FELLER'S RENEWAL THEOREM FOR SYSTEMS OF RENEWAL EQUATIONS

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ABSTRACT

In this paper, the renewal theorem of Feller is extended to the case of a system of renewal equations. Also a refinement of the renewal theorem is given and several open problems are listed.

Key words: Renewal theory; Perron—Frobenius root, Lattice distributions.

1. INTRODUCTION

In this paper we study the asymptotic behaviour (as $t \to \infty$) of solutions $M(t) = (M_1(t), \ldots, M_p(t))'$ of a system of renewal equations of the type

$$M_i(t) = Z_i(t) + \sum_{k=1}^{p} \int_{[0,t]} M_k(t-u) F_{ik}(du)$$

$$i = 1, 2, \ldots, p, (t > 0) \tag{1.1}$$

where $Z_1(t) = (Z(t), \ldots, Z_p(t))'$ is a vector of Borel-measurable functions bounded on compact sets and for each (i, j), $F_{ij}(.)$ is a non-decreasing bounded right-continuous on $[0 \infty)$ into itself.

The functions $M_i(t)$ arise in a natural way in many applications and especially in Branching processes [9]. Their behaviour as $t \to \infty$ is of great interest.

The case $p = 1$ and $F_{11}(\infty) = 1$ is, of course, the standard renewal equation and one has Feller's renewal theorm available for any directly Riemann integrable $Z_1(.)$ ([5] pp. 346–353). The object of this paper is to prove an extension of Feller's result to the present context.

In fact such a result is already available in the literature. K. S. Crump [3] following Feller's methods [5] extended Feller's theorem to obtain our theorem 2.2 below. However, Crump's proof as it is given in [3], though correct, does not give all the details and these, as we discovered, turned out to be non trivial. Besides streamlining and completing Crump's proof we also give a refinement of the renewal theorem (our theorem (2.4)) under second moment hypothesis. This latter result is new. From the point of view of applications the result most useful is in theoem 2.3 (Sec [1] and [9]).

The system (1.1) has also been studied by Chistyakov and Sevastyanov [2]. Their methods are Fourier analytic and involve Tauberian arguments. These, in turn, involve certain moment conditions which are much stronger than ours. Mode [9] too studied (1.1) and proved the result of theorem 2.1 below under moment conditions, absolute continuity of F_{ij}'s with their densities in some Lp, etc. Crump's arguments, on the other hand, are direct extensions of Feller's ideas which exploit the weak compactness of bounded measures on compact sets on the line. Our proof of theorem 2.1 is almost the same as Crump's with more details and is somewhat streamlined. Our proof of the refinement, *viz.*, theorem 2.4 is also based on an extension of Feller's ideas ([5] p. 357).

In section 2 we set up the basic machinery and state our results. The proofs are given in § 3. Some directions of future research are indicated in § 4.

§ 2. PRELIMINARIES AND STATEMENT OF RESULTS

Let $F(.) = \{F_{ij}(.) : 1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant p\}$ be a matrix of bounded non-decreasing right-continuous nonnegative functions on $[0\ \infty)$. For any $p \times r$ matrix $H(.)$ of Borel measurable real valued functions $H_{ij}(.)$ on $[0\ \infty)$ that are bounded on compact intervals, we define

$$(F * H)_{ij}(t) = \sum_{k=1}^{p} \int_{[0,\,t]} H_{kj}(t-u)\, F_{ik}(du) \tag{2.1}$$

for $t > 0$.

If, we make the convention that F and H are extended to the whole line by being made to vanish on $(-\infty\ 0)$, we may write the integral on the right side of (2.1) as over the whole real line. This convention shall stand wherever the domain of integration is not explicitly indicated.

We may now write (1.1) as

$$M(.) = Z(.) + (F * M)(.) \tag{2.2}$$

Now set

$$F^{\circ}(t) = ((\delta_{ij}(t)))$$

with

$$\delta_{ij}(t) = \begin{cases} 1 & \text{if } i = j \text{ and } t > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{2.3}$$

$$F^{(n)}(t) = (F * F^{(n-1)})(t) \; (n = 1, 2 \ldots)$$

$$U(t) = \sum_{n=0}^{\infty} F^{(n)}(t).$$

We shall refer to $F^{(n)}$ as the *n-fold convolution of F* and $U(.)$ as the *renewal function* associated with F.

For any matrix A with nonnegative entries, let $\rho(A)$ be its perron-Frobenius root. (see [8] for a definition).

Lemma 2.1

(*a*) $U(t) < \infty$ for each $t > 0$ if and only if $\rho(F(0)) < 1$.

(*b*) Let $\rho(F(0)) < 1$. Then $M(.) = (U * Z)(.)$ is a solution of (1.1). It is also the unique solution in the class of Borel measurable functions bounded on compact sets.

Proof

(*a*) Let

$$\hat{F}_{ij}(\alpha) = \int_{[0\ \infty]} \exp(-\alpha u) F_{ij}(du) \text{ for } \alpha \geqslant 0.$$

Since $\hat{F}_{ij}(\alpha) \downarrow F_{ij}(0)$ as $\alpha \uparrow \infty$, there must exist an $\alpha > 0$ such that $\rho(\hat{F}(\alpha)) < 1$, where

$$\hat{F}(\alpha) = ((\hat{F}_{ij}(\alpha))).$$

Thus,

$$\sum_{n=0}^{\infty} \{(\hat{F}(\alpha))^n\}_{(i,j)} < \infty \text{ for each } i \text{ and } j.$$

But

$$\begin{aligned} \{(\hat{F}(\alpha))^n\}_{i,j} &= \int_0^{\infty} e^{-\alpha u} F_{ij}^{(n)}(du) \\ &\geqslant \int_0^{\infty} e^{-\alpha u} F_{ij}^{(n)}(du) \\ &\geqslant \exp(-\alpha t) F_{ij}^{(n)}(t) \end{aligned}$$

Thus $\sum_{n=0}^{\infty} F_{ij}^{(n)}(t) = U_{ij}(t) < \infty$ for all i, j and $t > 0$. Conversely suppose $\lambda_0 = \rho(F(0)) \geqslant 1$. There exists an eigen vector $\xi = (\xi_1, \ldots, \xi_p)$ of $F(0)$ corresponding to the eigen-value λ_0 such that $\xi_i > 0$ for some i.

Since

$$\sum_{n=0}^{\infty} \sum_{j=1}^{p} \{(F(0))^n\}_{i,j}\, \xi_j = \sum_{n=0}^{\infty} \lambda^n \xi_i = \infty,$$

it follows that $\sum_{n=0}^{\infty} \{(F(0))^n\}_{i,j} = \infty$ for some j if i is such that $\xi_i > 0$.

Thus $U_{ij}(t) = \infty$ for some i and j and for all $t > 0$.

(*b*) That $M = U * Z$ satisfies (1.1) is straight forward. The uniqueness part follows as in the case $p = 1$, by using the fact that $F^{(n)}(t) \to 0$ as $n \to \infty$ for each t (see [5]).

We shall make the following hypothesis about $F(t)$ in the remainder of this paper:

$$\begin{aligned} &\text{(i) } \rho(F(0)) < 1 \\ &\text{(ii) } 0 < \lim_{t \uparrow \infty} F_{ij}(t) \equiv F_{ij}(\infty) < \infty \text{ for all } i \text{ and } j. \\ &\text{(iii) there exist } i \text{ and } j \text{ such that } F_{ij}(0) < F_{ij}(\infty) \end{aligned} \tag{2.4}$$

We know from Perron-Frobenius theory that if $\rho(F(\infty))$ is the Perron-Frobenius eigen value of $F(\infty)$, then the corresponding right and left eigen spaces are one-dimensional and that vectors m and u with strictly positive entries can be chosen so that

$$\begin{aligned} &F(\infty)\, m = \rho(F(\infty))\, m \\ &u' F(\infty) = \rho(F(\infty))\, u' \\ &\sum_{i=1}^{p} u_i m_i = 1 \\ &\sum_{i=1}^{p} m_i = 1. \end{aligned} \tag{2.5}$$

As in the case $p = 1$, there is a dichotomy in the behaviour of $M(t)$ between a lattice and non-lattice F. We make the following

Definition (2.1). $F(.)$ is *lattice* if

(i) $F_{ij}(.)$ is lattice with span λ_{ij} for any $i \neq j$ in the sense that $F_{ij}(.)$ is concentrated on a set of the form $\{b_{ij}, b_{ij} \pm \lambda, b_{ij} \pm 2\lambda, \ldots\}$ and λ_{ij} is the largest number λ with this property.

$F_{ii}(.)$ is arithmetic with span λ_{ii} for each i, in the sense that it is concentrated on a set of the form $\{0, \pm\lambda, \pm 2\lambda, \ldots\}$ and λ_{ii} is the largest number with this property.

(ii) each λ_{ii} is an integral multiple of some number. (We take λ to be the largest such number).

(iii) a_{ij}, a_{jk}, a_{ik} are points of increase of F_{ij}, F_{jk} and F_{ik} respectively implies that $a_{ij} + a_{jk} - a_{ik}$ is an integral multiple of λ.

We now introduce two moment matrices:

$$B = ((b_{ij})), \quad C = ((c_{ij}))$$

$$0 \leqslant b_{ij} = \int_{[0\ \infty)} uF_{ij}(du) \tag{2.6}$$

$$= \int_0^\infty (F_{ij}(\infty) - F_{ij}(u))\ (du) \leqslant \infty$$

$$0 \leqslant c_{ij} = \tfrac{1}{2} \int_{[0\ \infty)} u^2 F_{ij}(du)$$

$$= \int_0^\infty \left\{ \int_t^\infty (F_{ij}(\infty) - F_{ij}(u))\,(du) \right\} dt \leqslant \infty.$$

We are now ready to state our results.

Theorem 2.1

Assume that $\rho(F(\infty)) = 1$ and let $M(t) = (M_1(t) \ldots M_p(t))'$ be a vector or bounded continuous functions satisfying the system equations

$$M_i(t) = \sum_{k\ 1}^{p} \int_{-\infty}^{+\infty} M_k(t-u)\, F_{ik}(du)\ (1 \leq i \leq p). \tag{2.7}$$

Then,

(i) $F(.)$ non-lattice implies that $M(t)$ is a constant vector

(ii) $F(.)$ lattice implies that each $M_i(t)$ is periodic with period λ (see definition 2.1 for the meaning of λ). Further, for each i and j and for any point a_{ij} of increase of $F_{ij}(t)$, the vector $(M_1(t - a_{i1}), \ldots, M_p(t - a_{ip})$ is an eigen vector of $F(\infty)$ corresponding to the eigen value 1.

Theorem 2.2

Suppose $\rho(F(\infty)) = 1$.

(i) If $F(.)$ is non-lattice, then for each i, j and $h > 0$,

$$U_{ij}(t) - U_{ij}(t-h) \to cm_i u_j h \tag{2.8}$$

as $t \to \infty$ where

$$c = \frac{1}{\sum_{k=1}^{p} \sum_{r=1}^{p} m_r\, u_k\, b_{kr}}\,.$$

(If $b_{kr} = \infty$ for some k and r then c will be interpreted as zero).

(ii) If $F(.)$ is lattice, then (2.8) holds whenever h is a positive multiple of λ.

(iii) Let $Z(t) = \bigl(Z_1(t) \ldots Z_p(t)\bigr)'$ be a column vector of directly Riemann integrable functions on $[0\ \infty)$. (see Feller [5] for definition)

We set $Z_i(t) = 0$ for $t < 0$. Let $M(t) = (U^{*}Z)(t)$ be the solution of (2.1) unique in the sense of Lemma (2.1). If $F(.)$ is non-lattice, then, for each i,

$$M_i(t) \to c m_i \sum_{j=1}^{p} u_j \int_0^{\infty} Z_j(u)\, du, \tag{2.9}$$

as $t \to \infty$.

If $F(.)$ is lattice, then for each i,

$$M_i(t + n\lambda) \to c m_i \sum_{j=1}^{p} \lambda u_j \sum_{l=-\infty}^{+\infty} Z_j(t + l\lambda) \tag{2.10}$$

as $n \to \infty$.

Theorem 2.3

Let $\rho(F(\infty)) \neq 1$. Assume that there exists a real a such that $\rho(G(a)) = 1$, where

$$G_{ij}(a) = \int_0^{\infty} e^{-au} F_{ij}(du)$$

If $e^{-at}\ Z_i(t)$ is directly Riemann integrable for each i, then

$$M_i(t)\, e^{-at} \to \sum_{j=1}^{p} (a_{ij}) \int_0^{\infty} e^{-au} Z_j(u)\, du \tag{2.11}$$

as $t \to \infty$ (for each i) where

$$a_{ij} = \tilde{c}\ \tilde{m}_i\ \tilde{u}_j\,,$$

$$\tilde{m} = (\tilde{m}_1 \ldots \tilde{m}_p)$$

and

$$\tilde{u} = (\tilde{u}_1 \ldots \tilde{u}_r)$$

are positive right and left eigen vectors of $G(\alpha)$ corresponding to the eigen value one with the normalizations

$$\sum_{i=1}^{p} \tilde{m}_i \; \tilde{u}_i = 1$$

$$\sum_{i=1}^{p} \tilde{m}_i = 1,$$

$$\tilde{c} = \frac{1}{\sum\limits_{k=1}^{p} \sum\limits_{r=1}^{p} \tilde{m}_r \, \tilde{u}_k \, \tilde{b}_{kr}}$$

$$\tilde{b}_{kr} = \int_{(0\infty)} u e^{-\alpha u} F_{ij}(du) .$$

If $\tilde{b}_{ij} = \infty$ for some (i, j) then $\tilde{c}$ would be interpreted as zero.

Theorem 2.4

Assume that $F(.)$ is non-lattice and $\rho(F(\infty)) = 1$. Assume also that

$$c_{ij} = \tfrac{1}{2} \int_0^\infty t^2 F_{ij}(dt) < \infty \qquad \text{for each } (i, j).$$

Then

$$\begin{aligned} &U(t) - tA \\ &\rightarrow (I - AB + ACA)\, H^{-1} \; (t \rightarrow \infty) \end{aligned} \tag{2.12}$$

where

$$H = (I - F(\infty) + BA). \tag{2.13}$$

§ 3. *Proofs*

We begin with the following lemmata:

Lemma 3.1.

Let Σ_{ij} be the set of all points of increase of $F_{ij}(.)$, $F^{(2)}_{ij}(.)$, ..., ie..,
$\Sigma_{ij} = \{a\text{: for some } n \geq 1,\ F^{(n)}_{ij}(a + \epsilon) - F^{(n)}_{ij}(a - \epsilon) > 0 \text{ for each } \epsilon > 0\}$

Then

$$\Sigma_{ik} + \Sigma_{kj} \subset \Sigma_{ij} \text{ for all } i, j, k.$$

Lemma 3.2

(i) If $F(.)$ is non-lattice, then Σ_{ij} is asymptotically dense at ∞ for each (i, j) in the sense that for every $\epsilon > 0$ there exists $\Delta_\epsilon > 0$ such that

$x \geq \Delta_{\epsilon}$ implies $(x, x + \epsilon) \cap \Sigma_{ij} \neq \phi$ (ii) If $F(.)$ is lattice than Σ_{ij} contains only points of the form $a_{ij} + n\lambda$ and Σ_{ij} contains such points whenever n is sufficiently large.

Lemma 3.3

Let $K(t) = (K_1(t), \ldots, K_p(t))$ be a vector of uniformly continuous bounded functions such that

$$K_i(t) = \sum_{r=1}^{p} \int_{0}^{\infty} K_r(t - u)\, F_{ir}(du) \tag{3.1}$$

$(1 \leqslant i \leqslant p)$. Assume that $F(.)$ is non-lattice. Suppose that $a_{i_0} = \sup\limits_{t \in R} K_{i_0}(t)$ is strictly positive for some i_0. Then there exists $\delta_{i_0} > 0$ such that for any $h > 0$, there exists an interval $(t, t + h)$ of length h in which $K_{i_0}(x) > \delta_{i_0}$.

Proof of lemma 3.1

Let $x \epsilon \Sigma_{ik}$ and $y \epsilon \Sigma_{kj}$. Then x is a point of increase of $F_{ik}^{(n)}(.)$ and y is a point of increase of $F_{kj}^{(m)}$ for some n and m. Thus $x + y$ is a point of increase of $(F_{ik}^{(n)} * F_{kj}^{(m)})(.)$ and hence it is also a point of increase of $F_{ij}^{(n+m)}(.)$.

Proof of lemma 3.2

From lemma 3.1, it is clear that either each Σ_{ij} is asymptotically dense at ∞ or none of them is. Assume that none of the Σ_{ij}'s is asymptotically dense at ∞. Since Σ_{ii} is a subset of $[0\ \infty)$ closed under addition, $\Sigma_{ii} \neq \{0\}$ and Σ_{ii} is not asymptotically dense at ∞, it follows that

Σ_{ii} contains only multiples of some positive number δ_{ii} and it contains $n\,\delta_{ii}$ for all large n.

If $c \epsilon \Sigma_{ij}$, $d \epsilon \Sigma_{ji}$ and n is so large that $n\delta_{ii}$ and $(n + 1)\,\delta_{ii} \epsilon \Sigma_{ii}$ then $n\delta_{ii} + c + d \epsilon \Sigma_{jj}$ and

$$(n + 1)\,\delta_{ii} + c + d \epsilon \Sigma_{jj}$$

Thus $\delta_{ii} \geqslant \delta_{jj}$. By symmetry, $\delta_{ii} = \delta_{ij}$ for all i and j. Let

$$\delta = \delta_{11} = \delta_{22} = \ldots = \delta_{pp}.$$

By a similar argument we see that for $i \neq j$, Σ_{ij} contains only points of the form $b_{ij} + n\,\delta$ and Σ_{ij} contains such points for all large n. By lemma 3.1, we obtain,

$$b_{ij} + b_{jk} = b_{ik} + n\delta,$$

Thus $F(.)$ is lattice and λ (see definition 2.1) is a multiple of δ.

Thus, $F(.)$ is non lattice implies that each Σ_{ij} is asymptotically dense at ∞.

If $F(.)$ is lattice, then, by induction on n, we see that points of increase of each $F^{(n)}_{ij}(t)$ are contained in $\{b_{ij}, b_{ij} \pm \lambda, b_{ij} \pm 2\lambda, \ldots\}$ Q.E.D.

Proof of lemma 3.3

Let $K(t)$ be a solution of (2.14) which is bounded and uniformly continuous and let $a_{i_0} > 0$ where $a_i = \sup_t K_i(t)\,(1 \leqslant i \leqslant p)$. Fix j_0 such that

$$\frac{a_{j_0}}{m_{j_0}} = \max_{1 \leq j \leq p} \frac{a_j}{m_j}. \tag{3.2}$$

Note that for any integer $n \geqslant 1$,

$$\sum_{j=1}^{p} F^{n}_{ij}(\infty)\, m_j = m_i \qquad (1 \leqslant i \leqslant p).$$

Here and in the rest of this paper $F^{n}_{ij}(\infty)$ denotes the (i, j) element of $(F(\infty))^n$. We now have

$$\begin{aligned}
&\sum_{j=1}^{p} F^{n}_{j_0 j}(\infty)\, a_j \\
&= \sum_{j=1}^{p} F^{n}_{j_0 j}(\infty)\, m_j \frac{a_j}{m_j} \\
&\leq \sum_{j=1}^{p} F^{n}_{j_0 j}(\infty)\, m_j \frac{a_{j_0}}{m_{j_0}} \\
&= m_{j_0} \frac{a_{j_0}}{m_{j_0}} \\
&= a_{j_0}.
\end{aligned}$$

Thus

$$\sum_{j=1}^{p} F^{n}_{j_0 j}(\infty)\, a_j \leqslant a_{j_0}. \tag{3.3}$$

We divide the rest of the proof into two cases:

Case (*i*). There exists $t_0 \in R$ such that $K_{j_0}(t_0) = a_{j_0}$. In this case, we have

$$a_{j_0} = K_{j_0}(t_0) = \sum_{r=1}^{p} \int_0^{\infty} K_r(t_0 - u)\, F^{(n)}_{j_0 r}(du)$$

(by iteration of equation (3.1)).

Thus,

$$\begin{aligned} a_{j_0} &= \sum_{r=1}^{p} \int_0^\infty K_r(t_0-u)\, F_{j_0 r}^{(n)}(du) \\ &\leqslant \sum_{r=1}^{p} a_r \int_0^\infty F_{j_0 r}^{(n)}(du) \\ &= \sum_{r=1}^{p} a_r\, F_{j_0 r}^{(n)}(\infty) \\ &\leqslant \sum_{r=1}^{p} a_r\, F'_{j_0 r}(\infty) \\ &\leqslant a_{j_0}, \text{ by } (3.3) \end{aligned}$$

It follows that

$$\sum_{r=1}^{p} \int_0^\infty \{a_r - K_r(t_0-u)\}\, F_{j_0 r}^{(n)}(du) = 0$$

for each n and the integral being non-negative and continuous,

$K_r(t_0-u) = a_r$ whenever u is a point of increase of $F_{j_0 r}^{(n)}$ for some n.

(Note that if $K_r(t_0-u) \neq a_r$ for some u then $a_r - K_r(t_0-u)$ is bounded below by some $\delta > 0$ in some neighbourhood of u). By lemma 3.2, it follows that $\Sigma_{j_0 r}$ is asymptotically dense at ∞ for each r. The uniform continuity of the functions $K_r(.)$ now imply that

$$\lim_{t\to-\infty} K_i(t) \equiv K_i(-\infty) = a_r \ (1 \leqslant r \leqslant p). \tag{3.4}$$

Now, letting $t \to -\infty$ and using bounded convergence in

$$K_i(t) = \sum_{r=1}^{p} \int_0^\infty K_r(t-u)\, F_{ir}^{(n)}(du)\ (1 \leqslant i \leqslant p)$$

we obtain

$$a_i = \sum_{r=1}^{p} a_r F_{ir}^{(n)}(\infty). \tag{3.5}$$

Now, fix t and $r \epsilon \{1, 2, \ldots, p\}$. We have

$$\begin{aligned} & | K_r(t) - a_r | \\ &\quad \leqslant \sum_{m=1}^{p} \int_0^\infty | K_m(t-u) - a_m |\, F_{rm}^{(n)}(du) \qquad \text{(by 3.1 and 3.5)} \\ &\quad = \sum_{m=1}^{p} \int_0^T | K_m(t-u) - a_m |\, F_{rm}^{(n)}(du) \\ &\qquad + \sum_{m=1}^{p} \int_T^\infty | K_m(t-u) - a_m |\, F_{rm}^{(n)}(du). \end{aligned} \tag{3.6}$$

Observe that $F_{ij}^{(n)}(\infty) \leqslant F_{ij}^{n}(\infty)$ and $(F(\infty))^n \to ((m_i u_j))$ by Perron-Frobenius theory (see Karlin [8]). Hence

$$a = \sup_{i,j,n} F_{ij}^{(n)}(\infty) < \infty.$$

Thus, given $\epsilon > 0$ there exists T such that

$$\sum_{r=1}^{p} \int_{T}^{\infty} | K_r(t-u) - a_r | F_{ir}^{(n)}(du) < \epsilon \text{ for each } n.$$

Also, since $K_r(.)$ is bounded for each r and $F_{ir}^{(n)}(T) \to 0$ as $n \to \infty$ for each r, it follows that the first term on the right side of (3.6) approaches zero as $n \to \infty$. Thus, (3.6) implies that $K_r(t) = a_r$ for each t and r. The lemma is obvious from this.

Case (ii). $K_{j_0}(t) \neq a_{j_0}$ for each t. In this case, there is a sequence $t_n \to \pm\infty$ such that $K_{j_0}(t_n) \to a_{j_0}$. The function $\zeta_{n,i}(.)$ defined by $\zeta_{n,i}(x) = K_i(t_n + x)$ $(1 \leqslant i \leqslant p, n \geqslant 1)$ for a uniformly bounded equicontinuous family of functions and hence there is a subsequence $\{t_{n_j}\}$ of $\{t_j\}$ such that $\zeta_{n_j,i}(.)$ converges uniformly to a continuous bounded function $\eta_i(.)$ as $j \to \infty$ for each i. Since

$$\begin{aligned} \zeta_{n_j,i}(x) &= K_i(t_{n_j} + x) \\ &= \sum_{r=1}^{p} \int_{0}^{\infty} K_r(t_{n_j} + x - y) F_{ir}(du) \\ &= \sum_{r=1}^{p} \int_{0}^{\infty} \zeta_{n_j,r}(x-y) F_{ir}(dy) \end{aligned}$$

we have, by bounded convergence theorem,

$$\eta_i(x) = \sum_{r=1}^{p} \int_{0}^{\infty} \eta_r(x-y) F_{ir}(dy) \tag{3.7}$$

$(1 \leqslant i \leqslant p,\ x \in R)$.

Now, case (i) applies to the functions $\eta_i(x)$ since

$$\eta_i(x) \leqslant a_i \qquad (1 \leqslant i \leqslant p,\ x \in R)$$

and

$$\begin{aligned} \eta_{j_0}(0) &= \lim_{j \to \infty} \zeta_{n_j, j_0}(0) \\ &= \lim_{j \to \infty} K_{j_0}(t_{n_j}) \\ &= a_{j_0} \end{aligned}$$

so that

$$\eta_{j_0}(0) = \sup_{t \in R} \eta_{i_0}(t) = a_{j_0}.$$

Thus, each $\eta_i(.)$ is a constant c_i. Note that $c_{j_0} = a_{j_0}$. Now, (3.7) implies

$$c_i = \sum_{r=1}^{p} c_r F_{ir}(\infty) \qquad (1 \leqslant i \leqslant p).$$

By Perron-Frobenius theory [8] it follows that $(c_1, \ldots, c_p) = c(m_1, \ldots, m_p)$ for some c.

Since

$$\frac{a_{j_0}}{m_{j_0}} = \max_{1 \leqslant j \leqslant p} \frac{a_j}{m_j}$$

it follows that

$$\frac{a_{j_0}}{m_{j_0}} \geqslant \frac{a_{i_0}}{m_{i_0}} > 0.$$

Thus

$$c_{j_0} = a_{j_0} > 0$$

and hence

$$c = \frac{c_{j_0}}{m_{j_0}} > 0.$$

We now observe that

$K_{i_0}(t_{n_j} + x) \to c_{i_0}$ uniformly on $[0, h]$ for any fixed $h > 0$. Hence $K_{i_0}(x) > c_{i_0}/2$ for $x \in [t_{n_j}, t_{n_j} + h]$ whenever j is sufficiently large.

This completes the proof of lemma 3.3.

Proof of Theorem 2.1

Let $\rho(F(\infty)) = 1$ and $M(t)$ be a bounded continuous solution of (2.7). We shall prove the theorem only for the non lattice case. The lattice case is proved in a similar way by replacing derivatives by differences.

Set

$$\phi_\epsilon(t) = \frac{1}{\sqrt{2\pi}\,\epsilon} \exp\left\{-\frac{t^2}{2\epsilon^2}\right\} \; (t \in R),$$

and

$$f_{\epsilon, i}(t) = (\phi_\epsilon * M_i)(t)$$

i.e.,

$$f_{\epsilon, i}(t) = \int_{-\infty}^{\infty} \phi_\epsilon (t-y)\, M_i(y)\, dy$$

$$= \int_{-\infty}^{\infty} \phi_\epsilon (y)\, M_i(t-y)\, dy.$$

Then it is easily seen that for each $\epsilon > 0$, $f_{\epsilon, i}(.)$, $1 \leqslant i \leqslant p$ satisfy

$$f_{\epsilon, i}(t) = \sum_{r=1}^{p} \int_{-\infty}^{\infty} f_{\epsilon, r}(t-y)\, F_{ir}(dy) \qquad (1 \leqslant i \leqslant p).$$

Further, the functions $f_{\epsilon, i}(.)$ are infinitely differentiable and their derivatives $f'_{\epsilon, i}(.)$ satisfy the equations

$$f'_{\epsilon, i}(t) = \sum_{r=1}^{p} \int_{-\infty}^{+\infty} f'_{\epsilon, r}(t-y)\, F_{ir}(dy) \qquad (1 \leqslant i \leqslant p)$$

Since $f'_{\epsilon, i}(.)$ is bounded and uniformly continuous, we may apply lemma 3.3 to these. Let $a_i = \sup\limits_t f'_{\epsilon, i}(t)$. Suppose $a_i > 0$ for some i. By lemma 3.3, there exists $\delta_i > 0$ such that for any $h > 0$ there are intervals $(t, t+h)$ of length h in which $f'_{\epsilon, i}(x) > \delta_i$.

Thus

$$f_{\epsilon, i}(t+h) - f_{\epsilon, i}(t) > \delta_i h.$$

But the functions $f_{\epsilon, i}(t)$ are uniformly bounded and hence there exists a constant M such that $\delta_i h < M$ for all $h > 0$. This is impossible and hence

$a_i \leqslant 0$ for each i. Therefore

$f'_{\epsilon, i}(t) \leqslant 0$ for all t, i, ϵ.

Replacing $M_i(.)$ by the solution $(-M_1(.), \ldots, -M_p(.)$ of the system (2.7), we see that $f'_{\epsilon, i}(t) = 0$ for all i, t and ϵ. Thus $f_{\epsilon, i}$ is a constant for each i, ϵ; *i.e.*,

$f_{\epsilon, i}(t) = f_{\epsilon, i}(0)$ for all t, ϵ and i.

Letting $\epsilon \to 0$, we obtain ,

$M_i(t) = M_i(0)$ for each i and t.

This completes the proof of theorem 2.1.

Proof of Theorem 2.2

We break up the proof into four steps. The first step establishes the weak compactness of the translated measures $U_{ij}^{(t)}(I) = U_{ij}(t+b) - U_{ii}$

$(t + a)$ where $I = (a, b]$. The second step identifies any weak limit to be a multiple of Lebesgue measure. The third step shows that the mutiplying factor is of the form $cm_i u_j$ and the fourth one establishes the independence of the constant c on the particular subsequence $\{t_n\}$ of the t's.

Step I. The solution $M(t) = (M_1(t), \ldots, M_p(t))$ of the system of equations

$$M(t) = Z(t) + (F^*M)(t) \qquad (t > 0)$$

is given by

$$M(t) = \begin{cases} m \text{ if } t \geqslant 0 \\ 0 \text{ if } t < 0 \end{cases}$$

where $Z(t)$ is given by

$$Z(t = \begin{cases} (F(\infty) - F(t))\, m & \text{if } t \geqslant 0 \\ 0 & \text{if } t < 0. \end{cases}$$

Hence, by lemma 2.1,

$$m = (U^*Z)(t) \quad \text{for each } t > 0$$

Therefore,

$$\begin{aligned} m_i &= \int_0^t \sum_{k=1}^p Z_k(t-u)\, U_{ik}(du) \\ &\geqslant \int_{t-h}^t \sum_{k=1}^p Z_k(t-u)\, U_{ik}(du) \\ &\geqslant \sum_{k=1}^p Z_k(h)\,(U_{ik}(t) - U_{ik}(t-h)) \end{aligned}$$

since $Z_k(.)$ is monotone for each k.

It follows that for sufficiently small h, $U_{ij}(t) - U_{ij}(t-h)$ is bounded for all i and j. Since any bounded interval can be divided into a finite number of intervals of small length, it follows that the measures $U_{ij}^{(t)}(I)$ are weakly compact.

There is, therfore, a sequence $t_n \to \infty$ such that $U_{ij}^{(t_n)}(I) \to V_{ij}(I)$ for all i, j and for all intervals $I = (a\ b]$ such that $V_{ij}\{a\} = V_{ij}\{b\} = 0$ (as $n \to \infty$) where $V_{ij}(.)$ is a positive measure on R for each i and j (see the selection theorem, VIII. 6 in [5]).

Step II. Now, fix $k_0 \in \{1, 2, \ldots, p\}$. Let Z_{k_0} be a continuous function with support in $[0\ a]$ for some $a > 0$. We set $Z_k\ (t) = 0$ for all t and for $k \neq k_0$. Now $M(t) = (U^*Z)(t)$ is the solution of the system of equations

$$M_i(t) = Z_i(t) + \sum_{k=1}^{p} \int M_k(t-u) F_{ik}(du)\ (1 \leqslant i \leqslant p) \tag{3.8}$$

unique in the sense of lemma 2.1. By the weak convergence of $U_{ij}^{(t_n)}$ to V_{ij} it follows that

$$\begin{aligned} M_i(t_n + x) &= \int Z_{k_0}(t_n + x - y) U_{ik_0}(du) \\ &= \int Z_{k_0}(x-y) U_{ik_0}^{(t_n)}(du) \\ &\to \int Z_{k_0}(x-y) V_{ik_0}(du). \end{aligned}$$

Let

$$\zeta_i(x) = \int Z_{k_0}(x-y) V_{ik_0}(dy).$$

Clearly, ζ_i is a bounded continuous function. Since $M_i(t_n + x) \to \zeta_i(x)$ and $M_i(.)$ satisfies the system of equations (3.8), it follows by bounded convergence theorem that

$$\zeta_i(x) = \sum_{k=1}^{p} \int \zeta_k(x-y) F_{ik}(dy) \qquad (1 \leqslant i \leqslant p).$$

Theorem 2.1 now implies that $\zeta_i(.)$ is a constant for each i. Thus

$$\int Z_{k_0}(x-y) V_{ik_0}(du)$$

is independent of x for every continuous function $Z_{k_0}(t)$ with compact support (vanishing for $t < 0$) and hence the measure V_{ik_0} is proportional to Lebesgue measure for each i. Since k_0 is arbitrary it follows that each V_{ij} is proportional to Lebesgue measure, *i.e.*, there exist constants a_{ij} such that

$$V_{ij}(I) = a_{ij} m(I) \tag{3.9}$$

where m denotes Lebesgue measures.

Step III

We again fix k_0,

set $Z_k(t) = 0$ for all t if $k \neq k_0$

and set

$$Z_{k_0}(t) = \begin{cases} 1 & \text{if } 0 \leqslant t \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

The solution $M_i(t)\ (1 \leqslant i \leqslant p)$ of the equations

$$M_i(t) = Z_i(t) + \int_0^t \sum_{k=1}^{p} M_k(t-u) F_{ik}(du) \tag{3.10}$$

is now given by

$$M_i(t) = \int Z_{k_0}(t-u)\, U_{ik_0}(du)$$
$$= U_{ik_0}(t) - U_{ik_0}(t-1)$$

and hence

$$M_i(t_n - u) = U_{ik_0}(t_n - u) - U_{ik_0}(t_n - u - 1) \to a_{ik_0} \text{ for each } u.$$

Applying bounded convergence theorem in (3.10) we obtain

$$a_{ik0} = \sum_{k=1}^{p} a_{kk_0} F_{ik}(\infty)$$

i.e., $(a_{1k_0}, \ldots, a_{pk_0})$ is a right eigen vector of $F(\infty)$ with eigen value one. Hence, by Perron-Frobenius theory [8] there exists r_{k_0} such that

$$a_{ik_0} = r_{k_0} m_i \ (1 \leqslant i \leqslant p).$$

If we replace $((F_{ij}(.)))$ by $((F_{ji}(.)))$, $U_{ij}(.)$ becomes $U_{ji}(.)$ and hence, there exists s_{k_0} with

$$a_k = s_{k_0} u_i \ (1 \leqslant i \leqslant p).$$

The above argument can be applied for each k_0. Thus

$$a_{ik_0} = r_{k_0} m_i = s_i u_{k_0}.$$

Thus

$$\frac{r_{k_0}}{u_{k_0}} = \frac{s_i}{m_i} \text{ for each } i$$

and hence $r_{k_0}/u_{k_0} = s_i/m_i = c$ for all i and k_0 with c independent of both i and k_0. It now follows that $a_{ij} = r_j m_i = c m_i u_j$. We have thus shown that $((a_{ij}))$ is multiple of $((m_i u_j)$.

Step IV

To evaluate c we again consider the system

$$M_i(t) = Z_i(t) + \sum_{k=1}^{p} \int_0^{\infty} M_k(t-u)\, F_{ik}(du)$$

where

$$Z_i(t) = \sum_{k=1}^{p} (F_{ik}(\infty) - F_{ik}(t))\, m_k$$

i.e.,

$$Z(t) = (Z_1(t), \ldots Z_p(t))$$
$$= (F(\infty) - F(t))\, m, \ (t > 0).$$

The solution $M_i(.)$ is given by $M_i(t) = m_i$ for all i and t. Now, it is easily seen (as in [5]) that

$$M_i(t_n) \to \sum_{k=1}^{p} a_{ik} \int_0^\infty Z_k(u)\,du,$$

since $Z_i(.)$ is directly Riemann integrable for each i and $U_{ij}^{(t_n)}(I) \to a_{ij}\, m(I)$ for all i and j. Thus,

$$\begin{aligned} m_i &= M_i(t_n) \\ &\to \sum_{j=1}^{p} a_{ij} \int_0^\infty Z_j(u)\,du \\ &= \sum_{j=1}^{p} a_{ij} \sum_{k=1}^{p} b_{jk} m_k \end{aligned}$$

i.e.,

$$m_i = c \sum_{k=1}^{p} \sum_{j=1}^{p} m_i u_j b_{jk} m_k$$

i.e.,

$$1 = c \sum_{k=1}^{p} \sum_{j=1}^{p} u_j m_k b_{jk}.$$

i.e.,

$$c = \frac{1}{\sum_{k=1}^{p} \sum_{j=1}^{p} b_{jk} u_j m_k}.$$

This completes Step IV.

It now follows that the 'limit matrix' $((a_{ij}))$ is independent of the sequence $\{t_n\}$, since the argument above shows that every sequence $\{t_n\} \to \infty$ has a subsequence $\{t_{nk}\}$ for which

$$U_{ij}^{(t_{nk})}(I) \to a_{ij} m(I) \quad (K \to \infty).$$

Thus $U_{ij}^{(t)}(.)$ converges weakly to $a_{ij} m(.)$ as $t \to \infty$.

The proof of part (iii) of theorem 2.2 for the non-lattice case follows exactly as in [5]. The proof for the lattice case is similar and we omit the same.

An alternate form of a_{ij}:

Theorem 2.2 gives $((a_{ij}))$ in terms of the eigen vectors m, u and the mean matrix B. It is possible to specify a_{ij} exclusively in terms of the co-factors of $I - F(\infty)$ and the matrix B.

In fact, we have the following

Proposition

Let g_{ij} be the cofactor of the (i, j)-$^{\text{th}}$ element of $(I - F(\infty))$. Then

$$\alpha_{ij} = \frac{g_{ij}}{\sum\limits_{i,j} g_{ij}, b_{ij}}.$$

Proof.—Multiplying both sides of (1.1) by $e^{-\alpha t}$ and integrating over $[0\ \infty)$, we get,

$$\hat{M}_i(\alpha) = \hat{Z}_i(\alpha) + \sum_{j=1}^{p} \hat{M}_j(\alpha) \hat{F}_{ij}(\alpha) \ (\alpha > 0) \tag{3.11}$$

where, for any bounded function f,

$$\hat{f}(\alpha) = \int_0^\infty e^{-\alpha t} f(t)\, dt.$$

(Here we assume that $M_i(.)\ (1 \leqslant i \leqslant p)$ is the solution corresponding to continuous function $Z_i(.)$ with compact support so that the functions $Z_i(.)$, $M_i(.)$ are bounded). Equation (3.11) yields

$$\hat{M}_i(\alpha) = \sum_{j=1}^{p} \hat{Z}_j(\alpha) \frac{\hat{g}_{ij}(\alpha)}{\triangle(\alpha)}$$

where $\hat{g}_{ij}(\alpha)$ is the (i, j) cofactor of $(I - \hat{F}(\alpha))$ and $\triangle(\alpha)$ is its determinant.

Note that the invertibility of $(I - \hat{F}(\alpha))$ for $\alpha > 0$ is an easy consequence of the fact that $\rho(\hat{F}(\alpha)) < 1$. It is easy to show that if f is any bounded function on $[0\ \infty)$ such that $f(\infty) = \lim\limits_{t \uparrow \infty} f(t)$ exists, then

$$\lim_{\alpha \downarrow 0} \alpha \hat{f}(\alpha) = f(\infty).$$

Thus,

$$\begin{aligned} &\lim_{t \to \infty} M_i(t) \\ &\quad = \lim_{\alpha \downarrow 0} \alpha \hat{M}_i(\alpha) \\ &\quad = \sum_j \Big(\lim_{\alpha \downarrow 0} \hat{Z}_j(\alpha)\Big) \Big(\lim_{\alpha \downarrow 0} \hat{g}_{ij}(\alpha)\Big) \lim_{\alpha \downarrow 1} \Big(\frac{\alpha}{\triangle(\alpha)}\Big). \end{aligned}$$

Since $\hat{Z}_j$'s have compact support, $\lim\limits_{\alpha \downarrow 0} \hat{Z}_j(\alpha)$ exists and equals $\int_0^\infty Z_j(t)\, dt$. This forces $\lim\limits_{\alpha \downarrow 0} \alpha / \triangle(\alpha)$ to exist. We may now conclude that $((\alpha_{ij}))$ is proportional to g_{ij} (where g_{ij} is the (i, j) cofactor of $(I - \hat{F}(0))$, *i.e.*, of $(I - F(\infty))$.

Thus

$$\frac{g_{ij}}{\sum\limits_{ij=1}^{p}{}' \, g_{ij} b_{ij}} = \frac{m_i u_j}{\sum\limits_{i,\,j=1}^{p} m_i u_j b_{ij}} = a_{ij}. \qquad \text{Q.E.D.}$$

Remark

The above proof shows, incidentally, that $\lim\limits_{\alpha \downarrow 0} \triangle(\alpha)/\alpha$ always exists and equals $\sum\limits_{i,j=1}^{p} g_{ij} b_{ij}$. An independent proof is also possible directly using the definition of $\triangle(\alpha)$.

Proof of Theorem 2.3

This theorem follows easily from theorem 2.2, by writing (1.1) in the form

$$(M_i(t)\, e^{-\alpha t}) = (Z_i(t)\, e^{-\alpha t}) + \sum_{k=1}^{p} \int_{0}^{\infty} Z_k(t-u)\, e^{-\alpha(t-u)}\, G_{ik}(du)$$

where

$$G_{ij}(u) = \int_{0}^{u} e^{-\alpha v} F_{ij}(dv)$$

and observing that

$$\rho((G_{ij}(\infty))) = 1. \qquad \text{Q.E.D.}$$

Remark

If $\rho\big(F(\infty)\big) < 1$ then α certainly exists and is positive. If $\rho\big(F(\infty)\big) <$ 1 and α exists, then α is necessarily negative.

Proof of Theorem 2.4

If we set $M(t) = U(t) - tA\ (t > 0)$ and $M - F^{*}M = Z$ then we see that each column $M^{(j)}$ of M satisfies

$$M^{(j)}(t) = Z^{(j)}(t) + (F^{*}M^{(j)})(t)$$

and hence

$$M^{(j)}(.) = (U^{*}Z^{(j)})(.)$$

Thus

$$U_{ij}(t) - \alpha_{ij}\, t$$

$$= \sum_{k=1}^{p} \int_0^t [\delta_{kj} - (t-u)\, \alpha_{kj} + \sum_{r=1}^{p} \alpha_{rj} \int_0^{t-u} (t-u-v)\, F_{kr}(dv)]\; U_{ik} (du) \tag{3.12}$$

But since $\int_0^t (t-u)\, H(du) = \int_0^t H(u)\, du$ for any non-decreasing function H on $[0\ \infty)$ and since

$$\alpha_{kj} = \sum_{r=j}^{p} F_{kr}(\infty)\, \alpha_{rj} \text{ for all } k \text{ and } j,$$

we may rewrite (3.12) as

$$U_{ij}(t) - \alpha_{ij}\, t$$

$$= U_{ij}(t) - \sum_{k=1}^{p} \int_0^t \{\sum_{r=1}^{p} \alpha_{rj} \int_0^{t-u} \big(F_{kr}(\infty) - F_{kr}(v)\, (dv)\big)\}\, U_{ik}(du).$$

But $U(t) = I + (F^{*}U)(t)$ $(t > 0)$ where I is the $p \times p$ identity matrix. Hence,

$$U_{ij}(t) - \alpha_{ij}\, t$$

$$= \delta_{ij} + \sum_{k=1}^{p} \int_0^t [F_{kj}(t-u) - \sum_{r=1}^{p} \alpha_{rj} \int_0^{t-u} (F_{kr}(\infty) - F_{kr}(v)\, dv]\; U_{ik}(du)$$

$$= \delta_{ij} + \sum_{k=1}^{p} F_{kj}(\infty)\, U_{ik}(t)$$

$$+ \sum_{k=1}^{p} \int_0^t (F_{kj}(t-u) - F_{kj}(\infty))\, U_{ik}(du)$$

$$- \sum_{k=1}^{p} \int_0^t [\sum_{r=1}^{p} \alpha_{rj} \int_0^{t-u} (F_{kr}(\infty) - F_{kr}(v))\, dv]\, U_{ik}(du)$$

The last term on the right above may be rewritten as

$$\sum_{k=1}^{p} \int_0^t (\sum_{r=1}^{p} \alpha_{rj} \int_0^{\infty} (F_{kr}(\infty) - F_{kr}(v))\, dv))\, U_{ik}(du)$$

$$- \sum_{k=1}^{p} \int_0^t [\sum_{r=1}^{p} \alpha_{rj} \int_{t-u}^{\infty} (F_{kr}(\infty) - F_{kr}(v))\, dv]\, U_{ik}(du)$$

which is equal to

$$\sum_{k=1}^{p} U_{ik}(t) \sum_{r=1}^{p} F_{kr}\, a_{rj}$$

$$- \sum_{k=1}^{p} \int_{0}^{t} \Big[\sum_{r=1}^{p} a_{rj} \int_{t-u}^{\infty} \big(F_{kr}(\infty) - F_{kr}(v)\big)\, dv\Big]\, U_{ik}(du).$$

If we set

$$\tilde{Z}_{kr}(t) = F_{kr}(\infty) - F_{kr}(t)$$

and

$$\tilde{\tilde{Z}}_{kr}(t) = \int_{t}^{\infty} Z_{kr}(v)\, dv,$$

we get

$$U(t)(I - F(\infty) + BA) - tA = I - U^{*}\tilde{Z} + U^{*}(\tilde{\tilde{Z}}A)$$

under the hypothesis of finiteness of second moments, both $\tilde{Z}$ and $\tilde{\tilde{Z}}$ are matrices of directly Riemann integrable functions. By theorem 2.2, we conclude that

$$U(t)H - tA \to I - AB + ACA \tag{3.13}$$

where

$$H = I - F(\infty) + BA.$$

We now show that H is non-singular.

In fact, if $H\xi' = 0$ for some non-zero row vector $\xi = (\xi_1, \ldots, \xi_p)$ then, multiplying on the left by u we obtain $uH\xi' = 0$. But

$$uH = u - uF(\infty) + uBA = uBA$$

so that $UBA\xi' = 0$.

Now

$$A = c\,((m_i\, u_j)) \text{ and hence}$$

$$c\,(\sum_{j} u_j \xi_j)(\sum_{i,k} u_j b_{ik} m_k) = 0$$

i.e.,

$$\sum_{j=1}^{p} u_j\, \xi_j = 0.$$

But this implies that $BA\xi' = 0$ and hence

$$0 = H\xi' = \xi' - F(\infty)\,\xi' + BA\xi' = \xi' - F(\infty)\,\xi'.$$

Perron-Frobenius theory now tells us that ξ' is a multiple of m. But

$$\sum_{j=1}^{p} u_j\, m_j = 1$$

and

$$\sum_{j=1}^{p} u_i \, \xi_j = 0.$$

This forces ξ' to be the zero vector which is a contradiction.

Hence H is invertible.

We obtain, from (3.13),

$$U(t) - tAH^{-1} \to (I - AB + ACA)\, H^{-1}.$$

However, $\frac{U(t)}{t} \to A$ and hence we must have $AH^{-1} = A$.

This completes the proof of theorem 2.2.

Remark

For $p = 1$, the function $Z(.)$ is non-negative and hence we may conclude that $U(t) - t\,A$ is non-negative and converges to a strictly positive limit. No such conclusion is possible in the present context. The matrix H^{-1} will have negative entries as could the matrix $(I - AB + ACA)$.

§4. *Some Open Problems*

(*a*) *Infinite mean case*.—When $p = 1$, there is a body of results, due to K. B. Erickson [4] and others, for the case of infinite mean, *i.e.*,

$$\int_0^\infty t dF(t) = \infty.$$

These describe the behaviour of $U(t)$ in terms of the incomplete mean $m(t) = \int_0^t u d F(u)$ when F has a regularly varying tail. They also study $U(t) - U(t - h)$, as $t \to \infty$ as well as Z^*U for directly Riemann integrable Z. The corresponding theory for $p \geq 2$ is not available. The tools employed for $p = 1$ are Fourier analytic. Perhaps these could be useful for $p \geqslant 2$ also.

(*b*) *Proof of the basic lemma*.—A key step in our proof of the renewal theorem is the one asserting that if

$$\phi = F^*\phi$$

and ϕ is bounded, uniformly continuous, then ϕ is constant. Our proof here is a direct extension of Feller's [5]. In the case $p = 1$ there are two other proofs available. One uses martingable theory and the zero one law [5]. The other is *via* distributions and Wiener's Tauberian theory (see [10] p. 218). It should be possible to push these proofs to the present context of $p \geqslant 2$. This is open. Notice that if for all i, ϕ_i is in the rapidly decaying class, *i.e.*, $\sup \ | \phi_i (x)| e_i^{r\,|x|} < \infty$ for some $r_i > 0$ then the result is immediate by taking Fourier transforms as these will be analytic and vanish on a continuum.

(*c*) *Degenerate case.*—Even when $p = 1$, if z is d.r.i. but $\int_0^{+\infty} Z(t)\, dt = 0$ then all that the renewal theorem says that $(U^* Z)(t) \to 0$ as $t \to \infty$. But, the rate of convergence could be of interest. A partial result in this direction is available in Jagers [7] and Harris [6] (p. 162). The case $p \geqslant 2$ is completely open.

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After this paper was accepted for publication, another paper with the same purpose has appeared : T. A. RYAN, Jr. 'A multi-dimensional renewal theorem', *Annals of Probability*, Vol. 4, No. 4, 656-661, 1976. The proof of the renewal theorem is obtained in that paper by applying the one-dimensional renewal theorem to a decomposition of the solution of the renewal equation.