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FEJÉR-RIESZ INEQUALITY FOR HOLOMORPHIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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1. Introduction. If $f(z)$ is a holomorphic function on the closed unit disc $|z| \leqq 1$, then the inequality

$$(1) \qquad \int_{-1}^{1} |f(\rho e^{i\theta})|^p d\rho \leqq \frac{1}{2} \int_{0}^{2\pi} |f(e^{it})|^p dt$$

holds for any θ, $0 \leqq \theta < 2\pi$, and any $p > 0$, where the constant 1/2 is the best possible. This is the inequality mentioned in the title and was obtained by Fejér and Riesz [3]. The purpose of this note is to extend this result to holomorphic functions defined on the unit ball of the complex n-space $\boldsymbol{C}^n$ and then to apply it to obtain a certain geometric property of quasiconformal holomorphic mappings.

For the points of $\boldsymbol{C}^n$ we shall use the notation $z = (z_1, \cdots, z_n)$, where $z_k = x_{2k-1} + ix_{2k} \in \boldsymbol{C}$, $1 \leqq k \leqq n$, and x_l, $1 \leqq l \leqq 2n$, are real variables. Under the correspondence $z \to (x_1, \cdots, x_{2n})$ the space $\boldsymbol{C}^n$ is identified with the real Euclidean space $\boldsymbol{R}^{2n}$. The inner product $\langle z, w\rangle$ in $\boldsymbol{C}^n$ is defined by the expression $\sum_{k=1}^n z_k \bar{w}_k$. When z, w are viewed as vectors in $\boldsymbol{R}^{2n}$, their inner product $\langle z, w\rangle_r$ is given by the real part of $\langle z, w\rangle$, i.e., $\langle z, w\rangle_r = \mathrm{Re}\,(\langle z, w\rangle)$. Let B be the open unit ball $\{z \in \boldsymbol{C}^n \mid \sum_{k=1}^n |z_k|^2 < 1\}$ of $\boldsymbol{C}^n$ and ∂B be the boundary of B. The surface area element of the sphere ∂B will be denoted by $d\tau$. For any p, $0 < p < \infty$, the Hardy space $H^p(B)$ is then defined as the set of holomorphic functions f on B such that

$$\sup \left\{ \int_{\partial B} |f(rz)|^p d\tau(z) \,|\, 0 < r < 1 \right\} < \infty .$$

For $f \in H^p(B)$ the radial limit $f^*(z)$ is known to exist for almost every point $z \in \partial B$ and the resulting function f^* belongs to the L^p-space on ∂B with respect to the measure $d\tau$ (cf., Stein [6; Chapter II, Section 9]). In §2 we shall prove the following

THEOREM 1. *Let L be any hyperplane in the space $\boldsymbol{R}^{2n}$ passing through the origin, $d\sigma$ the surface area element of L, and w a unit vector in $\boldsymbol{C}^n$ which is orthogonal to L with respect to the real inner* *product* $\langle\ ,\ \rangle_r$. *Then the inequality*

$$\int_{L\cap B}|f(z)|^p d\sigma(z) \leqq \frac{1}{2}\int_{\partial B}|f^*(z)|^p|\langle z, w\rangle|d\tau(z) \tag{2}$$

holds for any p, $0 < p < \infty$, *and any* $f \in H^p(B)$. *In particular, we have*

$$\int_{L\cap B}|f(z)|^p d\sigma(z) \leqq \frac{1}{2}\int_{\partial B}|f^*(z)|^p d\tau(z)\ . \tag{3}$$

As is well known, the classical Fejér-Riesz theorem has a simple geometric meaning. Namely, if a univalent holomorphic function maps the unit disc $|z| < 1$ onto the interior of a domain bounded by a rectifiable Jordan curve C, then the image of any diameter is shorter than the half of the length of C. As an application of Theorem 1 it is possible to prove an analogous geometric result for K-quasiconformal holomorphic mappings from the closed unit ball in C^n. §3 is devoted to the proof of the following

THEOREM 2. *Let F be a univalent holomorphic mapping of the closed unit ball $\bar{B}$ into C^n, which is K-quasiconformal with a constant $K \geqq 1$ in the sense of Wu* [7] (cf., §3 *of this note*). *Let* Area(Γ) *denote the real* $(2n-1)$-*dimensional volume of a hypersurface Γ in the space R^{2n}. Then, for the hyperplane in R^{2n} of the form $L_n = \{z \in C^n \mid \operatorname{Im} z_n = 0\}$, we have*

$$\text{Area}\,(F(L_n \cap B)) \leqq 2^{-1}K^{2n}(1 + (2n-1)\alpha_K)^{1/2}\,\text{Area}\,(F(\partial B))\ , \tag{4}$$

where the consant α_K, $0 \leqq \alpha_K < 1$, is determined by the equation

$$(1-\alpha_K)^{2n-1}(1 + (2n-1)\alpha_K) = K^{-4n}\ .$$

In general, for any hyperplane L in R^{2n} passing through the origin, we have

$$\text{Area}\,(F(L \cap B)) \leqq 2^{-1}K'^{2n}(1 + (2n-1)\alpha_{K'})^{1/2}\,\text{Area}\,(F(\partial B))\ , \tag{5}$$

where $K' = K(1 + (2n-1)\alpha_K)^{1/2}$.

2. **Proof of Theorem 1.** First we shall prove a slightly more general result as a lemma. For $z = (z_1, \cdots, z_n) \in C^n$, $n \geqq 2$, we set $\tilde{z} = (z_1, \cdots, z_{n-1})$ and $\|\tilde{z}\| = (\sum_{k=1}^{n-1}|z_k|^2)^{1/2}$.

LEMMA 1. *Suppose that the function $f(z)$ is continuous on the closed unit ball $\bar{B}$ and, for each fixed $\tilde{z} \in C^{n-1}$ with $\|\tilde{z}\| < 1$, the function $z_n \to f(\tilde{z}, z_n)$ is holomorphic on the disc $|z_n| < (1 - \|\tilde{z}\|^2)^{1/2}$. Let $d\sigma_n$ be the surface area element of $L_n = \{z \in C^n \mid \operatorname{Im} z_n = 0\}$. Then*

$$\int_{L_n \cap B} |f(z)|^p d\sigma_n(z) \leqq \frac{1}{2} \int_{\partial B} |f(z)|^p |z_n| d\tau(z) \tag{6}$$

for every p, $0 < p < \infty$, where the constant 1/2 is the best possible.

PROOF. Note that the case $n = 1$ in (6) is the original Fejér-Riesz inequality (1), which is assumed to be known.

Let $n \geqq 2$. We define polar coordinates for ∂B as follows:

$$\begin{aligned} x_1 &= \cos\theta_1 , \\ x_2 &= \sin\theta_1 \cos\theta_2 , \\ &\cdots\cdots \\ x_{2n-1} &= \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{2n-2} \cos\theta_{2n-1} , \\ x_{2n} &= \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{2n-2} \sin\theta_{2n-1} , \end{aligned} \tag{7}$$

where $0 \leqq \theta_1, \cdots, \theta_{2n-2} \leqq \pi$ and $0 \leqq \theta_{2n-1} < 2\pi$. The surface area element of ∂B with respect to this parametrization is given by $d\tau = \prod_{k=1}^{2n-2} \sin^{2n-1-k}\theta_k d\theta_1 \cdots d\theta_{2n-1}$. Choose an arbitrary $\tilde{z} \in C^{n-1}$ with $\|\tilde{z}\| < 1$, which is fixed for a moment. If $z = (\tilde{z}, z_n) \in \partial B$, then $z_n = (1 - \|\tilde{z}\|^2)^{1/2} \exp(i\theta_{2n-1})$ for a unique θ_{2n-1}, $0 \leqq \theta_{2n-1} < 2\pi$, where $z_k = x_{2k-1} + ix_{2k}$, $1 \leqq k \leqq n-1$, and θ_k, $0 \leqq \theta_k \leqq \pi$, are fixed for k, $1 \leqq k \leqq 2n-2$. Now consider the function $\zeta \to f(\tilde{z}, (1 - \|\tilde{z}\|^2)^{1/2}\zeta)$ of a complex variable ζ. Since this function is holomorphic on the disc $|\zeta| < 1$ and continuous on $|\zeta| \leqq 1$, the Fejér-Riesz inequality (1) implies that

$$\int_{-1}^{1} |f(\tilde{z}, (1 - \|\tilde{z}\|^2)^{1/2} t)|^p dt \leqq \frac{1}{2} \int_0^{2\pi} |f(\tilde{z}, (1 - \|\tilde{z}\|^2)^{1/2} \exp(i\theta_{2n-1}))|^p d\theta_{2n-1} .$$

Putting $z_n = (1 - \|\tilde{z}\|^2)^{1/2} \exp(i\theta_{2n-1})$ and $|z_n| t = x$, we have

$$\int_{-|z_n|}^{|z_n|} |f(\tilde{z}, x)|^p dx \leqq \frac{1}{2} \int_0^{2\pi} |f(\tilde{z}, z_n)|^p |z_n| d\theta_{2n-1} . \tag{8}$$

Let $x = x_{2n-1} = \sin\theta_1 \cdots \sin\theta_{2n-2} \cos\theta_{2n-1}$, $0 \leqq \theta_{2n-1} \leqq \pi$, so that the left-hand side of (8) is equal to

$$\int_0^{\pi} |f(\tilde{z}, x_{2n-1})|^p \sin\theta_1 \cdots \sin\theta_{2n-1} d\theta_{2n-1} .$$

On the other hand, the mapping $(\theta_1, \cdots, \theta_{2n-1}) \to (x_1, x_2, \cdots, x_{2n-1})$ in (7) with $0 \leqq \theta_1, \cdots, \theta_{2n-1} \leqq \pi$ defines a parametrization for $L_n \cap B$, in which we can write $d\sigma_n = \prod_{k=1}^{2n-1} \sin^{2n-k}\theta_k d\theta_1 \cdots d\theta_{2n-1}$. It follows that

$$\begin{aligned} &\int_{L_n \cap B} |f(z)|^p d\sigma_n(z) \\ &\quad = \int_0^{\pi} \cdots \int_0^{\pi} \left(\int_0^{\pi} |f(\tilde{z}, x_{2n-1})|^p \prod_{k=1}^{2n-1} \sin\theta_k d\theta_{2n-1} \right) \prod_{k=1}^{2n-2} \sin^{2n-1-k}\theta_k d\theta_1 \cdots d\theta_{2n-2} \end{aligned} \tag{9}$$

$$\leqq \int_0^{\pi} \cdots \int_0^{\pi} \left(\frac{1}{2} \int_0^{2\pi} |f(\tilde{z}, z_n)|^p |z_n| d\theta_{2n-1} \right) \prod_{k=1}^{2n-2} \sin^{2n-1-k}\theta_k d\theta_1 \cdots d\theta_{2n-2}$$
$$= \frac{1}{2} \int_{\partial B} |f(z)|^p |z_n| d\tau(z) .$$

Finally, let $p > 0$ and let $\varepsilon > 0$. Since 1/2 is the best possible in the case $n = 1$, there exist a holomorphic function $h(z)$ on the disc $|z| \leqq 1$ and a constant ρ_0, $0 < \rho_0 < 1$, such that

$$\int_{-1}^{1} |h(\rho t)|^p dt > \left(\frac{1}{2} - \varepsilon \right) \int_0^{2\pi} |h(\rho e^{i\theta})|^p d\theta$$

for all ρ, $\rho_0 \leqq \rho \leqq 1$. Define a function f with $\varepsilon' > 0$ by

$$f(\tilde{z}, z_n) = h((1 + \varepsilon' - \|\tilde{z}\|^2)^{-1/2} z_n) .$$

Clearly, f satisfies the stated assumptions. Take $\tilde{z}$, $\|\tilde{z}\| \leqq \delta$, $\delta = ((1 - (1 + \varepsilon')\rho_0^2)(1 - \rho_0^2)^{-1})^{1/2}$, and consider $z = (\tilde{z}, z_n)$ on ∂B. Then

$$\int_{-|z_n|}^{|z_n|} |f(\tilde{z}, x)|^p dx > \left(\frac{1}{2} - \varepsilon \right) \int_0^{2\pi} |f(\tilde{z}, z_n)|^p |z_n| d\theta ,$$

where $z_n = |z_n| e^{i\theta}$. Now divide ∂B into S_1 and S_2, where S_1: $\|\tilde{z}\| \leqq \delta$ and S_2: $\|\tilde{z}\| > \delta$. It can be seen just as in the inequality (9) that

$$\int_{L_n \cap B} |f(z)|^p d\sigma_n(z) > \left(\frac{1}{2} - \varepsilon \right) \int_{S_1} |f(z)|^p |z_n| d\tau(z)$$
$$= \left(\frac{1}{2} - \varepsilon \right) \left(\int_{\partial B} |f(z)|^p |z_n| d\tau(z) - \int_{S_2} |f(z)|^p |z_n| d\tau(z) \right) ,$$

where the second term tends to 0 as $\varepsilon' \to 0$. It follows that

$$\int_{L_n \cap B} |f(z)|^p d\sigma_n(z) > \left(\frac{1}{2} - 2\varepsilon \right) \int_{\partial B} |f(z)|^p |z_n| d\tau(z)$$

for a sufficiently small ε'.

PROOF OF THEOREM 1. Choose a unitary transformation U in $\boldsymbol{C}^n$ in such a way that $Uw = (0, \cdots, 0, i)$. Then we have clearly $U(L) = L_n$. First assume that f is holomorphic in a neighborhood of the closed ball $\bar{B}$. In view of Lemma 1 we have

$$\int_{L_n \cap B} |(f \circ U^{-1})(z')|^p d\sigma_n(z') \leqq \frac{1}{2} \int_{\partial B} |(f \circ U^{-1})(z')|^p |z'_n| d\tau(z') .$$

Since $|z'_n| = |\langle z', (0, \cdots, 0, i)\rangle| = |\langle Uz, Uw\rangle| = |\langle z, w\rangle|$ with $z' = Uz$ and since unitary transformations in $\boldsymbol{C}^n$ do not change the surface area element of any surface, we have

$$\int_{L\cap B} |f(z)|^p d\sigma(z) \leqq \frac{1}{2}\int_{\partial B} |f(z)|^p |\langle z, w\rangle| d\tau(z) . \tag{10}$$

We now take an arbitrary $f \in H^p(B)$. Set $f_r(z) = f(rz)$ for $0 \leqq r < 1$. Since f_r are holomorphic in neighborhoods of $\bar{B}$, the inequality (10) holds for these functions. If we set $F(z) = \sup\{|f_r(z)|^p | 0 \leqq r < 1\}$ for $z \in \partial B$, then $F(z)$ is integrable with respect to the measure $d\tau$ as shown by Rauch [5; Theorem 1]. This implies that

$$\int_{\partial B} |f_r(z)|^p |\langle z, w\rangle| d\tau(z) \to \int_{\partial B} |f^*(z)|^p |\langle z, w\rangle| d\tau(z)$$

as r tends to 1. Hence, by means of Fatou's lemma, we have

$$\begin{aligned}\int_{L\cap B} |f(z)|^p d\sigma(z) &\leqq \liminf_{r\to 1} \int_{L\cap B} |f_r(z)|^p d\sigma(z) \\ &\leqq \lim_{r\to 1} \frac{1}{2}\int_{\partial B} |f_r(z)|^p |\langle z, w\rangle| d\tau(z) \\ &= \frac{1}{2}\int_{\partial B} |f^*(z)|^p |\langle z, w\rangle| d\tau(z) ,\end{aligned}$$

as was to be proved.

3. An application to quasiconformal holomorphic mappings. Let D be a domain in $\boldsymbol{C}^n$ and let $F: D \to \boldsymbol{C}^n$ be a holomorphic mapping, $F = (F_1, \cdots, F_n)$, where F_j are holomorphic functions defined in D. We say that F is K-quasiconformal in D if there exists a constant $K > 0$ such that

$$\|\partial F/\partial z_k\| \leqq K|\det J_F|^{1/n} \tag{11}$$

on D for $1 \leqq k \leqq n$. Here, $\|\ \|$ denotes the Euclidean norm of $\boldsymbol{C}^n$, $\partial F/\partial z_k = (\partial F_1/\partial z_k, \cdots, \partial F_n/\partial z_k)$ and J_F is the complex Jacobian matrix $(\partial F_j/\partial z_k)$ of F (cf., Wu [7; p. 229]).

We note that the K-quasiconformality has an equivalent formulation in terms of real coordinates. Namely, D can be considered as a domain in $\boldsymbol{R}^{2n}$, denoted by D_R, and F_j are expressed by real-valued functions $G_l(x_1, \cdots, x_{2n})$, $1 \leqq l \leqq 2n$, with the domain D_R such that

$$F_j(z_1, \cdots, z_n) = G_{2j-1}(x_1, \cdots, x_{2n}) + iG_{2j}(x_1, \cdots, x_{2n}) , \qquad 1 \leqq j \leqq n .$$

Setting $G = (G_1, \cdots, G_{2n})$, we get a mapping of D_R into $\boldsymbol{R}^{2n}$. Then F is K-quasiconformal if and only if the mapping G is K-quasiconformal in the sense that

$$\|\partial G/\partial x_l\| \leqq K|\det J_G|^{1/2n} \tag{11'}$$

on D_R for $1 \leqq l \leqq 2n$, where $\| \ \|$ denotes the Euclidean norm of $\boldsymbol{R}^{2n}$, $\partial G/\partial x_l = (\partial G_1/\partial x_l, \cdots, \partial G_{2n}/\partial x_l)$, and J_G is the Jacobian matrix $(\partial G_m/\partial x_l)$ of G. Indeed, it is easily checked by means of Cauchy-Riemann equations that $\|\partial G/\partial x_{2k-1}\| = \|\partial G/\partial x_{2k}\| = \|\partial F/\partial z_k\|$, $1 \leqq k \leqq n$, and $|\det J_G| = |\det J_F|^2$. So (11) and (11′) are equivalent. In order to prove Theorem 2 we need the following

LEMMA 2. *Let A be a nonsingular $N \times N$ matrix with real entries and regard it as a linear transformation in the real Euclidean N-space $\boldsymbol{R}^N$. Let $\boldsymbol{a}_j$, $1 \leqq j \leqq N$, be the j-th column vector of A so that $A = (\boldsymbol{a}_1 \cdots \boldsymbol{a}_n)$. The transformation A maps the unit sphere of $\boldsymbol{R}^N$ onto a hyperellipsoid, which is denoted by Σ_A. Let $l(A)$ be the length of maximum semi-axes of Σ_A. Given two numbers $J > 0$ and $K \geqq 1$, we denote by $l(K, J)$ the maximum of $l(A)$ when A varies over the collection of matrices satisfying the condition*

$$|\det A| = J \quad \text{and} \quad \|\boldsymbol{a}_j\| \leqq KJ^{1/N} \quad \text{for} \quad 1 \leqq j \leqq N .$$

Then we have

$$l(K, J) = KJ^{1/N}(1 + (N-1)\alpha_K)^{1/2} ,$$

where α_K is determined by the condition

$$(1 - \alpha_K)^{N-1}(1 + (N-1)\alpha_K) = K^{-2N} , \qquad 0 \leqq \alpha_K < 1 . \tag{12}$$

OUTLINE OF PROOF. Let $A = (\boldsymbol{a}_1 \cdots \boldsymbol{a}_N)$ be a matrix such that $l(A) = \|A\xi\| = l(K, J)$ for a $\xi = (\xi_1, \cdots, \xi_N) \in \boldsymbol{R}^N$, $\xi_1^2 + \cdots + \xi_N^2 = 1$. Let $\Sigma' = \Sigma_A \cap S'$ where S' denotes the subspace spanned by $\boldsymbol{a}_j$, $1 \leqq j \leqq N-1$. We can write $\boldsymbol{a}_N = \boldsymbol{y} + \boldsymbol{b}$, $\boldsymbol{y} \in S'$, $\boldsymbol{b} \perp S'$, and $A\xi = \xi'\boldsymbol{x} + \xi_N\boldsymbol{a}_N$, $\boldsymbol{x} \in \Sigma'$, $\xi' = (1 - \xi_N^2)^{1/2}$, so that $\|A\xi\|^2 = \xi'^2\|\boldsymbol{x}\|^2 + \xi_N^2\|\boldsymbol{y}\|^2 + 2\xi'\xi_N\langle \boldsymbol{x}, \boldsymbol{y}\rangle + \xi_N^2\|\boldsymbol{b}\|^2$. If $\xi_N\langle \boldsymbol{x}, \boldsymbol{y}\rangle < |\xi_N|\|\boldsymbol{x}\|\|\boldsymbol{y}\|$, then by rotating $\boldsymbol{a}_N$ we could take $\boldsymbol{a}'_N = \boldsymbol{y}' + \boldsymbol{b}$, $\boldsymbol{y}' \in S'$, $\|\boldsymbol{y}'\| = \|\boldsymbol{y}\|$, so that $\xi_N\langle \boldsymbol{x}, \boldsymbol{y}'\rangle = |\xi_N|\|\boldsymbol{x}\|\|\boldsymbol{y}'\|$, hence $\|A'\xi\| > \|A\xi\|$ with $|\det A'| = |\det A|$, where $A' = (\boldsymbol{a}_1 \cdots \boldsymbol{a}_{N-1}\boldsymbol{a}'_N)$. Thus $\xi_N\langle \boldsymbol{x}, \boldsymbol{y}\rangle = |\xi_N|\|\boldsymbol{x}\|\|\boldsymbol{y}\|$, which means that $\boldsymbol{x}$ and $\boldsymbol{y}$ lie on one and the same straight line in Σ', and we have

$$\|A\xi\|^2 = (\xi'\|\boldsymbol{x}\| + |\xi_N|\|\boldsymbol{y}\|)^2 + \xi_N^2\|\boldsymbol{b}\|^2 . \tag{13}$$

Now suppose $\|\boldsymbol{a}_N\| < KJ^{1/N}$. Then taking $\boldsymbol{a}'_N = \boldsymbol{y}' + \boldsymbol{b}$, $\|\boldsymbol{y}'\|^2 = (KJ^{1/N})^2 - \|\boldsymbol{b}\|^2 > \|\boldsymbol{y}\|^2$, we could have $\|A'\xi\| > \|A\xi\|$. It follows that $\|\boldsymbol{a}_j\| = KJ^{1/N}$, $1 \leqq j \leqq N$. It is seen from (13) that $\|\boldsymbol{x}\|$ must be equal to the length of maximum semi-axes of Σ'.

Next we shall show that A can be taken so that $\langle \boldsymbol{a}_j, \boldsymbol{a}_k\rangle$ is a non-negative constant for every pair of j, k, $j \neq k$. Let $\Sigma(i, \cdots, j) = \Sigma_A \cap S(i, \cdots, j)$, where $S(i, \cdots, j)$ denotes the subspace of $\boldsymbol{R}^N$ spanned by vectors $a_i, \cdots, a_j$, distinct from each other. Then, if $a_k \neq a_i, \cdots, a_j$, the projection of a_k to $S(i, \cdots, j)$ lies on a maximum semi-axis of $\Sigma(i, \cdots, j)$, as is easily seen in the same way as above. Suppose $\langle a_1, a_2\rangle \neq 0$. We may assume that $\langle a_1, a_2\rangle > 0$ by taking $-a_1$, if necessary. The projection of a_3 or $-a_3$ to $S(1, 2)$ lies on the line $t(a_1 + a_2)$, $t \in \boldsymbol{R}$, since $a_1 + a_2$ is a maximum semi-axis of $\Sigma(1, 2)$ by assumption, hence we have $a_3 = c(a_1 + a_2)$, $c > 0$. This implies that $\langle a_2, a_3\rangle > 0$ and $\langle a_3, a_1\rangle > 0$. Continuing this procedure by considering the projection of a_4 or $-a_4$ to $\Sigma(1, 2, 3)$ which has $a_1 + a_2 + a_3$ as one of its maximum semi-axes, we can finally conclude that $\langle a_j, a_k\rangle > 0$, $j \neq k$.

Take arbitrary three vectors, e.g., a_1, a_2, and a_3. If $\overrightarrow{OA_j}$ denotes the vector a_j, then the projection of $\overrightarrow{OA_1}$ onto the triangle $\triangle OA_2A_3$ bisects the angle $\angle A_2OA_3$. The situation is similar for A_2 and A_3, hence it can be seen that $\langle a_1, a_2\rangle = \langle a_2, a_3\rangle = \langle a_3, a_1\rangle$. Thus $\langle a_j, a_k\rangle$ is a positive constant for $j, k, j \neq k$. If $\langle a_j, a_k\rangle = 0$ for some j, k, then this holds for all $j, k, j \neq k$. Note that we can write $\langle a_j, a_k\rangle = \|a_j\|\|a_k\|\alpha = K^2J^{2/N}\alpha$ with $0 \leqq \alpha < 1$, for $j \neq k$, so the constant α can be computed from the following: $J^2 = \det(\langle a_j, a_k\rangle) = (K^2J^{2/N})^N(1-\alpha)^{N-1}(1+(N-1)\alpha))$. The constant $l(K, J)$ can be obtained by estimating $\|A\xi\|^2$, $\|\xi\| = 1$, in which $\sum_{j\neq k}\xi_j\xi_k$ takes on the maximum value $N-1$ on the sphere $\|\xi\| = 1$.

PROOF OF THEOREM 2. First we assume that $L = L_n$. Let $G = (G_1, \cdots, G_{2n})$, where $F_j = G_{2j-1} + iG_{2j}$. In order to estimate the left-hand side of the inequality (4), we consider the mapping $\Phi(t_1, \cdots, t_{2n-1}) = G(t_1, \cdots, t_{2n-1}, 0)$ of the unit ball $\varDelta = \{(t_1, \cdots, t_{2n-1}) \mid t_1^2 + \cdots + t_{2n-1}^2 < 1\}$ of $\boldsymbol{R}^{2n-1}$ into $\boldsymbol{R}^{2n}$, which is nothing other than the restriction of F to the set $L_n \cap B$. Then the surface area element of $\Phi(\varDelta)$ is given by $(\det(g_{lm}))^{1/2}dt_1 \cdots dt_{2n-1}$ where

$$g_{lm} = \sum_{s=1}^{2n} \frac{\partial G_s}{\partial x_l}\frac{\partial G_s}{\partial x_m}, \qquad 1 \leqq l, m \leqq 2n-1,$$

evaluated at the point $(t_1, \cdots, t_{2n-1}, 0)$. Since the matrix (g_{lm}) is positive semidefinite, we have

$$\det(g_{lm}) \leqq g_{11} \cdots g_{2n-1\,2n-1};$$

here we used the fact that, for any nonnegative hermitian matrix (h_{lm}) of any order n,

$$\det(h_{lm}) \leqq h_{11} \cdots h_{nn},$$

an inequality long known to be equivalent to Hadamard's determinant inequality. Now from the relations $g_{2k-1\,2k-1} = g_{2k\,2k} = \|\partial F/\partial z_k\|^2$, $1 \leqq k \leqq n$, stated in the paragraph preceding Lemma 2 as well as the inequality (11) it follows that

$$\begin{aligned}\text{Area}\,(F(L_n \cap B)) &= \text{Area}\,(\Phi(\Delta)) \\ &= \int_\Delta (\det\,(g_{lm}))^{1/2} dt_1 \cdots dt_{2n-1} \\ &\leqq \int_\Delta (g_{11} \cdots g_{2n-1\,2n-1})^{1/2} dt_1 \cdots dt_{2n-1} \\ &= \int_{L_n \cap B} \left\|\frac{\partial F}{\partial z_1}\right\|^2 \cdots \left\|\frac{\partial F}{\partial z_{n-1}}\right\|^2 \left\|\frac{\partial F}{\partial z_n}\right\| d\sigma_n(z) \\ &\leqq K^{2n-1} \int_{L_n \cap B} |\det J_F|^{(2n-1)/n} d\sigma_n(z)\,.\end{aligned}$$

Applying Theorem 1 (3), to the holomorphic function $\det J_F$ with $p = (2n-1)/n$, we get

$$\text{Area}\,(F(L_n \cap B)) \leqq \frac{1}{2} K^{2n-1} \int_{\partial B} |\det J_F|^{(2n-1)/n} d\tau(z)\,.$$

Next we should estimate Area $(F(\partial B))$. Let $z \in \partial B$, and let $\{e_1, \cdots, e_{2n-1}\}$ be an orthonormal frame of ∂B at the point z; then the surface area element of $F(\partial B)$ at the point $F(z)$ is given by $A(z)d\tau(z)$ where $A(z)$ denotes the area of the parallelopiped spanned by the vectors $J_G(z)e_j$, $1 \leqq j \leqq 2n-1$. Take the unit normal vector, e_{2n}, to ∂B at z. Since $|\det J_G(z)|$ represents the volume of the parallelopiped spanned by $J_G(z)e_j$, $1 \leqq j \leqq 2n$, we see $|\det J_G(z)| \leqq A(z) \| J_G(z)e_{2n} \|$. Here, we note that $\| J_G(z)e_{2n} \|$ does not exceed the length of maximum semi-axes of the hyperellipsoid Σ corresponding to the matrix $J_G(z)$. Applying Lemma 2 to the case $N = 2n$ and $J = |\det J_G(z)|$, we thus have $\| J_G(z)e_{2n} \| \leqq l(K, |\det J_G(z)|) = K(1 + (2n-1)\alpha_K)^{1/2} |\det J_G(z)|^{1/2n}$. It follows that $A(z) \geqq K^{-1}(1 + (2n-1)\alpha_K)^{-1/2} |\det J_G(z)|^{1-1/2n} = K^{-1}(1 + (2n-1)\alpha_K)^{-1/2} |\det J_F(z)|^{(2n-1)/n}$, and hence

$$\begin{aligned}\text{Area}\,(F(\partial B)) &= \int_{\partial B} A(z) d\tau(z) \\ &\geqq K^{-1}(1 + (2n-1)\alpha_K)^{-1/2} \int_{\partial B} |\det J_F(z)|^{(2n-1)/n} d\tau(z)\,.\end{aligned}$$

Thus we have the inequality (4): Area $(F(L_n \cap B)) \leqq 2^{-1}K^{2n}(1 + (2n - 1)\alpha_K)^{1/2}$ Area $(F(\partial B))$.

Finally, to prove the inequality (5), let U be the unitary transformation employed in the proof of Theorem 1. Let V denote the real representation of U, an orthogonal transformation in $\boldsymbol{R}^{2n}$, and let $V^{-1} = (v_{lj})$, $1 \leqq l, j \leqq 2n$, and $J_G = (\boldsymbol{a}_1 \cdots \boldsymbol{a}_{2n})$. Then the j-th column $\boldsymbol{c}_j$ of

$J_G J_{V^{-1}}$, the Jacobian matrix of the mapping GV^{-1}, is of the form $\boldsymbol{c}_j = \sum_{l=1}^{2n} v_{lj}\boldsymbol{a}_l$, $1 \leqq j \leqq 2n$. Since $\sum_{l=1}^{2n} v_{lj}^2 = 1$, $\boldsymbol{c}_j$ belongs to the hyperellipsoid spanned by the vectors $\boldsymbol{a}_k$, $1 \leqq k \leqq 2n$. So Lemma 2 shows that $\|\boldsymbol{c}_j\| \leqq l(K, |\det J_G|) = K(1 + (2n - 1)\alpha_K)^{1/2}|\det J_G|^{1/2n} = K(1 + (2n - 1)\alpha_K)^{1/2} \times |\det(J_G J_{V^{-1}})|^{1/2n}$, $1 \leqq j \leqq 2n$, which means that GV^{-1} is K'-quasiconformal with the constant $K' = K(1 + (2n - 1)\alpha_K)^{1/2}$. The inequality (4) can now be applied to yield the inequality (5).

4. **Remarks.** 1. We do not know whether the constant 1/2 in the inequalities (2), (3), (4), and (5) is the best possible or not when $n > 1$.

2. In the case of the unit disc there have been several extensions of the Fejér-Riesz inequality (cf., Carlson [2], Huber [4]). It may be of some interest to find corresponding generalizations in the case of the ball of $\boldsymbol{C}^n$.

3. A univalent holomorphic mapping is conformal if and only if $K = 1$ in (11), and it should be noted that α_K tends to zero as K tends to 1. There are a variety of (equivalent) definitions for the quasiconformality of mappings besides the one used here (cf., Caraman [1]). Other definitions will lead to different inequalities in place of (5).

4. Theorem 2 can be formulated for a wider class of mappings, e.g., nonsingular holomorphic mappings.

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