

$$\gamma_s p = M_s(p|_{\Gamma_s}) = p|_{\Gamma_s} - \frac{1}{|\Gamma_s|}\int_{\Gamma_s} p \quad \text{for all } p \in H^\sigma(\Omega) \quad \text{with } \sigma > 1/2.$$

Thus we have

$$M_s(p(t)|_{\Gamma_s}) = M_s\big((\pi(t) - q_t(t))|_{\Gamma_s}\big) = \nu\,\gamma_s N_0 \Delta P\mathbf{v}(t)\cdot\mathbf{n} - \gamma_s N_s \eta_{2,t}(t).$$

We can now rewrite the equation satisfied by η_2 in (2.6) in the form

$$(I + \rho_1\gamma_s N_s)\eta_{2,t} - \omega\eta_2 - \beta\eta_{1,xx} - \delta\eta_{2,xx} + \alpha M_s \eta_{1,xxxx} = \rho_1\nu\gamma_s N_0\Delta P\mathbf{v}(t)\cdot\mathbf{n} + M_s f \quad \text{on } \Sigma_\infty^s.$$

Lemma 3.1. *The operator $I + \rho_1\gamma_s N_s$ is an automorphism in $L^2_0(\Gamma_s)$.*

Proof. The operator $\gamma_s N_s$, considered as an operator belonging to $\mathcal{L}(L^2_0(\Gamma_s))$, is symmetric, positive, and compact. Indeed if $q = N_s\eta$ and $\tilde q = N_s\tilde\eta$, we have

$$0 = \int_\Omega \Delta q\,\tilde q = \int_{\Gamma_s} \eta\,\gamma_s N_s\tilde\eta - \int_{\Gamma_s} \gamma_s N_s\eta\,\tilde\eta,$$

for all η, $\tilde\eta \in L^2_0(\Gamma_s)$. Thus $\gamma_s N_s$ is symmetric. Moreover

$$0 = \int_\Omega \Delta q\, q = -\int_\Omega |\nabla q|^2 + \int_{\Gamma_s} \eta\,\gamma_s N_s \eta,$$

from which we deduce that $\gamma_s N_s$ is nonnegative. If

$$0 = \int_{\Gamma_s} \eta\,\gamma_s N_s\eta = \int_\Omega |\nabla q|^2,$$

we have $q = C = 0$ and $\frac{\partial q}{\partial \mathbf{n}} = \eta = 0$, which proves that $\gamma_s N_s$ is positive. Since $\gamma_s\, N_s \in \mathcal{L}(L^2_0(\Gamma_s), \mathcal{H}^1(\Gamma))$, it is clear that $\gamma_s N_s$ is a compact operator in $L^2_0(\Gamma_s)$. Thus $I + \rho_1\gamma_s N_s$ is symmetric and positive and it is an automorphism in $L^2_0(\Gamma_s)$. ◻

In order to write the system satisfied by $(P\mathbf{v}, \eta_1, \eta_2)$ as an evolution equation, we introduce the unbounded operator $(A_{\alpha,\beta}, D(A_{\alpha,\beta}))$ in $L^2_0(\Gamma_s)$ defined by

$$D(A_{\alpha,\beta}) = H^4(\Gamma_s)\cap H^2_0(\Gamma_s)\cap L^2_0(\Gamma_s), \quad A_{\alpha,\beta}\eta = \beta\eta_{xx} - \alpha M_s\eta_{xxxx}.$$

Let us notice that $(A_{\alpha,\beta}, D(A_{\alpha,\beta}))$ is a selfadjoint operator in $L^2_0(\Gamma_s)$. Since $A_{\alpha,\beta}$ is an isomorphism from $D(A_{\alpha,\beta})$ to $L^2_0(\Gamma_s)$, it can be extended as an isomorphism from $L^2_0(\Gamma_s)$ to $(D(A_{\alpha,\beta}))'$ (the dual of $D(A_{\alpha,\beta})$ with $L^2_0(\Gamma_s)$ as pivot space), and from $H^2_0(\Gamma_s)\cap L^2_0(\Gamma_s)$ into $(H^2_0(\Gamma_s)\cap L^2_0(\Gamma_s))'$. The space

$$\mathbf{H} = \mathbf{V}^0_n(\Omega)\times(H^2_0(\Gamma_s)\cap L^2_0(\Gamma_s))\times L^2_0(\Gamma_s)$$

will be equipped with the inner product

$$\big((\mathbf{v},\eta_1,\eta_2),(\mathbf{w},\zeta_1,\zeta_2)\big)_{\mathbf{H}} = \rho_1(\mathbf{v},\mathbf{w})_{\mathbf{V}^0_n(\Omega)} + (\eta_1,\zeta_1)_{H^2_0(\Gamma_s)} + (\eta_2,\zeta_2)_{L^2_0(\Gamma_s)},$$

with

$$(\eta_1,\zeta_1)_{H_0^2(\Gamma_s)} = \int_{\Gamma_s} (-A_{\alpha,\beta})^{1/2}\eta_1\,(-A_{\alpha,\beta})^{1/2}\zeta_1 = \int_{\Gamma_s} (\beta\eta_{1,x}\zeta_{1,x} + \alpha\eta_{1,xx}\zeta_{1,xx})\,dx.$$

We define the unbounded operator $(\mathcal{A}, D(\mathcal{A}))$ in $\mathbf{H}$ by

$$D(\mathcal{A}) = \Big\{(P\mathbf{v},\eta_1,\eta_2)\in \mathbf{V}_n^2(\Omega)\times(H^4\cap H_0^2\cap L_0^2)(\Gamma_s)\times(H_0^2\cap L_0^2)(\Gamma_s) \mid P\mathbf{v}-PD_s\eta_2\in \mathbf{V}^2(\Omega)\cap\mathbf{V}_0^1(\Omega)\Big\},$$

and

$$\mathcal{A} = \begin{pmatrix} I & 0 & 0\\ 0 & I & 0\\ 0 & 0 & (I+\rho_1\gamma_s N_s)^{-1}\end{pmatrix}\begin{pmatrix} A_0 & 0 & (-A_0)PD_s\\ 0 & 0 & I\\ \rho_1\,\nu\,\gamma_s N_0(\Delta(\cdot)\cdot\mathbf{n}) & A_{\alpha,\beta} & \delta\Delta_s\end{pmatrix},$$

where $\Delta_s = \frac{\partial^2}{\partial x_s^2}$. We define the unbounded operator $(A_s, D(A_s))$ in $H_s = (H_0^2(\Gamma_s)\cap L_0^2(\Gamma_s))\times L_0^2(\Gamma_s)$ by

$$A_s = \begin{pmatrix} 0 & I\\ A_{\alpha,\beta} & \delta\Delta_s\end{pmatrix},\quad D(A_s) = (H^4(\Gamma_s)\cap H_0^2(\Gamma_s)\cap L_0^2(\Gamma_s))\times(H_0^2(\Gamma_s)\cap L_0^2(\Gamma_s)).$$

It can be easily shown that A_s is an isomorphism from $D(A_s)$ into H_s.

Now, it is clear that, for $\omega = 0$, we can rewrite system (2.6) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} P\mathbf{v}\\ \eta_1\\ \eta_2\end{pmatrix} = \mathcal{A}\begin{pmatrix} P\mathbf{v}\\ \eta_1\\ \eta_2\end{pmatrix},\qquad \begin{pmatrix} P\mathbf{v}(0)\\ \eta_1(0)\\ \eta_2(0)\end{pmatrix} = \begin{pmatrix} P\mathbf{v}^0\\ \eta_1^0\\ \eta_2^0\end{pmatrix},$$
$$(I-P)\mathbf{v}(t) = (I-P)D(\eta_2(t)\,\vec{e}_2\,\chi_{\Gamma_s}).$$

The rewriting of system (2.6) when $\omega \neq 0$ is done in (4.1).

PROPOSITION 3.2. *The norm*

$$(P\mathbf{v},\eta_1,\eta_2)\longmapsto \|(P\mathbf{v},\eta_1,\eta_2)\|_{\mathbf{H}} + \|A_0P\mathbf{v}+(-A_0)PD_s\eta_2\|_{\mathbf{V}_n^0(\Omega)} + \|A_s(\eta_1,\eta_2)\|_{H_s}$$

is a norm on $D(\mathcal{A})$ equivalent to the norm

$$(P\mathbf{v},\eta_1,\eta_2)\longmapsto \|P\mathbf{v}\|_{\mathbf{V}_n^2(\Omega)} + \|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H_0^2(\Gamma_s)}.$$

Proof. For $\lambda > 0$, $\lambda I - A_s$ is an isomorphism from $D(A_s)$ to H_s (see e.g. section 3.4). Thus $(\eta_1,\eta_2)\mapsto \|(\eta_1,\eta_2)\|_{H_s} + \|A_s(\eta_1,\eta_2)\|_{H_s}$ is a norm equivalent to $(\eta_1,\eta_2)\mapsto\|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H_0^2(\Gamma_s)}$. Since $(-A_0)$ is an isomorphism from $\mathbf{V}^2(\Omega)\cap\mathbf{V}_0^1(\Omega)$ to $\mathbf{V}_n^0(\Omega)$, there exist positive constants C_1 and C_2 such that

$$C_1\|P\mathbf{v}-PD_s\eta_2\|_{\mathbf{V}_n^2(\Omega)} \le \|A_0P\mathbf{v}+(-A_0)PD_s\eta_2\|_{\mathbf{V}_n^0(\Omega)} \le C_2\|P\mathbf{v}-PD_s\eta_2\|_{\mathbf{V}_n^2(\Omega)}.$$

Moreover $D_s\in\mathcal{L}(H_0^{3/2}(\Gamma_s),\mathbf{V}^2(\Omega))$ (see Lemma 3.10) and $A_s\in\mathcal{L}(D(A_s),H_s)$, therefore we have

$$\begin{aligned}&\|(P\mathbf{v},\eta_1,\eta_2)\|_{\mathbf{H}} + \|A_0P\mathbf{v}+(-A_0)PD_s\eta_2\|_{\mathbf{V}_n^0(\Omega)} + \|A_s(\eta_1,\eta_2)\|_{H_s}\\ &\le C(\|P\mathbf{v}\|_{\mathbf{V}_n^2(\Omega)} + \|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H_0^2(\Gamma_s)}).\end{aligned}$$

To prove the reverse inequality we write

$$\begin{aligned}&\|P\mathbf{v}\|_{\mathbf{V}_n^2(\Omega)} + \|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H_0^2(\Gamma_s)}\\ &\le \tfrac{1}{C_1}\|A_0(P\mathbf{v}-PD_s\eta_2)\|_{\mathbf{V}_n^0(\Omega)} + \|PD_s\eta_2\|_{\mathbf{V}_n^2(\Omega)} + \|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H_0^2(\Gamma_s)}\\ &\le \tfrac{1}{C_1}\|A_0(P\mathbf{v}-PD_s\eta_2)\|_{\mathbf{V}_n^0(\Omega)} + \|\eta_1\|_{H^4(\Gamma_s)} + C\|\eta_2\|_{H_0^2(\Gamma_s)}.\end{aligned}$$

The proof is complete. ☐

THEOREM 3.3. *The operator $(\mathcal{A}, D(\mathcal{A}))$ is the infinitesimal generator of an analytic semigroup on* $\mathbf{H}$*, and the resolvent of $\mathcal{A}$ is compact.*

To prove this theorem, we rewrite $\mathcal{A}$ in the form $\mathcal{A} = \mathcal{A}_1 + B_0$, with

$$\mathcal{A}_1 = \begin{pmatrix} A_0 & 0 & (-A_0)PD_s \\ 0 & 0 & I \\ 0 & A_{\alpha,\beta} & \delta\Delta_s \end{pmatrix}$$

and

$$B_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \rho_1\nu(I + \rho_1\gamma_s N_s)^{-1}\gamma_s N_0(\Delta(\cdot)\cdot\mathbf{n})) & K_s\, A_{\alpha,\beta} & \delta K_s\,\Delta_s \end{pmatrix},$$

with $K_s = (I + \rho_1\gamma_s N_s)^{-1} - I$.

THEOREM 3.4. *The operator $(\mathcal{A}_1, D(\mathcal{A}_1))$ is the infinitesimal generator of a strongly continuous semigroup on* $\mathbf{H}$.

Proof. Step 1. We first show that the unbounded operator $(\widetilde{\mathcal{A}}_1, D(\widetilde{\mathcal{A}}_1))$ in $\mathbf{V}^{-1}(\Omega)\times H_s$, defined by

$$D(\widetilde{\mathcal{A}}_1) = \Big\{(P\mathbf{v},\eta_1,\eta_2)\in \mathbf{V}^1_n(\Omega)\times(H^4\cap H^2_0\cap L^2_0)(\Gamma_s)\times(H^2_0\cap L^2_0)(\Gamma_s) \mid P\mathbf{v} - PD_s\eta_2\in\mathbf{V}^1_0(\Omega)\Big\}$$

$$\text{and}\qquad \widetilde{\mathcal{A}}_1 = \begin{pmatrix} A_0 & 0 & (-A_0)PD_s \\ 0 & 0 & I \\ 0 & A_{\alpha,\beta} & \delta\Delta_s \end{pmatrix},$$

is the infinitesimal generator of a strongly continuous semigroup on $\mathbf{V}^{-1}(\Omega)\times H_s$. We endow $\mathbf{V}^{-1}(\Omega)$ with the norm

$$\mathbf{v} \longmapsto \Big(\big\langle(-A_0)^{-1}\mathbf{v},\mathbf{v}\big\rangle_{\mathbf{V}^1_0(\Omega),\mathbf{V}^{-1}(\Omega)}\Big)^{1/2},$$

and H_s with the norm $\|\cdot\|_{H^2_0(\Gamma_s)\times L^2_0(\Gamma_s)}$. For $\lambda>0$, we have

$$\begin{aligned}&\Big((\widetilde{\mathcal{A}}_1-\lambda I)(P\mathbf{v},\eta_1,\eta_2),(P\mathbf{v},\eta_1,\eta_2)\Big)_{\mathbf{V}^{-1}(\Omega)\times H_s}\\ &= -\|P\mathbf{v}\|^2_{\mathbf{V}^0_n(\Omega)} + (PD_s\eta_2,P\mathbf{v})_{\mathbf{V}^0_n(\Omega)} - \lambda\|P\mathbf{v}\|^2_{\mathbf{V}^{-1}(\Omega)} - \lambda\|(\eta_1,\eta_2)\|^2_{H_s} - \delta\|\eta_2\|_{L^2_0(\Gamma_s)}.\end{aligned}$$

Thus, for $\lambda>0$ big enough, $(\widetilde{\mathcal{A}}_1-\lambda I, D(\widetilde{\mathcal{A}}_1))$ is dissipative in $\mathbf{V}^{-1}(\Omega)\times H_s$. It can also be shown that it is maximal. Thus, for $\lambda>0$ big enough, $(\widetilde{\mathcal{A}}_1-\lambda I, D(\widetilde{\mathcal{A}}_1))$ is the infinitesimal generator of a semigroup of contractions on $\mathbf{V}^{-1}(\Omega)\times H_s$, and $(\widetilde{\mathcal{A}}_1, D(\widetilde{\mathcal{A}}_1))$ is the infinitesimal generator of a strongly continuous semigroup on $\mathbf{V}^{-1}(\Omega)\times H_s$.

Step 2. Let us consider the evolution equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} P\mathbf{v}\\ \eta_1\\ \eta_2\end{pmatrix} = \widetilde{\mathcal{A}}_1\begin{pmatrix} P\mathbf{v}\\ \eta_1\\ \eta_2\end{pmatrix},\qquad \begin{pmatrix} P\mathbf{v}(0)\\ \eta_1(0)\\ \eta_2(0)\end{pmatrix} = \begin{pmatrix} P\mathbf{v}^0\\ \eta_1^0\\ \eta_2^0\end{pmatrix}. \tag{3.3}$$

Let us recall that $(A_s, D(A_s))$ is the infinitesimal generator of an analytic semigroup on H_s (see e.g. [9, 29]). Let us notice that the solution $(P\mathbf{v},\eta_1,\eta_2)$ to equation (3.3) can be solved by first determining (η_1,η_2) and next $P\mathbf{v}$. Thus, if $(P\mathbf{v}^0,\eta_1^0,\eta_2^0)\in\mathbf{V}^{-1}(\Omega)\times H_s$, the solution $(P\mathbf{v},\eta_1,\eta_2)$ to equation (3.3) is such that $\eta_1\in H^{3,3/2}(\Sigma^s_T)$ and $\eta_2\in H^{1,1/2}(\Sigma^s_T)$ for all $T>0$ (see e.g. [1, Chapter 3, Corollary 2.1]).

From [22, Theorem 2.7] it follows that if $(P\mathbf{v}^0, \eta_1^0, \eta_2^0) \in \mathbf{H}$ then $P\mathbf{v} \in \mathbf{H}^{1,1/2}(Q_T) \cap C([0,T]; \mathbf{V}_n^0(\Omega))$, and $(P\mathbf{v}, \eta_1, \eta_2) \in C([0,T]; \mathbf{H})$. Therefore the restriction of the semigroup $(e^{t\widetilde{\mathcal{A}}_1})_{t\in\mathbb{R}^+}$ to $\mathbf{H}$ is a semigroup on $\mathbf{H}$. It is easy to verify that its domain is $D(\mathcal{A}_1) = D(\mathcal{A})$. □

We are going to prove the two following theorems.

THEOREM 3.5. *The operator $(\mathcal{A}_1, D(\mathcal{A}_1))$, with $D(\mathcal{A}_1) = D(\mathcal{A})$, is the infinitesimal generator of an analytic semigroup on $\mathbf{H} = \mathbf{V}_n^0(\Omega) \times (H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)) \times L_0^2(\Gamma_s)$.*

THEOREM 3.6. *The operator $(B_0, D(\mathcal{A}_1))$ is $\mathcal{A}_1$-bounded with relative bound zero.*

The first claim in Theorem 3.3 clearly follows from Theorems 3.5 and 3.6 (see [14, Chapter 9, Corollary 2.5]). The second claim is proved in section 3.4.

3.2. Proof of Theorem 3.5. Now we are going to estimate the resolvent of $\mathcal{A}_1$. We have

$$(\lambda I - \mathcal{A}_1)^{-1} = \begin{pmatrix} (\lambda I - A_0)^{-1} & 0 \quad \big((\lambda I - A_0)^{-1}(-A_0)PD_s\big)(\lambda I - A_s)^{-1} \\ 0 & (\lambda I - A_s)^{-1} \end{pmatrix}.$$

Since $(\lambda I - A_0)^{-1}(-A_0)PD_s = -\lambda(\lambda I - A_0)^{-1}PD_s + PD_s$, we obtain

$$(\lambda I - \mathcal{A}_1)^{-1} = \begin{pmatrix} (\lambda I - A_0)^{-1} & 0 \quad \big(-\lambda(\lambda I - A_0)^{-1}PD_s + PD_s\big)(\lambda I - A_s)^{-1} \\ 0 & (\lambda I - A_s)^{-1} \end{pmatrix}.$$

From [9] (see also [29, section 2.2]), we know that there exist $a \in \mathbb{R}$ and $\pi/2 < \theta_0 < \pi$ such that

$$\|(\lambda I - A_s)^{-1}\|_{\mathcal{L}(H_s)} \le \frac{C_s}{|\lambda - a|} \quad \text{for all } \lambda \in S_{a,\theta_0}, \tag{3.4}$$

where

$$S_{a,\theta_0} = \Big\{\lambda \in \mathbb{C} \mid \lambda \ne a, \quad |\arg(\lambda - a)| < \theta_0\Big\}.$$

For the Stokes resolvent we have

$$\|(\lambda I - A_0)^{-1}\mathbf{f}\|_{\mathbf{V}_n^0(\Omega)} \le \frac{C_0}{|\lambda|}\|\mathbf{\Theta}\|_{\mathbf{V}_n^0(\Omega)} \quad \text{for all } \lambda \in S_{0,\theta_1}, \tag{3.5}$$

with $\pi/2 < \theta_1 < \pi$. We can choose $\theta_0 = \theta_1$ and $a > 0$. Thus if $(\mathbf{f}, \mathbf{\Theta}) \in \mathbf{V}_n^0(\Omega) \times H_s$, we have

$$\begin{aligned} &(\lambda I - \mathcal{A}_1)^{-1} \begin{pmatrix} \mathbf{f} \\ \mathbf{\Theta} \end{pmatrix} \\ &= \begin{pmatrix} (\lambda I - A_0)^{-1}\mathbf{f} - \lambda(\lambda I - A_0)^{-1}PD_s\left((\lambda I - A_s)^{-1}\mathbf{\Theta}\right)_2 + PD\left((\lambda I - A_s)^{-1}\mathbf{\Theta}\right)_2 \\ (\lambda I - A_s)^{-1}\mathbf{\Theta} \end{pmatrix}. \end{aligned}$$

From (3.4) and (3.5), it follows that

$$\begin{aligned} &\|(\lambda I - A_s)^{-1}\mathbf{\Theta}\|_{H_s} \le \tfrac{C_s}{|\lambda - a|}\|\mathbf{\Theta}\|_{H_s}, \quad \|PD\left((\lambda I - A_s)^{-1}\mathbf{\Theta}\right)_2\|_{\mathbf{V}_n^0(\Omega)} \le \tfrac{C_{PD}\,C_s}{|\lambda - a|}\|\mathbf{\Theta}\|_{H_s}, \\ &\|\lambda(\lambda I - A_0)^{-1}PD\left((\lambda I - A_s)^{-1}\mathbf{\Theta}\right)_2\|_{\mathbf{V}_n^0(\Omega)} \le \tfrac{C_0\,C_{PD}\,C_s}{|\lambda - a|}\|\mathbf{\Theta}\|_{H_s} \quad \text{for all } \lambda \in S_{a,\theta_0}. \end{aligned}$$

By combining the previous estimates we obtain

$$\left\|(\lambda I - \mathcal{A}_1)^{-1}\begin{pmatrix} \mathbf{f} \\ \mathbf{\Theta} \end{pmatrix}\right\|_{\mathbf{V}_n^0(\Omega)\times H_s} \le \frac{C_0}{|\lambda|}\|\mathbf{f}\|_{\mathbf{V}_n^0(\Omega)} + \frac{C_0\,C_{PD}\,C_s}{|\lambda - a|}\|\mathbf{\Theta}\|_{H_s} + \frac{C_{PD}\,C_s}{|\lambda - a|}\|\mathbf{\Theta}\|_{H_s} + \frac{C_s}{|\lambda - a|}\|\mathbf{\Theta}\|_{H_s},$$

for all $\lambda \in S_{a,\theta_0}$, which proves the analyticity of the semigroup generated by $\mathcal{A}_1$.

3.3. Proof of Theorem 3.6. We set

$$B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \rho_1\nu\,(I+\rho_1\gamma_s N)^{-1}\gamma_s N_0(\Delta(\cdot)\cdot\mathbf{n}) & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & K_s\,A_{\alpha,\beta} & 0 \end{pmatrix},$$

and

$$B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \delta K_s\,\Delta_s \end{pmatrix}.$$

LEMMA 3.7. *The operator $(B_1, D(\mathcal{A}_1))$ is $\mathcal{A}_1$-bounded with relative bound zero.*

Proof. Let us prove that, for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\|\gamma_s N_0(\Delta\mathbf{v}\cdot\mathbf{n})\|_{L^2_0(\Gamma_s)} \le \varepsilon\|\mathbf{v}\|_{\mathbf{V}^2_n(\Omega)} + C_\varepsilon\|\mathbf{v}\|_{\mathbf{V}^0_n(\Omega)}, \tag{3.6}$$

for all $\mathbf{v}\in\mathbf{V}^2_n(\Omega)$. To prove (3.6), we argue by contradiction. We assume that there exists a sequence $(\mathbf{v}_k)_k \subset \mathbf{V}^2_n(\Omega)$ such that

$$\|\gamma_s N_0(\Delta\mathbf{v}_k\cdot\mathbf{n})\|_{L^2_0(\Gamma_s)} = 1, \quad \|\mathbf{v}_k\|_{\mathbf{V}^0_n(\Omega)} \longrightarrow 0 \quad\text{and}\quad \|\mathbf{v}_k\|_{\mathbf{V}^2_n(\Omega)} \le M,$$

for some $M > 0$. Therefore, without loss of generality, we can assume that there exists $\mathbf{v}\in\mathbf{V}^2_n(\Omega)$ such that

$$\mathbf{v}_k \rightharpoonup \mathbf{0} \quad\text{in } \mathbf{V}^2_n(\Omega), \qquad \Delta\mathbf{v}_k\cdot\mathbf{n} \rightharpoonup \mathbf{0} \quad\text{in } H^{-1/2}(\Gamma) \quad\text{and}\quad \Delta\mathbf{v}_k\cdot\mathbf{n} \longrightarrow 0 \quad\text{in } H^{-1/2-\varepsilon}(\Gamma),$$

for all $0 < \varepsilon \le 1/2$. From [7, Lemma A.5], we know that $\gamma_s N_0$ is bounded from $H^{-1}(\Gamma_s)$ to $L^2_0(\Gamma_s)$. Thus

$$\gamma_s N_0(\Delta\mathbf{v}_k\cdot\mathbf{n}) \longrightarrow 0 \quad\text{in } L^2_0(\Gamma_s),$$

which is in contradiction with

$$\|\gamma_s N_0(\Delta\mathbf{v}_k\cdot\mathbf{n})\|_{L^2_0(\Gamma_s)} = 1.$$

Thus (3.6) is proved. The lemma is a direct consequence of (3.6), of Lemma 3.1 and Proposition 3.2. ∎

LEMMA 3.8. *There exists $0 < \theta_2 < 1$ such that B_2 is bounded from $D((-\mathcal{A}_1)^{\theta_2})$ into $\mathbf{H}$.*

Proof. Let $(\phi_k)_{k\ge1}$ be an orthonormal basis in $L^2_0(\Gamma_s)$ constituted of eigenvectors of the operator $\rho_1\gamma_s N$ and let $\lambda_k > 0$ be the eigenvalue associated with ϕ_k. We have

$$(I+\rho_1\gamma_s N_s)f = \sum_{k=1}^{\infty}(1+\lambda_k)f_k\,\phi_k.$$

Thus

$$(I+\rho_1\gamma_s N_s)^{-1}f = \sum_{k=1}^{\infty}\frac{f_k}{1+\lambda_k}\,\phi_k,$$

and

$$K_s f = (I-(I+\rho_1\gamma_s N_s)^{-1})f = \sum_{k=1}^{\infty}\frac{\lambda_k}{1+\lambda_k}f_k\,\phi_k.$$

Since the operator $A_{\alpha,\beta}$ is an isomorphism from $H^4(\Gamma_s)\cap H^2_0(\Gamma_s)\cap L^2_0(\Gamma_s)$ into $L^2_0(\Gamma_s)$ and from $L^2_0(\Gamma_s)$ into $(H^4(\Gamma_s)\cap H^2_0(\Gamma_s)\cap L^2_0(\Gamma_s))'$, by interpolation it is also continuous from $H^{4-\varepsilon}(\Gamma_s)\cap H^2_0(\Gamma_s)\cap L^2_0(\Gamma_s)$ into $\mathcal{H}^{-\varepsilon}(\Gamma_s)$ for all $0\le\varepsilon\le 1$.

Denoting by $(A_{\alpha,\beta}f)_k$ the coefficient of $A_{\alpha,\beta}f$ in the basis $(\phi_k)_{k\geq 1}$, we have

$$\begin{aligned}
&\|K_s A_{\alpha,\beta}f\|^2_{L^2_0(\Gamma_s)} = \textstyle\sum_{k=1}^{\infty} \frac{\lambda_k^2}{(1+\lambda_k)^2}(A_{\alpha,\beta}f)^2_k \leq \sum_{k=1}^{\infty} \lambda_k^2 (A_{\alpha,\beta}f)^2_k = \|\rho_1\gamma_s N_s A_{\alpha,\beta}f\|^2_{L^2_0(\Gamma_s)} \\
&\leq C_\varepsilon \|A_{\alpha,\beta}f\|^2_{\mathcal{H}^{-\varepsilon}(\Gamma_s)} \leq C_\varepsilon \|f\|^2_{H^{4-\varepsilon}(\Gamma_s)},
\end{aligned}$$

for all $f \in H^{4-\varepsilon}(\Gamma_s) \cap H^2_0(\Gamma_s) \cap L^2_0(\Gamma_s)$ and all $0 \leq \varepsilon < 1/2$. Indeed $\gamma_s N_s$ is continuous from $\mathcal{H}^{-\varepsilon}(\Gamma_s)$ into $L^2_0(\Gamma_s)$ if $0 \leq \varepsilon < 1/2$ (see e.g. [7, Lemma A.5]). Since

$$\big(H^{4-\varepsilon}(\Gamma_s) \cap H^2_0(\Gamma_s) \cap L^2_0(\Gamma_s)\big) \supset D((-\mathcal{A}_1)^{(4-\varepsilon')/4}),$$

for all $0 \leq \varepsilon' < \varepsilon < 1/2$, the proof is complete. □

LEMMA 3.9. *There exists $0 < \theta_3 < 1$ such that B_3 is bounded from $D((-\mathcal{A}_1)^{\theta_3})$ into $\mathbf{H}$.*

Proof. The proof is very similar to that of the previous Lemma and is left to the reader. □

Theorem 3.6 is a direct consequence of Lemmas 3.7, 3.8 and 3.9.

3.4. Resolvent of $\mathcal{A}$. In this section we want to show that the resolvent of $\mathcal{A}$ is compact. For that we study the stationary problem

$$\begin{aligned}
&\lambda\mathbf{v} - \operatorname{div}\sigma(\mathbf{v},p) = \mathbf{f} \quad \text{and} \quad \operatorname{div}\mathbf{v} = 0 \quad \text{in } \Omega, \\
&\mathbf{v} = \eta_2\vec{e}_2 \ \text{ on } \ \Gamma_s, \quad \mathbf{v} = 0 \ \text{ on } \ \Gamma_0, \\
&\lambda\eta_1 - \eta_2 = g \quad \text{in } \Gamma_s, \\
&\lambda\eta_2 - \beta\eta_{1,xx} - \delta\eta_{2,xx} + \alpha M_s\eta_{1,xxxx} = M_s(\rho_1 p + h) \ \text{ in } \Gamma_s, \\
&\eta_1 = 0 \quad \text{and} \quad \eta_{1,x} = 0 \ \text{ on } \big\{0, L\big\},
\end{aligned} \tag{3.7}$$

where $\mathbf{f} \in \mathbf{V}^0_n(\Omega)$, $g \in H^2_0(\Gamma_s) \cap L^2_0(\Gamma_s)$, $h \in L^2_0(\Gamma_s)$, $\lambda \in \mathbb{R}$ and $\lambda > 0$. This system is equivalent to

$$\begin{aligned}
&\lambda\mathbf{v} - \operatorname{div}\sigma(\mathbf{v},p) = \mathbf{f} \quad \text{and} \quad \operatorname{div}\mathbf{v} = 0 \quad \text{in } \Omega, \\
&\mathbf{v} = (\lambda\eta_1 - g)\vec{e}_2 \ \text{ on } \ \Gamma_s, \quad \mathbf{v} = 0 \ \text{ on } \ \Gamma_0, \\
&\lambda\eta_1 - \eta_2 = g \quad \text{in } \Gamma_s, \\
&\lambda^2\eta_1 - \beta\eta_{1,xx} - \delta\lambda\eta_{1,xx} + \alpha M_s\eta_{1,xxxx} = M_s(\rho_1 p + h + \lambda g - \delta\lambda g_{xx}) \ \text{ in } \Gamma_s, \\
&\eta_1 = 0 \quad \text{and} \quad \eta_{1,x} = 0 \ \text{ on } \big\{0, L\big\}.
\end{aligned} \tag{3.8}$$

We denote by L the unbounded operator in $L^2_0(\Gamma_s)$ with domain $H^4(\Gamma_s) \cap H^2_0(\Gamma_s) \cap L^2_0(\Gamma_s)$ defined by

$$L\eta = \lambda^2\eta - \beta\eta_{xx} - \delta\lambda\eta_{xx} + \alpha M_s\eta_{xxxx}.$$

The operator L is also an isomorphism from $H^4(\Gamma_s) \cap H^2_0(\Gamma_s) \cap L^2_0(\Gamma_s)$ into $L^2_0(\Gamma_s)$ and from $H^2_0(\Gamma_s) \cap L^2_0(\Gamma_s)$ into $(H^2_0(\Gamma_s) \cap L^2_0(\Gamma_s))'$. Thus, we can rewrite the system (3.8) in the form

$$\begin{aligned}
&\lambda\mathbf{v} - \operatorname{div}\sigma(\mathbf{v},p) = \mathbf{f} \quad \text{and} \quad \operatorname{div}\mathbf{v} = 0 \quad \text{in } \Omega, \\
&\mathbf{v} = (\lambda L^{-1}M_s(\rho_1\gamma_s p + h + \lambda g - \delta\lambda g_{xx}) - g)\vec{e}_2 \ \text{ on } \ \Gamma_s, \quad \mathbf{v} = 0 \ \text{ on } \ \Gamma_0, \\
&\lambda\eta_1 - \eta_2 = g \quad \text{in } \Gamma_s, \\
&\lambda^2\eta_1 - \beta\eta_{1,xx} - \delta\lambda\eta_{1,xx} + \alpha M_s\eta_{1,xxxx} = \rho_1\gamma_s p + h + \lambda g - \delta\lambda g_{xx} \ \text{ in } \Gamma_s, \\
&\eta_1 = 0 \quad \text{and} \quad \eta_{1,x} = 0 \ \text{ on } \big\{0, L\big\}.
\end{aligned} \tag{3.9}$$

We consider the system

$$\begin{aligned}
&\lambda\mathbf{v} - \operatorname{div}\sigma(\mathbf{v},p) = \mathbf{f} \quad \text{and} \quad \operatorname{div}\mathbf{v} = 0 \quad \text{in } \Omega, \\
&\mathbf{v} = \lambda\rho_1 L^{-1}(\gamma_s p)\vec{e}_2 + f\vec{e}_2 \ \text{ on } \ \Gamma_s, \quad \mathbf{v} = 0 \ \text{ on } \ \Gamma_0,
\end{aligned} \tag{3.10}$$

where $f \in H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)$ stands for $\lambda L^{-1} M_s(h + \lambda g - \delta\lambda g_{xx}) - g$. We set

$$\mathbf{E} = \Big\{\mathbf{w} \in \mathbf{V}^1(\Omega) \mid \mathbf{v} = 0 \text{ on } \Gamma_0,\ \mathbf{v}_1 = 0 \text{ on } \Gamma_s,\ \mathbf{v}_2|_{\Gamma_s} \in H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)\Big\}.$$

The space $\mathbf{E}$, equipped with the norm

$$\|\mathbf{v}\|_{\mathbf{E}} = \Big(\|\mathbf{v}\|^2_{\mathbf{V}^1(\Omega)} + \|L^{1/2}\mathbf{v}_2|_{\Gamma_s}\|^2_{L_0^2(\Gamma_s)}\Big)^{1/2},$$

is a Hilbert space because $L^{1/2}$ is an isomorphism from $H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)$ onto $L_0^2(\Gamma_s)$.

Multiplying the first equation in (3.10) by $\mathbf{w} \in \mathbf{E}$, after integration we obtain

$$\int_\Omega (\lambda \mathbf{v} \cdot \mathbf{w} + \nu \nabla \mathbf{v} : \nabla \mathbf{w}) + \int_{\Gamma_s} p\, \mathbf{w}_2 = \int_\Omega \mathbf{f}\, \mathbf{w}.$$

Using

$$\lambda \rho_1 \gamma_s p = L\mathbf{v}_2 - Lf \quad \text{in } (H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s))',$$

we obtain

$$\int_\Omega (\lambda \mathbf{v} \cdot \mathbf{w} + \nu \nabla \mathbf{v} : \nabla \mathbf{w}) + \frac{1}{\lambda\rho_1} \int_{\Gamma_s} L^{1/2}\mathbf{v}_2\, L^{1/2}\mathbf{w}_2 = \int_\Omega \mathbf{f}\, \mathbf{w} + \frac{1}{\lambda\rho_1} \int_{\Gamma_s} L^{1/2} f\, L^{1/2}\mathbf{w}_2.$$

Next, we set

$$a(\mathbf{v}, \mathbf{w}) = \int_\Omega (\lambda \mathbf{v} \cdot \mathbf{w} + \nabla \mathbf{v} : \nabla \mathbf{w}) + \frac{1}{\lambda\rho_1} \int_{\Gamma_s} L^{1/2}\mathbf{v}_2\, L^{1/2}\mathbf{w}_2$$

and

$$\ell(\mathbf{w}) = \int_\Omega \mathbf{f}\, \mathbf{w} + \frac{1}{\lambda\rho_1} \int_{\Gamma_s} L^{1/2} f\, L^{1/2}\mathbf{w}_2.$$

Thus system (3.10) is equivalent to

$$\begin{aligned} &a(\mathbf{v}, \mathbf{w}) = \ell(\mathbf{w}) \quad \text{for all } \mathbf{w} \in \mathbf{E}, \\ &\lambda \rho_1 \gamma_s p = L\mathbf{v}_2 - Lf \quad \text{in } (H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s))'. \end{aligned} \tag{3.11}$$

With the Lax-Milgram theorem, we can prove that the variational problem

$$\text{Find } \mathbf{v} \in \mathbf{E} \text{ such that} \qquad a(\mathbf{v}, \mathbf{w}) = \ell(\mathbf{w}) \quad \text{for all } \mathbf{w} \in \mathbf{E}, \tag{3.12}$$

has a unique solution. Indeed, for all $H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)$, we have

$$\int_{\Gamma_s} L^{1/2}\eta\, L^{1/2}\eta = \int_{\Gamma_s} \big(\lambda^2|\eta|^2 + \beta|\eta_x|^2 + \alpha|\eta_{xx}|^2\big) \geq \rho\|\eta\|^2_{H_0^2(\Gamma_s)},$$

for some $\rho > 0$.

The solution $\mathbf{v} \in \mathbf{E}$ to the above variational problem obeys

$$\|\mathbf{v}\|_{\mathbf{E}} \leq C(\|\mathbf{f}\|_{\mathbf{V}_0^n(\Omega)} + \|L^{1/2} f\|_{L^2(\Gamma_s)}).$$

Since $f = \lambda L^{-1} M_s(h + \lambda g - \delta\lambda g_{xx}) - g$, we have

$$\|\mathbf{v}\|_{\mathbf{E}} \leq C(\|\mathbf{f}\|_{\mathbf{V}_0^n(\Omega)} + \|L^{-1/2} h\|_{L_0^2(\Gamma_s)} + \|L^{1/2} g\|_{L_0^2(\Gamma_s)}) \leq C(\|\mathbf{f}\|_{\mathbf{V}_0^n(\Omega)} + \|h\|_{L_0^2(\Gamma_s)} + \|g\|_{H_0^2(\Gamma_s)}).$$

Therefore

$$\|\mathbf{v}_2|_{\Gamma_s}\|_{H_0^2(\Gamma_s)} \leq C(\|\mathbf{f}\|_{\mathbf{V}_0^n(\Omega)} + \|h\|_{L_0^2(\Gamma_s)} + \|g\|_{H_0^2(\Gamma_s)}).$$

By taking $\mathbf{w} \in \mathbf{V}_0^1(\Omega)$ in the variational problem, we prove that $\mathbf{v} \in \mathbf{E}$ is the unique solution to the problem

$$\text{Find } \mathbf{v} \in \mathbf{E} \text{ such that } \quad \int_\Omega (\lambda \mathbf{v} \cdot \mathbf{w} + \nabla \mathbf{v} : \nabla \mathbf{w}) = \int_\Omega \mathbf{f}\, \mathbf{w} \quad \text{for all } \mathbf{w} \in \mathbf{V}_0^1(\Omega),$$
$$\mathbf{v} = 0 \quad \text{on} \quad \Gamma_0, \qquad \mathbf{v} = \mathbf{v}_2|_{\Gamma_s} \vec{e}_2 \quad \text{on} \quad \Gamma_s.$$

Since $\mathbf{v}|_{\Gamma_0} = \mathbf{0}$, $\mathbf{v}_1|_{\Gamma_s} = 0$, and $\mathbf{v}_2|_{\Gamma_s} \in H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)$, due to Lemma 3.10 below, it follows that $\mathbf{v} \in \mathbf{V}^2(\Omega) \cap \mathbf{E}$, and that

$$\|\mathbf{v}\|_{\mathbf{V}^2(\Omega)} \le C(\|\mathbf{f}\|_{\mathbf{V}_n^0(\Omega)} + \|g\|_{H_0^2(\Gamma_s)} + \|h\|_{L_0^2(\Gamma_s)}).$$

From the equation satisfied by $\mathbf{v}$ we also deduce that $p \in \mathcal{H}^1(\Omega)$, and

$$\|p\|_{\mathcal{H}^1(\Omega)} \le C(\|\mathbf{f}\|_{\mathbf{V}_n^0(\Omega)} + \|g\|_{H_0^2(\Gamma_s)} + \|h\|_{L_0^2(\Gamma_s)}).$$

Finally, with the equation satisfied by η_1 and η_2 in (3.9), we have shown that system (3.7) admits a unique solution $(\mathbf{v}, p, \eta_1, \eta_2) \in \mathbf{V}^2(\Omega) \times \mathcal{H}^1(\Omega) \times (H^4(\Gamma_s) \cap H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)) \times (H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s))$ and

$$\|\mathbf{v}\|_{\mathbf{V}^2(\Omega)} + \|p\|_{\mathcal{H}^1(\Omega)} + \|\eta_1\|_{H^4(\Gamma_s)} + \|\eta_2\|_{H_0^2(\Gamma_s)} \le C(\|\mathbf{f}\|_{\mathbf{V}_n^0(\Omega)} + \|g\|_{H_0^2(\Gamma_s)} + \|h\|_{L_0^2(\Gamma_s)}).$$

Thus the resolvent of $\mathcal{A}$ is compact in $\mathbf{H}$.

LEMMA 3.10. *If* $\mathbf{f} \in \mathbf{V}_n^0(\Omega)$, $g \in H_0^{3/2}(\Gamma_s) \cap L_0^2(\Gamma_s)$ *(with* $H_0^{3/2}(\Gamma_s) = [H_0^1(\Gamma_s), H_0^2(\Gamma_s)]_{1/2}$*), then the solution* $\mathbf{v}$ *to*

$$\lambda \mathbf{v} - \operatorname{div} \sigma(\mathbf{v}, p) = \mathbf{f} \quad \text{and} \quad \operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega, \quad \mathbf{v} = 0 \quad \text{on} \quad \Gamma_0, \quad \mathbf{v} = g\, \vec{e}_2 \quad \text{on} \quad \Gamma_s,$$

belongs to $\mathbf{V}^2(\Omega)$ *and*

$$\|\mathbf{v}\|_{\mathbf{V}^2(\Omega)} \le C(\|\mathbf{f}\|_{\mathbf{V}_n^0(\Omega)} + \|g\|_{H_0^{3/2}(\Gamma_s)}).$$

Proof. With a localization argument and regularity results in [11], we can show that $\mathbf{v}|_{(0,L)\times(0,1-\epsilon)}$ belongs to $\mathbf{V}^2((0,L) \times (0, 1-\epsilon))$ and that $\mathbf{v}|_{(\epsilon, L-\epsilon)\times(0,1)}$ belongs to $\mathbf{V}^2((\epsilon, L-\epsilon) \times (0,1))$ for all $0 < \epsilon < \min(1, L)$. Thus the only difficulty is at the corners $(0,1)$ and $(L,1)$. Let us set

$$\tilde{\Omega} = (-L, L) \times (0,1), \quad \tilde{\Gamma}_s = (-L, L) \times \{1\}, \quad \tilde{\mathbf{v}}(x,z) = \begin{cases} \mathbf{v}(x,z) & \text{if} \quad x \in (0,L), \\ -\mathbf{v}(x,-z) & \text{if} \quad x \in (-L,0), \end{cases}$$
$$\tilde{\mathbf{f}}(x,z) = \begin{cases} \mathbf{f}(x,z) & \text{if} \quad x \in (0,L), \\ -\mathbf{f}(x,-z) & \text{if} \quad x \in (-L,0), \end{cases} \qquad \tilde{g}(x) = \begin{cases} g(x) & \text{if} \quad x \in (0,L), \\ -g(-z) & \text{if} \quad x \in (-L,0). \end{cases}$$

It can be shown that $\tilde{g} \in H_0^{3/2}(\tilde{\Gamma}_s)$, and that $\tilde{\mathbf{v}}$ is solution to

$$\lambda \tilde{\mathbf{v}} - \operatorname{div} \sigma(\tilde{\mathbf{v}}, \tilde{p}) = \mathbf{f} \quad \text{and} \quad \operatorname{div} \tilde{\mathbf{v}} = 0 \quad \text{in } \tilde{\Omega},$$
$$\tilde{\mathbf{v}} = 0 \quad \text{on} \quad \{-L\} \times (0,1) \cup \{L\} \times (0,1) \cup (-L,L) \times \{0\}, \quad \tilde{\mathbf{v}} = \tilde{g}\, \vec{e}_2 \quad \text{on} \quad \tilde{\Gamma}_s.$$

Next as previously we can show that $\tilde{\mathbf{v}}|_{(-L+\epsilon, L-\epsilon)\times(0,1)}$ belongs to $\mathbf{V}^2((-L+\epsilon, L-\epsilon) \times (0,1))$ for all $0 < \epsilon < L$. Thus $\mathbf{v}|_{(0,L-\epsilon)\times(0,1)}$ belongs to $\mathbf{V}^2((0, L-\epsilon) \times (0,1))$ for all $0 < \epsilon < L$. We can proceed similarly with the other corner.
∎

3.5. Adjoint of $(\mathcal{A}, D(\mathcal{A}))$. THEOREM 3.11. *The adjoint of* $(\mathcal{A}, D(\mathcal{A}))$ *in* $\mathbf{H}$ *is defined by* $D(\mathcal{A}^*) = D(\mathcal{A})$ *and*

$$\mathcal{A}^* = \begin{pmatrix} A_0 & 0 & (-A_0)PD_s \\ 0 & 0 & -I \\ \rho_1 \nu \gamma_s N_0(\Delta(\cdot) \cdot \mathbf{n}) & -A_{\alpha,\beta} & \delta \Delta_s \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (I + \rho_1 \gamma_s N_s)^{-1} \end{pmatrix}.$$

Proof. Let $\mathbf{f}$ belong to $\mathbf{V}^0_n(\Omega)$, g belong to $H^2_0(\Gamma_s)\cap L^2_0(\Gamma_s)$, and h belong to $L^2_0(\Gamma_s)$. Let $(\mathbf{v}, p, \eta_1, \eta_2)$ be the solution to (3.7). Let $\boldsymbol{\Theta}$ belong to $\mathbf{V}^0_n(\Omega)$, ζ belong to $H^2_0(\Gamma_s)\cap L^2_0(\Gamma_s)$, ξ belong to $L^2_0(\Gamma_s)$ and let $(\boldsymbol{\Phi}, \psi, k_1, k_2)$ be the solution to system

$$\begin{aligned}
&\lambda\boldsymbol{\Phi} - \operatorname{div}\sigma(\boldsymbol{\Phi},\psi) = \boldsymbol{\Theta} \quad \text{and} \quad \operatorname{div}\boldsymbol{\Phi} = 0 \quad \text{in } \Omega,\\
&\boldsymbol{\Phi} = k_2\vec{e}_2 \ \text{ on } \ \Gamma_s, \quad \boldsymbol{\Phi} = 0 \ \text{ on } \ \Gamma_0,\\
&\lambda k_1 + k_2 = \zeta \quad \text{in } \Gamma_s,\\
&\lambda k_2 + \beta k_{1,xx} - \delta k_{2,xx} - \alpha M_s k_{1,xxxx} = M_s(\rho_1\psi) + \xi \ \text{ in } \Gamma_s,\\
&k_1 = 0 \quad \text{and} \quad k_{1,x} = 0 \ \text{ on } \{0, L\}.
\end{aligned} \tag{3.13}$$

With integration by parts we have

$$\begin{aligned}
&\int_\Omega \mathbf{f}\cdot\boldsymbol{\Phi} = \int_\Omega (\lambda\mathbf{v} - \operatorname{div}\sigma(\mathbf{v},p))\,\boldsymbol{\Phi}\\
&= \int_\Omega \mathbf{v}\,(\lambda\boldsymbol{\Phi} - \operatorname{div}\sigma(\boldsymbol{\Phi},\psi)) - \int_{\Gamma_s}\sigma(\mathbf{v},p)\mathbf{n}\cdot\boldsymbol{\Phi} + \int_{\Gamma_s}\sigma(\boldsymbol{\Phi},\psi)\mathbf{n}\cdot\mathbf{w}\\
&= \int_\Omega \mathbf{v}\cdot\boldsymbol{\Theta} + \int_{\Gamma_s} p\,\boldsymbol{\Phi}_2 - \int_{\Gamma_s}\psi\,\mathbf{v}_2\\
&= \int_\Omega \mathbf{v}\cdot\boldsymbol{\Theta} + \int_{\Gamma_s} p\,k_2 - \int_{\Gamma_s}\psi\,\eta_2,
\end{aligned}$$

$$\begin{aligned}
&\int_{\Gamma_s}\zeta\,(-A_{\alpha,\beta})\eta_1 = \int_{\Gamma_s}(\lambda k_1 + k_2)\,(-A_{\alpha,\beta})\eta_1\\
&= \int_{\Gamma_s}(\lambda(-A_{\alpha,\beta})k_1\,\eta_1 + k_2\,(-A_{\alpha,\beta})\eta_1)\\
&= \int_{\Gamma_s}\big((-\beta k_{1,xx} + \alpha k_{1,xxxx})(\eta_2 + g) + k_2(-\beta\eta_{1,xx} + \alpha\eta_{1,xxxx})\big)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Gamma_s}(\xi + \rho_1\psi)\,\eta_2 = \int_{\Gamma_s}(\lambda k_2 + \beta k_{1,xx} - \delta k_{2,xx} - \alpha k_{1,xxxx})\ \eta_2\\
&= \int_{\Gamma_s}(\lambda k_2\,\eta_2 + (\beta k_{1,xx} - \delta k_{2,xx} - \alpha k_{1,xxxx})\,\eta_2)\\
&= \int_{\Gamma_s}\big(k_2(\beta\eta_{1,xx} + \delta\eta_{2,xx} - \alpha\eta_{1,xxxx} + \rho_1 p + h) + (\beta k_{1,xx} - \delta k_{2,xx} - \alpha k_{1,xxxx})\,\eta_2\big)\\
&= \int_{\Gamma_s}\big(k_2(\beta\eta_{1,xx} - \alpha\eta_{1,xxxx} + \rho_1 p + h) + (\beta k_{1,xx} - \alpha k_{1,xxxx})\,\eta_2\big).
\end{aligned}$$

By combining the three identities, we obtain

$$\begin{aligned}
&\rho_1\int_\Omega \mathbf{f}\cdot\boldsymbol{\Phi} + \int_{\Gamma_s} g\,(-A_{\alpha,\beta})k_1 + \int_{\Gamma_s} k_2\,h\\
&= \rho_1\int_\Omega \mathbf{v}\cdot\boldsymbol{\Theta} + \rho_1\int_{\Gamma_s} p\,k_2 - \rho_1\int_{\Gamma_s}\psi\,\eta_2\\
&\quad + \int_{\Gamma_s}\zeta\,(-A_{\alpha,\beta})\eta_1 + \int_{\Gamma_s}\big((\beta k_{1,xx} - \alpha k_{1,xxxx})\eta_2 - k_2(-\beta\eta_{1,xx} + \alpha\eta_{1,xxxx})\big)\\
&\quad + \int_{\Gamma_s}(\xi + \rho_1\psi)\,\eta_2 + \int_{\Gamma_s}\big(k_2(-\beta\eta_{1,xx} + \alpha\eta_{1,xxxx} - \rho_1 p) - (\beta k_{1,xx} - \alpha k_{1,xxxx})\,\eta_2\big)\\
&= \rho_1\int_\Omega \mathbf{v}\cdot\boldsymbol{\Theta} + \int_{\Gamma_s}\zeta\,(-A_{\alpha,\beta})\eta_1 + \int_{\Gamma_s}\xi\,\eta_2.
\end{aligned}$$

To prove the theorem, we have to interpret the identity

$$\rho_1 \int_\Omega \mathbf{f}\cdot\mathbf{\Phi} + \int_{\Gamma_s} g\,(-A_{\alpha,\beta})k_1 + \int_{\Gamma_s} k_2\, h = \rho_1 \int_\Omega \mathbf{v}\cdot\mathbf{\Theta} + \int_{\Gamma_s} \zeta\,(-A_{\alpha,\beta})\eta_1 + \int_{\Gamma_s} \xi\, \eta_2. \tag{3.14}$$

For that we introduce the unbounded operator $(\mathcal{A}^\sharp, D(\mathcal{A}^\sharp))$ in $\mathbf{H}$ defined by $D(\mathcal{A}^\sharp) = D(\mathcal{A})$ and

$$\mathcal{A}^\sharp = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (I+\rho_1\gamma_s N_s)^{-1} \end{pmatrix} \begin{pmatrix} A_0 & 0 & (-A_0)PD_s \\ 0 & 0 & -I \\ \gamma_s N_0(\rho_1\nu\Delta(\cdot)\cdot\mathbf{n}) & -A_{\alpha,\beta} & \delta\Delta_s \end{pmatrix}.$$

We first notice that $(\mathbf{v}, p, \eta_1, \eta_2)$ is the solution to (3.7) if and only if it satisfies

$$(\lambda I - \mathcal{A}) \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ g \\ (I+\rho_1\gamma_s N_s)^{-1} h \end{pmatrix}, \qquad (I-P)\mathbf{v} = (I-P)D_s(\eta_2).$$

Similarly, we can show that $(\mathbf{\Phi}, \psi, k_1, k_2)$ is the solution to system (3.13) if and only if

$$(\lambda I - \mathcal{A}^\sharp) \begin{pmatrix} P\mathbf{\Phi} \\ k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} \mathbf{\Theta} \\ g \\ (I+\rho_1\gamma_s N_s)^{-1} \xi \end{pmatrix}, \qquad (I-P)\mathbf{\Phi} = (I-P)D_s(k_2).$$

Thus, identity (3.14) is equivalent to

$$\begin{aligned} &\big((\lambda I - \mathcal{A})(P\mathbf{v}, \eta_1, \eta_2), (\mathbf{\Phi}, k_1, (I+\rho_1\gamma_s N_s)k_2)\big)_{\mathbf{H}} \\ &= \big((\lambda I - \mathcal{A}^\sharp)(P\mathbf{\Phi}, k_1, k_2), (\mathbf{v}, \eta_1, (I+\rho_1\gamma_s N_s)\eta_2)\big)_{\mathbf{H}} \end{aligned} \tag{3.15}$$

for all $(P\mathbf{v}, \eta_1, \eta_2) \in D(\mathcal{A})$ and all $(P\mathbf{\Phi}, k_1, k_2) \in D(\mathcal{A})$. Let us denote by $\widehat{\mathbf{H}}$ the space $\mathbf{H}$ equipped with the inner product

$$\big((\mathbf{v}^0, \eta_1^0, \eta_2^0), (\mathbf{w}^0, \zeta_1^0, \zeta_2^0)\big)_{\widehat{\mathbf{H}}} = \rho_1 \big(\mathbf{v}^0, \mathbf{w}^0\big)_{\mathbf{V}_n^0(\Omega)} + \big(\eta_1^0, \zeta_1^0\big)_{H_0^2(\Gamma_s)} + \big(\eta_2^0, (I+\rho_1\gamma_s N_s)\zeta_2^0\big)_{L_0^2(\Gamma_s)}.$$

Thus identity (3.15) means that $(\mathcal{A}^\sharp, D(\mathcal{A}^\sharp))$ is the adjoint of $(\mathcal{A}, D(\mathcal{A}))$ in $\widehat{\mathbf{H}}$. We can easily deduce the theorem from this result. ▯

4. Regularity of solutions to the linearized system.

4.1. Studying system (2.6).

We introduce the operator $(\mathcal{A}_\omega, D(\mathcal{A}_\omega))$ defined by $D(\mathcal{A}_\omega) = D(\mathcal{A})$ and

$$\mathcal{A}_\omega = \mathcal{A} + \begin{pmatrix} \omega I & 0 & 0 \\ 0 & \omega I & 0 \\ 0 & 0 & \omega(I+\rho_1\gamma_s N_s)^{-1} \end{pmatrix}.$$

From calculations in section 3.1, it follows that, if $f \in L^2(0,\infty; L_0^2(\Gamma_s))$, system (2.6) can be rewritten in the following equivalent form:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \mathcal{A}_\omega \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} + \mathcal{B}f, \qquad \begin{pmatrix} P\mathbf{v}(0) \\ \eta_1(0) \\ \eta_2(0) \end{pmatrix} = \begin{pmatrix} P\mathbf{v}^0 \\ \eta_1^0 \\ \eta_2^0 \end{pmatrix}, \\ &(I-P)\mathbf{v}(t) = (I-P)D(\eta_2(t)\vec{e}_2\,\chi_{\Gamma_s}), \end{aligned} \tag{4.1}$$

where $\mathcal{B} \in \mathcal{L}(L^2_0(\Gamma_s), \mathbf{H})$ is defined by

$$\mathcal{B}f = \begin{pmatrix} \mathbf{0} \\ 0 \\ (I + \rho_1 \gamma_s N_s)^{-1} f \end{pmatrix}.$$

We have to study solutions to system (4.1) when $(P\mathbf{v}^0, \eta_1^0, \eta_2^0) \in [D(\mathcal{A}), \mathbf{H}]_{1/2}$. From the definition of $D(\mathcal{A})$ and $\mathbf{H}$, we can deduce that

$$[D(\mathcal{A}), \mathbf{H}]_{1/2} = \\ \Big\{ (P\mathbf{v}, \eta_1, \eta_2) \in \mathbf{V}^1_n(\Omega) \times (H^3 \cap H^2_0 \cap L^2_0)(\Gamma_s) \times (H^1_0 \cap L^2_0)(\Gamma_s) \mid P\mathbf{v} + PD_s\eta_2 \in \mathbf{V}^1_0(\Omega) \Big\}.$$

Equipped with the norm

$$(P\mathbf{v}, \eta_1, \eta_2) \longrightarrow \Big(\|\mathbf{v}\|^2_{\mathbf{V}^1_n(\Omega)} + \|\eta_1\|^2_{H^3(\Gamma_s)} + \|\eta_2\|^2_{H^1(\Gamma_s)} \Big)^{1/2}$$

$[D(\mathcal{A}), \mathbf{H}]_{1/2}$ is a Hilbert space.

If $(P\mathbf{v}^0, \eta_1^0, \eta_2^0)$ belongs to $\mathbf{H}$ no compatibility condition, between $P\mathbf{v}^0$ and η_2^0, is needed to define weak solutions of the evolution equation (4.1). But the mapping $t \mapsto (I - P)\mathbf{v}(t)$ which satisfies the second equation in (4.1) will be continuous only if $(I-P)\mathbf{v}^0$ and η_2^0 satisfy $(I-P)\mathbf{v}^0 \cdot \mathbf{n} = \eta_2^0\, \chi_{\Gamma_s}$. Notice that if $\mathbf{v}^0 \in \mathbf{V}^0(\Omega)$, then $\operatorname{div}(I-P)\mathbf{v}^0 = 0$ and $(I-P)\mathbf{v}^0 \cdot \mathbf{n}$ is well defined in $\mathcal{H}^{-1/2}(\Gamma)$. We define a space of initial conditions, satisfying the compatibility condition needed for the continuity of the mapping $t \mapsto (I-P)\mathbf{v}(t)$, as follows

$$\mathbf{H}_{cc} = \Big\{ (\mathbf{v}^0, \eta_1^0, \eta_2^0) \in \mathbf{V}^0(\Omega) \times H_s \mid \mathbf{v}^0 \cdot \mathbf{n} = \eta_2^0\, \chi_{\Gamma_s} \Big\}.$$

Recall that $H_s = (H^2_0(\Gamma_s) \cap L^2_0(\Gamma_s)) \times L^2_0(\Gamma_s)$ (see section 3.2). We equip $\mathbf{H}_{cc}$ with the inner product

$$\big((\mathbf{v}^0, \eta_1^0, \eta_2^0), (\mathbf{w}_0, \zeta_1^0, \zeta_2^0)\big)_{\mathbf{H}_{cc}} = \rho_1(\mathbf{v}^0, \mathbf{w}_0)_{\mathbf{L}^2(\Omega)} + (\eta_1^0, \zeta_1^0)_{H^2_0(\Gamma_s)} + (\eta_2^0, \zeta_2^0)_{L^2_0(\Gamma_s)}.$$

THEOREM 4.1. *(i) If $(P\mathbf{v}^0, \eta_1^0, \eta_2^0) \in [D(\mathcal{A}), \mathbf{H}]_{1/2}$, $(\mathbf{v}^0, \eta_1^0, \eta_2^0) \in \mathbf{H}_{cc}$, and $f \in L^2(0,T; L^2_0(\Gamma_s))$, then system (4.1) admits a unique strict solution satisfying*

$$\|P\mathbf{v}\|_{\mathbf{H}^{2,1}(Q_T)} + \|\eta_1\|_{H^{4,2}(\Sigma^s_T)} + \|\eta_2\|_{H^{2,1}(\Sigma^s_T)} \le C(\|(P\mathbf{v}^0, \eta_1^0, \eta_2^0)\|_{[D(\mathcal{A}),\mathbf{H}]_{1/2}} + \|f\|_{L^2(0,T;L^2_0(\Gamma_s))}),$$

and

$$\|(I-P)\mathbf{v}\|_{L^2(0,T;\mathbf{H}^2(\Omega))} + \|(I-P)\mathbf{v}\|_{H^1(0,T;\mathbf{H}^{1/2}(\Omega))} \le C(\|(P\mathbf{v}^0, \eta_1^0, \eta_2^0)\|_{[D(\mathcal{A}),\mathbf{H}]_{1/2}} + \|f\|_{L^2(0,T;L^2_0(\Gamma_s))}).$$

(ii) If $(\mathbf{v}^0, \eta_1^0, \eta_2^0) \in \mathbf{H}_{cc}$ and if $f \in L^2(0,T; L^2_0(\Gamma_s))$, then system (4.1) admits a unique weak solution (in the sense of semigroup theory) satisfying

$$\|P\mathbf{v}\|_{W(0,T;\mathbf{V}^1(\Omega),\mathbf{V}^{-1}(\Omega))} + \|\eta_1\|_{H^{2,1}(\Sigma^s_T)} + \|\eta_2\|_{L^2(0,T;H^1(\Gamma_s))} \le C(\|(P\mathbf{v}^0, \eta_1^0, \eta_2^0)\|_{\mathbf{H}} + \|f\|_{L^2(0,T;L^2_0(\Gamma_s))})$$

and

$$\|(I-P)\mathbf{v}\|_{L^2(0,T;\mathbf{H}^{3/2}(\Omega))} \le C(\|(P\mathbf{v}^0, \eta_1^0, \eta_2^0)\|_{\mathbf{H}} + \|f\|_{L^2(0,T;L^2_0(\Gamma_s))}).$$

(Here we use the terminology 'strict solution' and 'weak solution' in the sense of semigroup theory for the evolution equation satisfied by $(P\mathbf{v}, \eta_1, \eta_2)$ and not for the equation satisfied by $(I-P)\mathbf{v}$.)

Proof. (i) If $(P\mathbf{v}^0, \eta_1^0, \eta_2^0) \in [D(\mathcal{A}), \mathbf{H}]_{1/2}$ and $f \in L^2(0,T; L^2_0(\Gamma_s))$, the estimate of $(P\mathbf{v}, \eta_1, \eta_2)$ follows from [1, Chapter 1, Theorem 3.1]. The estimate of $(I-P)\mathbf{v}$ in $L^2(0,T;\mathbf{H}^2(\Omega))$ follows from Lemma 3.10 and from the estimate of η_2 in $H^{2,1}(\Sigma^s_T)$. The estimate of $(I-P)\mathbf{v}$ in $H^1(0,T;\mathbf{H}^{1/2}(\Omega))$ follows from the property of the operator D.

(ii) If $(P\mathbf{v}^0, \eta_1^0, \eta_2^0) \in \mathbf{H}$ and $f \in L^2(0,T; L^2_0(\Gamma_s))$, we know that system (4.1) admits a unique weak solution in $L^2(0,T;\mathbf{H})$ satisfying

$$\|(P\mathbf{v}, \eta_1, \eta_2)\|_{C([0,T];\mathbf{H})} \le C(\|(P\mathbf{v}^0, \eta_1^0, \eta_2^0)\|_{\mathbf{H}} + \|f\|_{L^2(0,T;L^2_0(\Gamma_s))}).$$

With this estimate and the equation $\eta_{1,t} = \eta_2 + \omega\eta_1$, we obtain

$$\|\eta_1\|_{H^1(0,T;L^2_0(\Gamma_s))} \le C(\|(P\mathbf{v}^0, \eta_1^0, \eta_2^0)\|_{\mathbf{H}} + \|f\|_{L^2(0,T;L^2_0(\Gamma_s))}).$$

To prove the other estimates, we have to write an energy estimate for strict solutions to system (2.6). We substitute η_2 by $\eta_{1,t} - \omega\eta_1$ in the equation of η_2:

$$\eta_{1,tt} - 2\omega\eta_{1,t} + \omega^2\eta_1 - \beta\eta_{1,xx} - \delta\eta_{1,txx} + \delta\omega\eta_{1,xx} + \alpha\eta_{1,xxxx} = \rho_1 p + f.$$

We multiply this equation by $\eta_{1,t} - \omega\eta_1$ and by $\rho_1\mathbf{v}$ the equation satisfied by $\mathbf{v}$. After integration and by adding the two identities, we obtain

$$\begin{aligned}
&\frac{\rho_1}{2}\int_\Omega |\mathbf{v}(t)|^2 + \nu\,\rho_1\int_{Q_t}|\nabla\mathbf{v}|^2 + \frac{1}{2}\int_{\Gamma_s}|(\eta_{1,t} - \omega\eta_1)(t)|^2 - \omega\int_0^t\int_{\Gamma_s}|\eta_{1,t} - \omega\eta_1|^2 + \frac{\beta}{2}\int_{\Gamma_s}|\eta_{1,x}(t)|^2 \\
&-\beta\omega\int_0^t\int_{\Gamma_s}|\eta_{1,x}|^2 + \delta\int_0^t\int_{\Gamma_s}|\eta_{1,tx} - \omega\eta_{1,x}|^2 + \frac{\alpha}{2}\int_{\Gamma_s}|\eta_{1,xx}(t)|^2 - \alpha\omega\int_0^t\int_{\Gamma_s}|\eta_{1,xx}|^2 + \omega\int_0^t\int_{\Gamma_s}\eta_1\,p \\
&= \frac{\rho_1}{2}\int_\Omega|\mathbf{v}^0|^2 + \frac{\beta}{2}\int_{\Gamma_s}|\eta_{1,x}^0|^2 + \frac{\alpha}{2}\int_{\Gamma_s}|\eta_{1,xx}^0|^2 + \frac{1}{2}\int_{\Gamma_s}|\eta_2^0 - \omega\eta_1^0|^2 + \int_0^t\int_{\Gamma_s} f(\eta_{1,t} - \omega\eta_1).
\end{aligned}$$

We also have

$$\begin{aligned}
&\omega\int_0^t\int_{\Gamma_s}\eta_1\,p \\
&= \omega\int_{\Gamma_s}\eta_1(t)\,\eta_{1,t}(t) - \omega\int_{\Gamma_s}\eta_1^0\,\eta_2^0 - \omega\int_0^t\int_{\Gamma_s}|\eta_{1,t}|^2 - \omega\int_{\Gamma_s}|\eta_1(t)|^2 + \omega\int_{\Gamma_s}|\eta_1^0|^2 + \omega^3\int_0^t\int_{\Gamma_s}|\eta_1|^2 \\
&\quad +\beta\omega\int_0^t\int_{\Gamma_s}|\eta_{1,x}|^2 + \frac{\omega\delta}{2}\int_{\Gamma_s}|\eta_{1,x}(t)|^2 - \frac{\omega\delta}{2}\int_{\Gamma_s}|\eta_{1,x}^0|^2 - \delta\omega^2\int_0^t\int_{\Gamma_s}|\eta_{1,x}|^2 \\
&\quad +\alpha\omega\int_0^t\int_{\Gamma_s}|\eta_{1,xx}|^2 - \omega\int_0^t\int_{\Gamma_s} f\,\eta_1.
\end{aligned}$$

From these identities and the previous estimates we deduce that

$$\|\mathbf{v}\|_{L^2(0,T;\mathbf{H}^1(\Omega))} + \|\eta_2\|_{L^2(0,T;H^1(\Gamma_s))} \le C(\|(\mathbf{v}^0, \eta_1^0, \eta_2^0)\|_{\mathbf{H}_{cc}} + \|f\|_{L^2(0,T;L^2_0(\Gamma_s))}),$$

not only for strict solutions but also for weak solutions. Next we obtain

$$\|(I-P)\mathbf{v}\|_{L^2(0,T;\mathbf{H}^{3/2}(\Omega))} \le C_\varepsilon\|\eta_2\|_{L^2(0,T;H^1(\Gamma_s))},$$

from the properties of the operator D (see e.g. [22]. We can also adapt the proof of Lemma 3.10). Thus we have

$$\|P\mathbf{v}\|_{L^2(0,T;\mathbf{V}^1(\Omega))} + \|(I-P)\mathbf{v}\|_{L^2(0,T;\mathbf{H}^{3/2}(\Omega))} \le C(\|(\mathbf{v}^0, \eta_1^0, \eta_2^0)\|_{\mathbf{H}_{cc}} + \|f\|_{L^2(0,T;L^2_0(\Gamma_s))}).$$

Finally using that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathbf{v}\cdot\mathbf{\Phi} = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega P\mathbf{v}\cdot\mathbf{\Phi} = -\nu\int_\Omega\nabla\mathbf{v}:\nabla\mathbf{\Phi} + \omega\int_\Omega\mathbf{v}\cdot\mathbf{\Phi},$$

for all $\mathbf{\Phi} \in \mathbf{V}^1_0(\Omega)$, we deduce that

$$\|P\mathbf{v}\|_{H^1(0,T;\mathbf{V}^{-1}(\Omega))} \le C\|\mathbf{v}\|_{L^2(0,T;\mathbf{V}^1(\Omega))} \le C(\|(\mathbf{v}^0, \eta_1^0, \eta_2^0)\|_{\mathbf{H}_{cc}} + \|f\|_{L^2(0,T;L^2_0(\Gamma_s))}),$$

and the proof is complete. □

4.2. Another nonhomogeneous system (2.6)**.** We now consider the system

$$
\begin{aligned}
&\mathbf{v}_t - \operatorname{div}\sigma(\mathbf{v},p) - \omega\mathbf{v} = F \quad \text{and} \quad \operatorname{div}\mathbf{v} = 0 \quad \text{in } Q_\infty,\\
&\mathbf{v} = \eta_2\vec{e}_2 \ \text{ on } \Sigma^s_\infty, \quad \mathbf{v} = 0 \ \text{ on } \Sigma^0_\infty, \quad \mathbf{v}(0) = \mathbf{v}^0 \text{ in } \Omega,\\
&\eta_{1,t} = \eta_2 + \omega\eta_1 \quad \text{on } \Sigma^s_\infty,\\
&\eta_{2,t} - \omega\eta_2 - \beta\eta_{1,xx} - \delta\eta_{2,xx} + \alpha M_s\eta_{1,xxxx} = M_s(\rho_1 p + f) \ \text{ on } \Sigma^s_\infty,\\
&\eta_1 = 0 \quad \text{and} \quad \eta_{1,x} = 0 \ \text{ on } \big\{0,L\big\}\times(0,\infty),\\
&\eta_1(0) = \eta_1^0 \quad \text{and} \quad \eta_2(0) = \eta_2^0 \ \text{ in } \Gamma_s,
\end{aligned}
\tag{4.2}
$$

where F belongs to $L^2(0,\infty;\mathbf{L}^2(\Omega))$. We shall need to write this system in the form:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} P\mathbf{v}\\ \eta_1\\ \eta_2\end{pmatrix} = \mathcal{A}_\omega\begin{pmatrix} P\mathbf{v}\\ \eta_1\\ \eta_2\end{pmatrix} + \mathcal{B}f + \mathcal{C}F, \qquad \begin{pmatrix} P\mathbf{v}(0)\\ \eta_1(0)\\ \eta_2(0)\end{pmatrix} = \begin{pmatrix} P\mathbf{v}^0\\ \eta_1^0\\ \eta_2^0\end{pmatrix},\\
&(I-P)\mathbf{v}(t) = (I-P)D(\eta_2(t)\vec{e}_2\,\chi_{\Gamma_s}),
\end{aligned}
\tag{4.3}
$$

where $\mathcal{C}\in\mathcal{L}(\mathbf{L}^2(\Omega),\mathbf{H})$ as to be determined. For that we decompose $F = PF + (I-P)F$, and we denote by $\pi_F\in L^2(0,\infty;\mathcal{H}^1(\Omega))$ the function defined by $\nabla\pi_F = (I-P)F$. We have

$$
p = \pi - q_t + \pi_F,
$$

where q is the solution to (3.1), π is the solution to (3.2), and $\pi_F = \pi_1 + \pi_2$ with

$$
\pi_1 \in H^1_0(\Omega), \quad \Delta\pi_1 = \operatorname{div}F \text{ in } \Omega \quad \text{and} \quad \Delta\pi_2 = 0 \text{ in } \Omega, \quad \frac{\partial\pi_2}{\partial n} = (F - \nabla\pi_1)\cdot\mathbf{n} \text{ on } \Gamma.
$$

If we set $\pi_1 = -(-\Delta_D)^{-1}(\operatorname{div}F)$, we have $\pi_2 = N((F+\nabla(-\Delta_D)^{-1}(\operatorname{div}F))\cdot\mathbf{n})$. Thus the term $M_s p$ in the equation satisfied by η_2 in system (4.2) is

$$
M_s p = \nu\,\gamma_s N_0\Delta P\mathbf{v}(t)\cdot\mathbf{n} - \gamma_s N_s\eta_{2,t}(t) + \gamma_s N((F+\nabla(-\Delta_D)^{-1}(\operatorname{div}F))\cdot\mathbf{n}).
$$

Therefore

$$
\mathcal{C}F = \begin{pmatrix} PF\\ 0\\ \rho_1\,(I+\rho_1\gamma_s N_s)^{-1}(\gamma_s N((F+\nabla(-\Delta_D)^{-1}(\operatorname{div}F))\cdot\mathbf{n}))\end{pmatrix}.
$$

5. Approximate controllability and stabilizability. In this section, we study the approximate controllability of system coupling the Stokes equation with the beam equation. Next we prove that system (2.6) is exponentially stabilizable.

Recall that the linearized system is

$$
\begin{aligned}
&\mathbf{v}_t - \operatorname{div}\sigma(\mathbf{v},p) = 0 \quad \text{and} \quad \operatorname{div}\mathbf{v} = 0 \quad \text{in } Q_T,\\
&\mathbf{v} = \eta_2\vec{e}_2 \ \text{ on } \Sigma^s_T, \quad \mathbf{v} = 0 \ \text{ on } \Sigma^0_T, \quad \mathbf{v}(0) = \mathbf{v}^0 \text{ in } \Omega,\\
&\eta_{1,t} = \eta_2,\\
&\eta_{2,t} - \beta\eta_{1,xx} - \delta\eta_{2,xx} + \alpha\eta_{1,xxxx} = \rho_1 p + f \ \text{ on } \Sigma^s_T,\\
&\eta_1 = 0 \quad \text{and} \quad \eta_{1,x} = 0 \ \text{ on } \big\{0,L\big\}\times(0,T),\\
&\eta_1(0) = \eta_1^0 \quad \text{and} \quad \eta_2(0) = \eta_2^0 \ \text{ in } \Gamma_s.
\end{aligned}
\tag{5.1}
$$

THEOREM 5.1. *System (5.1) is approximately controllable, in time $T > 0$, in the space $\mathbf{H}_{cc}$ by controls f belonging to $L^2(0,T;L^2_0(\Gamma_s))$.*

Proof. To prove the above approximate controllability result in $\mathbf{H}_{cc}$ we have to show that if $(\mathbf{v}^0, \eta_0, \eta_1) = (\mathbf{0}, 0, 0)$ then the reachable set $R(T)$ at time T, when the control f describes $L^2(0,T; L^2_0(\Gamma_s))$, is dense in $\mathbf{H}_{cc}$. To prove that result we assume that $(\mathbf{\Phi}^T, k_1^T, k_2^T) \in R(T)^\perp$. We want to show that $(\mathbf{\Phi}^T, k_1^T, k_2^T) = 0$.
We introduce the adjoint system

$$\begin{aligned}
&-\mathbf{\Phi}_t - \operatorname{div}\sigma(\mathbf{\Phi}, \psi) = \mathbf{0} \quad \text{and} \quad \operatorname{div}\mathbf{\Phi} = 0 \quad \text{in } Q_T,\\
&\mathbf{\Phi} = k_2 \vec{e}_2 \ \text{ on } \Sigma_T^s, \quad \mathbf{\Phi} = 0 \ \text{ on } \Sigma_T^0, \quad \mathbf{\Phi}(T) = \mathbf{\Phi}^T \text{ in } \Omega,\\
&-k_{1,t} = -k_2,\\
&-k_{2,t} + \beta k_{1,xx} - \delta k_{2,xx} - \alpha k_{1,xxxx} = \rho_1 \psi \ \text{ on } \Sigma_T^s,\\
&k_1 = 0 \quad \text{and} \quad k_{1,x} = 0 \ \text{ on } \{0, L\} \times (0, \infty),\\
&k(T) = k_1^T \quad \text{and} \quad k_t(T) = k_2^T \ \text{ in } \Gamma_s.
\end{aligned} \tag{5.2}$$

With an integration by parts we obtain

$$\rho_1 \int_\Omega \mathbf{v}(T) \cdot \mathbf{\Phi}^T + \int_{\Gamma_s} (-A_{\alpha,\beta})^{1/2} \eta_1(T)\, (-A_{\alpha,\beta})^{1/2} k_1^T + \int_{\Gamma_s} \eta_2(T)\, k_2^T = \int_0^T \int_{\Gamma_s} f\, k_2.$$

If $(\mathbf{\Phi}^T, k_1^T, k_2^T) \in R(T)^\perp$, we deduce that

$$\int_0^T \int_{\Gamma_s} f\, k_2 = 0$$

for all $f \in L^2(0,T; L^2_0(\Gamma_s))$, that is $k_2 = 0$. Thus we must show that if $k_2 = 0$ and if $\mathbf{\Phi}$ is solution to

$$\begin{aligned}
&-\mathbf{\Phi}_t - \operatorname{div}\sigma(\mathbf{\Phi}, \psi) = 0, \quad \operatorname{div}\mathbf{\Phi} = 0 \quad \text{in } Q_T,\\
&\mathbf{\Phi} = 0 \ \text{ on } \Sigma_T, \quad \mathbf{\Phi}(T) = \mathbf{\Phi}^T \text{ in } \Omega,
\end{aligned} \tag{5.3}$$

then $(\mathbf{\Phi}^T, k_1^T, k_2^T) = 0$.

By taking the time derivative in the equation

$$k_{2,t} - \beta k_{1,xx} + \delta k_{2,xx} - \alpha M_s k_{1,xxxx} = -\rho_1 M_s \psi$$

we deduce that $\psi_t|_{\Sigma_s} = C(t)$. Thus using an expansion of the solution $\mathbf{\Phi}$ to equation (5.3) in terms of the eigenfunctions of the Stokes operator, as in Osses-Puel [18], the approximate controllability problem reduces to show that if

$$\begin{aligned}
&-\nu \Delta \mathbf{v} + \nabla p = \mu \mathbf{v}, \quad \operatorname{div}\mathbf{v} = 0 \quad \text{in } \Omega,\\
&\mathbf{v} = 0 \ \text{ on } \Gamma \quad \text{and} \quad p = C \text{ on } \Gamma_s,
\end{aligned}$$

with $\mu \in \mathbb{R}$, then $\mathbf{v} = 0$. Thus we can use reults from [18, 19] to complete the proof. □

Theorem 5.2. *For all $\omega > 0$, and all $(\mathbf{v}^0, \eta_1^0, \eta_2^0) \in \mathbf{H}_{cc}$, there exists $f \in L^2(0, \infty; L^2_0(\Gamma_s))$ such that the solution to system* (2.6) *obeys*

$$\|(\mathbf{v}, \eta_1, \eta_2)\|_{L^2(0,\infty;\mathbf{H}_{cc})} < \infty.$$

Proof. Without loss of generality, we can choose ω in the resolvent set of $\mathcal{A}$. Due to Theorem 3.5, we know that the spectrum of $-\mathcal{A}$ is only a pointwise spectrum constituted of a countable number of distinct eigenvalues, that we can order as follows

$$\Re\lambda_1 \geq \Re\lambda_2 \geq \cdots \geq \Re\lambda_N > -\omega > \Re\lambda_{N+1} \geq \cdots .$$

Moreover the generalized eigenspace of each eigenvalue is of finite dimension (see [14]). Let us denote by $G(\lambda_i)$ the real generalized eigenspace associated to λ_i if $\lambda_i \in \mathbb{R}$ and to the pair $(\lambda_i, \bar{\lambda_i})$ if $\Im\lambda_i \neq 0$, and let us set $\mathbf{H}_u = \bigoplus_{i=1}^N G(\lambda_i)$ and $\mathbf{H}_s = \bigoplus_{i=N+1}^\infty G(\lambda_i)$. If $E(\lambda_i)$ denotes the complex generalized eigenspace associated to λ_i and if $(e_j(\lambda_i))_{1\le j\le m(\lambda_i)}$ is a basis of $E(\lambda_i)$, then $G(\lambda_i)$ is nothing else than the space generated by the family $\{\Re e_j(\lambda_i),\, \Im e_j(\lambda_i) \mid 1\le j\le m(\lambda_i)\}$. Let us observe that $\mathbf{H}_u$ is the unstable subspace of system (2.6) while $\mathbf{H}_s$ is the stable space. Let us denote by P_ω the projection onto the finite-dimensional unstable subspace $\mathbf{H}_u$ (parallel to the stable subspace $\mathbf{H}_s$). If we project system (5.1) on $\mathbf{H}_u$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} P_\omega \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \mathcal{A}\, P_\omega \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} + P_\omega \mathcal{B} f, \qquad P_\omega \begin{pmatrix} P\mathbf{v}(0) \\ \eta_1(0) \\ \eta_2(0) \end{pmatrix} = P_\omega \begin{pmatrix} P\mathbf{v}^0 \\ \eta_1^0 \\ \eta_2^0 \end{pmatrix}. \tag{5.4}$$

Due to Theorem 5.1, system (5.1) is approximately controllable in time $T > 0$. Thus the projected system (5.4) is also approximately controllable. Since it is of finite dimension, it is also controllable. Let $f_0 \in L^2(0,T; L^2_0(\Gamma_s))$ be a control such that $P_\omega(P\mathbf{v}, \eta_1, \eta_2)(T) = (\mathbf{0}, 0, 0)$, and still denote by f_0 its extension by zero to (T,∞). Now, we notice that $P_\omega(P\mathbf{v}, \eta_1, \eta_2)$ is the solution of system (5.4) corresponding to $f = f_0$ if and only if $P_\omega(P\hat{\mathbf{v}}, \hat\eta_1, \hat\eta_2) = e^{\omega t} P_\omega(P\mathbf{v}, \eta_1, \eta_2)$ is the solution of system

$$\frac{\mathrm{d}}{\mathrm{d}t} P_\omega \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \mathcal{A}_\omega\, P_\omega \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} + P_\omega \mathcal{B} f, \qquad P_\omega \begin{pmatrix} P\mathbf{v}(0) \\ \eta_1(0) \\ \eta_2(0) \end{pmatrix} = P_\omega \begin{pmatrix} P\mathbf{v}^0 \\ \eta_1^0 \\ \eta_2^0 \end{pmatrix}, \tag{5.5}$$

corresponding to the control $f = e^{\omega t} f_0$. Thus system (5.5) is stabilizable. System (5.5) is the projection of system (2.6) onto its unstable subspace. Due to [30, 17], system (2.6) is stabilizable by a control f belonging in $L^2(0,\infty; L^2_0(\Gamma_s))$, if and only if its projection onto its finite dimensional unstable subspace is stabilizable. The proof is complete. ☐

6. Feedback stabilization of system (2.6)**.** In this section, we study the feedback stabilization of system (2.6). There are several ways to do that. One way consists in studying the infinite time horizon control problem

$$(\mathcal{P}^\infty_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0}) \qquad \inf\Big\{ I(\mathbf{v}, \eta_1, \eta_2, f) \ \mid\ (\mathbf{v}, \eta_1, \eta_2, f) \text{ satisfies (2.6)},\ f \in L^2(0,\infty; L^2_0(\Gamma_s)) \Big\},$$

where

$$\begin{aligned} &I(\mathbf{v}, \eta_1, \eta_2, f) \\ &= \frac{\rho_1}{2}\int_0^\infty \int_\Omega |\mathbf{v}|^2 \,\mathrm{d}x\mathrm{d}z\mathrm{d}t + \frac12 \int_0^\infty \|\eta_1(t)\|^2_{H^2_0(\Gamma_s)}\,\mathrm{d}t + \frac12\int_0^\infty \int_{\Gamma_s} |\eta_2|^2 \,\mathrm{d}x\mathrm{d}t + \frac12 \int_0^\infty |f(t)|^2_{L^2(\Gamma_s)}\,\mathrm{d}t, \end{aligned}$$

and (see section 3)

$$\|\eta_1\|^2_{H^2_0(\Gamma_s)} = \int_{\Gamma_s} |(-A_{\alpha,\beta})^{1/2}\eta_1|^2.$$

From Theorem 5.2 we know that system (2.6) is stabilizable in $\mathbf{H}_{cc}$. Thanks to this stabilizability result, and following the approach in [23], the next theorem can be proved.

THEOREM 6.1. *For all $(\mathbf{v}^0, \eta_1^0, \eta_2^0) \in \mathbf{H}_{cc}$, Problem $(\mathcal{P}^\infty_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$ admits a unique solution $(\mathbf{v}_{\mathbf{v}^0,\eta_1^0,\eta_2^0}, \eta_{1,\mathbf{v}^0,\eta_1^0,\eta_2^0}, \eta_{2,\mathbf{v}^0,\eta_1^0,\eta_2^0}, f_{\mathbf{v}^0,\eta_1^0,\eta_2^0})$. There exists $\Pi \in \mathcal{L}(\mathbf{H}_{cc})$, obeying $\Pi = \Pi^* \ge 0$, such that the optimal cost is given by*

$$\inf(\mathcal{P}^\infty_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0}) = \frac12 \Big(\Pi(\mathbf{v}^0, \eta_1^0, \eta_2^0), (\mathbf{v}^0, \eta_1^0, \eta_2^0) \Big)_{\mathbf{H}_{cc}}. \tag{6.1}$$

Theorem 6.1 will be proved in section 8.1.

The operator $\Pi \in \mathcal{L}(\mathbf{H}_{cc})$, which defines the value function of $(\mathcal{P}^\infty_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$ through formula (6.1), is obtained as the limit of the operator $\Pi(T) \in \mathcal{L}(\mathbf{H}_{cc})$ when T tends to infinity, where $\Pi(T) \in \mathcal{L}(\mathbf{H}_{cc})$ is the operator defining the value function of the corresponding finite time horizon control problem

$$(\mathcal{P}^T_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0}) \qquad \inf\Big\{I_0^T(\mathbf{v},\eta_1,\eta_2,f) \ \mid\ (\mathbf{v},\eta_1,\eta_2,f) \text{ satisfies (2.6)},\ f\in L^2(0,T;L^2_0(\Gamma_s))\Big\},$$

where

$$\begin{aligned}&I_0^T(\mathbf{v},\eta_1,\eta_2,f)\\ &= \frac{\rho_1}{2}\int_0^T\int_\Omega |\mathbf{v}|^2\,\mathrm{d}x\mathrm{d}z\mathrm{d}t + \frac{1}{2}\int_0^T \|\eta_1(t)\|^2_{H^2_0(\Gamma_s)}\,\mathrm{d}t + \frac{1}{2}\int_0^T\int_{\Gamma_s}|\eta_2|^2\,\mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_0^T |f(t)|^2_{L^2_0(\Gamma_s)}\,\mathrm{d}t.\end{aligned}$$

We are going to see in section 8.1 that the solution $\Big(\mathbf{v}_{\mathbf{v}^0,\eta_1^0,\eta_2^0},\eta_{1,\mathbf{v}^0,\eta_1^0,\eta_2^0},\eta_{2,\mathbf{v}^0,\eta_1^0,\eta_2^0},f_{\mathbf{v}^0,\eta_1^0,\eta_2^0}\Big)$ of problem $(\mathcal{P}^\infty_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$ obeys the feedback law

$$f_{\mathbf{v}^0,\eta_1^0,\eta_2^0}(t) = -\Pi_3\Big(\mathbf{v}_{\mathbf{v}^0,\eta_1^0,\eta_2^0}(t),\eta_{1,\mathbf{v}^0,\eta_1^0,\eta_2^0}(t),\eta_{2,\mathbf{v}^0,\eta_1^0,\eta_2^0}(t)\Big),$$

where $\Pi_3 \in \mathcal{L}(\mathbf{H}_{cc}, L^2_0(\Gamma_s))$ is the third component of the mapping Π:

$$\Pi = \begin{pmatrix}\Pi_1\\ \Pi_2\\ \Pi_3\end{pmatrix} = \begin{pmatrix}\Pi_{11} & \Pi_{12} & \Pi_{13}\\ \Pi_{21} & \Pi_{22} & \Pi_{23}\\ \Pi_{31} & \Pi_{32} & \Pi_{33}\end{pmatrix} \in \mathcal{L}(\mathbf{H}_{cc}).$$

We would like to find an equation characterizing the operator Π. For that the most natural way is to follow [2] or [15].

The classical approach to find a feedback control law, as developed in [2] or in [15], consists in considering the evolution equation (4.1) rather than the system (2.6). In that case $(I-P)\mathbf{v}$ must be considered as an auxiliary variable and must be removed from the control problem. For that we set

$$\begin{aligned}\widetilde{I}(P\mathbf{v},\eta_1,\eta_2,f) &= \frac{\rho_1}{2}\int_0^\infty\int_\Omega |P\mathbf{v}|^2\,dxdzdt + \frac{1}{2}\int_0^\infty \|\eta_1(t)\|^2_{H^2_0(\Gamma_s)}\,dt\\ &+\frac{1}{2}\int_0^\infty\int_{\Gamma_s}|(I+\rho_1 D_s^*(I-P)D_s)^{1/2}\eta_2|^2\,dxdt + \frac{1}{2}\int_0^\infty \|f(t)\|^2_{L^2(\Gamma_s)}\,dt.\end{aligned}$$

We can notice that $\tilde{I}(P\mathbf{v},\eta_1,\eta_2,f) = I(\mathbf{v},\eta_1,\eta_2,f)$ if $(\mathbf{v},\eta_1,\eta_2,f)$ obeys (4.1). We consider the problem

$$(\mathcal{Q}^\infty_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0}) \qquad \inf\Big\{\tilde{I}(P\mathbf{v},\eta_1,\eta_2,f)\ \mid\ (\mathbf{v},\eta_1,\eta_2,f) \text{ satisfies (4.1)},\ f\in L^2(0,\infty;L^2_0(\Gamma_s))\Big\}.$$

As in [2] or in [15], the following theorem can be proved.

THEOREM 6.2. *For all $(P\mathbf{v}^0,\eta_1^0,\eta_2^0)\in\mathbf{H}$, problem $(\mathcal{Q}^\infty_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0})$ admits a unique solution $(\mathbf{v}_{P\mathbf{v}^0,\eta_1^0,\eta_2^0}, \eta_{1,P\mathbf{v}^0,\eta_1^0,\eta_2^0},\eta_{2,P\mathbf{v}^0,\eta_1^0,\eta_2^0},f_{P\mathbf{v}^0,\eta_1^0,\eta_2^0})$. There exists $\widetilde{\Pi}\in\mathcal{L}(\mathbf{H})$, obeying $\widetilde{\Pi}=\widetilde{\Pi}^*\geq 0$, such that the optimal cost is given by*

$$\inf(\mathcal{Q}^\infty_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0}) = \frac{1}{2}\Big(\widetilde{\Pi}(P\mathbf{v}^0,\eta_1^0,\eta_2^0),(P\mathbf{v}^0,\eta_1^0,\eta_2^0)\Big)_{\mathbf{H}}.$$

Moreover, $\widetilde{\Pi}$ is the solution to the algebraic Riccati equation

$$\widetilde{\Pi}\in\mathcal{L}(\mathbf{H}),\quad \widetilde{\Pi}=\widetilde{\Pi}^*\geq 0,\qquad \widetilde{\Pi}\mathcal{A}_\omega + \mathcal{A}_\omega^*\widetilde{\Pi} - \widetilde{\Pi}\mathcal{B}\mathcal{B}^*\widetilde{\Pi} + C^*C = 0, \tag{6.2}$$

with

$$C = \begin{pmatrix} I & 0 & 0\\ 0 & I & 0\\ 0 & 0 & (I+\rho_1 D_s^*(I-P)D_s)^{1/2}\end{pmatrix} \quad\text{and}\quad C^*C = \begin{pmatrix} I & 0 & 0\\ 0 & I & 0\\ 0 & 0 & (I+\rho_1 D_s^*(I-P)D_s)\end{pmatrix}.$$

The operator $\mathcal{B}^* \in \mathcal{L}(\mathbf{H}, L^2_0(\Gamma_s))$ is the adjoint of $\mathcal{B} \in \mathcal{L}(L^2_0(\Gamma_s), \mathbf{H})$, and one can easily verify that

$$\mathcal{B}^*(\mathbf{f}, g, h)^T = (\mathbf{0}, 0, (I + \rho_1\gamma_s N_s)^{-1}h)^T.$$

The optimal control of problem $(\mathcal{Q}^\infty_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0})$ obeys the feedback law

$$f = -\mathcal{B}^*\widetilde{\Pi}(P\mathbf{v}(t), \eta_1(t), \eta_2(t)) = -(I + \rho_1\gamma_s N_s)^{-1}\widetilde{\Pi}_3(P\mathbf{v}(t), \eta_1(t), \eta_2(t)).$$

(Here $\widetilde{\Pi}_3$ is the third component of the mapping $\widetilde{\Pi}$.) Since $(\mathcal{Q}^\infty_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0})$ is an equivalent formulation of problem $(\mathcal{P}^\infty_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0})$, we have

$$(I + \rho_1\gamma_s N_s)^{-1}\widetilde{\Pi}_3(P\mathbf{v}(t), \eta_1(t), \eta_2(t)) = \Pi_3(\mathbf{v}(t), \eta_1(t), \eta_2(t)), \tag{6.3}$$

along the optimal trajectory. This does not give any precise relationship between $\widetilde{\Pi}$ and Π.

If we compare both approaches we can say that the drawback of problem $(\mathcal{P}^\infty_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$ is that the operator Π is not characterized by an algebraic Riccati equation. But from the numerical viewpoint, it can be more interesting to work with a system of partial differential equations rather than with a system in the form (4.1) in which the numerical approximation of $\mathcal{A}^*$ may be more tricky. The numerical approximation of the adjoint system may be also useful to design reduced order models (see e.g. [27]). This is why it is interesting to determine a feedback control law by solving an optimal control problem for which the adjoint system may be easily interpreted as a system of partial differential equations.

To address this issue we introduce a third problem leading to another feedback law that we can link with the one expressed with Π. We consider the problem

$$(\mathcal{R}^\infty_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0}) \qquad \inf\Big\{\widehat{I}(P\mathbf{v}, \eta_1, \eta_2, f) \ \mid\ (P\mathbf{v}, \eta_1, \eta_2, f) \text{ satisfies (4.1)},\ f \in L^2(0,\infty; L^2_0(\Gamma_s))\Big\},$$

where

$$\widehat{I}(P\mathbf{v}, \eta_1, \eta_2, f) = \frac{\rho_1}{2}\int_{Q_\infty} |P\mathbf{v}|^2 + \frac{1}{2}\int_0^\infty \|\eta_1\|^2_{H^2_0(\Gamma_s)} + \frac{1}{2}\int_{\Sigma^s_\infty} |(I + \rho_1\gamma_s N_s)\eta_2|^2 + \frac{1}{2}\int_{\Sigma^s_\infty} |f|^2.$$

Observe that

$$\widehat{I}(P\mathbf{v}, \eta_1, \eta_2, f) = \frac{1}{2}\int_0^\infty \|(P\mathbf{v}(t), \eta_1(t), \eta_2(t))\|^2_{\widehat{\mathbf{H}}}\, dt + \frac{1}{2}\int_{\Sigma^s_\infty} |f|^2.$$

Problem $(\mathcal{R}^\infty_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0})$ will be studied in section 8.2. The following analogue of Theorem 6.2 can be proved, still with [2] or [15].

THEOREM 6.3. *For all $(P\mathbf{v}^0, \eta_1^0, \eta_2^0) \in \widehat{\mathbf{H}}$, problem $(\mathcal{R}^\infty_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0})$ admits a unique solution $(P\mathbf{v}_{P\mathbf{v}^0,\eta_1^0,\eta_2^0}, \eta_{1,P\mathbf{v}^0,\eta_1^0,\eta_2^0}, \eta_{2,P\mathbf{v}^0,\eta_1^0,\eta_2^0}, f_{P\mathbf{v}^0,\eta_1^0,\eta_2^0})$. There exists $\widehat{\Pi} \in \mathcal{L}(\widehat{\mathbf{H}})$, obeying $\widehat{\Pi} = \widehat{\Pi}^* \geq 0$, such that the optimal cost is given by*

$$\inf(\mathcal{R}^\infty_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0}) = \frac{1}{2}\Big(\widehat{\Pi}(P\mathbf{v}^0, \eta_1^0, \eta_2^0), (P\mathbf{v}^0, \eta_1^0, \eta_2^0)\Big)_{\widehat{\mathbf{H}}}.$$

Moreover, $\widehat{\Pi}$ is the solution to the algebraic Riccati equation

$$\widehat{\Pi} \in \mathcal{L}(\widehat{\mathbf{H}}), \quad \widehat{\Pi} = \widehat{\Pi}^* \geq 0, \qquad \widehat{\Pi}\mathcal{A}_\omega + \mathcal{A}^\sharp_\omega\widehat{\Pi} - \widehat{\Pi}\mathcal{B}\mathcal{B}^\sharp\widehat{\Pi} + I = 0,$$

where $(\mathcal{A}^\sharp_\omega, D(\mathcal{A}^\sharp_\omega))$ is the adjoint of $(\mathcal{A}_\omega, D(\mathcal{A}_\omega))$ in $\widehat{\mathbf{H}}$, and $\mathcal{B}^\sharp \in \mathcal{L}(\widehat{\mathbf{H}}, L^2_0(\Gamma_s))$ is the adjoint of $\mathcal{B} \in \mathcal{L}(L^2_0(\Gamma_s), \widehat{\mathbf{H}})$.

One can verify that $D(\mathcal{A}^\sharp_\omega) = D(\mathcal{A}_\omega) = D(\mathcal{A})$ and

$$\mathcal{A}^\sharp_\omega = \mathcal{A}^\sharp + \begin{pmatrix} \omega I & 0 & 0 \\ 0 & \omega I & 0 \\ 0 & 0 & \omega(I + \rho_1\gamma_s N_s)^{-1} \end{pmatrix}.$$

Moreover

$$\mathcal{B}^\sharp(\mathbf{f}, g, h)^T = (\mathbf{0}, 0, h)^T.$$

We are able to prove the following relationship between Π and $\widehat{\Pi}$.

THEOREM 6.4. *The operator* $\Pi \in \mathcal{L}(\mathbf{H}_{cc})$ *can be expressed in terms of*

$$\widehat{\Pi} = \begin{pmatrix} \widehat{\Pi}_1 \\ \widehat{\Pi}_2 \\ \widehat{\Pi}_3 \end{pmatrix} = \begin{pmatrix} \widehat{\Pi}_{11} & \widehat{\Pi}_{12} & \widehat{\Pi}_{13} \\ \widehat{\Pi}_{21} & \widehat{\Pi}_{22} & \widehat{\Pi}_{23} \\ \widehat{\Pi}_{31} & \widehat{\Pi}_{32} & \widehat{\Pi}_{33} \end{pmatrix} \in \mathcal{L}(\widehat{\mathbf{H}})$$

as follows

$$\begin{aligned} &P\Pi_1(\mathbf{v}^0, \eta_1^0, \eta_2^0) = \widehat{\Pi}_1(P\mathbf{v}^0, \eta_1^0, \eta_2^0), \qquad \Pi_2(\mathbf{v}^0, \eta_1^0, \eta_2^0) = \widehat{\Pi}_2(P\mathbf{v}^0, \eta_1^0, \eta_2^0), \\ &\Pi_3(\mathbf{v}^0, \eta_1^0, \eta_2^0) = \widehat{\Pi}_3(P\mathbf{v}^0, \eta_1^0, \eta_2^0), \qquad (I-P)\Pi_1(\mathbf{v}^0, \eta_1^0, \eta_2^0) = (I-P)D_s\widehat{\Pi}_3(P\mathbf{v}^0, \eta_1^0, \eta_2^0), \end{aligned}$$

for all $(\mathbf{v}^0, \eta_1^0, \eta_2^0) \in \mathbf{H}_{cc}$.

The main interest of problem $(\mathcal{R}^\infty_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0})$ is that its optimality system is the same one as for problem $(\mathcal{P}^\infty_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$ (see section 8.2).

7. Studying problem $(\mathcal{P}^T_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$**.** THEOREM 7.1. *For all* $(\mathbf{v}^0, \eta_1^0, \eta_2^0) \in \mathbf{H}_{cc}$*, Problem* $(\mathcal{P}^T_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$ *admits a unique solution* $(\bar{\mathbf{v}}, \bar{\eta}_1, \bar{\eta}_2, \bar{f})$ *and the optimal control is*

$$\bar{f} = -k_2,$$

where $(\mathbf{\Phi}, k_1, k_2)$ *is the solution of the following adjoint system*

$$\begin{aligned} &-\mathbf{\Phi}_t - \operatorname{div}\sigma(\mathbf{\Phi}, \psi) - \omega\mathbf{\Phi} = \bar{\mathbf{v}} \quad \text{and} \quad \operatorname{div}\mathbf{\Phi} = 0 \quad \text{in } Q_T, \\ &\mathbf{\Phi} = k_2\vec{e}_2 \text{ on } \Sigma^s_T, \quad \mathbf{\Phi} = 0 \text{ on } \Sigma^0_T, \quad \mathbf{\Phi}(T) = 0 \text{ in } \Omega, \\ &-k_{1,t} = -k_2 + \omega k_1 + \bar{\eta}_1, \\ &-k_{2,t} - \omega k_2 + \beta k_{1,xx} - \delta k_{2,xx} - \alpha k_{1,xxxx} = \rho_1\psi + \bar{\eta}_2 \text{ on } \Sigma^s_T, \\ &k_1 = 0 \quad \text{and} \quad k_{1,x} = 0 \text{ on } \{0, L\} \times (0, \infty), \\ &k_1(T) = 0 \quad \text{and} \quad k_2(T) = 0 \text{ in } \Gamma_s. \end{aligned} \tag{7.1}$$

Conversely, the system

$$\begin{aligned} &\mathbf{v}_t - \operatorname{div}\sigma(\mathbf{v}, p) - \omega\mathbf{v} = 0 \quad \text{and} \quad \operatorname{div}\mathbf{v} = 0 \quad \text{in } Q_T, \\ &\mathbf{v} = \eta_2\vec{e}_2 \text{ on } \Sigma^s_T, \quad \mathbf{v} = 0 \text{ on } \Sigma^0_T, \quad \mathbf{v}(0) = \mathbf{v}^0 \text{ in } \Omega, \\ &\eta_{1,t} = \eta_2 + \omega\eta_1, \\ &\eta_{2,t} - \omega\eta_2 - \beta\eta_{1,xx} - \delta\eta_{2,xx} + \alpha\eta_{1,xxxx} = \rho_1 p - k_2 \text{ on } \Sigma^s_T, \\ &\eta_1 = 0 \quad \text{and} \quad \eta_{1,x} = 0 \text{ on } \{0, L\} \times (0, \infty), \\ &\eta_1(0) = \eta_1^0 \quad \text{and} \quad \eta_2(0) = \eta_2^0 \quad \textit{in } \Gamma_s, \\ &-\mathbf{\Phi}_t - \operatorname{div}\sigma(\mathbf{\Phi}, \psi) - \omega\mathbf{\Phi} = \mathbf{v} \quad \text{and} \quad \operatorname{div}\mathbf{\Phi} = 0 \quad \text{in } Q_T, \\ &\mathbf{\Phi} = k_2\vec{e}_2 \text{ on } \Sigma^s_T, \quad \mathbf{\Phi} = 0 \text{ on } \Sigma^0_T, \quad \mathbf{\Phi}(T) = 0 \text{ in } \Omega, \\ &-k_{1,t} = -k_2 + \omega k_1 + \eta_1, \\ &-k_{2,t} - \omega k_2 + \beta k_{1,xx} - \delta k_{2,xx} - \alpha k_{1,xxxx} = \rho_1\psi + \eta_2 \text{ on } \Sigma^s_T, \\ &k_1 = 0 \quad \text{and} \quad k_{1,x} = 0 \text{ on } \{0, L\} \times (0, \infty), \\ &k_1(T) = 0 \quad \text{and} \quad k_2(T) = 0 \text{ in } \Gamma_s \end{aligned} \tag{7.2}$$

admits a unique solution $(\mathbf{v}, p, \eta_1, \eta_2, \mathbf{\Phi}, \psi, k_1, k_2)$ *and the optimal solution to* $(\mathcal{P}^T_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$ *is*

$$f = -k_2.$$

The operator $\Pi(T) \in \mathcal{L}(\mathbf{H}_{cc})$ *defined by*

$$\Pi(T)(\mathbf{v}^0, \eta_1^0, \eta_2^0) = (\mathbf{\Phi}(0), k_1(0)), k_2(0))\,,$$

is linear and continuous in $\mathbf{H}_{cc}$*, it is symmetric and semidefinite positive, and the optimal cost is given by*

$$\inf(\mathcal{P}^T_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0}) = \frac{1}{2}\big(\Pi(T)(\mathbf{v}^0, \eta_1^0, \eta_2^0), (\mathbf{v}^0, \eta_1^0, \eta_2^0)\big)_{\mathbf{H}_{cc}}.$$

Proof. The existence of a unique optimal control can be proved in a classical way.

Let us establish the first order optimality conditions for $(\mathcal{P}^T_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$. Let us denote by $(\mathbf{v}(f), p(f), \eta_1(f), \eta_2(f))$ the solution of (2.6) corresponding to f and let us set $J(f) = I_0^T(\mathbf{v}(f), \eta_1(f), \eta_2(f), f)$. We have

$$J'(f)g = \rho_1 \int_0^T \int_\Omega \mathbf{v}\cdot\mathbf{w}\,dxdzdt + \int_0^T \int_{\Gamma_s} (-A_{\alpha,\beta})\eta_1\,\zeta_1\,dxdt + \int_0^T \int_{\Gamma_s} \eta_2\,\zeta_2\,dxdt + \int_0^T \int_{\Gamma_s} fg\,dxdt,$$

where $(\mathbf{w}, q, \zeta_1, \zeta_2)$ is the solution to

$$\begin{aligned}
&\mathbf{w}_t - \operatorname{div}\sigma(\mathbf{w}, q) - \omega\mathbf{w} = 0 \quad \text{and} \quad \operatorname{div}\mathbf{w} = 0 \quad \text{ in } Q_T,\\
&\mathbf{w} = \zeta_2\vec{e}_2 \;\text{ on } \Sigma_T^s, \quad \mathbf{w} = 0 \;\text{ on } \Sigma_T^0, \quad \mathbf{w}(0) = 0 \text{ in } \Omega,\\
&\zeta_{1,t} = \zeta_2 + \omega\zeta_1,\\
&\zeta_{2,t} - \omega\zeta_2 - \beta\zeta_{1,xx} - \delta\zeta_{2,xx} + \alpha\zeta_{1,xxxx} = \rho_1 q + g \;\text{ on } \Sigma_T^s,\\
&\zeta_1 = 0 \quad \text{and} \quad \zeta_{1,x} = 0 \;\text{ on } \big\{0, L\big\} \times (0, T),\\
&\zeta_1(0) = 0 \quad \text{and} \quad \zeta_2(0) = 0 \;\text{ in } \Gamma_s.
\end{aligned} \tag{7.3}$$

Let us notice that

$$\begin{aligned}
&\int_0^T \int_{\Gamma_s} (-A_{\alpha,\beta})^{1/2}\eta_1\,(-A_{\alpha,\beta})^{1/2}\zeta_1\,dxdt = \int_0^T \int_{\Gamma_s} (\beta\eta_{1,x}\zeta_{1,x} + \alpha\eta_{1,xx}\zeta_{1,xx})\,dxdt\\
&= \int_0^T \langle(-A_{\alpha,\beta})\eta_1, \zeta_1\rangle_{(H_0^2(\Gamma_s)\cap L_0^2(\Gamma_s))', H_0^2(\Gamma_s)\cap L_0^2(\Gamma_s)}\,dt\\
&= \int_0^T \langle\eta_1, (-A_{\alpha,\beta})\zeta_1\rangle_{H_0^2(\Gamma_s)\cap L_0^2(\Gamma_s), (H_0^2(\Gamma_s)\cap L_0^2(\Gamma_s))'}\,dt
\end{aligned}$$

for all η_1 and ζ_1 belonging to $H_0^2(\Gamma_s) \cap L_0^2(\Gamma_s)$. Actually the writing $\int_0^T \int_{\Gamma_s} (-A_{\alpha,\beta})\eta_1\,\zeta_1\,dxdt$ is an abuse of notation which must be understood in the above sense. In what follows, we shall do this type of abuse below in order to simplify the writing.

Let $(\mathbf{\Phi}, \psi, k_1, k_2)$ be the solution to the adjoint system (7.1) corresponding to $(\mathbf{v}(f), p(f), \eta_1(f), \eta_2(f)) = (\mathbf{v}, p, \eta_1, \eta_2)$. We have

$$\begin{aligned}
0 &= \int_{Q_T} (\mathbf{w}_t - \operatorname{div}\sigma(\mathbf{w}, q) - \omega\mathbf{w})\,\mathbf{\Phi}\\
&= \int_{Q_T} \mathbf{w}\,(-\mathbf{\Phi}_t - \operatorname{div}\sigma(\mathbf{\Phi}, \psi) - \omega\mathbf{\Phi}) - \int_{\Sigma_T^s} \sigma(\mathbf{w}, q)\mathbf{n}\cdot\mathbf{\Phi} + \int_{\Sigma_T^s} \sigma(\mathbf{\Phi}, \psi)\mathbf{n}\cdot\mathbf{w}\\
&= \int_{Q_T} \mathbf{w}\cdot\mathbf{v} + \int_{\Sigma_T^s} q\,\mathbf{\Phi}_2 - \int_{\Sigma_T^s} \psi\,\mathbf{w}_2\\
&= \int_{Q_T} \mathbf{w}\cdot\mathbf{v} + \int_{\Sigma_T^s} q\,k_2 - \int_{\Sigma_T^s} \psi\,\zeta_2,
\end{aligned}$$

$$\int_{\Sigma_T^s} \eta_1\,(-A_{\alpha,\beta})\zeta_1 = \int_{\Sigma_T^s} (-k_{1,t} + k_2 - \omega k_1)\,(-A_{\alpha,\beta})\zeta_1$$
$$= \int_{\Sigma_T^s} ((-A_{\alpha,\beta})k_1\,\zeta_{1,t} + (k_2 - \omega k_1)\,(-A_{\alpha,\beta})\zeta_1)$$
$$= \int_{\Sigma_T^s} \big((-\beta k_{1,xx} + \alpha k_{1,xxxx})(\zeta_2 + \omega\zeta_1) + (k_2 - \omega k_1)(-\beta\zeta_{1,xx} + \alpha\zeta_{1,xxxx})\big)$$
$$= \int_{\Sigma_T^s} \big((-\beta k_{1,xx} + \alpha k_{1,xxxx})\zeta_2 + k_2(-\beta\zeta_{1,xx} + \alpha\zeta_{1,xxxx})\big)$$

and

$$\int_{\Sigma_T^s} (\eta_2 + \rho_1\psi)\,\zeta_2 = \int_{\Sigma_T^s} (-k_{2,t} - \omega k_2 + \beta k_{1,xx} - \delta k_{2,xx} - \alpha k_{1,xxxx})\;\zeta_2$$
$$= \int_{\Sigma_T^s} (k_2\,\zeta_{2,t} - (\omega k_2 - \beta k_{1,xx} + \delta k_{2,xx} + \alpha k_{1,xxxx})\,\zeta_2)$$
$$= \int_{\Sigma_T^s} \big(k_2(\omega\zeta_2 + \beta\zeta_{1,xx} + \delta\zeta_{2,xx} - \alpha\zeta_{1,xxxx} + \rho_1 q + g) - (\omega k_2 - \beta k_{1,xx} + \delta k_{2,xx} + \alpha k_{1,xxxx})\,\zeta_2\big)$$
$$= \int_{\Sigma_T^s} \big(k_2(\beta\zeta_{1,xx} - \alpha\zeta_{1,xxxx} + \rho_1 q + g) + (\beta k_{1,xx} - \alpha k_{1,xxxx})\,\zeta_2\big).$$

By combining the three identities, we obtain

$$\rho_1 \int_{Q_T} \mathbf{w}\cdot\mathbf{v} + \int_{\Sigma_T^s} \eta_1\,(-A_{\alpha,\beta})\zeta_1 + \int_{\Sigma_T^s} \eta_2\,\zeta_2$$
$$= \int_{\Sigma_T^s} \big((-\beta k_{1,xx} + \alpha k_{1,xxxx})\zeta_2 + k_2(-\beta\zeta_{1,xx} + \alpha\zeta_{1,xxxx})\big)$$
$$+ \int_{\Sigma_T^s} \big(k_2(\beta\zeta_{1,xx} - \alpha\zeta_{1,xxxx} + g) + (\beta k_{1,xx} - \alpha k_{1,xxxx})\,\zeta_2\big) = \int_{\Sigma_T^s} k_2\,g.$$

Thus we have

$$J'(f)g = \int_0^T \int_{\Gamma_s} k_2\,g + \int_0^T \int_{\Gamma_s} f\,g,$$

and the optimal solution to $(\mathcal{P}^T_{0,\mathbf{v}^0,\eta_0,\eta_1})$ is given by

$$f = -k_2,$$

where $(\mathbf{\Phi}, \psi, k_1, k_2)$ is the solution to (7.1) associated with the optimal state $(\mathbf{v}, \eta_1, \eta_2)$.

The converse statement consisting in showing that the optimal control is characterized by system (7.2) is classical. We refer for example to [23, Theorem 3.1].

The fact that the operator $\Pi(T)$ introduced in the statement of the theorem belongs to $\mathcal{L}(\mathbf{H}_{cc})$ and obeys $\Pi(T) = \Pi(T)^* \geq 0$ is also classical.

To prove the last statement, we can see that the solution $(\mathbf{v}, p, \eta_1, \eta_2, \boldsymbol{\Phi}, \psi, k_1, k_2)$ to system (7.2) obeys

$$
\begin{aligned}
0 &= \int_{Q_T} (\mathbf{v}_t - \operatorname{div}\sigma(\mathbf{v},p) - \omega\mathbf{v})\,\boldsymbol{\Phi} \\
&= \int_{Q_T} \mathbf{v}\,(-\boldsymbol{\Phi}_t - \operatorname{div}\sigma(\boldsymbol{\Phi},\psi) - \omega\boldsymbol{\Phi}) - \int_{\Sigma_T^s} \sigma(\mathbf{v},p)\mathbf{n}\cdot\boldsymbol{\Phi} + \int_{\Sigma_T^s} \sigma(\boldsymbol{\Phi},\psi)\mathbf{n}\cdot\mathbf{v} - \int_\Omega \mathbf{v}^0\,\boldsymbol{\Phi}_0 \\
&= \int_{Q_T} |\mathbf{v}|^2 + \int_{\Sigma_T^s} p\,\boldsymbol{\Phi}_2 - \int_{\Sigma_T^s} \psi\,\mathbf{v}_2 - \int_\Omega \mathbf{v}^0\,\boldsymbol{\Phi}_0 \\
&= \int_{Q_T} |\mathbf{v}|^2 + \int_{\Sigma_T^s} p\,k_2 - \int_{\Sigma_T^s} \psi\,\eta_2 - \int_\Omega \mathbf{v}^0\,\boldsymbol{\Phi}_0,
\end{aligned}
$$

$$
\begin{aligned}
0 &= \int_{\Sigma_T^s} (\eta_{1,t} - \eta_2 - \omega\eta_1)(-A_{\alpha,\beta})k_1 \\
&= \int_{\Sigma_T^s} (-(-A_{\alpha,\beta})\eta_1 k_{1,t} - (-A_{\alpha,\beta})\eta_2\,k_1 - \omega\eta_1(-A_{\alpha,\beta})k_1) - \int_{\Gamma_s} (-A_{\alpha,\beta})^{1/2}\eta_1^0\,(-A_{\alpha,\beta})^{1/2}k_1(0) \\
&= \int_{\Sigma_T^s} ((-A_{\alpha,\beta})\eta_1(-k_2 + \omega k_1 + \eta_1) - (-A_{\alpha,\beta})\eta_2\,k_1 - \omega\eta_1(-A_{\alpha,\beta})k_1) \\
&\qquad\qquad - \int_{\Gamma_s} (-A_{\alpha,\beta})^{1/2}\eta_1^0\,(-A_{\alpha,\beta})^{1/2}k_1(0) \\
&= \int_{\Sigma_T^s} ((-A_{\alpha,\beta})\eta_1(-k_2 + \eta_1) - (-A_{\alpha,\beta})\eta_2\,k_1) - \int_{\Gamma_s} (-A_{\alpha,\beta})^{1/2}\eta_1^0\,(-A_{\alpha,\beta})^{1/2}k_1(0)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\Sigma_T^s} (f + \rho_1 p)k_2 = \int_{\Sigma_T^s} (\eta_{2,t} - \omega\eta_2 - \beta\eta_{1,xx} - \delta\eta_{2,xx} + \alpha\eta_{1,xxxx})\,k_2 \\
&= \int_{\Sigma_T^s} (-\eta_2\,k_{2,t} - \omega\eta_2\,k_2 - (\beta\eta_{1,xx} + \delta\eta_{2,xx} - \alpha\eta_{1,xxxx})\,k_2) - \int_{\Gamma_s} k_2(0)\,\eta_2^0 \\
&= \int_{\Sigma_T^s} \big(\eta_2(-\beta k_{1,xx} + \delta k_{2,xx} + \alpha k_{1,xxxx} + \rho_1\psi + \eta_2) - (\beta\eta_{1,xx} + \delta\eta_{2,xx} - \alpha\eta_{1,xxxx})\,k_2\big) - \int_{\Gamma_s} k_2(0)\,\eta_2^0 \\
&= \int_{\Sigma_T^s} \big(\eta_2(-\beta k_{1,xx} + \alpha k_{1,xxxx} + \rho_1\psi + \eta_2) - (\beta\eta_{1,xx} - \alpha\eta_{1,xxxx})\,k_2\big) - \int_{\Gamma_s} k_2(0)\,\eta_2^0.
\end{aligned}
$$

By combining the three identities, we obtain

$$
\begin{aligned}
&\rho_1 \int_{Q_T} |\mathbf{v}|^2 + \int_{\Sigma_T^s} |(-A_{\alpha,\beta})^{1/2}\eta_1|^2 + \int_{\Sigma_T^s} |\eta_2|^2 + \int_{\Sigma_T^s} |f|^2 \\
&= \rho_1 \int_\Omega \mathbf{v}^0\,\boldsymbol{\Phi}_0 + \int_{\Gamma_s} (-A_{\alpha,\beta})\eta_1^0\,k_1(0) + \int_{\Gamma_s} \eta_2^0\,k_2(0).
\end{aligned}
$$

Thus the optimal state, the optimal control, and the corresponding adjoint states obey

$$
\begin{aligned}
&\rho_1 \int_{Q_T} |\mathbf{v}|^2 + \int_{\Sigma_T^s} |\eta_2|^2 + \int_{\Sigma_T^s} |(-A_{\alpha,\beta})^{1/2}\eta_1|^2 + \int_{\Sigma_T^s} |f|^2 \\
&= \rho_1 \int_\Omega \mathbf{v}^0\cdot\boldsymbol{\Phi}(0) + \int_{\Gamma_s} (-A_{\alpha,\beta})\eta_1^0\,k_1(0) + \int_{\Gamma_s} \eta_2^0\,k_2(0) \\
&= \big(\Pi(T)(\mathbf{v}^0,\eta_1^0,\eta_2^0), (\mathbf{v}^0,\eta_1^0,\eta_2^0)\big)_{\mathbf{H}_{cc}}.
\end{aligned}
$$

This ends the proof. □

8. Studying problems $(\mathcal{P}^\infty_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$ and $(\mathcal{R}^\infty_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0})$.

8.1. Problem $(\mathcal{P}^\infty_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$. *Proof of Theorem 6.1.* The existence of admissible controls follows from Theorem 5.2. Next the existence of an optimal control can be proved in a classical way. The operator Π is obtained as the limit of $\Pi(T)$ when T tends to infinity (see e.g. [23, Theorem 4.1]). □

Following the approach of [23, Lemma 4.2], we can obtain an optimality system for Problem $(\mathcal{P}^\infty_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$ in the form

$$\begin{aligned}
&\mathbf{v}_t - \operatorname{div}\sigma(\mathbf{v},p) - \omega\mathbf{v} = 0 \quad \text{and} \quad \operatorname{div}\mathbf{v} = 0 \quad \text{in } Q_\infty,\\
&\mathbf{v} = \eta_2\vec{e}_2 \ \text{ on } \Sigma^s_\infty, \quad \mathbf{v} = 0 \ \text{ on } \Sigma^0_\infty, \quad \mathbf{v}(0) = \mathbf{v}^0 \text{ in } \Omega,\\
&\eta_{1,t} = \eta_2 + \omega\eta_1,\\
&\eta_{2,t} - \omega\eta_2 - \beta\eta_{1,xx} - \delta\eta_{2,xx} + \alpha\eta_{1,xxxx} = \rho_1 p - k_2 \ \text{ on } \Sigma^s_\infty,\\
&\eta_1 = 0 \quad \text{and} \quad \eta_{1,x}x = 0 \ \text{ on } \big\{0, L\big\} \times (0,\infty),\\
&\eta_1(0) = \eta_1^0 \quad \text{and} \quad \eta_2(0) = \eta_2^0 \ \text{ in } \Gamma_s,\\
&-\mathbf{\Phi}_t - \operatorname{div}\sigma(\mathbf{\Phi},\psi) - \omega\mathbf{\Phi} = \mathbf{v} \quad \text{and} \quad \operatorname{div}\mathbf{\Phi} = 0 \quad \text{in } Q_\infty,\\
&\mathbf{\Phi} = k_2\vec{e}_2 \ \text{ on } \Sigma^s_\infty, \quad \mathbf{\Phi} = 0 \ \text{ on } \Sigma^0_\infty, \quad \mathbf{\Phi}(\infty) = 0 \text{ in } \Omega,\\
&-k_{1,t} = -k_2 + \omega k_1 - \eta_1,\\
&-k_{2,t} - \omega k_2 + \beta k_{1,xx} - \delta k_{2,xx} - \alpha k_{1,xxxx} = \rho_1\psi + \eta_2 \ \text{ on } \Sigma^s_\infty,\\
&k_1(\infty) = 0 \quad \text{and} \quad k_2(\infty) = 0 \ \text{ in } \Gamma_s,\\
&(\mathbf{\Phi}(t), k_1(t), k_2(t)) = \Pi(\mathbf{v}(t), \eta_1(t), \eta_2(t)).
\end{aligned} \tag{8.1}$$

More precisely the following theorem can be proved by adapting the proof of [23, Lemma 4.2] to problem $(\mathcal{P}^\infty_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$.

THEOREM 8.1. *For all $(\mathbf{v}^0,\eta_1^0,\eta_2^0) \in \mathbf{H}_{cc}$, system (8.1) admits a unique solution $(\mathbf{v}, p, \eta_1, \eta_2, \mathbf{\Phi}, \psi, k_1, k_2)$ in $W(0,\infty;\mathbf{V}^1(\Omega),\mathbf{V}^{-1}(\Omega)) \times L^2(0,\infty;L^2_0(\Omega)) \times H^{2,1}(\Sigma^s_\infty) \times L^2(0,\infty;\mathcal{H}^1(\Gamma_s)) \times \mathbf{V}^{2,1}(Q_\infty) \times L^2(0,\infty;\mathcal{H}^1(\Omega)) \times H^{4,2}(\Sigma^s_\infty) \times H^{2,1}(\Sigma^s_\infty)$, and the optimal control to $(\mathcal{P}^\infty_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$ is*

$$f = -k_2.$$

THEOREM 8.2. *If $(P\mathbf{v}^0,\eta_1^0,\eta_2^0) \in [D(\mathcal{A}),\mathbf{H}]_{1/2}$, $(\mathbf{v}^0,\eta_1^0,\eta_2^0) \in \mathbf{H}_{cc}$, then the optimal solution to Problem $(\mathcal{P}^\infty_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$ belongs to $\mathbf{H}^{2,1}(Q_\infty) \times H^{4,2}(\Sigma^s_\infty) \times H^{2,1}(\Sigma^s_\infty)$ and*

$$\begin{aligned}
&\|\mathbf{v}\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|\eta_1\|_{H^{4,2}(\Sigma^s_\infty)} + \|\eta_2\|_{H^{2,1}(\Sigma^s_\infty)}\\
&\le C(\|P\mathbf{v}^0\|_{\mathbf{V}^1_n(\Omega)} + \|\eta_1^0\|_{H^3(\Gamma_s)\cap H^2_0(\Gamma_s)} + \|\eta_2^0\|_{H^1_0(\Gamma_s)}).
\end{aligned}$$

The proof is postponed to subsection 8.3.

8.2. Problem $(\mathcal{R}^\infty_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0})$. In order to prove Theorem 6.4, we first need to compare the solutions to $(\mathcal{P}^T_{0,\mathbf{v}^0,\eta_1^0,\eta_2^0})$ and $(\mathcal{R}^T_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0})$, where

$$(\mathcal{R}^T_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0}) \qquad \inf\Big\{\widehat{I}^T_0(P\mathbf{v},\eta_1,\eta_2,f) \ \mid \ (P\mathbf{v},\eta_1,\eta_2,f) \text{ satisfies (4.1)}, \ f \in L^2(0,T;L^2_0(\Gamma_s))\Big\},$$

and

$$\widehat{I}^T_0(P\mathbf{v},\eta_1,\eta_2,f) = \frac{\rho_1}{2}\int_{Q_T}|P\mathbf{v}|^2 + \frac{1}{2}\int_0^T \|\eta_1(t)\|^2_{H^2_0(\Gamma_s)} + \frac{1}{2}\int_{\Sigma^s_T}|(I+\rho_1\gamma_s N_s)\eta_2|^2 + \frac{1}{2}\int_{\Sigma^s_T}|f|^2.$$

The following theorem is a classical result in control theory.

THEOREM 8.3. *For all $(P\mathbf{v}^0,\eta_1^0,\eta_2^0) \in \widehat{\mathbf{H}}$, Problem $(\mathcal{R}^T_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0})$ admits a unique solution.*

The system

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \mathcal{A}_\omega \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} - \mathcal{B}\mathcal{B}^\sharp \begin{pmatrix} P\mathbf{\Phi} \\ k_1 \\ k_2 \end{pmatrix}, \qquad \begin{pmatrix} P\mathbf{v}(0) \\ \eta_1(0) \\ \eta_2(0) \end{pmatrix} = \begin{pmatrix} P\mathbf{v}^0 \\ \eta_1^0 \\ \eta_2^0 \end{pmatrix}, \\
&-\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} P\mathbf{\Phi} \\ k_1 \\ k_2 \end{pmatrix} = \mathcal{A}_\omega^\sharp \begin{pmatrix} P\mathbf{\Phi} \\ k_1 \\ k_2 \end{pmatrix} + \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, \qquad \begin{pmatrix} P\mathbf{\Phi}(T) \\ k_1(T) \\ k_2(T) \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \\ 0 \end{pmatrix},
\end{aligned} \tag{8.2}
$$

admits a unique solution $(P\mathbf{v}, \eta_1, \eta_2, P\mathbf{\Phi}, k_1, k_2)$ *and the optimal control to* $(\mathcal{R}^T_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0})$ *is*

$$f(t) = -\mathcal{B}^\sharp(P\mathbf{\Phi}(t), k_1(t), k_2(t)) = -k_2(t).$$

The operator $\widehat{\Pi}(T) \in \mathcal{L}(\widehat{\mathbf{H}})$ *defined by*

$$\widehat{\Pi}(T)(P\mathbf{v}^0, \eta_1^0, \eta_2^0) = (P\mathbf{\Phi}(0), k_1(0)), k_2(0)),$$

is linear and continuous in $\widehat{\mathbf{H}}$*, it is symmetric and semidefinite positive, and the optimal cost is given by*

$$\inf(\mathcal{R}^T_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0}) = \frac{1}{2}\big(\widehat{\Pi}(T)(P\mathbf{v}^0, \eta_1^0, \eta_2^0), (P\mathbf{v}^0, \eta_1^0, \eta_2^0)\big)_{\widehat{\mathbf{H}}}.$$

Using the expression of $\mathcal{A}_\omega^\sharp$ determined in section 3.5, it can be shown that the solution $(P\mathbf{v}, \eta_1, \eta_2, P\mathbf{\Phi}, k_1, k_2)$ to system (8.2) and the solution $(\bar{\mathbf{v}}, \bar{p}, \bar{\eta}_1, \bar{\eta}_2, \bar{\mathbf{\Phi}}, \bar{\psi}, \bar{k}_1, \bar{k}_2)$ to system (7.2) obey

$$(P\bar{\mathbf{v}}, \bar{\eta}_1, \bar{\eta}_2, P\bar{\mathbf{\Phi}}, \bar{k}_1, \bar{k}_2) = (P\mathbf{v}, \eta_1, \eta_2, P\mathbf{\Phi}, k_1, k_2).$$

Therefore we have

$$
\begin{aligned}
&\widehat{\Pi}(T)(P\mathbf{v}^0, \eta_1^0, \eta_2^0) = (P\mathbf{\Phi}(0), k_1(0), k_2(0)) = (P\bar{\mathbf{\Phi}}(0), \bar{k}_1(0), \bar{k}_2(0)) \\
&= \begin{pmatrix} P & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \Pi(T)(\mathbf{v}^0, \eta_1^0, \eta_2^0) \quad \text{for all } (\mathbf{v}^0, \eta_1^0, \eta_2^0) \in \mathbf{H}_{cc}.
\end{aligned} \tag{8.3}
$$

The first part of Theorem 6.3 can be proved as in Theorem 6.1. For the existence of a unique solution to the Riccati equation (6.2), we may proceed in a usual way as in [2] or in [15].

The following analogue of Theorem 8.1 can be proved for problem $(\mathcal{R}^\infty_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0})$.

THEOREM 8.4. *For all* $(P\mathbf{v}^0, \eta_1^0, \eta_2^0) \in \mathbf{H}$*, we consider the system*

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} = \mathcal{A}_\omega \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix} - \mathcal{B}\mathcal{B}^\sharp \begin{pmatrix} P\mathbf{\Phi} \\ k_1 \\ k_2 \end{pmatrix}, \qquad \begin{pmatrix} P\mathbf{v}(0) \\ \eta_1(0) \\ \eta_2(0) \end{pmatrix} = \begin{pmatrix} P\mathbf{v}^0 \\ \eta_1^0 \\ \eta_2^0 \end{pmatrix}, \\
&-\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} P\mathbf{\Phi} \\ k_1 \\ k_2 \end{pmatrix} = \mathcal{A}_\omega^\sharp \begin{pmatrix} P\mathbf{\Phi} \\ k_1 \\ k_2 \end{pmatrix} + \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, \qquad \begin{pmatrix} P\mathbf{\Phi}(\infty) \\ k_1(\infty) \\ k_2(\infty) \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \\ 0 \end{pmatrix}, \\
&(P\mathbf{\Phi}(t), k_1(t), k_2(t)) = \widehat{\Pi}(P\mathbf{v}(t), \eta_1(t), \eta_2(t)).
\end{aligned} \tag{8.4}
$$

System (8.4) *admits a unique solution* $(P\mathbf{v}, \eta_1, \eta_2, P\mathbf{\Phi}, k_1, k_2)$ *in* $W(0,\infty; \mathbf{V}^1(\Omega), \mathbf{V}^{-1}(\Omega)) \times H^{2,1}(\Sigma^s_\infty) \times \mathbf{V}^{2,1}(Q_\infty) \times H^{4,2}(\Sigma^s_\infty) \times H^{2,1}(\Sigma^s_\infty)$*, and the optimal control to* $(\mathcal{R}^\infty_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0})$ *is*

$$f = -k_2.$$

This theorem may be proved, as in [23], by passing to the limit in the optimality system of the finite time horizon control problem $(\mathcal{R}^T_{0,P\mathbf{v}^0,\eta_1^0,\eta_2^0})$.

Proof of Theorem 6.4. Since Π and $\widehat{\Pi}$ are defined as the respective limits of $\Pi(T)$ and $\widehat{\Pi}(T)$ when T tends to infinity, with (8.3), we obtain

$$\begin{pmatrix} P & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \Pi(\mathbf{v}^0,\eta_1^0,\eta_2^0) = \widehat{\Pi}(P\mathbf{v}^0,\eta_1^0,\eta_2^0),$$

for all $(\mathbf{v}^0,\eta_1^0,\eta_2^0) \in \mathbf{H}_{cc}$. This equality gives the expression for $P\Pi_1$, Π_2 and Π_3. The expression for $(I-P)\Pi_1$ follows from the equalities

$$(I-P)\Pi_1(\mathbf{v}^0,\eta_1^0,\eta_2^0) = (I-P)\mathbf{\Phi}(0) = (I-P)D_s k_2(0) = (I-P)D_s\widehat{\Pi}_3(P\mathbf{v}^0,\eta_1^0,\eta_2^0).$$

□

8.3. Proof of Theorem 8.2. The proof is based on the fact that system (8.1) is equivalent to system (8.4) with the additional equations $(I-P)\mathbf{v} = (I-P)D(\eta_2\vec{e}_2\chi_{\Gamma_s})$ and $(I-P)\mathbf{\Phi} = (I-P)D(k_2\vec{e}_2\chi_{\Gamma_s})$. Since we can use, for system (8.4), the maximal regularity result stated in [1, Chapter 1, Theorem 3.1], we can derive the same estimates for the solution to system (8.1).

We already know that

$$\begin{aligned} &\|P\mathbf{v}\|_{L^2(0,\infty;\mathbf{V}^0_n(\Omega))} + \|\eta_1\|_{L^2(0,\infty;H^2_0(\Gamma_s))} + \|\eta_2\|_{L^2(0,\infty;L^2_0(\Gamma_s))} \\ &\qquad + \|P\mathbf{\Phi}\|_{L^2(0,\infty;\mathbf{V}^0_n(\Omega))} + \|k_1\|_{L^2(0,\infty;H^2_0(\Gamma_s))} + \|k_2\|_{L^2(0,\infty;L^2_0(\Gamma_s))} \\ &\le C(\|P\mathbf{v}^0\|_{\mathbf{V}^0_n(\Omega)} + \|\eta_1^0\|_{H^2_0(\Gamma_s)} + \|\eta_2^0\|_{L^2_0(\Gamma_s)}). \end{aligned} \tag{8.5}$$

We can rewrite the adjoint equation of (8.4) in the form

$$-\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} P\mathbf{\Phi} \\ k_1 \\ k_2 \end{pmatrix} = (\mathcal{A}^\sharp_\omega - \lambda I)\begin{pmatrix} P\mathbf{\Phi} \\ k_1 \\ k_2 \end{pmatrix} + \lambda\begin{pmatrix} P\mathbf{\Phi} \\ k_1 \\ k_2 \end{pmatrix} + \begin{pmatrix} P\mathbf{v} \\ \eta_1 \\ \eta_2 \end{pmatrix}, \qquad \begin{pmatrix} P\mathbf{\Phi}(\infty) \\ k_1(\infty) \\ k_2(\infty) \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \\ 0 \end{pmatrix}, \tag{8.6}$$

We choose $\lambda > 0$ such that $(e^{t(\mathcal{A}^\sharp_\omega - \lambda I)})_{t\ge 0}$ is exponentially stable. From [1, Chapter 1, Theorem 3.1], with estimate (8.5), it can be shown that the solution $(P\mathbf{\Phi}, k_1, k_2)$ of system (8.6) obeys

$$\begin{aligned} &\|P\mathbf{\Phi}\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|k_1\|_{H^{4,2}(\Sigma^s_\infty)} + \|k_2\|_{H^{2,1}(\Sigma^s_\infty)} \\ &\le C(\|P\mathbf{\Phi}\|_{L^2(0,\infty;\mathbf{V}^0_n(\Omega))} + \|k_1\|_{L^2(0,\infty;H^2_0(\Gamma_s))} + \|k_2\|_{L^2(0,\infty;L^2_0(\Gamma_s))}) \\ &\le C(\|P\mathbf{v}^0\|_{\mathbf{V}^0_n(\Omega)} + \|\eta_1^0\|_{H^2_0(\Gamma_s)} + \|\eta_2^0\|_{L^2_0(\Gamma_s)}). \end{aligned} \tag{8.7}$$

Next, with estimates (8.5) and (8.7), still with [1, Chapter 1, Theorem 3.1], and with [23], we can show that

$$\begin{aligned} &\|P\mathbf{v}\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|\eta_1\|_{H^{4,2}(\Sigma^s_\infty)} + \|\eta_2\|_{H^{2,1}(\Sigma^s_\infty)} \\ &\le C(\|P\mathbf{v}^0\|_{\mathbf{V}^1_n(\Omega)} + \|\eta_1^0\|_{(H^3\cap H^2_0)(\Gamma_s)} + \|\eta_2^0\|_{H^1_0(\Gamma_s)} + \|k_2\|_{L^2_0(\Sigma^s_\infty)}). \end{aligned} \tag{8.8}$$

This completes the proof.

9. Nonhomogeneous system. We now consider the nonhomogeneous linearized system

$$
\begin{aligned}
&\mathbf{v}_t - \operatorname{div}\sigma(\mathbf{v},p) - \omega\mathbf{v} = F \quad \text{and} \quad \operatorname{div}\mathbf{v} = G = \operatorname{div}\bar{\mathbf{w}} \quad \text{in } Q_\infty,\\
&\mathbf{v} = \eta_2\vec{e}_2 \ \text{ on } \Sigma^s_\infty, \quad \mathbf{v} = 0 \ \text{ on } \Sigma^0_\infty, \quad \mathbf{v}(0) = \mathbf{v}^0 \text{ in } \Omega,\\
&\eta_{1,t} = \eta_2 + \omega\eta_1 \quad \text{on } \Sigma^s_\infty,\\
&\eta_{2,t} - \omega\eta_2 - \beta\eta_{1,xx} - \delta\eta_{2,xx} + \alpha\eta_{1,xxxx} = \rho_1 p - 2\nu\rho_2\mathbf{v}_{2,z} + H - \Pi_3(\mathbf{v},\eta_1,\eta_2) \ \text{ on } \Sigma^s_\infty,\\
&\eta_1 = 0 \quad \text{and} \quad \eta_{1,x} = 0 \ \text{ on } \big\{0,L\big\}\times(0,\infty),\\
&\eta_1(0) = \eta^0_1 \quad \text{and} \quad \eta_2(0) = \eta^0_2 \ \text{ in } \Gamma_s.
\end{aligned}
\tag{9.1}
$$

We can look for a solution to system (9.1) in the form $\mathbf{v} = \mathbf{w} + \bar{\mathbf{w}}$, where $(\mathbf{w}, p, \eta)$ is the solution to

$$
\begin{aligned}
&\mathbf{w}_t - \operatorname{div}\sigma(\mathbf{w},p) - \omega\mathbf{w} = F - \bar{\mathbf{w}}_t + \nu\Delta\bar{\mathbf{w}} + \nu\nabla\operatorname{div}\bar{\mathbf{w}} + \omega\bar{\mathbf{w}} \quad \text{and} \quad \operatorname{div}\mathbf{w} = 0 \quad \text{in } Q_\infty,\\
&\mathbf{w} = \eta_2\vec{e}_2 \ \text{ on } \Sigma^s_\infty, \quad \mathbf{w} = 0 \ \text{ on } \Sigma^0_\infty, \quad \mathbf{w}(0) = \mathbf{v}^0 - \bar{\mathbf{w}}(0) \text{ in } \Omega,\\
&\eta_{1,t} = \eta_2 + \omega\eta_1 \quad \text{on } \Sigma^s_\infty,\\
&\eta_{2,t} - \omega\eta_2 - \beta\eta_{1,xx} - \delta\eta_{2,xx} + \alpha\eta_{1,xxxx}\\
&\qquad\qquad = \rho_1 p - 2\nu\rho_2(\mathbf{w}_{2,z} + \bar{\mathbf{w}}_{2,z}) + H - \Pi_3(\bar{\mathbf{w}},0,0) - \Pi_3(\mathbf{w},\eta_1,\eta_2) \ \text{ on } \Sigma^s_\infty,\\
&\eta_1 = 0 \quad \text{and} \quad \eta_{1,x} = 0 \ \text{ on } \big\{0,L\big\}\times(0,\infty),\\
&\eta_1(0) = \eta^0_1 \quad \text{and} \quad \eta_2(0) = \eta^0_2 \ \text{ in } \Gamma_s.
\end{aligned}
\tag{9.2}
$$

Since $\operatorname{div}\mathbf{w} = 0$, the term $2\nu\rho_2\mathbf{w}_{2,z}$ can be dropped out in the equation satisfied by η_2, but not the term $2\nu\rho_2\bar{\mathbf{w}}_{2,z}$. We introduce the operator unbounded operator $(\mathcal{A}_{\omega,\widehat{\Pi}}, D(\mathcal{A}_{\omega,\widehat{\Pi}}))$ in $\mathbf{H}$, defined by $D(\mathcal{A}_{\omega,\widehat{\Pi}}) = D(\mathcal{A})$ and

$$\mathcal{A}_{\omega,\widehat{\Pi}} = \mathcal{A}_\omega - \mathcal{B}\mathcal{B}^\sharp\widehat{\Pi}.$$

System (9.2) can be written in the form

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} P\mathbf{w}\\ \eta_1\\ \eta_2\end{pmatrix} = \mathcal{A}_{\omega,\widehat{\Pi}}\begin{pmatrix} P\mathbf{w}\\ \eta_1\\ \eta_2\end{pmatrix} + \mathcal{B}\bar{H} + \begin{pmatrix} P\bar{F}\\ 0\\ 0\end{pmatrix}, \qquad \begin{pmatrix} P\mathbf{w}(0)\\ \eta_1(0)\\ \eta_2(0)\end{pmatrix} = \begin{pmatrix} P(\mathbf{v}^0 - \bar{\mathbf{w}}(0))\\ \eta^0_1\\ \eta^0_2\end{pmatrix},\\
&(I-P)\mathbf{w} = (I-P)D(\eta_2\vec{e}_2\chi_{\Gamma_s}),
\end{aligned}
\tag{9.3}
$$

where

$$
\begin{aligned}
&\bar{F} = F - \bar{\mathbf{w}}_t + \nu\Delta\bar{\mathbf{w}} + \nu\nabla\operatorname{div}\bar{\mathbf{w}} + \omega\bar{\mathbf{w}},\\
&\bar{H} = -2\nu\rho_2\bar{\mathbf{w}}_{2,z} + H - \Pi_3(\bar{\mathbf{w}},0,0) + \rho_1(I+\rho_1\gamma_s N_s)^{-1}\gamma_s N((\bar{F} + \nabla(-\Delta_D)^{-1}\operatorname{div}\bar{F})\cdot\mathbf{n}).
\end{aligned}
$$

We assume that $\bar{\mathbf{w}}$ belongs to $\mathbf{H}^{2,1}(Q_\infty)$, $F \in L^2(0,\infty;\mathbf{L}^2(\Omega))$, and $H \in L^2(0,\infty;L^2_0(\Gamma_s))$. Thus $P\bar{F}$ belongs to $L^2(0,\infty;\mathbf{V}^0_n(\Omega))$. Moreover $(\bar{F}+\nabla(-\Delta_D)^{-1}\operatorname{div}\bar{F})\cdot\mathbf{n}$ belongs to $L^2(0,\infty;\mathcal{H}^{-1/2}(\Gamma))$, $\gamma_s N((\bar{F}+\nabla(-\Delta_D)^{-1}\operatorname{div}\bar{F})\cdot\mathbf{n})$ belongs to $L^2(0,\infty;\mathcal{H}^{1/2}(\Gamma_s))$, and $\bar{H}$ belongs to $L^2(0,\infty;L^2_0(\Gamma_s))$. Since the semigroup generated by $(\mathcal{A}_{\omega,\widehat{\Pi}}, D(\mathcal{A}_{\omega,\widehat{\Pi}}))$ is exponentially stable on $\mathbf{V}^0_n(\Omega)\times(H^2_0(\Gamma_s)\cap L^2_0(\Gamma_s))\times L^2_0(\Gamma_s)$, system (9.3) admits a unique solution $(P\mathbf{w},\eta_1,\eta_2)$ in $L^2(0,\infty;\mathbf{V}^0_n(\Omega)\times(H^2_0(\Gamma_s)\cap L^2_0(\Gamma_s))\times L^2_0(\Gamma_s))$.

Theorem 9.1. *If* $(P\mathbf{v}^0 - P\bar{\mathbf{w}}(0),\eta^0_1,\eta^0_2) \in [D(\mathcal{A}),\mathbf{H}]_{1/2}$, $(\mathbf{v}^0 - \bar{\mathbf{w}}(0),\eta^0_1,\eta^0_2) \in \mathbf{H}_{cc}$, $F \in L^2(0,\infty;\mathbf{L}^2(\Omega))$, $\bar{\mathbf{w}} \in \mathbf{H}^{2,1}(Q_\infty)$, $H \in L^2(0,\infty;L^2_0(\Gamma_s))$, *then system* (9.1) *admits a unique solution, which belongs to* $\mathbf{H}^{2,1}(Q_\infty)\times H^{4,2}(\Sigma^s_\infty)\times H^{2,1}(\Sigma^s_\infty)$ *and*

$$
\begin{aligned}
&\|\mathbf{v}\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|p\|_{L^2(0,1;H^1(\Omega))} + \|\eta_1\|_{H^{4,2}(\Sigma^s_\infty)} + \|\eta_1\|_{L^\infty(\Sigma^s_\infty)} + \|\eta_2\|_{H^{2,1}(\Sigma^s_\infty)}\\
&\le C_1(\|P\mathbf{v}^0\|_{\mathbf{V}^1_n(\Omega)} + \|\eta_0\|_{H^3(\Gamma_s)} + \|\eta_1\|_{H^1(\Gamma_s)} + \|F\|_{\mathbf{L}^2(Q_\infty)} + \|\bar{\mathbf{w}}\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|H\|_{L^2(\Sigma^s_\infty)}).
\end{aligned}
$$

Proof. We first consider system (9.3). We know that $(P\mathbf{v}^0 - P\bar{\mathbf{w}}(0), \eta_1^0, \eta_2^0) \in [D(\mathcal{A}_{\omega,\widehat{\Pi}}), \mathbf{H}]_{1/2}$, $(P\bar{F}, 0, \bar{H}) \in L^2(0,\infty;\mathbf{H})$, and that the semigroup generated by $(\mathcal{A}_{\omega,\widehat{\Pi}}, D(\mathcal{A}_{\omega,\widehat{\Pi}}))$ is exponentially stable on $\mathbf{H}$. Thus arguing as in (8.6), from [1, Chapter 1, Theorem 3.1] it follows that

$$
\begin{aligned}
&\|\mathbf{w}\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|\eta_1\|_{H^{4,2}(\Sigma^s_\infty)} + \|\eta_2\|_{H^{2,1}(\Sigma^s_\infty)} \\
&\le C_1(\|P\mathbf{v}^0\|_{\mathbf{V}^1_n(\Omega)} + \|\eta_1^0\|_{H^3(\Gamma_s)} + \|\eta_2^0\|_{H^1(\Gamma_s)} + \|F\|_{\mathbf{L}^2(Q_\infty)} + \|\bar{\mathbf{w}}\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|H\|_{L^2(\Sigma^s_\infty)}).
\end{aligned}
$$

Since $\mathbf{v} = \mathbf{w} + \bar{\mathbf{w}}$ and $\bar{\mathbf{w}} \in \mathbf{H}^{2,1}(Q_\infty)$, we recover the estimate for $\mathbf{v}$. The estimate for the pressure can be obtained from the estimate for $\mathbf{v}$ and from the first equation of system (9.1). ∎

10. Stabilization of the coupled system. In this section we study the nonlinear closed loop system

$$
\begin{aligned}
&\tilde{\mathbf{u}}_t - \operatorname{div}\sigma(\tilde{\mathbf{u}},\tilde{p}) - \omega\tilde{\mathbf{u}} = e^{-\omega t}\tilde{F}(\tilde{\mathbf{u}},\tilde{p},\tilde{\eta}_1,\tilde{\eta}_2), \quad \operatorname{div}\tilde{\mathbf{u}} = e^{-\omega t}\tilde{G}(\tilde{\eta}_1,\tilde{\mathbf{u}}) \quad \text{in } Q_\infty,\\
&\tilde{\mathbf{u}} = \tilde{\eta}_2\vec{e}_2 \ \text{ on } \Sigma^s_\infty, \quad \tilde{\mathbf{u}} = 0 \ \text{ on } \Sigma^0_\infty, \quad \tilde{\mathbf{u}}(0) = \hat{\mathbf{u}}^0 \text{ in } \Omega,\\
&\tilde{\eta}_{1,t} = \tilde{\eta}_2 + \omega\tilde{\eta}_1 \quad \text{on } \Sigma^s_\infty,\\
&\tilde{\eta}_{2,t} - \omega\tilde{\eta}_2 - \beta\tilde{\eta}_{1,xx} - \delta\tilde{\eta}_{2,xx} + \alpha\tilde{\eta}_{1,xxxx}\\
&\qquad\qquad = \rho_1\tilde{p} - 2\nu\rho_2\tilde{\mathbf{u}}_{2,z} + e^{-\omega t}\tilde{H}(\tilde{\mathbf{u}},\tilde{\eta}_1) - \Pi_3(\tilde{\mathbf{u}},\tilde{\eta}_1,\tilde{\eta}_2) \ \text{ on } \Sigma^s_\infty,\\
&\tilde{\eta}_1 = 0 \quad \text{and} \quad \tilde{\eta}_{1,x} = 0 \ \text{ on } \{0,L\}\times(0,\infty),\\
&\tilde{\eta}_1(0) = \eta_1^0 \quad \text{and} \quad \tilde{\eta}_2(0) = \eta_2^0 \ \text{ in } \Gamma_s,
\end{aligned}
\tag{10.1}
$$

with

$$
\begin{aligned}
&\tilde{F}(\tilde{\mathbf{u}},\tilde{p},\tilde{\eta}_1,\tilde{\eta}_2)\\
&= -\tilde{\eta}_1(\tilde{\mathbf{u}}_t - \omega\tilde{\mathbf{u}}) + \left(z\tilde{\eta}_2 + \nu z\left(\frac{\tilde{\eta}_{1,x}^2}{e^{\omega t}+\tilde{\eta}_1} - \tilde{\eta}_{1,xx}\right)\right)\tilde{\mathbf{u}}_z - (\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}}\\
&\quad +\nu\left(-2z\tilde{\eta}_{1,x}\tilde{\mathbf{u}}_{xz} + \tilde{\eta}_1\tilde{\mathbf{u}}_{xx} + \left(\frac{z^2\tilde{\eta}_{1,x}^2 - e^{-\omega t}\tilde{\eta}_1}{e^{\omega t}+\tilde{\eta}_1}\right)\tilde{\mathbf{u}}_{zz}\right)\\
&\quad +z(\tilde{\eta}_{1,x}\tilde{p}_z - \tilde{\eta}_1\tilde{p}_x)\vec{e}_1 - (1+e^{-\omega t}\tilde{\eta}_1)\tilde{\mathbf{u}}_1\tilde{\mathbf{u}}_x + (ze^{-\omega t}\tilde{\eta}_{1,x}\tilde{\mathbf{u}}_1 - \tilde{\mathbf{u}}_2)\tilde{\mathbf{u}}_z,
\end{aligned}
\tag{10.2}
$$

$$
\tilde{G}(\tilde{\mathbf{u}},\tilde{\eta}_1) = -\tilde{\eta}_1\tilde{\mathbf{u}}_{1,x} + z\tilde{\eta}_{1,x}\tilde{\mathbf{u}}_{1,z} = \operatorname{div}\tilde{\mathbf{w}}, \quad \tilde{\mathbf{w}} = -\tilde{\eta}_1\tilde{\mathbf{u}}_1\vec{e}_1 + z\tilde{\eta}_{1,x}\tilde{\mathbf{u}}_1\vec{e}_2,
\tag{10.3}
$$

and

$$
\tilde{H}(\tilde{\mathbf{u}},\tilde{\eta}_1) = \nu\rho_2\left(\frac{\tilde{\eta}_{1,x}}{e^{\omega t}+\tilde{\eta}_1}\tilde{\mathbf{u}}_{1,z} + e^{-\omega t}\tilde{\eta}_{1,x}\tilde{\mathbf{u}}_{2,x} - \frac{\tilde{\eta}_{1,x}^2}{e^{\omega t}+\tilde{\eta}_1}\tilde{\mathbf{u}}_{2,z} + \frac{\tilde{\eta}_1}{e^{\omega t}+\tilde{\eta}_1}\tilde{\mathbf{u}}_{2,z}\right).
\tag{10.4}
$$

We want to show the following theorem.

THEOREM 10.1. *There exist $0 < \mu_0 < 1$ and an increasing function θ_0 from $\mathbb{R}^+$ into itself such that if $\mu \in (0,\mu_0)$, $(P(\hat{\mathbf{u}}^0 + \eta_1^0\hat{\mathbf{u}}_1^0\vec{e}_1 - z\eta_{1,x}^0\hat{\mathbf{u}}_1^0\vec{e}_2), \eta_1^0, \eta_2^0) \in [D(\mathcal{A}), \mathbf{H}]_{1/2}$, $(\hat{\mathbf{u}}^0 + \eta_1^0\hat{\mathbf{u}}_1^0\vec{e}_1 - z\eta_{1,x}^0\hat{\mathbf{u}}_1^0\vec{e}_2)|_\Gamma = \eta_2^0\,\vec{e}_2\,\chi_{\Gamma_s}$, $\|P\hat{\mathbf{u}}^0\|_{\mathbf{V}^1_n(\Omega)} + \|\eta_1^0\|_{H^3(\Gamma_s)\cap H^2_0(\Gamma_s)} + \|\eta_2^0\|_{H^1_0(\Gamma_s)} \le \theta_0(\mu)$ then system* (10.1) *admits a unique solution in the set*

$$
\begin{aligned}
\tilde{D}_\mu = \Big\{&(\tilde{\mathbf{u}},\tilde{p},\tilde{\eta}_1,\tilde{\eta}_2) \mid \\
&\|\tilde{\mathbf{u}}\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|\tilde{p}\|_{L^2(0,\infty;H^1(\Omega))} + \|\tilde{\eta}_1\|_{H^{4,2}(\Sigma^s_\infty)} + \|\tilde{\eta}_1\|_{L^\infty(\Sigma^s_\infty)} + \|\tilde{\eta}_2\|_{H^{2,1}(\Sigma^s_\infty)} \le \mu\Big\}.
\end{aligned}
$$

Next we consider the system

$$
\begin{aligned}
&\hat{\mathbf{u}}_t - \operatorname{div}\sigma(\hat{\mathbf{u}},\hat{p}) = \hat{F}(\hat{\mathbf{u}},\hat{p},\eta_1,\eta_2), \quad \operatorname{div}\hat{\mathbf{u}} = \hat{G}(\eta_1,\hat{\mathbf{u}}) \quad \text{ in } Q_\infty,\\
&\hat{\mathbf{u}} = \eta_2\vec{e}_2 \ \text{ on } \Sigma_\infty^s, \quad \hat{\mathbf{u}} = 0 \ \text{ on } \Sigma_\infty^0, \quad \hat{\mathbf{u}}(0) = \hat{\mathbf{u}}^0 \text{ in } \Omega,\\
&\eta_{1,t} = \eta_2 \quad \text{ on } \Sigma_\infty^s,\\
&\eta_{2,t} - \beta\eta_{1,xx} - \delta\eta_{2,xx} + \alpha\eta_{1,xxxx} = \rho_1\hat{p} + \hat{H}(\hat{\mathbf{u}},\eta_1) - \Pi_3(\hat{\mathbf{u}},\eta_1,\eta_2) \ \text{ on } \Sigma_\infty^s,\\
&\eta_1 = 0 \quad \text{and} \quad \eta_{1,x} = 0 \ \text{ on } \big\{0,L\big\}\times(0,\infty),\\
&\eta_1(0) = \eta_1^0 \quad \text{and} \quad \eta_2(0) = \eta_2^0 \ \text{ in } \Gamma_s,
\end{aligned}
\tag{10.5}
$$

where

$$
\begin{aligned}
&\hat{F}(\hat{\mathbf{u}},\hat{p},\eta_1,\eta_2)\\
&= -\eta_1\hat{\mathbf{u}}_t + \left(z\eta_2 + \nu z\left(\tfrac{\eta_{1,x}^2}{1+\eta_1} - \eta_{1,xx}\right)\right)\hat{\mathbf{u}}_z - (\hat{\mathbf{u}}\cdot\nabla)\hat{\mathbf{u}}\\
&\quad +\nu\left(-2z\eta_{1,x}\hat{\mathbf{u}}_{xz} + \eta_1\hat{\mathbf{u}}_{xx} + \left(\tfrac{z^2\eta_{1,x}^2-\eta_1}{1+\eta_1}\right)\hat{\mathbf{u}}_{zz}\right)\\
&\quad +z(\eta_{1,x}\hat{p}_z - \eta_1\hat{p}_x)\vec{e}_1 - (1+\eta_1)\hat{\mathbf{u}}_1\hat{\mathbf{u}}_x + (z\eta_{1,x}\hat{\mathbf{u}}_1 - \hat{\mathbf{u}}_2)\hat{\mathbf{u}}_z,
\end{aligned}
$$

$$
\hat{G}(\hat{\mathbf{u}},\eta_1) = -\eta_1\hat{\mathbf{u}}_{1,x} + z\eta_{1,x}\hat{\mathbf{u}}_{1,z} = \operatorname{div}(\hat{\mathbf{w}}), \quad \hat{\mathbf{w}} = (-\eta_1\hat{\mathbf{u}}_1\vec{e}_1 + z\eta_{1,x}\hat{\mathbf{u}}_1\vec{e}_2),
$$

and

$$
\hat{H}(\hat{\mathbf{u}},\eta_1) = \nu\rho_2\left(\tfrac{\eta_{1,x}}{1+\eta_1}\hat{\mathbf{u}}_{1,z} + \eta_{1,x}\hat{\mathbf{u}}_{2,x} - \tfrac{2+\eta_{1,x}^2}{1+\eta_1}\hat{\mathbf{u}}_{2,z}\right).
$$

From calculations in section 2 it follows that $(\tilde{\mathbf{u}},\tilde{p},\tilde{\eta}_1,\tilde{\eta}_2)$ is a solution to system (10.1) if and only if

$$
\hat{\mathbf{u}} = e^{-\omega t}\tilde{\mathbf{u}}, \quad \hat{p} = e^{-\omega t}\tilde{p}, \quad \eta_1 = e^{-\omega t}\tilde{\eta}_1, \quad \eta_2 = e^{-\omega t}\tilde{\eta}_2,
$$

is a solution to system (10.5). Therefore from Theorem 10.1, we deduce:

THEOREM 10.2. *There exist $0<\mu_0<1$ and an increasing function θ_0 from $\mathbb{R}^+$ into itself such that if $\mu\in(0,\mu_0)$, $(P(\hat{\mathbf{u}}^0+\eta_1^0\hat{\mathbf{u}}_1^0\vec{e}_1 - z\eta_{1,x}^0\hat{\mathbf{u}}_1^0\vec{e}_2),\eta_1^0,\eta_2^0)\in[D(\mathcal{A}),\mathbf{H}]_{1/2}$, $(\hat{\mathbf{u}}^0+\eta_1^0\hat{\mathbf{u}}_1^0\vec{e}_1 - z\eta_{1,x}^0\hat{\mathbf{u}}_1^0\vec{e}_2)|_{\Gamma_s} = \eta_2^0\,\vec{e}_2\,\chi_{\Gamma_s}$, $\|P\hat{\mathbf{u}}^0\|_{\mathbf{V}_n^1(\Omega)} + \|\eta_1^0\|_{H^3(\Gamma_s)\cap H_0^2(\Gamma_s)} + \|\eta_2^0\|_{H_0^1(\Gamma_s)} \le \theta_0(\mu)$ then system (10.5) admits a unique solution in the set*

$$
\begin{aligned}
D_\mu = \Big\{(\hat{\mathbf{u}},\hat{p},\eta_1,\eta_2) \mid\ & \|e^{\omega\cdot}\hat{\mathbf{u}}\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|e^{\omega\cdot}\hat{p}\|_{L^2(0,\infty;H^1(\Omega))} + \|e^{\omega\cdot}\eta_1\|_{H^{4,2}(\Sigma_\infty^s)}\\
&+\|e^{\omega\cdot}\eta_1\|_{L^\infty(\Sigma_\infty^s)} + \|e^{\omega\cdot}\eta_2\|_{H^{2,1}(\Sigma_\infty^s)} \le \mu\Big\}.
\end{aligned}
$$

Still from calculations in section 2 we know that $(\hat{\mathbf{u}},\hat{p},\eta_1,\eta_2)$ is a solution to system (10.5) if and only if $(\mathbf{u},p,\eta,\eta_t) = (\hat{\mathbf{u}}\circ\mathcal{T}_{\eta_1},\hat{p}\circ\mathcal{T}_{\eta_1},\eta_1,\eta_2)$ is solution to system (1.1) with $\mathbf{u}^0 = \hat{\mathbf{u}}^0\circ\mathcal{T}_{\eta_1^0}$. Thus from Theorem 10.2, we deduce:

THEOREM 10.3. *There exist $0<\mu_0<1$ and an increasing function θ_0 from $\mathbb{R}^+$ into itself such that if $\mu\in(0,\mu_0)$, $(P(\hat{\mathbf{u}}^0+\eta_1^0\hat{\mathbf{u}}_1^0\vec{e}_1 - z\eta_{1,x}^0\hat{\mathbf{u}}_1^0\vec{e}_2),\eta_1^0,\eta_2^0)\in[D(\mathcal{A}),\mathbf{H}]_{1/2}$, $(\hat{\mathbf{u}}^0+\eta_1^0\hat{\mathbf{u}}_1^0\vec{e}_1 - z\eta_{1,x}^0\hat{\mathbf{u}}_1^0\vec{e}_2)|_{\Gamma_s} = \eta_2^0\,\vec{e}_2\,\chi_{\Gamma_s}$, $\|P\hat{\mathbf{u}}^0\|_{\mathbf{V}_n^1(\Omega)} + \|\eta_1^0\|_{H^3(\Gamma_s)\cap H_0^2(\Gamma_s)} + \|\eta_2^0\|_{H_0^1(\Gamma_s)} \le \theta_0(\mu)$, where $\hat{\mathbf{u}}^0 = (\hat{\mathbf{u}}_1^0,\hat{\mathbf{u}}_2^0) = \mathbf{u}^0\circ\mathcal{T}_{\eta_1^0}^{-1}$, then system (1.1) with the feedback law $f = -\Pi_3(\mathbf{u}\circ\mathcal{T}_\eta^{-1}(x,z,t),\eta,\eta_t)$ admits a unique solution in the set*

$$
\begin{aligned}
F_\mu = \Big\{(\mathbf{u},p,\eta,\eta_t) \mid\ & \|e^{\omega\cdot}\mathbf{u}\circ\mathcal{T}_\eta^{-1}\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|e^{\omega\cdot}p\circ\mathcal{T}_\eta^{-1}\|_{L^2(0,\infty;H^1(\Omega))} + \|e^{\omega\cdot}\eta\|_{H^{4,2}(\Sigma_\infty^s)}\\
&+\|e^{\omega\cdot}\eta\|_{L^\infty(\Sigma_\infty^s)} + \|e^{\omega\cdot}\eta_t\|_{H^{2,1}(\Sigma_\infty^s)} \le \mu\Big\},
\end{aligned}
$$

where $\mathcal{T}_\eta$ is defined in (2.2).

11. Some Lipschitz properties. THEOREM 11.1. *The mapping*

$$(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2) \longmapsto (\tilde{F}(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2), \tilde{\mathbf{w}}(\tilde{\mathbf{u}}, \tilde{\eta}_1), \tilde{H}(\tilde{\mathbf{u}}, \tilde{\eta}_1)),$$

where $\tilde{F}$, $\tilde{\mathbf{w}}$, *and* $\tilde{H}$ *are respectively defined by* (10.2), (10.3) *and* (10.4), *is locally Lipschitz from* $\mathbf{H}^{2,1}(Q_\infty)\times L^2(0,\infty; H^1(\Omega))\times H^{4,2}(\Sigma_\infty^s)\times H^{2,1}(\Sigma_\infty^s)$ *into* $\mathbf{L}^2(Q_\infty)\times \mathbf{H}^{2,1}(Q_\infty)\times L^2(\Sigma_\infty^s)$. *More precisely, for all* $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2)$, $(\tilde{\mathbf{u}}^1, \tilde{p}^1, \tilde{\eta}_1^1, \tilde{\eta}_2^1)$, $(\tilde{\mathbf{u}}^2, \tilde{p}^2, \tilde{\eta}_1^2, \tilde{\eta}_2^2)$ *belonging to* $\mathbf{H}^{2,1}(Q_\infty) \times L^2(0,\infty; H^1(\Omega)) \times H^{4,2}(\Sigma_\infty^s)$ $\times H^{2,1}(\Sigma_\infty^s)$ *and such that* $\max(\|(1+\tilde{\eta}_1)^{-1}\|_{L^\infty(\Sigma_\infty^s)}, \|(1+\tilde{\eta}_1^1)^{-1}\|_{L^\infty(\Sigma_\infty^s)}, \|(1+\tilde{\eta}_1^2)^{-1}\|_{L^\infty(\Sigma_\infty^s)}) \le \mu_1$ *and* $\max(\|\tilde{\eta}_{1,x}\|_{L^\infty(\Sigma_\infty^s)}, \|\tilde{\eta}_{1,x}^1\|_{L^\infty(\Sigma_\infty^s)}, \|\tilde{\eta}_{1,x}^2\|_{L^\infty(\Sigma_\infty^s)}) \le 1$, *we have*

$$\begin{aligned}\|\tilde{F}(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2)\|_{L^\infty(0,\infty;\mathbf{L}^2(\Omega))} \le C_2(\mu_1)(&\|\tilde{\eta}_1\|_{H^{4,2}(\Sigma_\infty^s)}\|\tilde{\mathbf{u}}_1\|_{H^{2,1}(Q_\infty)} + \|\tilde{\eta}_2\|_{H^{2,1}(\Sigma_\infty^s)}\|\tilde{\mathbf{u}}_1\|_{H^{2,1}(Q_\infty)} \\ &+\|\tilde{\eta}_1\|_{H^{4,2}(\Sigma_\infty^s)}\|\tilde{p}\|_{L^2(0,\infty;\mathbf{H}^1(\Omega))} + \|\tilde{\mathbf{u}}\|_{\mathbf{H}^{2,1}(Q_\infty)}\|\tilde{\mathbf{u}}\|_{\mathbf{H}^{2,1}(Q_\infty)}),\end{aligned} \tag{11.1}$$

$$\begin{aligned}&\|\tilde{F}(\tilde{\mathbf{u}}^1, \tilde{p}^1, \tilde{\eta}_1^1, \tilde{\eta}_2^1) - \tilde{F}(\tilde{\mathbf{u}}^2, \tilde{p}^2, \tilde{\eta}_1^2, \tilde{\eta}_2^2)\|_{L^\infty(0,\infty;\mathbf{L}^2(\Omega))} \\ &\le C_2(\mu_1)\big(\|(\tilde{\mathbf{u}}^1, \tilde{p}^1, \tilde{\eta}_1^1, \tilde{\eta}_2^1)\|_{\mathbf{W}}\|(\tilde{\mathbf{u}}^1, \tilde{p}^1, \tilde{\eta}_1^1, \tilde{\eta}_2^1) - (\tilde{\mathbf{u}}^2, \tilde{p}^2, \tilde{\eta}_1^2, \tilde{\eta}_2^2)\|_{\mathbf{W}} \\ &\quad +\|(\tilde{\mathbf{u}}^2, \tilde{p}^2, \tilde{\eta}_1^2, \tilde{\eta}_2^2)\|_{\mathbf{W}}\|(\tilde{\mathbf{u}}^1, \tilde{p}^1, \tilde{\eta}_1^1, \tilde{\eta}_2^1) - (\tilde{\mathbf{u}}^2, \tilde{p}^2, \tilde{\eta}_1^2, \tilde{\eta}_2^2)\|_{\mathbf{W}}\big),\end{aligned} \tag{11.2}$$

with $\mathbf{W} = \mathbf{H}^{2,1}(Q_\infty) \times L^2(0,\infty; H^1(\Omega)) \times H^{4,2}(\Sigma_\infty^s) \times H^{2,1}(\Sigma_\infty^s)$,

$$\|\tilde{\mathbf{w}}(\tilde{\mathbf{u}}, \tilde{\eta}_1)\|_{\mathbf{H}^{2,1}(Q_\infty)} \le C_2(\mu_1)\|\tilde{\eta}_1\|_{H^{4,2}(\Sigma_\infty^s)}\|\tilde{\mathbf{u}}_1\|_{\mathbf{H}^{2,1}(Q_\infty)}, \tag{11.3}$$

$$\begin{aligned}&\|\tilde{\mathbf{w}}(\tilde{\mathbf{u}}^1, \tilde{\eta}_1^1) - \tilde{\mathbf{w}}(\tilde{\mathbf{u}}^2, \tilde{\eta}_1^2)\|_{\mathbf{H}^{2,1}(Q_\infty)} \\ &\le C_2(\mu_1)(\|\tilde{\eta}_1\|_{H^{4,2}(\Sigma_\infty^s)}\|\tilde{\mathbf{u}}_1^1 - \tilde{\mathbf{u}}_1^2\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|\tilde{\eta}_1^1 - \tilde{\eta}_1^2\|_{H^{4,2}(\Sigma_\infty^s)}\|\tilde{\mathbf{u}}_1^2\|_{\mathbf{H}^{2,1}(Q_\infty)}),\end{aligned} \tag{11.4}$$

$$\|\tilde{H}(\tilde{\mathbf{u}}, \tilde{\eta}_1)\|_{L^2(\Sigma_\infty^s)} \le C_2(\mu_1)\|\tilde{\mathbf{u}}_1\|_{H^{2,1}(Q_\infty)}\|\tilde{\eta}_1\|_{H^{4,2}(\Sigma_\infty^s)}, \tag{11.5}$$

and

$$\begin{aligned}&\|\tilde{H}(\tilde{\mathbf{u}}^1, \tilde{\eta}_1^1) - \tilde{H}(\tilde{\mathbf{u}}^2, \tilde{\eta}_1^2)\|_{L^2(\Sigma_\infty^s)} \\ &\le C_2(\mu_1)(\|\tilde{\mathbf{u}}_1\|_{H^{2,1}(Q_\infty)}\|\tilde{\eta}_1^1 - \tilde{\eta}_1^2\|_{H^{4,2}(\Sigma_\infty^s)} + \|\tilde{\mathbf{u}}_1^1 - \tilde{\mathbf{u}}_1^2\|_{H^{2,1}(Q_\infty)}\|\tilde{\eta}_1\|_{H^{4,2}(\Sigma_\infty^s)}).\end{aligned} \tag{11.6}$$

(*In these estimates the constant* C_2 *depends in an explicit manner of* μ_1.)

Proof.

(i) *Proof of* (11.3) *and* (11.4). If $(\tilde{\mathbf{u}}, \tilde{\eta}_1) \in \mathbf{H}^{2,1}(Q_\infty) \times H^{4,2}(\Sigma_\infty^s)$, then we have

$$\begin{aligned}&\|\tilde{\eta}_1\tilde{\mathbf{u}}_1\|_{L^2(0,\infty;\mathbf{H}^2(\Omega))} + \|\tilde{\eta}_{1,x}\tilde{\mathbf{u}}_1\|_{L^2(0,\infty;\mathbf{H}^2(\Omega))} \\ &\le \big(\|\tilde{\eta}_1\|_{L^\infty(0,\infty;H^3(\Gamma_s))}\|\tilde{\mathbf{u}}_1\|_{L^2(0,\infty;\mathbf{H}^2(\Omega))} + \|\tilde{\eta}_1\|_{L^\infty(0,\infty;H^2(\Gamma_s))}\|\tilde{\mathbf{u}}_1\|_{L^2(0,\infty;\mathbf{H}^2(\Omega))}\big) \\ &\le C\|\tilde{\eta}_1\|_{H^{4,2}(\Sigma_\infty^s)}\|\tilde{\mathbf{u}}_1\|_{\mathbf{H}^{2,1}(Q_\infty)}.\end{aligned}$$

We also have

$$\begin{aligned}&\|\tilde{\eta}_1\tilde{\mathbf{u}}_1\|_{H^1(0,\infty;\mathbf{L}^2(\Omega))} + \|\tilde{\eta}_{1,x}\tilde{\mathbf{u}}_1\|_{H^1(0,\infty;\mathbf{L}^2(\Omega))} \\ &\le \big(\|\tilde{\eta}_1\|_{H^1(0,\infty;L^\infty(\Gamma_s))}\|\tilde{\mathbf{u}}_1\|_{H^1(0,\infty;\mathbf{L}^2(\Omega))} + \|\tilde{\eta}_1\|_{H^1(0,\infty;H^1(\Gamma_s))}\|\tilde{\mathbf{u}}_1\|_{L^2(0,\infty;\mathbf{H}^2(\Omega))}\big) \\ &\le C\|\tilde{\eta}_1\|_{H^{4,2}(\Sigma_\infty^s)}\|\tilde{\mathbf{u}}_1\|_{\mathbf{H}^{2,1}(Q_\infty)}.\end{aligned}$$

In these estimates we have used that

$$\|\tilde{\eta}_1\|_{H^{3/2}(0,\infty;H^1(\Gamma_s))} \le C\|\tilde{\eta}_1\|_{H^{4,2}(\Sigma_\infty^s)}.$$

Thus we have

$$\|\tilde{\eta}_1 \tilde{\mathbf{u}}_1\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|\tilde{\eta}_{1,x}\tilde{\mathbf{u}}_1\|_{\mathbf{H}^{2,1}(Q_\infty)} \le C_2 \|\tilde{\eta}_1\|_{H^{4,2}(\Sigma^s_\infty)} \|\tilde{\mathbf{u}}_1\|_{\mathbf{H}^{2,1}(Q_\infty)}.$$

Now, we assume that $(\tilde{\mathbf{u}}^1, \tilde{\eta}_1^1) \in \mathbf{H}^{2,1}(Q_\infty) \times H^{4,2}(\Sigma^s_\infty)$ and $(\tilde{\mathbf{u}}^2, \tilde{\eta}_1^2) \in \mathbf{H}^{2,1}(Q_\infty) \times H^{4,2}(\Sigma^s_\infty)$. Let us estimate

$$\tilde{\eta}^1_{1,x} \tilde{\mathbf{u}}^1_1 - \tilde{\eta}^2_{1,x} \tilde{\mathbf{u}}^2_1.$$

The other component, that is $\tilde{\eta}^1_1 \tilde{\mathbf{u}}^1_1 - \tilde{\eta}^2_1 \tilde{\mathbf{u}}^2_1$, can be estimated in the same way. We have

$$\tilde{\eta}^1_{1,x} \tilde{\mathbf{u}}^1_1 - \tilde{\eta}^2_{1,x} \tilde{\mathbf{u}}^2_1 = \tilde{\eta}^1_{1,x}(\tilde{\mathbf{u}}^1_1 - \tilde{\mathbf{u}}^2_1) + (\tilde{\eta}^1_{1,x} - \tilde{\eta}^2_{1,x})\tilde{\mathbf{u}}^2_1.$$

As above, we estimate these terms as follows

$$\begin{aligned} &\|\tilde{\eta}^1_{1,x}(\tilde{\mathbf{u}}^1_1 - \tilde{\mathbf{u}}^2_1)\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|(\tilde{\eta}^1_{1,x} - \tilde{\eta}^2_{1,x})\tilde{\mathbf{u}}^2_1\|_{\mathbf{H}^{2,1}(Q_\infty)} \\ &\le C_2 (\|\tilde{\eta}_1\|_{H^{4,2}(\Sigma^s_\infty)} \|\tilde{\mathbf{u}}^1_1 - \tilde{\mathbf{u}}^2_1\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|\tilde{\eta}^1_1 - \tilde{\eta}^2_1\|_{H^{4,2}(\Sigma^s_\infty)} \|\tilde{\mathbf{u}}^2_1\|_{\mathbf{H}^{2,1}(Q_\infty)}). \end{aligned}$$

(ii) *Proof of* (11.1) *and* (11.2). To estimate the different terms in $\tilde{F}$, we firstly write

$$\begin{aligned} &\|\tilde{\eta}_1 \tilde{\mathbf{u}}_t\|_{L^2(Q)} \le \|\tilde{\eta}_1\|_{L^\infty(\Sigma^s_\infty)} \|\tilde{\mathbf{u}}_t\|_{L^2(Q)}, \\ &\|\tilde{\eta}_1 \omega \tilde{\mathbf{u}}\|_{L^2(Q)} \le \|\tilde{\eta}_1\|_{L^\infty(\Sigma^s_\infty)} \|\omega \tilde{\mathbf{u}}\|_{L^2(Q)}, \\ &\|z \tilde{\eta}_2 \tilde{\mathbf{u}}_z\|_{L^2(Q)} \le \|\tilde{\eta}_2\|_{L^\infty(\Sigma^s_\infty)} \|\tilde{\mathbf{u}}_z\|_{L^2(Q)} \le C \|\tilde{\eta}_2\|_{L^\infty(0,\infty;H^2_0(\Gamma_s))} \|\tilde{\mathbf{u}}_z\|_{L^2(0,\infty;\mathbf{H}^1(\Omega))}, \\ &\left\| \nu z \frac{2\tilde{\eta}^2_{1,x}}{e^{\omega t} + \tilde{\eta}_1} \tilde{\mathbf{u}}_z \right\|_{L^2(Q)} \le C \|(1+\tilde{\eta}_1)^{-1}\|_{L^\infty(\Sigma^s_\infty)} \|\tilde{\eta}_{1,x}\|^2_{L^\infty(\Sigma^s_\infty)} \|\tilde{\mathbf{u}}_z\|_{L^2(Q)} \le C \|\tilde{\eta}_{1,x}\|_{L^\infty(\Sigma^s_\infty)} \|\tilde{\mathbf{u}}_z\|_{L^2(Q)}, \\ &\|\nu z \tilde{\eta}_{1,xx} \tilde{\mathbf{u}}_z\|_{L^2(Q)} \le C \|\tilde{\eta}_{1,xx}\|_{L^\infty(\Sigma^s_\infty)} \|\tilde{\mathbf{u}}_z\|_{L^2(Q)}. \end{aligned}$$

In these estimates we have used that $\|(1+\tilde{\eta}_1)^{-1}\|_{L^\infty(\Sigma^s_\infty)} \le \mu_1$, $\|\tilde{\eta}_{1,x}\|_{L^\infty(\Sigma^s_\infty)} \le 1$ and that

$$\tilde{\eta}_{1,xx} \in H^{2,1}(\Sigma^s_\infty) \subset L^\infty(0,\infty;H^1(\Gamma_s)) \subset L^\infty(\Sigma^s_\infty),$$

because Γ_s is of dimension one.

We continue as follows

$$\begin{aligned} &\|\tilde{\eta}_{1,x} \tilde{\mathbf{u}}_{xz}\|_{L^2(Q)} \le \|\tilde{\eta}_{1,x}\|_{L^\infty(\Sigma^s_\infty)} \|\tilde{\mathbf{u}}_{xz}\|_{L^2(Q)}, \\ &\|\tilde{\eta}_1 \tilde{\mathbf{u}}_{xx}\|_{L^2(Q)} \le \|\tilde{\eta}_1\|_{L^\infty(\Sigma^s_\infty)} \|\tilde{\mathbf{u}}_{xx}\|_{L^2(Q)}, \\ &\left\| \frac{z^2 \tilde{\eta}^2_{1,x}}{e^{\omega t} + \tilde{\eta}_1} \tilde{\mathbf{u}}_{zz} \right\|_{L^2(Q)} \le C \|(1+\tilde{\eta}_1)^{-1}\|_{L^\infty(\Sigma^s_\infty)} \|\tilde{\eta}_{1,x}\|^2_{L^\infty(\Sigma^s_\infty)} \|\tilde{\mathbf{u}}_{zz}\|_{L^2(Q)}, \\ &\left\| \frac{e^{-\omega t} \tilde{\eta}_1}{e^{\omega t} + \tilde{\eta}_1} \tilde{\mathbf{u}}_{zz} \right\|_{L^2(Q)} \le C \|\tilde{\eta}_1\|_{L^\infty(\Sigma^s_\infty)} \|\tilde{\mathbf{u}}_{zz}\|_{L^2(Q)}, \\ &\|\tilde{\eta}_{1,x} \tilde{p}_z\|_{L^2(Q)} \le \|\tilde{\eta}_{1,x}\|_{L^\infty(\Sigma^s_\infty)} \|\tilde{p}_z\|_{L^2(Q)}, \\ &\|\tilde{\eta}_1 \tilde{p}_x\|_{L^2(Q)} \le \|\tilde{\eta}_1\|_{L^\infty(\Sigma^s_\infty)} \|\tilde{p}_x\|_{L^2(Q)}, \\ &\|(1 + e^{-\omega t}\tilde{\eta}_1) \tilde{\mathbf{u}}_1 \tilde{\mathbf{u}}_x\|_{L^2(Q)} \le C \|1 + \tilde{\eta}_1\|_{L^\infty(\Sigma^s_\infty)} \|\tilde{\mathbf{u}}_1\|_{L^\infty(0,\infty;\mathbf{H}^1(\Omega))} \|\tilde{\mathbf{u}}_x\|_{L^2(0,\infty;\mathbf{H}^1(\Omega))}, \\ &\|e^{-\omega t} \tilde{\eta}_{1,x} \tilde{\mathbf{u}}_1\|_{L^2(Q)} \le C \|\tilde{\eta}_{1,x}\|_{L^\infty(\Sigma^s_\infty)} \|\tilde{\mathbf{u}}_1\|_{L^2(0,\infty;\mathbf{L}^2(\Omega))}, \\ &\|\tilde{\mathbf{u}}_2 \tilde{\mathbf{u}}_z\|_{L^2(Q)} \le \|\tilde{\mathbf{u}}_2\|_{L^\infty(0,\infty;\mathbf{H}^1(\Omega))} \|\tilde{\mathbf{u}}_z\|_{L^2(0,\infty;\mathbf{H}^1(\Omega))}, \\ &\|(\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{u}}\|_{L^2(Q)} \le \|\tilde{\mathbf{u}}\|_{L^\infty(0,\infty;\mathbf{H}^1(\Omega))} \|\tilde{\mathbf{u}}\|_{L^2(0,\infty;\mathbf{H}^1(\Omega))}. \end{aligned}$$

Thus

$$\begin{aligned} \|\tilde{F}(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\eta}_1, \tilde{\eta}_2)\|_{L^\infty(0,\infty;\mathbf{L}^2(\Omega))} \le C_2 (&\|\tilde{\eta}_1\|_{H^{4,2}(\Sigma^s_\infty)} \|\tilde{\mathbf{u}}_1\|_{H^{2,1}(Q_\infty)} + \|\tilde{\eta}_2\|_{H^{2,1}(\Sigma^s_\infty)} \|\tilde{\mathbf{u}}_1\|_{H^{2,1}(Q_\infty)} \\ &+ \|\tilde{\eta}_1\|_{H^{4,2}(\Sigma^s_\infty)} \|\tilde{p}\|_{L^2(0,\infty;\mathbf{H}^1(\Omega))} + \|\tilde{\mathbf{u}}\|_{\mathbf{H}^{2,1}(Q_\infty)} \|\tilde{\mathbf{u}}\|_{\mathbf{H}^{2,1}(Q_\infty)}). \end{aligned}$$

Estimate (11.2) can be proved in the same way.

(iii) *Proof of* (11.5) *and* (11.6). We have

$$\left\|\frac{\tilde\eta_{1,x}}{e^{\omega t}+\tilde\eta_1}\tilde{\mathbf{u}}_{1,z}\right\|_{L^2(\Sigma^s_\infty)} \le C\|(1+\tilde\eta_1)^{-1}\|_{L^\infty(\Sigma^s_\infty)}\|\tilde\eta_{1,x}\|_{L^\infty(\Sigma^s_\infty)}\|\tilde{\mathbf{u}}_{1,z}\|_{L^2(\Sigma^s_\infty)},$$

$$\|\tilde\eta_{1,x}\tilde{\mathbf{u}}_{2,x}\|_{L^2(\Sigma^s_\infty)} \le C\|\tilde\eta_{1,x}\|_{L^\infty(\Sigma^s_\infty)}\|\tilde{\mathbf{u}}_{2,x}\|_{L^2(\Sigma^s_\infty)},$$

$$\left\|\frac{\tilde\eta_{1,x}^2}{e^{\omega t}+\tilde\eta_1}\tilde{\mathbf{u}}_{2,z}\right\|_{L^2(\Sigma^s_\infty)} \le C\|(1+\tilde\eta_1)^{-1}\|_{L^\infty(\Sigma^s_\infty)}\|\tilde\eta_{1,x}\|^2_{L^\infty(\Sigma^s_\infty)}\|\tilde{\mathbf{u}}_{2,z}\|_{L^2(\Sigma^s_\infty)},$$

$$\left\|\frac{\tilde\eta_{1}}{e^{\omega t}+\tilde\eta_1}\tilde{\mathbf{u}}_{2,z}\right\|_{L^2(\Sigma^s_\infty)} \le C\|\tilde\eta_1\|^2_{L^\infty(\Sigma^s_\infty)}\|\tilde{\mathbf{u}}_{2,z}\|_{L^2(\Sigma^s_\infty)}.$$

(We have used that $\|(1+\tilde\eta_1)^{-1}\|_{L^\infty(\Sigma^s_\infty)} \le \mu_1$ and $\|\tilde\eta_{1,x}\|_{L^\infty(\Sigma^s_\infty)} \le 1$.) With these estimates we can show that

$$\|\tilde H(\tilde{\mathbf{u}},\tilde\eta_1)\|_{L^2(\Sigma^s_\infty)} \le C_2\|\tilde{\mathbf{u}}_1\|_{H^{2,1}(Q_\infty)}\|\tilde\eta_1\|_{H^{4,2}(\Sigma^s_\infty)},$$

and that

$$\|\tilde H(\tilde{\mathbf{u}}^1,\tilde\eta_1^1)-\tilde H(\tilde{\mathbf{u}}^2,\tilde\eta_1^2)\|_{L^2(\Sigma^s_\infty)} \le C_2(\|\tilde{\mathbf{u}}_1\|_{H^{2,1}(Q_\infty)}(\|\tilde\eta_1^1-\tilde\eta_1^2\|_{H^{4,2}(\Sigma^s_\infty)}+\|\tilde{\mathbf{u}}_1^1-\tilde{\mathbf{u}}_1^2\|_{H^{2,1}(Q_\infty)}\|\tilde\eta_1\|_{H^{4,2}(\Sigma^s_\infty)}).$$

□

12. Proof of Theorem 10.1. To prove Theorem 10.1, we consider the nonhomogeneous closed loop linear system

$$\begin{aligned}
&\mathbf{v}_t-\operatorname{div}\sigma(\mathbf{v},p)-\omega\mathbf{v}=e^{-\omega t}\tilde F \quad\text{and}\quad \operatorname{div}\mathbf{v}=e^{-\omega t}\tilde G=e^{-\omega t}\operatorname{div}\tilde{\mathbf{w}} \quad \text{in } Q_\infty,\\
&\mathbf{v}=\eta_2\vec e_2 \ \text{ on } \Sigma^s_\infty,\quad \mathbf{v}=0 \ \text{ on } \Sigma^0_\infty,\quad \mathbf{v}(0)=\hat{\mathbf{u}}^0 \text{ in } \Omega,\\
&\eta_{1,t}=\eta_2+\omega\eta_1 \quad \text{on } \Sigma^s_\infty,\\
&\eta_{2,t}-\omega\eta_2-\beta\eta_{1,xx}-\delta\eta_{2,xx}+\alpha\eta_{1,xxxx}=\rho_1 p-2\nu\rho_2\mathbf{v}_{2,z}+e^{-\omega t}\tilde H-\Pi_3(\mathbf{v},\eta_1,\eta_2) \ \text{ on } \Sigma^s_\infty,\\
&\eta_1=0 \quad\text{and}\quad \eta_{1,x}=0 \ \text{ on } \{0,L\}\times(0,\infty),\\
&\eta_1(0)=\eta_1^0 \quad\text{and}\quad \eta_2(0)=\eta_2^0 \ \text{ in } \Gamma_s,
\end{aligned}\tag{12.1}$$

where $\tilde F$, $\tilde G$ and $\tilde H$ stand respectively for the mappings $\tilde F(\tilde\eta_1,\tilde\eta_2,\tilde{\mathbf{u}},\nabla\tilde p)$, $\tilde G(\tilde\eta_1,\tilde{\mathbf{u}})$ and $\tilde H(\tilde{\mathbf{u}},\tilde p,\tilde\eta_1)$ defined in (10.2), (10.3) and (10.4).

We first choose $0<\mu_1$. Without loss of generality, we can assume that $C_1\ge 1$ and $C_2(\mu_1)\ge 1$. We set

$$\mu_0=\min\left(\frac{1}{2C_1\,C_2(\mu_1)},1-\frac{1}{\mu_1}\right) \quad\text{and}\quad \theta_0(\mu)=\frac{\mu}{2C_1}.$$

Let us notice that if $(\tilde{\mathbf{u}},\tilde p,\tilde\eta_1,\tilde\eta_2)$ belongs to $\tilde D_\mu$, then $\|(1+\tilde\eta_1)^{-1}\|_{L^\infty(\Sigma^s_\infty)} \le \frac{1}{1-\mu} \le \frac{1}{1-\mu_0} \le \mu_1$ and $\|\tilde\eta_{1,x}\|_{L^\infty(\Sigma^s_\infty)} \le \mu < 1$. Thus estimates of Theorem 11.1 may be used for elements in $\tilde D_\mu$.

We are going to prove that the mapping

$$\mathcal F \ : \ (\tilde{\mathbf{u}},\tilde p,\tilde\eta_1,\tilde\eta_2)\longmapsto(\mathbf{v},p,\eta_1,\eta_2),$$

where $(\mathbf{v},p,\eta_1,\eta_2)$ is the solution to system (12.1), in which $\tilde F$, $\tilde G$ and $\tilde H$ are the functions of $(\tilde{\mathbf{u}},\tilde p,\tilde\eta_1,\tilde\eta_2)$ defined by (10.2), (10.3), (10.4), is a contraction in $\tilde D_\mu$.

If $(\mathbf{v},p,\eta_1,\eta_2)=\mathcal F(\tilde{\mathbf{u}},\tilde p,\tilde\eta_1,\tilde\eta_2)$, due to Theorems 9.1 and 11.1, we have

$$\begin{aligned}
&\|\mathbf{v}\|_{\mathbf{H}^{2,1}(Q_\infty)}+\|p\|_{L^2(0,\infty;H^1(\Omega))}+\|\eta_1\|_{H^{4,2}(\Sigma^s_\infty)}+\|\eta_1\|_{L^\infty(\Sigma^s_\infty)}+\|\eta_2\|_{H^{2,1}(\Sigma^s_\infty)}\\
&\le C_1(\|P\mathbf{v}^0\|_{\mathbf{V}^1_0(\Omega)}+\|\eta_1^0\|_{H^3(\Gamma_s)\cap H^2_0(\Gamma_s)}+\|\eta_2^0\|_{H^1_0(\Gamma_s)}\\
&\qquad+\|e^{-\omega t}\tilde F\|_{L^2(Q_\infty)}+\|e^{-\omega t}\tilde{\mathbf{w}}\|_{\mathbf{H}^{2,1}(Q_\infty)}+\|e^{-\omega t}\tilde H\|_{L^2(\Sigma^s_\infty)})\\
&\le C_1(\tfrac{1}{2C_1}\mu+C_2\mu^2)\le\mu.
\end{aligned}$$

Thus $\mathcal{F}$ is a mapping from $\tilde{D}_\mu$ into itself.

Let $(\tilde{\mathbf{u}}^1, \tilde{p}^1, \tilde{\eta}_1^1, \eta_2^1)$ and $(\tilde{\mathbf{u}}^2, \tilde{p}^2, \tilde{\eta}_1^2, \eta_2^2)$ belong to $\tilde{E}_\mu$. For $i = 1, 2$, we set $(\mathbf{v}^i, p^i, \eta_1^i, \eta_2^i) = \mathcal{F}(\tilde{\mathbf{u}}^i, \tilde{p}^i, \tilde{\eta}_1^i, \eta_2^i)$. Due to Theorems 9.1 and 11.1 , we also have

$$
\begin{aligned}
&\|\mathbf{v}^1 - \mathbf{v}^2\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|p^1 - p^2\|_{L^2(0,\infty;H^1(\Omega))} + \|\eta_1^1 - \eta_1^2\|_{H^{4,2}(\Sigma_\infty^s)} + \|\eta_2^1 - \eta_2^2\|_{H^{2,1}(\Sigma_\infty^s)} \\
&\le C_1(\|e^{-\omega t}(\tilde{F}^1 - \tilde{F}^2)\|_{L^2(Q_\infty)} + \|e^{-\omega t}(\tilde{\mathbf{w}}^1 - \tilde{\mathbf{w}}^2)\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|e^{-\omega t}(\tilde{H}^1 - \tilde{H}^2)\|_{L^2(\Sigma_\infty^s)}) \\
&\le C_1 C_2 \mu(\|\mathbf{v}^1 - \mathbf{v}^2\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|p^1 - p^2\|_{L^2(0,\infty;H^1(\Omega))} + \|\eta_1^1 - \eta_1^2\|_{H^{4,2}(\Sigma_\infty^s)} + \|\eta_2^1 - \eta_2^2\|_{H^{2,1}(\Sigma_\infty^s)}) \\
&\le \tfrac{1}{2}(\|\mathbf{v}^1 - \mathbf{v}^2\|_{\mathbf{H}^{2,1}(Q_\infty)} + \|p^1 - p^2\|_{L^2(0,\infty;H^1(\Omega))} + \|\eta_1^1 - \eta_1^2\|_{H^{4,2}(\Sigma_\infty^s)} + \|\eta_2^1 - \eta_2^2\|_{H^{2,1}(\Sigma_\infty^s)}).
\end{aligned}
$$

Thus $\mathcal{F}$ is a contraction in $\tilde{D}_\mu$ and the proof is complete.

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