and, for a fixed i, the maximum value assumed by $|\alpha_i(k)|$ is $\gamma\nu^{-1}\|x(0)\|\beta^k$.

Observe that when $|\alpha_i(k)|$ assumes its maximum value, the remainder components of the vector $\alpha(k)$ have to be zero due to inequality (57). Hence, for the worst case, $\alpha_i(k) \neq 0$ and $\alpha_j(k) = 0$ for $j \neq i$.

From (47)

$$\lim_{k\to\infty} h^T(k)(\theta(k) - p) = 0. \tag{58}$$

Without loss of generality, we will assume for the worst case that $\alpha_i(k)$ is equal to $\gamma\nu^{-1}\|x(0)\|\beta^k$, which implies

$$h(k) = \gamma\nu^{-1}\|x(0)\|\beta^k h_i. \tag{59}$$

Substituting (59) into (58), yields

$$\lim_{k\to\infty} \gamma\nu^{-1}\|x(0)\|\beta^k h_i^T[p - \theta(k)] = 0. \tag{60}$$

Using (52) and (60), we can write, for the worst case, that

$$\begin{aligned}\lim_{k\to\infty} &\left[y(k+d) - h^T(k+d)\theta(k)\right]\\ &= \lim_{k\to\infty} \alpha_i(k+d)h_i^T[p - \theta(k)]\\ &= \lim_{k\to\infty} \gamma\nu^{-1}\|x(0)\|\beta^{(k+d)}h_i^T[p - \theta(k)]\\ &= \beta^d \lim_{k\to\infty} \gamma\nu^{-1}\|x(0)\|\beta^k h_i^T[p - \theta(k)] = 0. \end{aligned} \tag{61}$$

This completes the proof.

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REFERENCES

[1] I. D. Landau and H. M. Silveira, "A stability theorem with applications to adaptive control," *IEEE Trans. Automat. Contr.,* vol. AC-24, pp. 305–312, Apr. 1979.

[2] I. D. Landau, "An extension of stability theorem applicable to adaptive control," *IEEE Trans. Automat. Contr.,* vol. AC-25, pp. 814–817, Aug. 1980.

[3] G. C. Goodwin, P. J. Ramadge, and P. E. Caines, "Discrete-time multivariable adaptive control," *IEEE Trans. Automat. Contr.,* vol. AC-25, pp. 449–456, June 1980.

[4] I. D. Landau, R. Lozano, and M. M'Saad, *Adaptive Control.* New York: Springer-Verlag, 1997.

[5] C. T. Chen, *Linear System Theory and Design.* New York: Holt, Rinehart, and Winston, 1984.

[6] L. Guo and L. Ljung, "Exponential stability of general tracking algorithms," *IEEE Trans. Automat. Contr.,* pp. 1376–1387, June 1980.

[7] O. Macchi and E. Eweda, "Second-order convergence analysis of stochastic adaptive linear filtering," *IEEE Trans. Automat. Contr.,* vol. AC-28, pp. 76–85, Jan. 1983.

Feedback Stabilization of MIMO 3-D Linear Systems

Zhiping Lin

***Abstract*— In this paper, the authors solve the open problem of the existence of double coprime factorizations for a large class of multi-input/multi-output (MIMO) three-dimensional (3-D) linear systems. It is proven that if all the unstable zeros of the contents associated with a left and a right matrix fraction descriptions of a given feedback stabilizable causal MIMO 3-D plant are simple, then the plant has a double coprime factorization. The authors then give a parameterization of all stabilizing compensators for an MIMO 3-D system in this class. The key result developed in the paper is a novel and constructive technique of "replacing" an unstable polynomial with a stable polynomial step by step. An illustrative example is also provided.**

***Index Terms*— Coprime factorization, feedback stabilization, multidimensional systems.**

I. INTRODUCTION

The problem of feedback stabilization of multi-input/multi-output (MIMO) linear systems has drawn much attention in the past years because of its importance in control and systems (see, e.g., [1]–[11] and the references therein). Consider the feedback system shown in Fig. 1, where P represents a plant and C represents a compensator. The relationship between $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{e}_1$, $\mathbf{e}_2$ can be expressed as

$$\begin{bmatrix}\mathbf{e}_1\\ \mathbf{e}_2\end{bmatrix} = \underbrace{\begin{bmatrix}(I+PC)^{-1} & -P(I+CP)^{-1}\\ C(I+PC)^{-1} & (I+CP)^{-1}\end{bmatrix}}_{H_{eu}}\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix}. \tag{1}$$

A given plant P is said to be feedback stabilizable if and only if there exists a compensator C such that the feedback system H_{eu} is stable, i.e., each entry of H_{eu} has no poles in the unstable region [3], [4]. For linear discrete multidimensional (n-D) systems, the feedback system is structurally stable[1] if and only if each entry of H_{eu} has no poles in $\overline{U}^n = \{(z_1, \cdots, z_n)\colon |z_1| \le 1, \cdots, |z_n| \le 1\}$ [12], [13].

The problem of feedback stabilization of MIMO two-dimensional (2-D) systems using the matrix fraction description (MFD) approach has been investigated by a number of researchers (see, e.g., [5]–[8] and the references therein). It is now well known that by decomposing a given plant P into an MFD $P = \tilde{D}^{-1}\tilde{N}$, where $\tilde{D}$ and $\tilde{N}$ are minor coprime[2] 2-D polynomial matrices, a necessary and sufficient condition for feedback stabilizability of P is that the matrix $[\tilde{D} \quad \tilde{N}]$ is of full rank in $\overline{U}^2$ [5], [6]. Constructive algorithms for the feedback stabilizability and stabilization problem have also been presented for MIMO 2-D systems [5]–[8]. Furthermore, the parameterization of *all* stabilizing compensators for a given stabilizable 2-D plant has been given in [5], which is a generalization of the celebrating result on the parameterization of *all* stabilizing compensators for a given one-dimensional (1-D) plant [1]–[4].

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[1] In this paper, stability means structural stability rather than bounded-input/bounded-output (BIBO) stability [12].

[2] See [14] and [16] for definitions of minor left coprime (MLC), minor right coprime (MRC), factor left coprime (FLC), factor right coprime (FRC), minor left prime (MLP), minor right prime (MRP), factor left prime (FLP), and factor right prime (FRP).

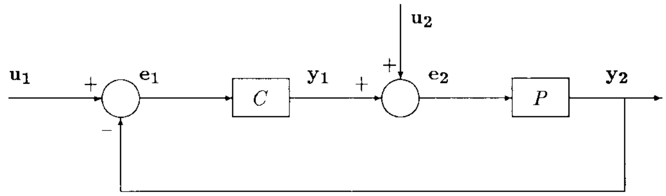

Fig. 1. Feedback system.

For n-D ($n \geq 3$) systems, it becomes much more difficult to tackle the feedback stabilization problem because of some fundamental differences between MIMO 2-D systems and their n-D ($n \geq 3$) counterparts [14]–[18]. In particular, since a given MIMO n-D ($n \geq 3$) system P may not always admit a minor coprime MFD [14], [16], existing criterion for feedback stabilizability of MIMO 2-D systems is not applicable to an n-D ($n \geq 3$) system P that does not admit a minor coprime MFD. Suppose that P admits a factor coprime (but not minor coprime) MFD $P = \tilde{D}^{-1}\tilde{N}$; Sule [9] and Lin [10] have recently shown that the condition $[\tilde{D} \quad \tilde{N}]$ being of full rank in $\overline{U}^n$ is not necessary for P to be feedback stabilizable. By introducing the new concept of "reduced minors" [9] and "generating polynomials"[3] [16], it has been shown that a necessary and sufficient condition for P to be feedback stabilizable is that the reduced minors of the matrix $[\tilde{D} \quad \tilde{N}]$ have no common zeros in $\overline{U}^n$ [9], [10].

However, it is still unknown whether or not there exists a double coprime factorization (DCF) for a given stabilizable MIMO n-D ($n \geq 3$) linear system [9]–[11]. This problem is in fact a special case of a more general problem posed by Vidyasagar *et al.* in [3] "*Is it always necessary that C and P individually have coprime factorizations when the closed loop is stable?*" The answer to this question is affirmative for MIMO 1-D and 2-D linear systems [1]–[5]. However, for general linear systems, the above question is not easy to answer. Anantharam showed via an example that for linear systems over an arbitrary integral domain, it is possible to stabilize plants which have no coprime factorizations [19]. Lin conjectured in [10] that a stabilizable MIMO n-D linear system also has a DCF, but a proof is not available currently.

Another closely related important problem is the parameterization of *all* stabilizing compensators for a given stabilizable MIMO n-D plant. Sule gave a characterization of stabilizing compensators for a stabilizable n-D plant in [9]. However, as to be discussed in Section III, his characterization [9] is not equivalent to the well-known Q-parameterization [1]–[5] in the sense that the characterization given in [9] is *not* constructive.

In this paper, we show that for a large class of MIMO 3-D linear systems, it is always possible to construct DCF's, thus proving in part the conjecture raised in [10]. The parameterization of *all* stabilizing compensators for this class of MIMO 3-D systems is also given.

The organization of the paper is as follows. In the next section, we give a constructive proof for the existence of DCF's for a large class of MIMO 3-D linear systems. In Section III, the problem of parameterization of *all* stabilizing compensators for a given stabilizable n-D plant is discussed. An example is illustrated in Section IV, and conclusions are given in Section V. To save space, we refer the reader to the cited references for some definitions which require rather lengthy descriptions such as content, coprimeness, and reduced minors.

[3] Both are in fact equivalent; see [9], [10], and [16] for the definitions and more details.

II. Double Coprime Factorizations

In the following, we shall denote: $\mathbf{C}(\mathbf{R})$ the field of complex (real) numbers; $\mathbf{C}(\mathbf{z}) = \mathbf{C}(z_1, \cdots, z_n)$ the set of rational functions in complex variables $z_1, \cdots, z_n$ with coefficients in $\mathbf{C}$; $\mathbf{C}[\mathbf{z}]$ the set of polynomials in complex variables $z_1, \cdots, z_n$ with coefficients in $\mathbf{C}$; $\mathbf{C_s}(\mathbf{z})$ the set of rational functions in $\mathbf{C}(\mathbf{z})$ having no poles in $\overline{U}^n$; $\mathbf{C}^{m\times l}[\mathbf{z}]$ the set of $m \times l$ matrices with entries in $\mathbf{C}[\mathbf{z}]$, $\mathbf{C_s}^{m\times l}(\mathbf{z})$ the set of $m \times l$ matrices with entries in $\mathbf{C_s}(\mathbf{z})$, etc. Throughout this paper, a zero of an n-D polynomial is called a stable zero if it is not in $\overline{U}^n$; otherwise, it is called an unstable zero. An n-D polynomial is called a stable polynomial if it has no zeros in $\overline{U}^n$; otherwise, it is called an unstable polynomial.

We now reproduce the definition of DCF.

Definition 1 [3], [4]: Let $P \in \mathbf{C}^{m\times l}(\mathbf{z})$. Then P is said to have a DCF if:

1) there exist $\tilde{D}_s \in \mathbf{C_s}^{m\times m}(\mathbf{z})$, $D_s \in \mathbf{C_s}^{l\times l}(\mathbf{z})$, and $\tilde{N}_s, N_s \in \mathbf{C_s}^{m\times l}(\mathbf{z})$;
2) there exist $\tilde{X}_s \in \mathbf{C_s}^{l\times l}(\mathbf{z})$, $X_s \in \mathbf{C_s}^{m\times m}(\mathbf{z})$, and $\tilde{Y}_s, Y_s \in \mathbf{C_s}^{l\times m}(\mathbf{z})$;
3) $\tilde{D}_s, D_s, \tilde{X}_s, X_s$ are all nonsingular;
4) $P = \tilde{D}_s^{-1}\tilde{N}_s = N_s D_s^{-1}$ and the identity holds[4]:

$$\begin{bmatrix} \tilde{X}_s & \tilde{Y}_s \\ -\tilde{N}_s & \tilde{D}_s \end{bmatrix} \begin{bmatrix} D_s & -Y_s \\ N_s & X_s \end{bmatrix} = \begin{bmatrix} I_l & 0_{l,m} \\ 0_{m,l} & I_m \end{bmatrix}.$$

In this section, we solve the DCF problem constructively for a large class of MIMO 3-D systems. Some lemmas are first required.

Lemma 1: Let $P \in \mathbf{C}^{m\times l}(z_1, z_2, z_3)$. Then $P(z_1, z_2, z_3)$ can be decomposed into MFD's $P = \tilde{D}^{-1}\tilde{N} = ND^{-1}$, where $\tilde{D} \in \mathbf{C}^{m\times m}[z_1, z_2, z_3]$, $D \in \mathbf{C}^{l\times l}[z_1, z_2, z_3]$, and $\tilde{N}, N \in \mathbf{C}^{m\times l}[z_1, z_2, z_3]$, such that the greatest common divisor (g.c.d.), denoted by $d(z_1, z_2, z_3)$, of the $l \times l$ minors of $F = [D^T \quad N^T]^T$ and the g.c.d. $\tilde{d}(z_1, z_2, z_3)$ of the $m \times m$ minors of $\tilde{F} = [-\tilde{N} \quad \tilde{D}]$ are both independent of the variables z_2, z_3.

A proof can be given by considering $P(z_1, z_2, z_3)$ as a rational matrix in two variables z_2, z_3 over the field of rational functions $\mathbf{C}(z_1)$, applying the known result on factorization of 2-D polynomial matrices over an arbitrary coefficient field [20] and then using a renormalization technique proposed in [21]. The details are omitted here to save space. In the remainder of this paper and particularly in Theorem 2, we assume that the MFD's of a given MIMO 3-D system always possess the property stated in Lemma 1, i.e., $d(z_1, z_2, z_3)$ ($\tilde{d}(z_1, z_2, z_3)$) is equal to its content[5] $g(z_1)$ ($\tilde{g}(z_1)$). For this reason, we simply call $g(z_1)$ ($\tilde{g}(z_1)$) the content of F ($\tilde{F}$).

Lemma 2: Let $F(z_1) \in \mathbf{C}^{k\times l}[z_1]$ be of normal full rank with $k \geq l$. Let $a(z_1)$ be the g.c.d. of the $l \times l$ minors of $F(z_1)$. If z_{11} is a simple zero[6] of $a(z_1)$, then rank $F(z_{11}) = l - 1$.

Proof: By transforming $F(z_1)$ into its Smith form [22], the result follows immediately. □

Lemma 3: Let $F(z_1, z_2, z_3) \in \mathbf{C}^{k\times l}[z_1, z_2, z_3]$ be of normal full rank, with $k \geq l$. Let $a_1(z_1, z_2, z_3), \cdots, a_\beta(z_1, z_2, z_3)$ be the $l \times l$ minors of $F(z_1, z_2, z_3)$, and $a_i(z_1, z_2, z_3) = g(z_1)b_i(z_1, z_2, z_3)$ ($i = 1, \cdots, \beta$) such that $b_1(z_1, z_2, z_3), \cdots, b_\beta(z_1, z_2, z_3)$ have no common divisors of the form $(z_1 - z_{10})$ for any $z_{10} \in \mathbf{C}$. Assume that z_{11} is a simple zero of $g(z_1)$. If for some fixed $z_2 = z_{21}, z_3 = z_{31}$, (z_{11}, z_{21}, z_{31}) is not a common zero of $b_1(z_1, z_2, z_3), \cdots, b_\beta(z_1, z_2, z_3)$, then rank $F(z_{11}, z_{21}, z_{31}) = l - 1$. Furthermore, the normal rank of $F(z_{11}, z_2, z_3)$ is equal to $l - 1$.

[4] I_l is the $l \times l$ identity matrix and 0_l, $0_{m,l}$ denote the $l \times l$ and $m \times l$ zero matrices, respectively.

[5] See [13] and [17] for the definition of content for an n-D polynomial.

[6] z_{11} is called a simple zero of $a(z_1)$ if $z_1 - z_{11}$ is a divisor of $a(z_1)$, but $(z_1 - z_{11})^2$ is not a divisor of $a(z_1)$.

Proof: Consider the 1-D polynomial matrix $F(z_1, z_{21}, z_{31})$. Let $a'_1(z_1), \cdots, a'_\beta(z_1)$ denote the $l \times l$ minors of $F(z_1, z_{21}, z_{31})$. We have

$$\begin{aligned} a'_i(z_1) &= a_i(z_1, z_{21}, z_{31}) \\ &= g(z_1) b_i(z_1, z_{21}, z_{31}), \qquad i = 1, \cdots, \beta. \end{aligned} \tag{2}$$

Let $c(z_1)$ denote the g.c.d. of $b_1(z_1, z_{21}, z_{31}), \cdots, b_\beta(z_1, z_{21}, z_{31})$. The assumption that (z_{11}, z_{21}, z_{31}) is not a common zero of $b_1(z_1, z_2, z_3), \cdots, b_\beta(z_1, z_2, z_3)$ implies that z_{11} is *not* a zero of $c(z_1)$. Hence, z_{11} is a simple zero of $g(z_1)\, c(z_1)$. From (2) and the fact that $c(z_1)$ is the g.c.d. of $b_1(z_1, z_{21}, z_{31}), \cdots, b_\beta(z_1, z_{21}, z_{31})$, it follows that $g(z_1)\, c(z_1)$ is the g.c.d. of $a'_1(z_1), \cdots, a'_\beta(z_1)$. By Lemma 2, rank $F(z_{11}, z_{21}, z_{31}) = l - 1$. Furthermore, we claim that there indeed exists some $z_2 = z_{21}$, $z_3 = z_{31}$ such that (z_{11}, z_{21}, z_{31}) is not a common zero of $b_1(z_1, z_2, z_3), \cdots, b_\beta(z_1, z_2, z_3)$. For otherwise, $(z_1 - z_{11})$ would be a common divisor of $b_1(z_1, z_2, z_3), \cdots, b_\beta(z_1, z_2, z_3)$, contradicting to the assumption. Therefore, the normal rank of $F(z_{11}, z_2, z_3)$ is equal to $l - 1$. □

Lemma 4 [14], [18]: Let $F \in \mathbf{C}^{m \times l}[z_2, z_3]$ be of normal rank r with $r < \min\{m, l\}$. Then $F(z_2, z_3)$ can be factorized as $F = F_1 F_2$, for some $F_1 \in \mathbf{C}^{m \times r}[z_2, z_3]$ and $F_2 \in \mathbf{C}^{r \times l}[z_2, z_3]$, with $F_2(z_2, z_3)$ being MLP.

Lemma 5 [5]: Let $F \in \mathbf{C}^{r \times l}[z_2, z_3]$ be of normal rank r with $r < l$. If F is MLP and all the common zeros of the $r \times r$ minors of $F(z_2, z_3)$ are not in $\overline{U}^2 = \{(z_2, z_3) : |z_2| \le 1, |z_3| \le 1\}$,[7] then there exists $W \in \mathbf{C}^{l \times l}[z_2, z_3]$, with $w = \det W \neq 0$ in $\overline{U}^2$, such that $FW = B$, with the first $l - r$ columns of $B(z_2, z_3)$ being identically zero. □

We are now in a position to present the key results of this paper in the following two theorems.

Theorem 1: Let $F \in \mathbf{C}^{k \times l}[z_1, z_2, z_3]$, with $k \ge l$, and let $a_1(z_1, z_2, z_3), \cdots, a_\beta(z_1, z_2, z_3)$ denote the $l \times l$ minors of $F(z_1, z_2, z_3)$. Suppose that $a_i(z_1, z_2, z_3) = g(z_1) b_i(z_1, z_2, z_3)$ $(i = 1, \cdots, \beta)$ such that $b_1(z_1, z_2, z_3), \cdots, b_\beta(z_1, z_2, z_3)$ have no common divisors of the form $(z_1 - z_{10})$ for any $z_{10} \in \mathbf{C}$. If z_{11} is a simple zero of $g(z_1)$ with $|z_{11}| \le 1$, and $b_1(z_1, z_2, z_3), \cdots, b_\beta(z_1, z_2, z_3)$ have no common zeros in $\overline{U}^3$, then there exists $W \in \mathbf{C}^{l \times l}[z_2, z_3]$, with $w(z_2, z_3) = \det W(z_2, z_3) \neq 0$ in $\overline{U}^2$, such that

$$F(z_1, z_2, z_3)\, W(z_2, z_3) = F_1(z_1, z_2, z_3)\, E_1(z_1) \tag{3}$$

where $F_1 \in \mathbf{C}^{k \times l}[z_1, z_2, z_3]$, and $E_1 = \operatorname{diag}\{(z_1 - z_{11}), 1, \cdots, 1\}$.

Proof: Let $F^{(0)}(z_2, z_3) = F(z_{11}, z_2, z_3)$. By Lemma 3, the normal rank of $F^{(0)}(z_2, z_3)$ is $l - 1$. By Lemma 4, $F^{(0)}(z_2, z_3)$ can be factorized as

$$F^{(0)}(z_2, z_3) = F^{(1)}(z_2, z_3)\, F^{(2)}(z_2, z_3) \tag{4}$$

where $F^{(1)} \in \mathbf{C}^{k \times (l-1)}[z_2, z_3]$, $F^{(2)} \in \mathbf{C}^{(l-1) \times l}[z_2, z_3]$, with $F^{(2)}(z_2, z_3)$ being MLP. Since $F^{(2)}(z_2, z_3)$ is MLP, there are only a finite number of points (z_{2j}, z_{3j}) $(j = 1, \cdots, J)$ such that rank $F^{(2)}(z_2, z_3)$ is smaller than $l - 1$. From (4), rank $F^{(0)}(z_{2j}, z_{3j})$ is also smaller than $l - 1$. This, in turn, implies that rank $F(z_{11}, z_{2j}, z_{3j})$ is smaller than $l - 1$ since $F^{(0)}(z_2, z_3) = F(z_{11}, z_2, z_3)$. By Lemma 3, (z_{11}, z_{2j}, z_{3j}) must be a common zero of $b_1(z_1, z_2, z_3), \cdots, b_\beta(z_1, z_2, z_3)$. The assumptions that $b_1(z_1, z_2, z_3), \cdots, b_\beta(z_1, z_2, z_3)$ have no common zeros in $\overline{U}^3$ and $|z_{11}| \le 1$ imply that we cannot have $|z_{2j}| \le 1$ *and* $|z_{3j}| \le 1$, for $j = 1, \cdots, J$. By Lemma 5, there exists $W \in \mathbf{C}^{l \times l}[z_2, z_3]$, with $w(z_2, z_3) = \det W(z_2, z_3) \neq 0$ in $\overline{U}^2$, such that

$$F^{(2)}(z_2, z_3)\, W(z_2, z_3) = B(z_2, z_3) \tag{5}$$

with the first column of $B(z_2, z_3)$ being identically zero. Combining (4) and (5) gives

$$F^{(0)}(z_2, z_3)\, W(z_2, z_3) = B_1(z_2, z_3) \tag{6}$$

with the first column of $B_1(z_2, z_3)$ being identically zero. Since $F^{(0)}(z_2, z_3) = F(z_{11}, z_2, z_3)$, (6) leads to $F(z_{11}, z_2, z_3)\, W(z_2, z_3) = B_1(z_2, z_3)$, or

$$F(z_1, z_2, z_3)\, W(z_2, z_3) = F_1(z_1, z_2, z_3)\, E_1(z_1) \tag{7}$$

where $F_1 \in \mathbf{C}^{k \times l}[z_1, z_2, z_3]$, and $E_1 = \operatorname{diag}\{(z_1 - z_{11}), 1, \cdots, 1\}$. □

Theorem 2: Let a causal[8] 3-D plant $P = \tilde{D}^{-1}\tilde{N} = ND^{-1} \in \mathbf{C}^{m \times l}(z_1, z_2, z_3)$, where $\tilde{D} \in \mathbf{C}^{m \times m}[z_1, z_2, z_3]$, $D \in \mathbf{C}^{l \times l}[z_1, z_2, z_3]$, and $\tilde{N}, N \in \mathbf{C}^{m \times l}[z_1, z_2, z_3]$, such that the g.c.d. of the family of $l \times l$ minors of $F = [D^T \;\; N^T]^T \in \mathbf{C}^{k \times l}[z_1, z_2, z_3]$ $(k = m + l)$ is equal to the content $g(z_1)$ of F and the g.c.d. of the family of $m \times m$ minors of $\tilde{F} = [-\tilde{N} \;\; \tilde{D}]$ is equal to the content $\tilde{g}(z_1)$ of $\tilde{F}$. Let $b_1(z_1, z_2, z_3), \cdots, b_\beta(z_1, z_2, z_3)$ be the reduced minors of $F = [D^T \;\; N^T]^T$. If $b_1(z_1, z_2, z_3), \cdots, b_\beta(z_1, z_2, z_3)$ have no common zeros in $\overline{U}^3$ and all the unstable zeros of $g(z_1)$ and $\tilde{g}(z_1)$ are simple, then $P(z_1, z_2, z_3)$ has a DCF.

Proof: Let $g(z_1) = \prod_{n=1}^{N'} (z_1 - z_{1n})\, g_0(z_1)$, where $|z_{1n}| \le 1$ $(n = 1, \cdots, N')$ and $g_0(z_1)$ is a stable polynomial. Since $b_1(z_1, z_2, z_3), \cdots, b_\beta(z_1, z_2, z_3)$ have no common zeros in $\overline{U}^3$ and z_{11} is a simple unstable zero of $g(z_1)$, by Theorem 1 there exists $W_1 \in \mathbf{C}^{l \times l}[z_2, z_3]$, with $w_1(z_2, z_3) = \det W_1(z_2, z_3) \neq 0$ in $\overline{U}^2$, such that

$$F\, W_1 = F_1\, E_1 \tag{8}$$

where $F_1 \in \mathbf{C}^{k \times l}[z_1, z_2, z_3]$, and $E_1 = \operatorname{diag}\{(z_1 - z_{11}), 1, \cdots, 1\}$.

Rewrite (8) as

$$F_1 = F\, W_1 E_1^{-1}. \tag{9}$$

Let $a_{11}(z_1, z_2, z_3), \cdots, a_{1\beta}(z_1, z_2, z_3)$ denote the $l \times l$ minors of $F_1(z_1, z_2, z_3)$. From (9), we have

$$\begin{aligned} a_{1i}(z_1, z_2, z_3) &= a_i(z_1, z_2, z_3)\, w_1(z_2, z_3)\, (z_1 - z_{11})^{-1}, \\ & i = 1, \cdots, \beta \end{aligned} \tag{10}$$

or

$$\begin{aligned} a_{1i}(z_1, z_2, z_3) &= g_1(z_1)\, b_i(z_1, z_2, z_3)\, w_1(z_2, z_3), \\ & i = 1, \cdots, \beta \end{aligned} \tag{11}$$

where $g_1(z_1) = \prod_{n=2}^{N'} (z_1 - z_{1n})\, g_0(z_1)$. Notice that the unstable 1-D polynomial $(z_1 - z_{11})$ has been replaced by the stable 2-D polynomial $w_1(z_2, z_3)$.

Let $b_{1i}(z_1, z_2, z_3) = b_i(z_1, z_2, z_3) w_1(z_2, z_3)$ for $i = 1, \cdots, \beta$. Clearly, $b_{11}(z_1, z_2, z_3), \cdots, b_{1\beta}(z_1, z_2, z_3)$ have no common zeros in $\overline{U}^3$. Since $w_1(z_2, z_3)$ is independent of z_1, $b_{11}(z_1, z_2, z_3), \cdots, b_{1\beta}(z_1, z_2, z_3)$ have no common divisors of the form $(z_1 - z_{10})$ for any $z_{10} \in \mathbf{C}$. Hence, Theorem 1 can be again applied to $F_1(z_1, z_2, z_3)$.

Repeating the above procedure N' times, we finally obtain the desired factorization

$$F_{N'} = F \prod_{n=1}^{N'} W_n E_n^{-1} \tag{12}$$

where $F_{N'} \in \mathbf{C}^{k \times l}[z_1, z_2, z_3]$, $W_n \in \mathbf{C}^{l \times l}[z_2, z_3]$, with $w_n(z_2, z_3) = \det W_n(z_2, z_3) \neq 0$ in $\overline{U}^2$ and $E_n = \operatorname{diag}\{(z_1 -$

[7] In the rest of this paper, we define $\overline{U}^2 = \{(z_2, z_3) : |z_2| \le 1, |z_3| \le 1\}$.

[8] See [5] and [10] for the definition of causality of n-D systems.

$z_{1n}), 1, \cdots, 1\}$. Let $a_{N'1}(z_1, z_2, z_3), \cdots, a_{N'\beta}(z_1, z_2, z_3)$ denote the $l \times l$ minors of $F_{N'}(z_1, z_2, z_3)$. It can be easily shown that

$$a_{N'i}(z_1, z_2, z_3) = g_0(z_1)\, b_i(z_1, z_2, z_3) \prod_{n=1}^{N'} w_n(z_2, z_3), \quad i = 1, \cdots, \beta \tag{13}$$

where $\prod_{n=1}^{N'} w_n(z_2, z_3)$ is a stable 2-D polynomial. Since $g_0(z_1)$ is also a stable polynomial and $b_1(z_1, z_2, z_3), \cdots, b_\beta(z_1, z_2, z_3)$ have no common zeros in $\overline{U}^3$, it follows immediately that $a_{N'1}(z_1, z_2, z_3), \cdots, a_{N'\beta}(z_1, z_2, z_3)$ have no common zeros in $\overline{U}^3$, i.e., $F_{N'}(z_1, z_2, z_3)$ is of full rank in $\overline{U}^3$.

Let $F_{N'} = [D_s^T \quad N_s^T]^T$, where $D_s \in \mathbf{C}^{l\times l}[z_1, z_2, z_3]$ and $N_s \in \mathbf{C}^{m\times l}[z_1, z_2, z_3]$. We next show that $P = N_s D_s^{-1}$. Let $E = \prod_{n=1}^{N'} W_n E_n^{-1}$. Clearly, $\det E \not\equiv 0$. From (12), we have $F_{N'} = FE$, or $D_s = DE$, $N_s = NE$. Since $P = ND^{-1}$, it follows that $P = NEE^{-1}D^{-1} = \{NE\}\{DE\}^{-1} = N_s D_s^{-1}$.

It can be similarly shown that $P = \tilde{D}_s^{-1}\tilde{N}_s$ for some $\tilde{D}_s \in \mathbf{C}^{m\times m}[z_1, z_2, z_3]$ and $\tilde{N}_s \in \mathbf{C}^{m\times l}[z_1, z_2, z_3]$ such that $[-\tilde{N}_s \quad \tilde{D}_s]$ is of full rank in $\overline{U}^3$.

Applying a result in [23], we can find $\tilde{X}_1 \in \mathbf{C_s}^{l\times l}(z_1, z_2, z_3)$, $X_s \in \mathbf{C_s}^{m\times m}(z_1, z_2, z_3)$, and $\tilde{Y}_1, Y_s \in \mathbf{C_s}^{l\times m}(z_1, z_2, z_3)$ such that

$$\tilde{X}_1 D_s + \tilde{Y}_1 N_s = I_l \tag{14}$$

$$\tilde{N}_s Y_s + \tilde{D}_s X_s = I_m. \tag{15}$$

Let $\Delta = -\tilde{X}_1 Y_s + \tilde{Y}_1 X_s$, and $\tilde{X}_s = \tilde{X}_1 + \Delta\tilde{N}_s$, $\tilde{Y}_s = \tilde{Y}_1 - \Delta\tilde{D}_s$. It can be easily checked that the following identity holds:

$$\begin{bmatrix} \tilde{X}_s & \tilde{Y}_s \\ -\tilde{N}_s & \tilde{D}_s \end{bmatrix} \begin{bmatrix} D_s & -Y_s \\ N_s & X_s \end{bmatrix} = \begin{bmatrix} I_l & 0_{l,m} \\ 0_{m,l} & I_m \end{bmatrix}. \tag{16}$$

We have shown that D_s and $\tilde{D}_s$ are nonsingular. It remains to show that X_s and $\tilde{X}_s$ are nonsingular. Since P is causal by assumption, using a technique similar to the one in [7] and [11], it is easy to show that $\det \tilde{X}_s(0, 0, 0) \neq 0$ and $\det X_s(0, 0, 0) \neq 0$, implying that $\tilde{X}_s$ and X_s are nonsingular. □

III. Parameterization of Stabilizing Compensators

There are apparently two methods for characterizing all stabilizing compensators for a given stabilizable n-D plant. The first method is the celebrated Q-parameterization formula [1]–[5], while the second method is the characterization formula proposed recently by Sule [9]. We briefly review these two methods.

1) First Method: For a stabilizable n-D plant $P \in \mathbf{C}^{m\times l}(\mathbf{z})$, we first obtain a DCF given in Definition 1. Then all stabilizing compensators for P are parameterized as

$$C = (\tilde{X}_s - Q\tilde{N}_s)^{-1}(\tilde{Y}_s + Q\tilde{D}_s): \quad Q \in \mathbf{C_s}^{l\times m}(\mathbf{z}) \text{ and } \det(\tilde{X}_s - Q\tilde{N}_s) \neq 0. \tag{17}$$

The beauty of the above Q-parameterization formula is that we only need to obtain a particular solution for the DCF problem and then derive all stabilizing compensators according to (17). This method is constructive as one can obtain all stabilizing compensators by varying Q freely in $\mathbf{C_s}^{l\times m}(\mathbf{z})$ (with the constraint that $(\tilde{X}_s - Q\tilde{N}_s)$ is nonsingular). A limitation of this method is that it requires P to have a DCF.

2) Second Method: Consider an MFD $P = N_s d_s^{-1}$ for a stabilizable P, where $N_s \in \mathbf{C_s}^{m\times l}(\mathbf{z})$ and $d_s \in \mathbf{C_s}(\mathbf{z})$. Then solve the following equation:

$$\begin{aligned} X_s N_s &= U_s d_s \\ Y_s N_s &= W_s d_s \\ N_s Y_s &= (I_m - X_s)\, d_s \end{aligned} \tag{18}$$

where $X_s \in \mathbf{C_s}^{m\times m}(\mathbf{z})$, $Y_s \in \mathbf{C_s}^{l\times m}(\mathbf{z})$, $U_s \in \mathbf{C_s}^{m\times l}(\mathbf{z})$, $W_s \in \mathbf{C_s}^{l\times l}(\mathbf{z})$. It was shown in [9] that all stabilizing compensators for P are given by $C = Y_s X_s^{-1}$ where X_s and Y_s satisfy (18). The advantage of this method is that it does not require P to have a DCF. However, unlike the Q-parameterization, such a characterization is *not* constructive since there are no free parameters to choose in (18)! In fact, even when a particular stabilizing compensator is available, we still have to resolve (18) in order to obtain another stabilizing compensator.

From the above discussion, it is clear that the Q-parameterization (17) is preferred if P has a DCF. We have shown in the previous section that a MIMO 3-D plant has a DCF when all the unstable zeros of its associated contents $g(z_1)$ and $\tilde{g}(z_1)$ are simple. Therefore, following [2]–[5], we are able to give a Q-parameterization of all stabilizing compensators for a large class of MIMO 3-D systems as follows.

Theorem 3: Let $P(z_1, z_2, z_3)$ be given as in Theorem 2. Then all stabilizing compensators for P are parameterized by

$$C = (\tilde{X}_s - Q\tilde{N}_s)^{-1}(\tilde{Y}_s + Q\tilde{D}_s): \; Q \in \mathbf{C_s}^{l\times m}(z_1, z_2, z_3) \text{ and } \det(\tilde{X}_s - Q\tilde{N}_s) \neq 0. \tag{19}$$

IV. Example

Consider a causal unstable 3-D system represented by

$$P(z_1, z_2, z_3) = \frac{1}{\Delta_p}\begin{bmatrix} z_1 + 0.5 & 0 \\ z_2 & z_3 + 2 \end{bmatrix} \tag{20}$$

where $\Delta_p = z_1 + 0.5$. Applying Lemma 1, we can decompose P into MFD's $P = \tilde{D}^{-1}\tilde{N} = ND^{-1}$, where

$$D = \tilde{D} = \begin{bmatrix} z_1 + 0.5 & 0 \\ 0 & z_1 + 0.5 \end{bmatrix}$$

$$N = \tilde{N} = \begin{bmatrix} z_1 + 0.5 & 0 \\ z_2 & z_3 + 2 \end{bmatrix}.$$

Let $F = [D^T \quad N^T]^T$, $\tilde{F} = [-\tilde{N} \quad \tilde{D}]$, and let $a_1(z_1, z_2, z_3), \cdots, a_6(z_1, z_2, z_3)$ denote the 2×2 minors of F. We have $a_i(z_1, z_2, z_3) = g(z_1)b_i(z_1, z_2, z_3)$, for $i = 1, \cdots, 6$, where $g(z_1) = \tilde{g}(z_1) = z_1 + 0.5$, and $b_1(z_1, z_2, z_3), \cdots, b_6(z_1, z_2, z_3)$ are the reduced minors of F given by

$$z_1 + 0.5, \quad 0, \quad z_3 + 2, \quad -(z_1 + 0.5), \quad -z_2, \quad z_3 + 2.$$

It is easy to see that $b_1, \cdots, b_6$ have no common zeros in $\overline{U}^3$. Hence, $P(z_1, z_2, z_3)$ is feedback stabilizable [9], [10]. However, since $g(z_1)$ has a zero at $z_1 = -0.5$ inside the unit disc $\overline{U}^1$, F is not of full rank in $\overline{U}^3$. Applying a criterion for the existence of primitive factorizations for 3-D polynomial matrices [17], it can be easily tested that F is already FRP. Thus, unlike the 2-D case [20], we cannot extract a right divisor with determinant equal to $g(z_1)$ from F. As a result, a DCF of P is not readily available. Since the only unstable zero of $g(z_1)$ ($\tilde{g}(z_1)$) is simple, by Theorem 2, P has a DCF. For $z_1 = -0.5$, we have

$$F(-0.5, z_2, z_3) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ z_2 & z_3+2 \end{bmatrix}$$

$$F(-0.5, z_2, z_3)W_1(z_2, z_3) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ z_2 & z_3+2 \end{bmatrix} \begin{bmatrix} z_3+2 & 0 \\ -z_2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & z_3+2 \end{bmatrix}.$$

This implies

$$F(z_1, z_2, z_3)W_1(z_2, z_3) = F_s(z_1, z_2, z_3)E_1(z_1)$$
$$F_s(z_1, z_2, z_3) = \begin{bmatrix} z_3+2 & 0 \\ -z_2 & z_1+0.5 \\ z_3+2 & 0 \\ 0 & z_3+2 \end{bmatrix}$$
$$E_1(z_1) = \begin{bmatrix} z_1+0.5 & 0 \\ 0 & 1 \end{bmatrix}.$$

It can be easily tested that F_s is of full rank in $\overline{U}^3$. Similarly, we can obtain

$$\tilde{F}_s(z_1, z_2, z_3) = \begin{bmatrix} -1 & 0 & 1 & 0 \\ -z_2 & -(z_3+2) & 0 & z_1+0.5 \end{bmatrix}$$

with $\tilde{F}_s(z_1, z_2, z_3)$ being of full rank in $\overline{U}^3$. Partition F_s and $\tilde{F}_s$ as $F_s = [D_s^T \quad N_s^T]^T$ and $\tilde{F}_s = [-\tilde{N}_s \quad \tilde{D}_s]$

$$\tilde{D}_s = \begin{bmatrix} 1 & 0 \\ 0 & z_1+0.5 \end{bmatrix}, \qquad \tilde{N}_s = \begin{bmatrix} 1 & 0 \\ z_2 & z_3+2 \end{bmatrix}$$
$$D_s = \begin{bmatrix} z_3+2 & 0 \\ -z_2 & z_1+0.5 \end{bmatrix}, \qquad N_s = \begin{bmatrix} z_3+2 & 0 \\ 0 & z_3+2 \end{bmatrix}.$$

We next construct

$$\tilde{X}_s = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \tilde{Y}_s = \frac{1}{\Delta_s} \begin{bmatrix} z_3+3 & 0 \\ -z_2 & z_1+1.5 \end{bmatrix}$$
$$X_s = \begin{bmatrix} -(z_3+2) & 0 \\ z_2 & -1 \end{bmatrix}$$
$$Y_s = \frac{1}{\Delta_s} \begin{bmatrix} (z_3+3)(z_3+2) & 0 \\ -z_2(z_3+z_1+3.5) & z_1+1.5 \end{bmatrix}$$

where $\Delta_s = z_3 + 2$, such that

$$\begin{bmatrix} \tilde{X}_s & \tilde{Y}_s \\ -\tilde{N}_s & \tilde{D}_s \end{bmatrix} \begin{bmatrix} D_s & -Y_s \\ N_s & X_s \end{bmatrix} = \begin{bmatrix} I_2 & 0_2 \\ 0_2 & I_2 \end{bmatrix}. \tag{21}$$

Notice that $\tilde{D}_s$, D_s, $\tilde{N}_s$, N_s, $\tilde{X}_s$, X_s, $\tilde{Y}_s$, Y_s are all in $\mathbf{R_s}^{2\times 2}(z_1, z_2, z_3)$, D_s, $\tilde{D}_s$, X_s, $\tilde{X}_s$ are all nonsingular, and $\det \tilde{X}_s(0, 0, 0) \neq 0$, $\det X_s(0, 0, 0) \neq 0$. Finally, all stabilizing compensators for the given unstable 3-D plant $P(z_1, z_2, z_3)$ are parameterized by

$$C = (\tilde{X}_s - Q\tilde{N}_s)^{-1}(\tilde{Y}_s + Q\tilde{D}_s): \ Q \in \mathbf{C_s}^{2\times 2}(z_1, z_2, z_3) \quad \text{and} \quad \det(\tilde{X}_s - Q\tilde{N}_s) \neq 0. \tag{22}$$

If real stabilizing compensators are desired, we can restrict $Q \in \mathbf{R_s}^{2\times 2}(z_1, z_2, z_3)$ in (22). For this simple example, the details for obtaining $X_s, Y_s, \tilde{X}_s, \tilde{Y}_s$ are just routine algebra, but rather lengthy, and hence are omitted here to save space.

V. Conclusions

In this paper, we have solved the open problem of the existence of DCF's for a large class of MIMO 3-D linear systems. We have proven that if all the unstable zeros of the associated contents $g(z_1)$ and $\tilde{g}(z_1)$ of a feedback stabilizable causal MIMO 3-D plant $P(z_1, z_2, z_3)$ are simple, then $P(z_1, z_2, z_3)$ has a DCF, thus proving in part a conjecture raised in [10]. We have also given a parameterization of all stabilizing compensators for an MIMO 3-D system in this class. We hope our results stimulate further research in this direction.

The key result developed in this paper is a novel and constructive technique of "replacing" an unstable polynomial with a stable one, as presented in Theorem 1. This technique is in some sense similar to the technique presented in [7] but is much more complicated, since we have to deal with 3-D polynomial matrices here instead of 2-D ones in [7]. The main contribution is that we are able to construct a DCF for a given MIMO 3-D plant in the class discussed in a finite number of steps. However, it is nontrivial to extend the proposed technique to the general n-D ($n > 3$) case, as in general we are not able to decompose a give MIMO n-D ($n > 3$) linear system $P(\mathbf{z})$ into MFD's $P = \tilde{D}^{-1}\tilde{N} = ND^{-1}$ such as the g.c.d. $d(\mathbf{z})$ of the $l \times l$ minors of $[D^T \quad N^T]^T$ is equal to its content $g(z_1)$. The problem of the existence of a DCF for a general stabilizable MIMO n-D ($n > 3$) linear system remains open at this stage.

Throughout this paper, we have assumed the ground field to be $\mathbf{C}$. It is natural to ask if $P \in \mathbf{R}^{m\times l}(z_1, z_2, z_3)$, and is it possible to decompose P into MFD's $P = \tilde{D}_s^{-1}\tilde{N}_s = N_s D_s^{-1}$, with $\tilde{D}_s \in \mathbf{R}^{m\times m}[z_1, z_2, z_3]$, $D_s \in \mathbf{R}^{l\times l}[z_1, z_2, z_3]$, and $\tilde{N}_s, N_s \in \mathbf{R}^{m\times l}[z_1, z_2, z_3]$, such that the matrices $[D_s^T \quad N_s^T]^T$ and $[\tilde{N}_s \quad \tilde{D}_s]$ are of full rank in $\overline{U}^3$? The answer is affirmative if all the unstable zeros of $g(z_1)$ and $\tilde{g}(z_1)$ are simple and real, since in this case the ground field of all the definitions, lemmas, and theorems can be restricted to $\mathbf{R}$. However, if $g(z_1)$ or $\tilde{g}(z_1)$ has some unstable complex zeros, we cannot guarantee that P has a DCF with coefficients over $\mathbf{R}$. More research is required before an answer could be given. Another unresolved open problem arising from this paper is to investigate whether or not $P(z_1, z_2, z_3)$ has a DCF when $g(z_1)$ or $\tilde{g}(z_1)$ has a multiple zero.

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References

[1] D. C. Youla, H. A. Jabr, and J. J. Bongiorno, Jr., "Modern Wiener-Hopf design of optimal controllers—Part II: The multivariable case," *IEEE Trans. Automat. Contr.,* vol. 21, pp. 319–338, 1976.

[2] C. A. Desoer, R.-W. Liu, J. Murray, and R. Saeks, "Feedback system design: The fractional representation approach to analysis and synthesis," *IEEE Trans. Automat. Contr.,* vol. 25, pp. 399–412, 1980.

[3] M. Vidyasagar, H. Schneider, and B. A. Francis, "Algebraic and topological aspects of feedback stabilization," *IEEE Trans. Automat. Contr.,* vol. 27, pp. 880–894, 1982.

[4] M. Vidyasagar, *Control System Synthesis: A Factorization Approach.* Cambridge, MA: MIT Press, 1985.

[5] J. P. Guiver and N. K. Bose, "Causal and weakly causal 2-D filters with applications in stabilizations," in *Multidimensional Systems Theory: Progress, Directions, and Open Problems,* N. K. Bose, Ed. Dordrecht: Reidel, 1985, p. 52.

[6] M. Bisiacco, E. Fornasini, and G. Mar.esini, "Controller design for 2D systems," *Frequency Domain and State Space Methods for Linear Systems,* C. I. Byrnes and A. Lindquist, Eds. North-Holland: Elsevier, 1986, pp. 99–113.

[7] Z. Lin, "Feedback stabilization of multivariable two-dimensional linear systems," *Int. J. Contr.,* vol. 48, pp. 1301–1317, 1988.

[8] L. Xu, O. Saito, and K. Abe, "Output feedback stabilizability and stabilization algorithms for 2-D systems," *Multidimensional Systems and Signal Processing,* vol. 5, pp. 41–60, 1994.

[9] V. R. Sule, "Feedback stabilization over commutative rings: The matrix case," *SIAM J. Contr. Optim.,* vol. 32, pp. 1675–1695, 1994.

[10] Z. Lin, "Feedback stabilizability of MIMO n-D linear systems," *Multidimensional Systems and Signal Processing,* vol. 9, pp. 149–172, 1998.

[11] ——, "Output feedback stabilizability and stabilization of linear n-D systems," presented at the *First Int. Workshop on Multidimensional Systems*, Poland, July 1998.

[12] E. I. Jury, "Stability of multidimensional systems and related problems," in *Multidimensional Systems: Techniques and Applications,* S. G. Tzafestas, Ed. New York: Marcel Dekker, 1986, p. 89.

[13] N. K. Bose, *Applied Multidimensional Systems Theory.* New York: Van Nostrand Reinhold, 1982.

[14] D. C. Youla and G. Gnavi, "Notes on n-dimensional system theory," *IEEE Trans. Circuits Syst.,* vol. 26, pp. 105–111, 1979.

[15] D. C. Youla and P. F. Pickel, "The Quillen–Suslin theorem and the structure of n-dimensional elementary polynomial matrices," *IEEE Trans. Circuits Syst.,* vol. 31, pp. 513–518, 1984.

[16] Z. Lin, "On matrix fraction descriptions of multivariable linear n-D systems," *IEEE Trans. Circuits Syst.,* vol. 35, pp. 1317–1322, 1988.

[17] ——, "On primitive factorizations for 3-D polynomial matrices," *IEEE Trans. Circuits Syst.,* vol. 39, pp. 1024–1027, 1992.

[18] E. Fornasini and M. E. Valcher, "nD polynomial matrices with applications to multidimensional signal analysis," *Multidimensional Syst. Signal Processing,* vol. 8, pp. 387–408, 1997.

[19] V. Anantharam, "On stabilization and the existence of coprime factorizations," *IEEE Trans. Automat. Contr.,* vol. 30, pp. 1030–1031, 1985.

[20] J. P. Guiver and N. K. Bose, "Polynomial matrix primitive factorization over arbitrary coefficient field and related results," *IEEE Trans. Circuits Syst.,* vol. 30, pp. 649–657, 1982.

[21] M. Morf, B. C. Lévy, and S. Y. Kung, "New results in 2-D systems theory—Part I: 2-D polynomial matrices, factorization and coprimeness," *Proc. IEEE,* vol. 65, pp. 861–872, 1977.

[22] F. R. Gantmacher, *Theory of Matrices.* New York: Chelsea, 1959, vol. I/II.

[23] C. A. Berenstein and D. C. Struppa, "1-Inverses for polynomial matrices of nonconstant rank," *Syst. Contr. Lett.,* vol. 6, pp. 309–314, 1986.

On the Mortensen Equation for Maximum Likelihood State Estimation

Shin Ichi Aihara and Arunabha Bagchi

***Abstract*— The main purpose of the present paper is to formulate the maximum likelihood state estimation problem correctly for a continuous-time nonlinear stochastic dynamical system. By using the Onsager–Machlup functional, a modified likelihood is introduced. The basic equation for the maximum likelihood state estimate is derived with the aid of a dynamic programming approach. The numerical procedure for realizing the recursive filtering is also proposed with some numerical results.**

***Index Terms*—Finitely additive white noise, HJB-equation, maximum likelihood estimate, Onsager–Machlup functional.**

I. Introduction

Consider the following noisy plant:

$$dx(t) = f(x(t))\,dt + G\,dw(t), \qquad 0 < t \leq T \tag{1}$$

with $x(0) = x_o \in R^n$, where w is a standard n-dimensional Brownian motion process and G is an $n \times n$ constant matrix. Corresponding to (1), the noisy observation mechanism is modeled by

$$y(t) = h(x(t)) + e(t) \tag{2}$$

where e is a finitely additive white noise in $L^2(0, T;\ R^d)$ independent of w. (See Kallianpur and Karandikar [1] for a general introduction to the finitely additive white noise theory.) The minimum variance nonlinear filtering problem for the above setting has been studied and the related Zakai equation has also been derived in [1].

Here we shall consider the maximum likelihood state estimation problem instead. If the observation mechanism (2) is modeled by an Ito equation, the existence of a maximum *a posteriori* estimator has already been proposed by Dembo and Zeitouni [2]. Intuitively, this procedure computes the most probable state trajectory in function space that maximizes some "likelihood functional" of the data (observation process) $y(t)$ in some time interval, say $0 \leq t \leq T$. This gives us a nonlinear maximum likelihood smoother. The fundamental difficulty here is a suitable notion of some likelihood functional of the observation process in continuous time. This was resolved in [2] by using the Onsager–Machlup functional which gives a sort of density of a stochastic process with respect to another standard process (like a Brownian motion). This is achieved by calculating the ratio of the probability of a stochastic process lying inside an ϵ tube of some deterministic trajectory and the probability of the standard process lying inside an ϵ-ball and letting ϵ go to zero. This can be visualized as a large deviation-type argument.

Our objective of this paper is twofold. We first reformulate the maximum likelihood state estimation problem as introduced in [2] in the finitely additive white noise setup. The other and the main contribution of the paper is to obtain, instead of the MAP estimator, the nonlinear maximum likelihood filter. This is accomplished as

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