Open access • Journal Article • DOI:10.1088/1126-6708/2004/04/001

Fermion BMN operators, the dilatation operator of N=4 SYM, and pp-wave string interactions — Source link

George Georgiou, Gabriele Travaglini

Institutions: Durham University, Queen Mary University of London

Published on: 18 Mar 2004 - arXiv: High Energy Physics - Theory

Topics: String field theory, Non-critical string theory, Heterotic string theory, Compactification (physics) and Type I string theory

Fermion BMN operators, the dilatation operator of $\mathcal{N} = 4$ SYM, and pp-wave string interactions

George Georgiou[a] **and Gabriele Travaglini**[b]

[a] *Centre for Particle Theory, Department of Physics and IPPP, University of Durham, Durham, DH1 3LE, UK*

[b] *Department of Physics, Queen Mary College, London E1 4NS, UK*

george.georgiou@durham.ac.uk, g.travaglini@qmul.ac.uk

Abstract

The goal of this paper is to study the BMN correspondence in the fermionic sector. On the field theory side, we compute matrix elements of the dilatation operator in $\mathcal{N} = 4$ Super Yang-Mills for BMN operators containing two fermion impurities. Our calculations are performed up to and including $\mathcal{O}(\lambda')$ in the 't Hooft coupling and $\mathcal{O}(g_2)$ in the Yang-Mills genus counting parameter. On the string theory side, we compute the corresponding matrix elements of the interacting string Hamiltonian in string field theory, using the three-string interaction vertex constructed by Spradlin and Volovich (and subsequently elaborated by Pankiewicz and Stefanski). In string theory we use the natural string basis, and in field theory the basis which is isomorphic to it. We find that the matrix elements computed in field theory and the corresponding string amplitudes derived from the three-string vertex are, in all cases, in perfect agreement.

1 Introduction

In [1], Berenstein, Maldacena and Nastase (BMN) proposed an intriguing correspondence between type IIB superstring theory on a pp-wave background geometry and a sector of $\mathcal{N} = 4$ Super Yang-Mills (SYM). BMN compared the exact expression [2] for the mass spectrum of the string states in free string theory, $g_{\rm st} = 0$, to the planar anomalous dimension of certain field theory operators - since then called the BMN operators - to the first order in the 't Hooft coupling of the theory λ', finding remarkable agreement. Shortly after it was shown in [3] that, thanks to the superconformal invariance of the $\mathcal{N} = 4$ theory, it was actually possible to reproduce from field theory the full (all orders in λ') expression for the masses of string states at $g_{\rm st} = 0$. An important step forward was subsequently taken in [4,5], where the correspondence was expressed as [5]

$$\frac{1}{\mu} H_{\rm string} = \Delta - J \ , \tag{1.1}$$

where $H_{\rm string}$ is the *interacting* string Hamiltonian, μ is the scale parameter of the pp-wave metric, and $\Delta - J$ is the difference between the gauge theory dilatation operator Δ and the R-charge J. The relation (1.1) is conjectured to hold in the double-scaling limit $N \sim J^2 \to \infty$ to all orders in the two parameters of the theory, g_2 and λ', where

$$\lambda' \;=\; \frac{g^2_{\rm YM} N}{J^2} \;=\; \frac{1}{(\mu p^+ \alpha')^2} \ , \tag{1.2}$$

$$g_2 \;=\; \frac{J^2}{N} \;=\; 4\pi \, g_{\rm st} \, (\mu p^+ \alpha')^2 \ . \tag{1.3}$$

Here λ' is the effective 't Hooft coupling of the BMN sector [1], and g_2 is the genus counting parameter of Feynman diagrams [1,6,7]. The right hand sides of (1.2), (1.3) express λ' and g_2 in terms of the parameters in pp-wave string theory, so that $1/\lambda' \propto \mu$ measures the deviation from flat space $\mu \to 0$ and, importantly, the Yang-Mills genus counting parameter g_2 is proportional to the string coupling $g_{\rm st}$.

Tests of the relation (1.1) rely on the careful comparison of string amplitudes obtained in pp-wave string field theory to the matrix elements of the dilatation operator Δ in Yang-Mills, and have been performed, for the bosonic sector of the theory, in a variety of cases:

a. in [8], the case of BMN operators with two scalar impurities of different flavour was studied;

b. in [9,10], BMN operators with an arbitrary number of scalar impurities was considered; and finally,

c. in [10], all the $SO(4) \times SO(4)$ representations of two scalar impurity and two vector impurity BMN operators were studied, as well as BMN operators with mixed (one scalar/one vector) impurities.

In all cases, precise agreement was found between the string amplitudes obtained using the superstring vertex[1] and the corresponding matrix elements obtained in field theory [8–10]. In particular, the analysis of [10] clarified a puzzle concerning the realisation of the $\mathbb{Z}_2 \subset SO(8)$ symmetry of the pp-wave background geometry.[2] Apparently, the $\mathbb{Z}_2$ part of the bosonic symmetry of the pp-wave background is not respected by the Spradlin-Volovich three-string interactions [12, 15–18]: a relative minus sign appears in the string amplitude involving states with two oscillators along the first $SO(4)$ compared to that with two oscillators along the second $SO(4)$. However, the string vertex is invariant under $\mathbb{Z}_2$, and the puzzle is solved, if one makes the parity assignment under the $\mathbb{Z}_2$ symmetry: $|0\rangle \to |0\rangle$, $|\mathrm{v}\rangle \to -|\mathrm{v}\rangle$, where $|\mathrm{v}\rangle$ is the true ground state of the pp-wave theory, whereas $|0\rangle$ is the string state corresponding to the ground state for flat background (but not for $\mu \neq 0$, where its energy is proportional to μ). In [10], the emergence of the previous parity assignment was explicitly shown in field theory.

An important point should now be emphasised. In order to use and in particular to test (1.1) at the level of matrix elements, it is essential to construct the isomorphism which connects the basis of states on which the string theory Hamiltonian acts to the corresponding basis of field theory operators. Lightcone string field theory is naturally equipped with an orthonormal basis of single- and multi-string states, which does not correspond to either the "natural" basis in field theory, i.e. the basis of operators with well-defined conformal dimension (henceforth called the Δ-BMN basis), or to the basis originally considered by BMN. This subtle issue required some time to be appreciated and fully understood [5, 8, 19], but the isomorphism was finally constructed and used in [8–10] to successfully test the correspondence. Specifically, in [10] the natural emergence of the isomorphic to string basis was explained by constructing the proper overlap of states in terms of two-point function of BMN operators where the conjugated BMN operator is defined through hermitian conjugation *plus* an inversion [20]. This procedure, which was applied in [10] to BMN operators with scalar and/or vector impurities, will be used in section 3 in the case of fermion BMN operators.

Fermion BMN operators have recently been studied, with an emphasis on mixing issues, in [21, 22], however the equivalence relation (1.1) has not been investigated so far in the fermionic part of the BMN sector of $\mathcal{N} = 4$ Yang-Mills. The aim of this paper is to fill this important gap. One of the main motivations for our analysis lies in the fact that fermionic matrix elements of H_{string} have never been compared to any field theory result. Moreover, string field theory in the fermionic sector is not just a simple extension of its bosonic counterpart, and the construction of the fermionic prefactor of the string field Hamiltonian is not straightforward. Therefore, it is particularly compelling to investigate the BMN correspondence in the fermionic sector.

[1]The expression for this vertex was originally obtained by Spradlin and Volovich [11, 12], and further studied and clarified by Pankiewicz and Stefanski in [13, 14].

[2]The presence of a five-form flux in the pp-wave background breaks the lightcone $SO(8)$ symmetry down to $SO(4) \times SO(4) \times \mathbb{Z}_2$. The first (second) $SO(4)$ rotates the first (last) four directions $\{x^i\}$ ($\{x^a\}$), while the $\mathbb{Z}_2$ symmetry swaps $\{x^i\} \to \{x^a\}$.

In this paper we will make use of the superstring vertex in the $SO(4) \times SO(4)$ formalism, whose construction was given in [23]. It was shown in [24] that the $SO(8)$ formalism of [11–14] is actually completely equivalent to the $SO(4) \times SO(4)$ construction, as it was already conjectured in [23]. The $SO(8)$ and the $SO(4) \times SO(4)$ formalism differ in that, in the former, the string interaction vertex is built upon the state $|0\rangle$ (the ground state of the theory in flat background); whereas in the latter, the vertex is constructed on the true pp-wave ground state $|\mathrm{v}\rangle$. In both formalisms the external string states are built on the true pp-wave ground state $|\mathrm{v}\rangle$.

Our analysis in field theory will be performed up to and including $\mathcal{O}(\lambda')$ in the pp-wave 't Hooft coupling and $\mathcal{O}(g_2)$ in the genus counting parameter, and hence incorporates string interactions at the first nontrivial order.[3] Our result is simple: for all cases we consider, the matrix elements of the string field theory Hamiltonian derived from the superstring vertex of [11, 12, 14, 23] agree perfectly with the corresponding field theory quantities. This result allows us to confirm the validity of the conjectured duality relation (1.1) in the fermionic sector, and at the level of string interactions.

The plan of the rest of this paper is as follows. In the next section we introduce and discuss BMN operators with two fermion impurities. In section 3, we review in detail the procedure which leads to the identification of the basis in field theory which is isomorphic to the natural string basis of single- and multi-string states. We also summarise the strategy we follow in order to compare the matrix elements of the Yang-Mills dilatation operator to the matrix elements of the interacting string Hamiltonian – with particular attention to the new features arising from considering fermion BMN operators. In section 4 we derive the desired fermionic matrix elements of the string Hamiltonian in light-cone string field theory. Section 5 and 6 contain the calculation in field theory and the comparison to the string results previously derived. We conclude with a few appendices containing details of the calculations, as well as our notation and conventions in field and string theory.

2 Fermion BMN operators

In order to study the BMN sector of $\mathcal{N} = 4$ Super Yang-Mills, we need to pick an R-charge subgroup $U(1)_J \subset SU(4)$. Hence we need to decompose $SU(4) \to SO(4) \times U(1)_J \sim SU(2) \times SU(2) \times U(1)_J$. The branching rule for the fermion representation $\mathbf{4}$ of $SU(4)$, to which the fermions λ^A_α of the $\mathcal{N} = 4$ theory belong, is [27]

$$\mathbf{4} \longrightarrow (\mathbf{2}, \mathbf{1})_+ \; + \; (\mathbf{1}, \mathbf{2})_- \;, \tag{2.1}$$

[3] This result, as well as the results of [8–10, 25] are in contradistinction with [26], where it was argued that nonplanar effects (corresponding to string interactions in string theory) should not be incorporated into the pp-wave string theory/$\mathcal{N} = 4$ SYM duality.

from which we have, in terms of fields,

$$\lambda^A_\alpha \longrightarrow \big(\lambda_{r\alpha,\,(1/2)}\,,\,\lambda_{\dot r\alpha,\,(-1/2)}\big)\ , \tag{2.2}$$

$$\bar\lambda_{A\dot\alpha} \longrightarrow \big(\bar\lambda_{r\dot\alpha,\,(-1/2)}\,,\,\bar\lambda_{\dot r\dot\alpha,\,(1/2)}\big)\ . \tag{2.3}$$

Here $\alpha, \dot\alpha = 1, 2$ are spin indices, $A = 1 \ldots 4$, and $r = 3, 4$, $\dot r = 1, 2$. Notice that there are four fermions with positive R-charge, $\lambda_{r\alpha,\,(1/2)}$ and $\bar\lambda_{\dot r\dot\alpha,\,(1/2)}$, and four with negative R-charge, $\lambda_{\dot r\alpha,\,(-1/2)}$, $\bar\lambda_{r\dot\alpha,\,(-1/2)}$. We will refer to the former as to the BMN fermions, and to the latter as to the anti-BMN fermions. To simplify the notation, we will omit from now on the $U(1)$ R-charge subscript in the fermion fields.

The following table summarises the field content (scalar, vector and fermion fields) of the $\mathcal{N} = 4$ SYM theory participating in the BMN correspondence, together with their canonical dimensions, R-charge and decomposition into irreducible representations of $SO(4) \times SO(4) \sim SU(2) \times SU(2) \times SU(2) \times SU(2)$.

field	Δ_0	J	$\Delta_0 - J$	$SO(4) \times SO(4)$
Z	1	1	0	$(\mathbf{1}, \mathbf{1})$
$\bar Z$	1	-1	2	$(\mathbf{1}, \mathbf{1})$
ϕ^i	1	0	1	$(\mathbf{1}, \mathbf{4})$
D_μ	1	0	1	$(\mathbf{4}, \mathbf{1})$
$\lambda_{r\alpha}$	3/2	1/2	1	$((\mathbf{2}, \mathbf{1}), (\mathbf{2}, \mathbf{1}))$
$\bar\lambda_{\dot r\dot\alpha}$	3/2	1/2	1	$((\mathbf{1}, \mathbf{2}), (\mathbf{1}, \mathbf{2}))$
$\bar\lambda_{r\dot\alpha}$	3/2	-1/2	2	$((\mathbf{2}, \mathbf{1}), (\mathbf{1}, \mathbf{2}))$
$\lambda_{\dot r\alpha}$	3/2	-1/2	2	$((\mathbf{1}, \mathbf{2}), (\mathbf{2}, \mathbf{1}))$

Table 1: *In this table we list the canonical dimension Δ_0, R-charge J and $SO(4) \times SO(4)$ representations for the fields of $\mathcal{N} = 4$ Super Yang-Mills. For convenience, we also write the corresponding $\Delta_0 - J$ for each field.*

We now discuss the two-impurity fermion BMN operators. Their precise form can be obtained by acting with two supersymmetry transformations on the scalar BMN operators

$$\mathcal{O}^J_{ij,m} = \mathcal{C}\Big[\sum_{l=0}^{J} e^{\frac{2\pi i m l}{J}} \mathrm{Tr}\left(\phi_i Z^l \phi_j Z^{J-l}\right) \ - \ \delta_{ij}\,\mathrm{Tr}(\bar Z Z^{J+1})\Big]\ , \tag{2.4}$$

where $i, j = 1, \ldots, 4$ and we have defined

$$\mathcal{C} := \frac{1}{\sqrt{J N_0^{J+2}}}\ , \qquad N_0 := \frac{g^2}{2}\frac{N}{4\pi^2}\ . \tag{2.5}$$

The normalisation of the operators is such that their two-point functions take the canonical form in the planar limit. This procedure correctly identifies the possible compensating terms which may be present in the expression of the operators. We would like to remind the reader that these compensating terms are crucial for a correct understanding of the dynamics of the BMN sector.

Specifically, we will be considering

$$\mathcal{O}^J_{\rm vac} = \frac{1}{\sqrt{JN_0^J}}{\rm Tr}Z^J \ , \tag{2.6}$$

$$\mathcal{O}^{\alpha\beta;J}_{33;m} = \frac{\mathcal{C}}{2}\Big[\sum_{l=0}^{J} e^{\frac{2\pi i m l}{J}}\,{\rm Tr}(\lambda_3^\alpha\, Z^l\, \lambda_3^\beta\, Z^{J-l})\Big] \ , \tag{2.7}$$

$$\mathcal{O}^{\alpha\beta;J}_{34;m} = \frac{\mathcal{C}}{2}\Big[\sum_{l=0}^{J} e^{\frac{2\pi i m l}{J}}\,{\rm Tr}(\lambda_3^\alpha\, Z^l\, \lambda_4^\beta\, Z^{J-l}) \ - \ \frac{\sqrt{2}}{4}{\rm Tr}\Big((F_{\mu\nu}\sigma^{\mu\nu})_\gamma^{\ \beta}\epsilon^{\alpha\gamma}\, Z^{J+1}\Big)\Big] \ . \tag{2.8}$$

Very similar expressions can be written for operators where $(3,4) \to (1,2)$, i.e. undotted $SU(2)$ indices are replaced by dotted ones.

In the following, we will also make extensive use of the expressions for the double-trace operators

$$\mathcal{T}^{J,y}_{r\alpha,s\beta;\,m} = \ :\mathcal{O}^{y\cdot J}_{r\alpha,s\beta;\,m}: \, :O^{(1-y)\cdot J}_{\rm vac}: \ , \tag{2.9}$$

$$\mathcal{T}^{J,y}_{\dot r\dot\alpha,\dot s\dot\beta;\,m} = \ :\mathcal{O}^{y\cdot J}_{\dot r\dot\alpha,\dot s\dot\beta;\,m}: \, :O^{(1-y)\cdot J}_{\rm vac}: \ , \tag{2.10}$$

where $y \in (0,1)$.

A few important comments are in order.

1. First, note the appearance on right hand side of (2.8) of an all-important compensating term which modifies the naïve expression for $\mathcal{O}^{\alpha\beta;J}_{34;m}$. Compensating terms are required in order for the corresponding operators to be conformal primaries in the BMN limit, and are present also in the case of scalar BMN operators [28] (as the right hand side of (2.4) for $i=j$ shows) and vector BMN operators [29, 30].

2. Second, we would like to stress that these compensating terms play a key rôle in the evaluation of the conformal three-point functions of vector and mixed BMN operators. Indeed, had they not been taken into account, one would erroneously conclude that the three-point functions for scalar, vector and mixed BMN operators take actually all the same form. The three-point function coefficients for vector and for mixed BMN operators were computed in [20] and [10], respectively, and found to be different from that of the scalar case [31].[4] Of course, this is striking evidence against a direct correspondence between the conformal *three-point functions* and the superstring vertex at the nontrivial, interacting level.

[4]Their expressions are given in Eqs. (3.29)-(3.31) of [10].

3. Furthermore, the analysis of [10] showed that precisely thanks to the differences between the three-point function coefficients for scalar, vector and mixed impurity BMN operators it is possible to reproduce, from the field theory point of view, two key properties of the three-string vertex of Spradlin and Volovich, namely:
a. the vanishing of the three-string amplitude for string states with one vector and one scalar impurity; and
b. the relative minus sign in the string amplitude involving states with two vector impurities compared to that with two scalar impurities.
Once this is taken into account, perfect agreement between the string and field theory predictions is found.

To further clarify the rôle of the compensating terms, we consider the flavour-singlet and flavour-triplet combinations:

$$\mathcal{O}^{\alpha\beta;J}_{34,\mathbf{S};m} \;=\; \frac{\mathcal{C}}{2\sqrt{2}}\Big[\sum_{l=0}^{J} e^{\frac{2\pi i m l}{J}}\mathrm{Tr}(\lambda_3^\alpha\, Z^l\, \lambda_4^\beta\, Z^{J-l} - \lambda_4^\alpha\, Z^l\, \lambda_3^\beta Z^{J-l}) \;-\; \frac{\sqrt{2}}{2}\mathrm{Tr}\Big((F_{\mu\nu}\sigma^{\mu\nu})_\gamma^{\ \beta}\epsilon^{\alpha\gamma}\; Z^{J+1}\Big)\Big] \tag{2.11}$$

$$\mathcal{O}^{\alpha\beta;J}_{34,\mathbf{T};m} \;=\; \frac{\mathcal{C}}{2\sqrt{2}}\Big[\sum_{l=0}^{J} e^{\frac{2\pi i m l}{J}}\mathrm{Tr}(\lambda_3^\alpha\, Z^l\, \lambda_4^\beta\, Z^{J-l} \,+\, \lambda_4^\alpha\, Z^l\, \lambda_3^\beta\, Z^{J-l})\Big]\;. \tag{2.12}$$

We can further decompose each of the two operators in (2.11) and (2.12) into singlet and triplet of the spin, that is

$$\mathcal{O}^{\alpha\beta;J}_{34,\mathbf{S};m} \;\longrightarrow\; (\mathbf{1},\mathbf{3}^+) \;+\; (\mathbf{1},\mathbf{1})\;, \tag{2.13}$$

$$\mathcal{O}^{\alpha\beta;J}_{34,\mathbf{T};m} \;\longrightarrow\; (\mathbf{3}^+,\mathbf{3}^+) \;+\; (\mathbf{3}^+,\mathbf{1})\;. \tag{2.14}$$

It is immediately seen that the compensating term on the right hand side of (2.11) is symmetric under the exchange of the spin indices α and β. This means that this compensating term will affect only the $(\mathbf{1},\mathbf{3}^+)$ representation. This is perfectly consistent with the decomposition of the two-impurity BMN operators with *vector* impurities according to irreducible representations of $SO(4)\times SO(4)$. Indeed, by combining two vector impurities we can form the following representations:

$$(\mathbf{1},\mathbf{4})\times(\mathbf{1},\mathbf{4}) \;=\; (\mathbf{1},\mathbf{1}) \;+\; (\mathbf{1},\mathbf{9}) \;+\; (\mathbf{1},\mathbf{3}^+) \;+\; (\mathbf{1},\mathbf{3}^-)\;. \tag{2.15}$$

The only representation the right hand sides of (2.13) and (2.14) have in common with the right hand side of (2.15) are precisely $(\mathbf{1},\mathbf{3}^+)$, which receives a compensating term, and $(\mathbf{1},\mathbf{1})$, for which however no compensating term is generated as this order.[5]

For completeness, we mention here what are the possible irreducible representations of $SO(4)\times SO(4)\sim SU(2)\times SU(2)\times SU(2)\times SU(2)$ that can be obtained by combining

[5]For a discussion along similar lines on the possible mixing of fermion BMN operators with scalar operators, see [22].

two fermion impurities:[6]

$$
\begin{aligned}
((\mathbf{2},\mathbf{1}),(\mathbf{2},\mathbf{1}))\times((\mathbf{2},\mathbf{1}),(\mathbf{2},\mathbf{1})) &= (\mathbf{1},\mathbf{1}) + (\mathbf{3}^+,\mathbf{3}^+) + (\mathbf{3}^+,\mathbf{1}) + (\mathbf{1},\mathbf{3}^+)\ , &(2.16)\\
((\mathbf{1},\mathbf{2}),(\mathbf{1},\mathbf{2}))\times((\mathbf{1},\mathbf{2}),(\mathbf{1},\mathbf{2})) &= (\mathbf{1},\mathbf{1}) + (\mathbf{3}^-,\mathbf{3}^-) + (\mathbf{3}^-,\mathbf{1}) + (\mathbf{1},\mathbf{3}^-)\ , &(2.17)\\
((\mathbf{2},\mathbf{1}),(\mathbf{2},\mathbf{1}))\times((\mathbf{1},\mathbf{2}),(\mathbf{1},\mathbf{2})) &= (\mathbf{4},\mathbf{4})\ . &(2.18)
\end{aligned}
$$

3 Comparing matrix elements of H_{string} and Δ

In this section we briefly review the strategy adopted in [8,10] to compare matrix elements of the dilatation operator Δ in SYM to matrix elements of the fully interacting string Hamiltonian H_{string}.

To begin with, we notice that H_{string} and Δ act on the states of two different theories. Hence, in order to test the duality (1.1) we need to construct an isomorphism between the Hilbert spaces of the lightcone pp-wave string field theory and of the BMN sector of $\mathcal{N}=4$ Yang-Mills. The choice of the basis in string theory and in field theory is in principle arbitrary, however we remember that string field theory is naturally equipped with an orthonormal basis of single- and multi-string states, given by the tensor product of single-string states. This natural string basis, $\{|s_\alpha\rangle^{\text{string}}\}$, diagonalises the free string Hamiltonian. Once interactions are taken into account, they allow the strings to split and join, i.e. the states in the natural string basis are not eigenstates of the interacting pp-wave string Hamiltonian H_{string}. Matrix elements of H_{string} are known in the natural string basis, hence our goal will be the identification of the field theory basis $\{|s_\alpha\rangle^{\text{SYM}}\}$ which is isomorphic to it. This, in turns, will enable us to recast (1.1) at the level of the matrix elements as

$$
{}^{\text{string}}\langle s_\alpha|\mu^{-1}H_{\text{string}}|s_\beta\rangle^{\text{string}} = {}^{\text{SYM}}\langle s_\alpha|\Delta - J|s_\beta\rangle^{\text{SYM}}\ . \tag{3.1}
$$

But what is the situation in field theory? In the conformal $\mathcal{N}=4$ Yang-Mills there is also a privileged basis of states, the basis of conformal primary BMN operators $\mathcal{O}_{\Delta_\alpha}(x)$, or Δ-BMN basis. This is the basis of the eigenstates of the SYM dilatation operator, and its eigenkets can be expressed as linear combinations of the original BMN operators $\mathcal{O}_\alpha(x)$ proposed in [1]. For BMN operators with scalar impurities, the Δ-BMN basis was explicitly constructed in [31], and extended to include vector and mixed impurities in [20]. Conformal invariance guarantees that, in the Δ-BMN basis, two- and three-point functions of Δ-BMN operators take the canonical form, with a universal x-dependence. Specifically, for scalar conformal primary operators one has

$$
\langle \mathcal{O}^\dagger_{\Delta_\alpha}(x)\mathcal{O}_{\Delta_\beta}(0)\rangle = \frac{\delta_{\alpha\beta}}{(x^2)^{\Delta_\alpha}}\ , \tag{3.2}
$$

[6]See also the discussion in section IV of [35].

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}^\dagger_{\Delta_3}(x_3)\rangle = \frac{C_{123}}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}}(x_{13}^2)^{\frac{\Delta_1+\Delta_3-\Delta_2}{2}}(x_{23}^2)^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}} \ . \tag{3.3}$$

Correlation functions of conformal primary operators with vector or mixed impurities appear to be harder to interpret, however it was noted in [20] that this problem is eliminated, and the correlators for all types of impurities can be expressed in a form similar to (3.2) and (3.3), if on the left hand sides of (3.2) and (3.3) we use a different notion of conjugation $\bar{\mathcal{O}}$ instead of $\mathcal{O}^\dagger$ [20, 26]. This different notion of operator conjugation is defined as *hermitian conjugation* followed by an *inversion* of the insertion point of the operator $x'_\mu = x_\mu/x^2$.

Let us now briefly review the transformation properties under an inversion of scalar and fermion operators [36]. A scalar operator $\mathcal{O}_\Delta(x)$ of conformal dimension Δ transforms as

$$\mathcal{O}_\Delta(x) \to \mathcal{O}'_\Delta(x') = x^{2\Delta}\mathcal{O}_\Delta(x) \quad , \quad x_\mu \to x'_\mu = \frac{x_\mu}{x^2} \ , \tag{3.4}$$

while vector or tensor operators (i.e. operator with vector impurities) pick a factor $J_{\mu\nu}(x) = \delta_{\mu\nu} - 2x_\mu x_\nu/x^2$ on the right hand side for each vector index of the operator. $J_{\mu\nu}(x)$ is the usual inversion tensor, in terms of which the Jacobian of the inversion is expressed $\partial x'_\mu/\partial x_\nu = J_{\mu\nu}(x)/x^2$. This prescription is essential in order to make vector Δ-BMN operators orthonormalisable, see section 2 of [20] for more details. Let us now concentrate on the fermionic operators which are of direct concern for this paper.[7] It is well known that, under conformal inversion, a Dirac spinor field ψ of dimension d transforms as [36]

$$\psi(x) \to \psi'(x') = \eta\,\frac{\hat{x}}{|x|}\,x^{2d}\,\psi(x) \ , \qquad \eta^4 = 1 \ , \tag{3.5}$$

where $\hat{x} = x_\mu\gamma^\mu$, and γ^μ are the Euclidean gamma matrices. In terms of Weyl spinors λ_α, $\bar{\xi}^{\dot\alpha}$, (3.5) implies

$$\lambda_\alpha(x) \to \lambda'_\alpha(x') \ = \ \eta\,\frac{(x\bar{\xi})_\alpha}{|x|}\,x^{2d} \ , \tag{3.6}$$

$$\bar{\xi}^{\dot\alpha}(x) \to \bar{\xi}^{\dot\alpha\prime}(x') \ = \ \eta\,\frac{(\bar{x}\lambda)^{\dot\alpha}}{|x|}\,x^{2d} \ , \tag{3.7}$$

where we set $x = x_\mu\sigma^\mu$, $\bar{x} = x_\mu\bar{\sigma}^\mu$. Hence, an operator of conformal dimension Δ with $f = p + q$ fermionic insertions transforms under inversion as:

$$\mathcal{O}^{\dot\alpha_1\ldots\dot\alpha_q}_{\alpha_1\ldots\alpha_p}(x) \ \to \ \eta^f\,x^{2\Delta}\,J_{\alpha_1\dot\beta_1}\cdots J_{\alpha_p\dot\beta_p}\cdots\bar{J}^{\dot\alpha_1\beta_1}\ldots\bar{J}^{\dot\alpha_q\beta_q}\,\mathcal{O}^{\dot\beta_1\ldots\dot\beta_p}_{\beta_1\ldots\beta_q}(x) \ , \tag{3.8}$$

where

$$J_{\alpha\dot\beta}(x) := \frac{x_{\alpha\dot\beta}}{|x|} \ , \qquad \bar{J}^{\dot\alpha\beta}(x) := \frac{\bar{x}^{\dot\alpha\beta}}{|x|} \ . \tag{3.9}$$

[7] The transformation under inversion of fermionic BMN operators has also been considered in [26].

The notion of conjugation [20], as ordinary hermitian conjugation followed by an inversion, can then applied to a generic operator with conformal dimension Δ with scalar, vector or fermion impurities; and the conjugated operator $\bar{\mathcal{O}}$ can then be written (schematically) as:

$$\bar{\mathcal{O}}_\Delta(x) \equiv x^{2\Delta}\, J \cdot \mathcal{O}^\dagger_\Delta(x) \ , \qquad (3.10)$$

where by J we mean the tensor product by the appropriate inversion operators $J_{\mu\nu}(x)$, for each vector index, and $J_{\alpha\dot{\alpha}}(x)$ (or $\bar{J}^{\dot{\alpha}\alpha}(x)$), for each spinor index. It was noticed in [20] that the advantage of this new conjugation resides in that the two-point function for scalar, vector and fermion Δ-BMN operators take all the same canonical form when the $\bar{\mathcal{O}}$ operator is employed:

$$\langle \bar{\mathcal{O}}_{\Delta_a}(x)\mathcal{O}_{\Delta_b}(0)\rangle = \delta_{ab} \ . \qquad (3.11)$$

The right hand side of (3.11) does not depend on x, and represents the overlap of the corresponding states in conformal $\mathcal{N}=4$ Super Yang-Mills.[8]

The operators of a generic basis in field theory will not enjoy the simple interpretation in terms of states given by (3.11), nevertheless we can always decompose these operators along the eigenvectors of the Δ-BMN basis. Specifically, for any operator basis $\tilde{\mathcal{O}}_\alpha$ such that

$$\tilde{\mathcal{O}}_\alpha = U_{\alpha\beta}\,\mathcal{O}_{\Delta_\beta} \ , \qquad (3.12)$$

where $U_{\alpha\beta}$ is a constant matrix, we can easily compute the overlap of the basis vectors: this is simply given by [10]

$$\langle \bar{\tilde{\mathcal{O}}}_\alpha(x)\tilde{\mathcal{O}}_\beta(0)\rangle = U_{\beta\gamma}U^\dagger_{\gamma\alpha} \equiv S_{\beta\alpha} \ . \qquad (3.13)$$

The previous relation is highly nontrivial: despite the fact that the operators $\tilde{\mathcal{O}}_\alpha$ do not have definite scaling dimensions, the overlap (3.13) does not depend on x! The key step in this procedure is to be found in the inversion procedure. Indeed, notice that in (3.4) the *full* conformal dimension Δ of the conformal BMN operator $\mathcal{O}_{\Delta_\alpha}$ is used in order to define the proper notion of conjugation. This expression will have in general a perturbative (and, in principle, nonperturbative) expansion in λ', which is eventually responsible for the simple form of (3.13) (and (3.11)).

A simple and practical way of calculating simultaneously the overlaps of states and the matrix elements of the anomalous dimension operator $\delta = \Delta - \Delta_{\rm cl}$ (where $\Delta_{\rm cl}$ is the canonical dimension in the free theory) was described in [10]. We will apply this procedure in the following, therefore we briefly outline its steps. First, one defines the barred-operator $\bar{\tilde{\mathcal{O}}}(x)$ as the Hermitian conjugation of $\tilde{\mathcal{O}}(x)$ followed by an inversion of the resulting operator, defined as if it was free, i.e. instead of the factor $x^{2\Delta}$ in (3.10) we put $x^{2\Delta_{\rm cl}}$, such that

$$\bar{\mathcal{O}}_\Delta(x) \equiv x^{2\Delta_{\rm cl}}\, J \cdot \mathcal{O}^\dagger_\Delta(x) \ . \qquad (3.14)$$

[8] A side comment: for two-impurity fermion BMN operators, an additional minus sign should be included in the definition of the hermitian conjugation of the operators in order to get the two-point functions normalised as in (3.11).

The two-point function takes now the form:

$$\begin{aligned}\langle\bar{\tilde{\mathcal{O}}}_\alpha(x)\tilde{\mathcal{O}}_\beta(0)\rangle &= U_{\beta\gamma}\, e^{\delta_\gamma \log(\Lambda x)^{-2}}\, U^\dagger_{\gamma\alpha} \\ &= S_{\beta\alpha} + T_{\beta\alpha}\log(\Lambda x)^{-2} + \mathcal{O}\left((\log(\Lambda x)^{-2})^2\right) , \end{aligned} \tag{3.15}$$

where we have expanded the full result in powers of $\log x^{-2}$. From (3.15) we can read off the overlap of the two states, defined as the zeroth-order term of the expansion,

$$S_{\beta\alpha} = U_{\beta\gamma}U^\dagger_{\gamma\alpha} , \tag{3.16}$$

as well as the matrix of anomalous dimensions in this basis, given by the first order term,

$$T_{\beta\alpha} = U_{\beta\gamma}\delta_\gamma U^\dagger_{\gamma\alpha} . \tag{3.17}$$

We now consider the original BMN basis, for which we have

$$\langle\bar{\mathcal{O}}_\alpha(x)\,\mathcal{O}_\beta\rangle = S_{\beta\alpha} + T_{\beta\alpha}\log x^{-2} + \cdots , \tag{3.18}$$

and relate this basis to the isomorphic to string basis via a transformation U as in (3.12):

$$\mathcal{O}^{\rm string}_\beta = U_{\beta\gamma}\mathcal{O}_\gamma \quad , \qquad \bar{\mathcal{O}}^{\rm string}_\alpha = \bar{\mathcal{O}}_\delta U^\dagger_{\delta\alpha} . \tag{3.19}$$

In the basis isomorphic to the natural string basis, we have

$$S^{\rm string} = \mathbb{1} = USU^\dagger , \qquad T^{\rm string} = UTU^\dagger . \tag{3.20}$$

Notice that S is a Hermitian, positive matrix, therefore $S^{-1/2}$ is well-defined.[9] S is then diagonalised by choosing

$$U := S^{-1/2}\cdot V , \tag{3.21}$$

where $V^\dagger V = \mathbb{1}$:

$$S \longrightarrow USU^\dagger = \mathbb{1} , \tag{3.22}$$

$$T \longrightarrow UTU^\dagger = V^\dagger(S^{-\frac{1}{2}}TS^{-\frac{1}{2}})V . \tag{3.23}$$

At this point, there is still an arbitrariness contained in V. This was fixed in [5, 8] by requiring that (3.1) holds and that the known three-string interaction vertex of the pp-wave light-cone string field theory [11, 12] is reproduced from gauge theory matrix elements involving BMN states (operators) with two scalar impurities. This condition implies $V = \mathbb{1}$. Hence, we conclude that the matrix of anomalous dimensions in the string basis is given by [8, 10]

$$\Gamma := T^{\rm string} = S^{-\frac{1}{2}}\,T\,S^{-\frac{1}{2}} . \tag{3.24}$$

This is the result we wanted to derive. The procedure for obtaining the matrix elements of the anomalous dimension operator (and hence of the dilatation operator in Yang-Mills) in the basis in field theory which is isomorphic to the natural string basis is therefore

[9] The matrix $S^{-\frac{1}{2}}$ also appears in [19] and [40].

clear: choose an arbitrary operator basis in Yang-Mills (in particular the simple basis originally considered by BMN in [1]), and compute two-point functions as in (3.15) (or (3.18)); work out the expressions for the matrices S and T; and, finally, apply (3.24) to derive the desired matrix elements in the isomorphic to string basis.

Eq. (3.24) was employed in [10] to explore and successfully test the bosonic (scalar and vector) sector of the BMN correspondence, finding precise agreement between the matrix elements in of the interacting string Hamiltonian in string field theory and the dual matrix elements (3.24) of the field theory dilatation operator. In the next sections we will pursue this programme and carry out a number of tests in the fermionic sector of the string and field theory. More precisely, we will show that, with the same choice of $V = \mathbb{1}$, the matrix elements of Γ between BMN operators with two fermion impurities precisely agree with the corresponding matrix elements of the interacting string Hamiltonian.

Other studies of the dilatation operator in $\mathcal{N} = 4$ Yang-Mills and its interpretation in quantum mechanical and integrable models can be found in the recent papers [37–46].

4 Matrix elements of H_{string} in string field theory

There are two equivalent ways to describe superstring interactions in string field theory, known as the $SO(8)$ and the $SO(4) \times SO(4)$ formalism. In the former approach, the three-string vertex in string field theory is built upon a state $|0\rangle$ with energy equal to 4μ. This state is therefore not the ground state, but it has the advantage that, as $\mu \to 0$, the $SO(8)$ construction flows smoothly to string field theory in flat space [47–50]. In the $SO(4) \times SO(4)$ formalism, the Hilbert space of states in string field theory is built on the true vacuum $|\text{v}\rangle$ of pp-wave string theory (see Appendix A for details). Remarkably, the two formalisms have been shown to be completely equivalent [23], hence it is only a matter of convenience which one to use. In this paper we will make use of the $SO(4) \times SO(4)$ vertex, since there it is more straightforward to compute string amplitudes involving fermionic oscillators.

The string amplitude has the form [11, 12][10]

$$\langle 1|\langle 2|\, H_{\text{string}}\, |3\rangle \;\; = \;\; \langle \Phi|\mathsf{P}|V_B\rangle|V_F\rangle \;, \qquad (4.1)$$

where $\langle\Phi| := \langle 1|\langle 2|\langle 3|'$ is the external three–string state, and $|V_B\rangle$ and $|V_F\rangle$ are the kinematical part of the bosonic and fermionic vertex, (B.6) and (B.7) respectively. Finally, the prefactor P is written in (B.9).

We will be interested in external states with fermionic impurities,

$$\beta^{\alpha\beta\,\dagger}_{n(\mathrm{r})}\,\beta^{\alpha'\beta'\,\dagger}_{-n(\mathrm{r})}\,|\text{v}\rangle_{\mathrm{r}} \;, \qquad (4.2)$$

[10]The reader is referred to Appendix A for more details on the string field theory vertex.

where the fermionic operators β's are related to the oscillators in the string basis by [14]

$$\beta_n = \frac{1}{\sqrt{2}}(b_n + ib_{-n}) , \quad \beta_{-n} = \frac{1}{\sqrt{2}}(b_n - ib_{-n}) . \tag{4.3}$$

Specifically, we will compute matrix elements of the form

$$\mathcal{H}^{r\alpha,s\beta;\,m,J}_{r'\alpha',s'\beta';n,J,y} := \frac{1}{\mu}\langle \mathcal{T}^{J,y}_{r'\alpha',s'\beta';\,n}|H_{\rm string}|\mathcal{O}^{r\alpha,s\beta;m}_J\rangle , \tag{4.4}$$

for all possible values of the indices. After a lengthy but straightforward algebra, we obtain:

$$\begin{aligned}\langle {\rm v}|\beta_{\alpha\beta,-m(3)}\beta_{\gamma\delta,m(3)}\beta_{\alpha'\beta',n(1)}\beta_{\gamma'\delta',-n(1)}|H_3\rangle &= C_{\rm norm}\frac{\beta+1}{3\pi^2\mu}\sin^2\pi m\beta \qquad (4.5)\\ \cdot\Big[\epsilon_{\alpha'\alpha}\epsilon_{\gamma'\gamma}(\epsilon_{\delta\beta}\epsilon_{\delta'\beta'}+\epsilon_{\delta\beta'}\epsilon_{\delta'\beta}) + \epsilon_{\gamma'\alpha}\epsilon_{\alpha'\gamma}&(\epsilon_{\delta\beta}\epsilon_{\delta'\beta'}+\epsilon_{\delta\delta'}\epsilon_{\beta\beta'}) - \epsilon_{\alpha\gamma}\epsilon_{\alpha'\gamma'}(\epsilon_{\beta\beta'}\epsilon_{\delta\delta'}-\epsilon_{\delta\beta'}\epsilon_{\delta'\beta})\Big] ,\end{aligned}$$

where $C_{\rm norm}$ is given in (B.14). An expression similar to (4.5) holds when the fermions in (4.5) have both dotted indices. Notice that from (4.5) it follows that
a. the string amplitude vanishes whenever a fermion appears more than once, whereas
b. it is nonvanishing when all fermions are different, and gives always the same result (up to a minus sign).

It is more illuminating to write (4.5) for a few basic cases:

$$\begin{aligned}\mathcal{H}^{12,11;\,m,J}_{11,12;n,J,y} &:= \mu^{-1}\langle {\rm v}|\beta^{12}_{-m(3)}\beta^{11}_{m(3)}\beta_{11,n(1)}\beta_{12,-n(1)}|H_3\rangle \qquad (4.6)\\ &= -\mu^{-1}\langle {\rm v}|\beta_{21,-m(3)}\beta_{22,m(3)}\beta_{11,n(1)}\beta_{12,-n(1)}|H_3\rangle = -\lambda' C_{\rm norm}\frac{\beta+1}{\pi^2}\sin^2\pi m\beta ,\end{aligned}$$

$$\begin{aligned}\mathcal{H}^{11,11;\,m,J}_{11,11;n,J,y} &:= \mu^{-1}\langle {\rm v}|\beta^{11}_{-m(3)}\beta^{11}_{m(3)}\beta_{11,n(1)}\beta_{11,-n(1)}|H_3\rangle\\ &= \mu^{-1}\langle {\rm v}|\beta_{22,-m(3)}\beta_{22,m(3)}\beta_{11,n(1)}\beta_{11,-n(1)}|H_3\rangle = 0 , \qquad (4.7)\end{aligned}$$

$$\begin{aligned}\mathcal{H}^{22,11;\,m,J}_{11,22;n,J,y} &:= \mu^{-1}\langle {\rm v}|\beta^{22}_{-m(3)}\beta^{11}_{m(3)}\beta_{11,n(1)}\beta_{22,-n(1)}|H_3\rangle \qquad (4.8)\\ &= \mu^{-1}\langle {\rm v}|\beta_{11,-m(3)}\beta_{22,m(3)}\beta_{11,n(1)}\beta_{22,-n(1)}|H_3\rangle = 0 ,\end{aligned}$$

$$\begin{aligned}\mathcal{H}^{21,11;\,m,J}_{11,21;n,J,y} &:= \mu^{-1}\langle {\rm v}|\beta^{21}_{-m(3)}\beta^{11}_{m(3)}\beta_{11,n(1)}\beta_{21,-n(1)}|H_3\rangle \qquad (4.9)\\ &= -\mu^{-1}\langle {\rm v}|\beta_{12,-m(3)}\beta_{22,m(3)}\beta_{11,n(1)}\beta_{21,-n(1)}|H_3\rangle = \lambda' C_{\rm norm}\frac{\beta+1}{\pi^2}\sin^2\pi m\beta .\end{aligned}$$

We can directly compare the expressions (4.6)-(4.9) to the analogous matrix elements obtained from the three-string vertex of [11,12] for scalar and for vector and mixed (scalar-vector) BMN states:

$$\frac{1}{\mu}\langle \mathcal{T}^{J,y}_{ij,n}|H_{\rm string}|\mathcal{O}^J_{ij,m}\rangle = -\lambda' C_{\rm norm}\frac{\beta+1}{\pi^2}\sin^2\pi m\beta , \tag{4.10}$$

$$\frac{1}{\mu}\langle \mathcal{T}^{J,y}_{i\mu,n}|H_{\rm string}|\mathcal{O}^J_{i\mu,m}\rangle \;=\; 0\;, \tag{4.11}$$

$$\frac{1}{\mu}\langle \mathcal{T}^{J,y}_{\mu\nu,n}|H_{\rm string}|\mathcal{O}^J_{\mu\nu,m}\rangle \;=\; \lambda'\, C_{\rm norm}\frac{\beta+1}{\pi^2}\sin^2\pi m\beta\;. \tag{4.12}$$

Notice hat the amplitude in (4.6) is identical to the amplitude (4.10) involving two scalars of different flavour, whereas that in (4.9) is equal to the amplitude (4.12) involving two different vectors. Similar considerations apply to the vanishing of the mixed amplitude (4.11) and (4.7), (4.8).

The equality of the string amplitude between two BMN states with scalar impurities and two BMN states with fermion impurities had already been derived, in the $SO(8)$ formalism, in [14]. We also notice that our amplitudes, derived in the $SO(4)\times SO(4)$ formalism, precisely coincide with those of [14].

5 Matrix elements of Δ in $\mathcal{N}=4$ Yang-Mills

In [10], a general technique was devised for deriving the matrix of overlaps S (3.16), the matrix of anomalous dimension T (3.17), and hence the desired matrix elements of the SYM dilatation operator in the isomorphic to string basis, (3.24), from the coefficients of the *three*-point functions of BMN operators. Here we report the results of the analysis of [10], referring the curious reader to section 3 of that paper for more details.[11]

The matrices S and T have an expansion in powers of g_2, but in our analysis we will only need their expressions up to and including $\mathcal{O}(g_2)$ terms. We will also work at one loop in the effective 't Hooft coupling λ'. Notice that the matrix T is of $\mathcal{O}(\lambda')$, whereas S is of $\mathcal{O}(1)$. In this case, (3.15) is simply

$$\langle \mathcal{O}_\alpha(0)\bar{\mathcal{O}}_\beta(x)\rangle = S_{\alpha\beta} + T_{\alpha\beta}\log(x\Lambda)^{-2}\;. \tag{5.1}$$

Let us focus on the following *three*-point correlators,

$$G(x_1,x_2,x_3) = \langle \mathcal{O}^{y\cdot J}_{AB,n}(x_1)\mathcal{O}^{(1-y)\cdot J}_{\rm vac}(x_2)\bar{\mathcal{O}}^J_{AB,m}(x_3)\rangle\;, \tag{5.2}$$

where A can be a scalar, vector or fermion index, and $A\neq B$. On general grounds, these three-point function take the form [7, 31–34]

$$G(x_1,x_2,x_3) = g_2 C_{m,ny}\left[1-\lambda'\left(a_{m,ny}\log(x_{31}\Lambda)^2 + b_{m,ny}\log(x_{32}x_{31}\Lambda/x_{12})\right)\right]\;, \tag{5.3}$$

where $g_2 C_{m,ny}$ is the tree-level contribution, with

$$C_{m,ny} := \frac{\sqrt{(1-y)/y}\,\sin^2(\pi m y)}{\sqrt{J}\,\pi^2(m-n/y)^2}\;, \tag{5.4}$$

[11] Our notation and conventions in Yang-Mills are reviewed in Appendix A.

and the coefficients $a_{m,ny}$ and $b_{m,ny}$ must be calculated in perturbation theory at $\mathcal{O}(\lambda')$. The *two*-point function $\langle \mathcal{T}^{J,y}_{AB,n}(0)\, \bar{\mathcal{O}}^J_{AB,m}(x)\rangle$ can easily be derived from (5.2) by simply setting $x_{13} = x_{23} = x$ and $x_{12} = \Lambda^{-1}$ [34],

$$\langle \mathcal{T}^{J,y}_{AB,n}(0)\, \bar{\mathcal{O}}^J_{AB,m}(x)\rangle = g_2 C_{m,ny} \left[1 + \lambda' \left(a_{m,ny} + b_{m,ny}\right) \log(x\Lambda)^{-2}\right] \ . \tag{5.5}$$

The analysis of section 3 of [10] then showed that the matrices S and T are then given, up to $\mathcal{O}(g_2)$, by the following expressions:

$$S \ = \ \begin{pmatrix} \delta_{mn} & g_2\, C_{m,qz} \\ g_2\, C_{py,n} & \delta_{pq} \end{pmatrix} \ + \ \mathcal{O}(g_2^2) \ = \ \mathbb{1} + g_2 s + \mathcal{O}(g_2^2) \ , \tag{5.6}$$

$$(5.7)$$

$$\begin{aligned} T \ &= \ \lambda' \begin{pmatrix} m^2\, \delta_{mn} & g_2\, C_{m,ny}\, (a+b)_{m,qz} \\ g_2\, C_{py,n}\, (a+b)_{py,n} & (p^2/y^2)\, \delta_{pq}\delta_{yz} \end{pmatrix} \ + \ \mathcal{O}(g_2^2) \qquad (5.8) \\ &\equiv \ d \ + \ g_2 t \ + \ \mathcal{O}(g_2^2) \ , \end{aligned}$$

with

$$d \ = \ \lambda' \begin{pmatrix} m^2\, \delta_{mn} & 0 \\ 0 & (p^2/y^2)\, \delta_{pq}\delta_{yz} \end{pmatrix} \ , \tag{5.9}$$

$$t \ = \ \lambda' \begin{pmatrix} 0 & C_{m,ny}\, (a+b)_{m,qz} \\ C_{py,n}\, (a+b)_{py,n} & 0 \end{pmatrix} \ . \tag{5.10}$$

It then follows that

$$S^{-1/2} = \mathbb{1} - g_2(s/2) + \mathcal{O}(g_2^2) \tag{5.11}$$

diagonalises S at $\mathcal{O}(g_2)$, and hence (3.24) leads to

$$\Gamma \ = \ d + g_2 \Big[t - \frac{1}{2}\{s\, ,\, d\}\Big] \ . \tag{5.12}$$

We now need to compute the explicit expressions for a^y_{mn} and b^y_{mn} for the fermion case. First, it was observed in [10] that, at $\mathcal{O}(\lambda')$ in planar perturbation theory, the coefficient a^y_{mn} is simply given by the $\mathcal{O}(\lambda')$ anomalous dimension of the "small" BMN operator at x_1. In accordance with supersymmetry, it turns out that the $\mathcal{O}(\lambda')$ anomalous dimension of BMN operators with two arbitrary impurities is given by

$$a_{m,ny} \ = \ \frac{n^2}{y^2} \ , \tag{5.13}$$

independently of the type of impurity considered. This result was first obtained for the case of two scalar impurities in [1], for one scalar and one vector impurity in [29], for two vector impurities in [52]. We have also explicitly derived (5.13) in the fermion case with a perturbative calculation in Yang-Mills.

We now move on to consider $b_{m,ny}$. From (5.3), we see that $b_{m,ny}$ is the coefficient which multiplies the $\log x_{12}$ contribution in the three-point function $G(x_1, x_2, x_3)$ in (5.2), where the "large" ("small") BMN operator $\bar{\mathcal{O}}^J_{AB,m}$ ($\mathcal{O}^{y\cdot J}_{AB,n}$) is inserted at x_3 (x_1), and the vacuum operator $\mathcal{O}_{\rm vac}$ at point x_2. Hence, the tactic we will follow in the next section will consist in computing the $\log x_{12}$ term of the three-point function[12] $G(x_1, x_2, x_3)$.

Let us quote here the result for $b_{m,ny}$ in the case of scalar, mixed, or vector BMN operators in (5.2) [10]:

$$[b_{m,ny}]_{\rm scalar} = m^2 - \frac{mn}{y} , \tag{5.14}$$

$$[b_{m,ny}]_{\rm scalar-vector} = \frac{1}{2}\left(m^2 - \frac{n^2}{y^2}\right) , \tag{5.15}$$

$$[b_{m,ny}]_{\rm vector} = -\frac{n^2}{y^2} + \frac{mn}{y} . \tag{5.16}$$

We conclude this section with one important comment, which anticipates our results for the fermions to be derived in the next section: we will show that, for fermion BMN operators, the coefficients $b_{m,ny}$ for the various representations precisely take one of the three expressions (5.14)-(5.16).

5.1 The three-point function of fermion BMN operators

In the previous section we explained how to obtain the matrix elements of the SYM dilatation operator in any arbitrary basis of SYM operators, and specifically in the isomorphic to string basis. Here we present the field theory computation of the coefficients $b_{m,ny}$ appearing in (5.3), from which the coefficient of the conformal three-point function of two-impurity fermion BMN operators can also be derived. The matrix elements (5.12) of the SYM dilatation operators in the natural string basis will then be obtained using the expressions for $b_{m,ny}$ and (5.6)-(5.10) and (5.13). The reader not interested in the details of this calculations may skip a few pages, and proceed directly to section 6.

Let us consider the three-point function of the operators in (2.8), i.e.

$$\langle \mathcal{O}^{y\cdot J}_{3\alpha,4\beta;n}(x_1)\, \mathcal{O}^{(1-y)\cdot J}_{\rm vac}(x_2)\, \bar{\mathcal{O}}^J_{3\dot{\alpha},4\dot{\beta};m}(x_3)\rangle . \tag{5.17}$$

[12] It was shown in [10,53] that from the knowledge of the coefficient $b_{m,ny}$ it is also possible to determine the coefficient of the *conformal* three-point function, despite the fact that, due to mixing effects [31,54], the correlator (5.2) does not take the conformal form (3.3), since the original BMN operators in (5.3) are not conformal primaries for $g_2 \neq 0$.

We start off by evaluating the Feynman diagrams which originate from the pure BMN parts of both the barred and unbarred operators. These diagrams are represented in Figure 1, where we draw only the diagrams where the impurities $\lambda_{4\alpha}$ and $\bar{\lambda}_{4\dot{\beta}}$ participate in the interactions, and the other impurity propagates freely. In diagram $1a$ (type I), the interacting impurity goes across, while in $1b$ the interacting impurity goes straight (type II). The latter diagram has a minus sign relative to the former from the Yukawa vertex (see (A.3)). The result for the type I diagram is, concentrating on the interacting part:

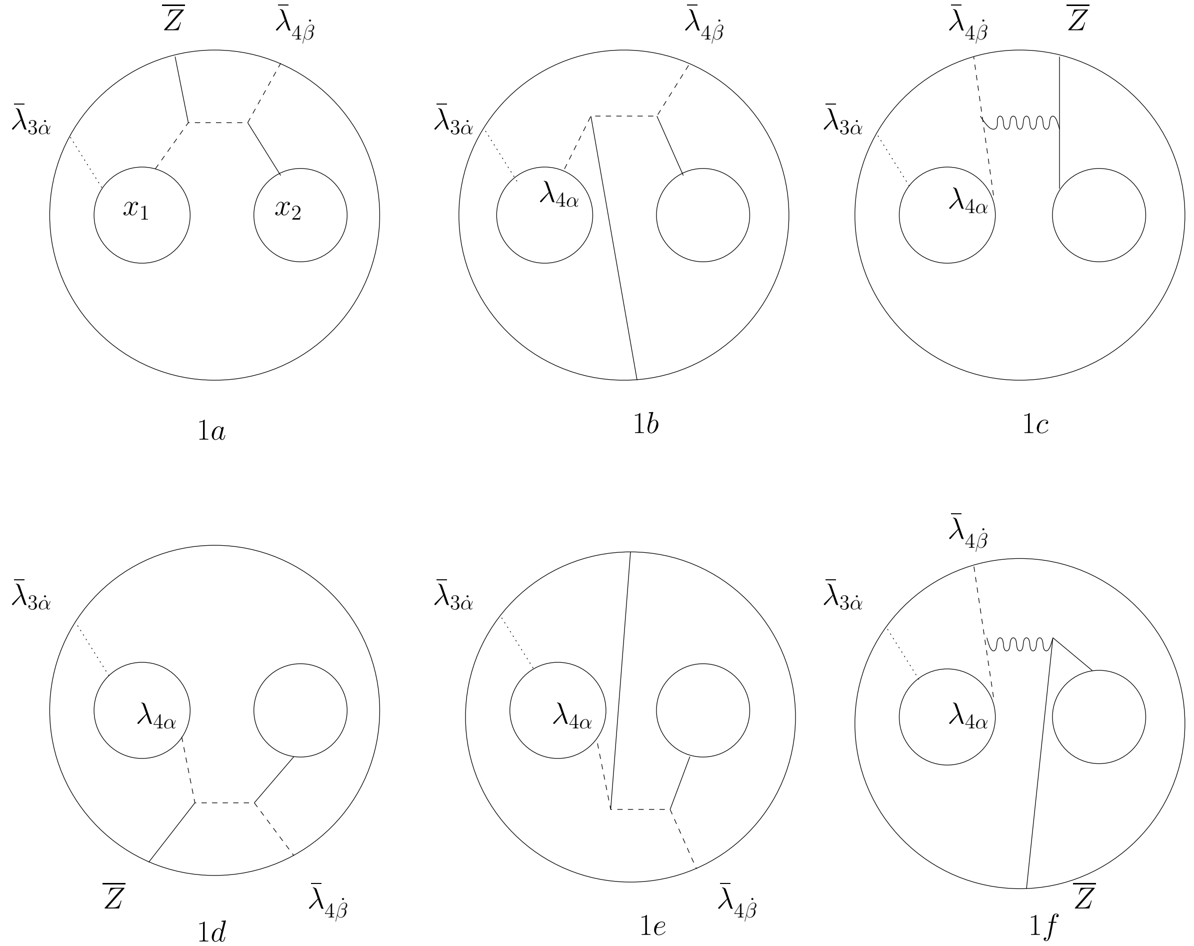

Figure 1: *Feynman diagrams from the pure BMN parts: of type I (in 1a and 1d) and type II (1b and 1e). Diagrams 1d, 1e, are the mirrors of 1a, 1b. Diagrams 1d and 1e have phase factors which are the complex conjugate of those of 1a and 1b. The gluon interaction diagrams in 1c and 1f have the same BMN factor and cancel each other.*

$$J^I_{\alpha\dot{\beta}} = (\partial^1_\nu \sigma^\nu_{\alpha\dot{\alpha}})(-\sqrt{2} i \epsilon^{\dot{\psi}\dot{\alpha}})\sigma^\rho_{\chi\dot{\psi}}(-\sqrt{2} i \epsilon^{\beta\chi})(-\partial^3_\mu \sigma^\mu_{\beta\dot{\beta}}) H_{\rho 1423} \ , \tag{5.18}$$

where

$$\begin{aligned} H_{\rho1423} &= \int d^4x\, d^4y\ \Delta(x_1 - x)\Delta(x_4 - x)\left[-\partial^x_\rho \Delta(x-y)\right]\Delta(x_3 - y)\Delta(x_4 - y) \\ &= -(\partial^1_\rho + \partial^4_\rho) H_{1432}\ . \end{aligned} \tag{5.19}$$

Notice that in order to be able to perform the inversion in the conjugated operator, we momentarily split the insertion points of the fermion and the Z impurity, respectively x_3 and x_4. The last equality is obtained after integrating by parts with respect to x and then converting the x derivative acting on $\Delta(x_1 - x)$ and $\Delta(x_4 - x)$ to derivatives with respect to x_1 and x_4, respectively. The partial derivatives in (5.18) come from the fermion propagator $S_{\alpha\dot{\alpha}}(x) = -\partial_{\alpha\dot{\alpha}}\Delta(x)$, where $\partial_{\alpha\dot{\alpha}} := \partial_\mu \sigma^\mu_{\alpha\dot{\alpha}}$.

Using now (A.9) and ignoring the ϵ term which eventually does not contribute to the $\log x_{12}^2$ term, one obtains:

$$J^I_{\alpha\dot{\beta}} \;=\; 2\big[\partial^3_{\alpha\dot{\beta}} J_A + \partial^1_{\alpha\dot{\beta}} J_B + (\partial^1 + \partial^4)_{\alpha\dot{\beta}} J_C\big] \;, \tag{5.20}$$

where

$$\begin{aligned}
J_A &= \partial^1 \cdot (\partial^1 + \partial^4)\, H_{1423} \;, &(5.21)\\
J_B &= \partial^3 \cdot (\partial^1 + \partial^4)\, H_{1423} \;, &(5.22)\\
J_C &= -\partial^3 \cdot \partial^1\, H_{1423} \;. &(5.23)
\end{aligned}$$

The explicit expressions for J_A, J_B and J_C are worked out in (E.8)-(E.10). After some algebra, one realises that the only non-zero contribution to $J^I_{\alpha\dot{\beta}}$ comes from the term involving J_A. Keeping track of the relevant to us terms which contain $\log x_{12}^2$, we get:

$$J^I_{\alpha\dot{\beta}} \;=\; \frac{1}{2^3\pi^2} \log x_{12}^2\, \Delta(x_4 - x_1) \left[\partial^4_{\alpha\dot{\beta}} \Delta(x_4 - x_1)\right] \;. \tag{5.24}$$

We note that (5.24) is precisely of the form of (a first-order correction to) two freely propagating fields, one Z boson and one fermion, as it is expected. In order to make the comparison with the string amplitude, we should now apply the inversion on the conjugated operator, that is on the scalar Z field and the interacting fermion. For the scalar field this is rather trivial: according to (3.4), one has to multiply $J^I_{\alpha\dot{\beta}}$ by x_4^2. For the fermion, (3.6) instructs us to multiply by $\eta x_4^2 \bar{\sigma}_\mu^{\dot{\beta}\beta} x_4^\mu$. Taking into account the identity $(\sigma_\mu)_{\alpha\dot{\beta}} \bar{\sigma}_\nu^{\dot{\beta}\beta} x^\mu x^\nu = \delta_\alpha^\beta\, x^2$ one obtains:

$$-\,4\Big(\frac{g^2}{2}\Big)^3 P_I \frac{\log x_{12}^2}{2^8\pi^6}\, \eta^2 \delta_\alpha^\beta \;. \tag{5.25}$$

The overall factor $(g^2/2)^3$ comes from the insertion of two vertices, which give $(2/g^2)^2$, and five propagators, which give $(g^2/2)^5$. P_I is the phase factor associated with the diagrams of type I, and is explicitly calculated in (C.2). In order to obtain the final result for the type I diagrams, we have still to multiply (5.25) by a factor of 2 from the free contraction of the non-interacting impurity, and by a factor of $1/4$ from the normalisation of the two fermion BMN operators. Doing so, and setting $\eta^2 = 1$ we get:

$$\text{type I} - \text{fermions}: \qquad -2\Big(\frac{g^2}{2}\Big)^3 P_I \frac{\log x_{12}^2}{2^8\pi^6} \delta_\alpha^\beta \;. \tag{5.26}$$

Notice that this result is precisely the same as the result for the case of two different scalar impurities. Similarly, one gets, for the type II diagrams:

$$\text{type II} - \text{fermions}: \qquad 2\Big(\frac{g^2}{2}\Big)^3 P_{II}\frac{\log x_{12}^2}{2^8\pi^6}\delta_\alpha^\beta\ . \tag{5.27}$$

To the type II diagrams is associated the phase factor P_{II} of (C.2). Moreover, in order to get the final expression for the coefficient $b_{m,ny}$, one should also include the diagrams where the other impurity, λ_3, participates in the interaction.

If this was the whole story, we would conclude that three-point functions of fermions take the same form as the three-point functions for scalars. But the correct expressions for BMN operators often contain compensating terms, and so is the case for the operator in (2.8). Importantly, these compensating terms do contribute to the three-point functions and must be taken into account. In our case, the compensating terms affect the fermion operators that have a projection in the $(\mathbf{1},\mathbf{3}^+)$ representation (see the discussion after (2.14)), and the corresponding contribution is important and we will now compute it.

In Figure 2 we draw the Feynman diagrams obtained by taking the compensating term on the right hand side of (2.8) for the operator sitting at x_1 . To the diagrams $2a$ and $2c$ a BMN phase factor equal to 1 is associated. From the first diagram in Figure 2 we get:[13]

$$I_{\hat\alpha\hat\gamma,\dot\alpha\dot\beta} \;=\; -2\cdot(-1)^3\partial_\mu^4\sigma^\mu_{\beta\dot\beta}\partial_\nu^3\sigma^\nu_{\alpha\dot\alpha}(-i\bar\sigma_\rho^{\dot\gamma\beta})\sigma^\tau_{\gamma\dot\gamma}(-\sqrt{2}i\epsilon^{\gamma\alpha})\partial_\delta^1(\sigma_{\delta\rho})_{\hat\gamma}^{\hat\beta}\epsilon_{\hat\alpha\hat\beta}(-H_{\tau1432})\ . \tag{5.28}$$

The factor of $(-1)^3$ comes from three propagators, while the factor of 2 arises from the two terms in the field strength $F_{\rho\delta}$. The $\sigma_{\rho\delta}$ is related to the compensating term of the operator at x_1 while the $\bar\sigma_\rho$ matrix comes from the gluon-fermion interaction vertex. Finally, the minus sign in front of (5.28) comes from Wick contracting fermions.

We can now elaborate (5.28) using the completeness relation (A.8), to obtain:

$$I_{\hat\alpha\hat\gamma,\dot\alpha\dot\beta} = 2\sqrt{2}\big[(\sigma^\delta\bar\sigma^\tau\sigma^\nu)_{\hat\gamma\dot\alpha}\sigma^\mu_{\hat\alpha\dot\beta} + (\sigma^\delta\bar\sigma^\tau\sigma^\nu)_{\hat\alpha\dot\alpha}\sigma^\mu_{\hat\gamma\dot\beta}\big]\,\partial_\mu^4\partial_\nu^3\partial_\delta^1\,(-H_{\tau,1432})\ . \tag{5.29}$$

Using (A.9), and discarding the irrelevant ϵ-terms as before, we get:

$$I_{\hat\alpha\hat\gamma,\dot\alpha\dot\beta} = 2\sqrt{2}\partial^4_{\hat\alpha\dot\beta}\big(\partial^3_{\hat\gamma\dot\alpha}J_A + \partial^1_{\hat\gamma\dot\alpha}J_B + (\partial^1+\partial^4)_{\hat\gamma\dot\alpha}J_C\big) + \hat\alpha \longleftrightarrow \hat\gamma\ . \tag{5.30}$$

Similarly to the diagrams in Figure 1, the $\log x_{12}^2$ contributions from J_B and J_C cancel out, and we are left with:

$$I_{\hat\alpha\hat\gamma,\dot\alpha\dot\beta} \;=\; 8\sqrt{2}\frac{1}{2^8\pi^6}\log x_{12}^2\,(\sigma^\mu_{\hat\alpha\dot\beta}\sigma^\nu_{\hat\gamma\dot\alpha} + \sigma^\mu_{\hat\gamma\dot\beta}\sigma^\nu_{\hat\alpha\dot\alpha})\frac{x_3^\mu x_3^\nu}{(x_3^2)^4}\ . \tag{5.31}$$

[13] As in [10], the diagrams where the compensating term is taken in the external operator, or both in the external and internal operator, do not contribute.

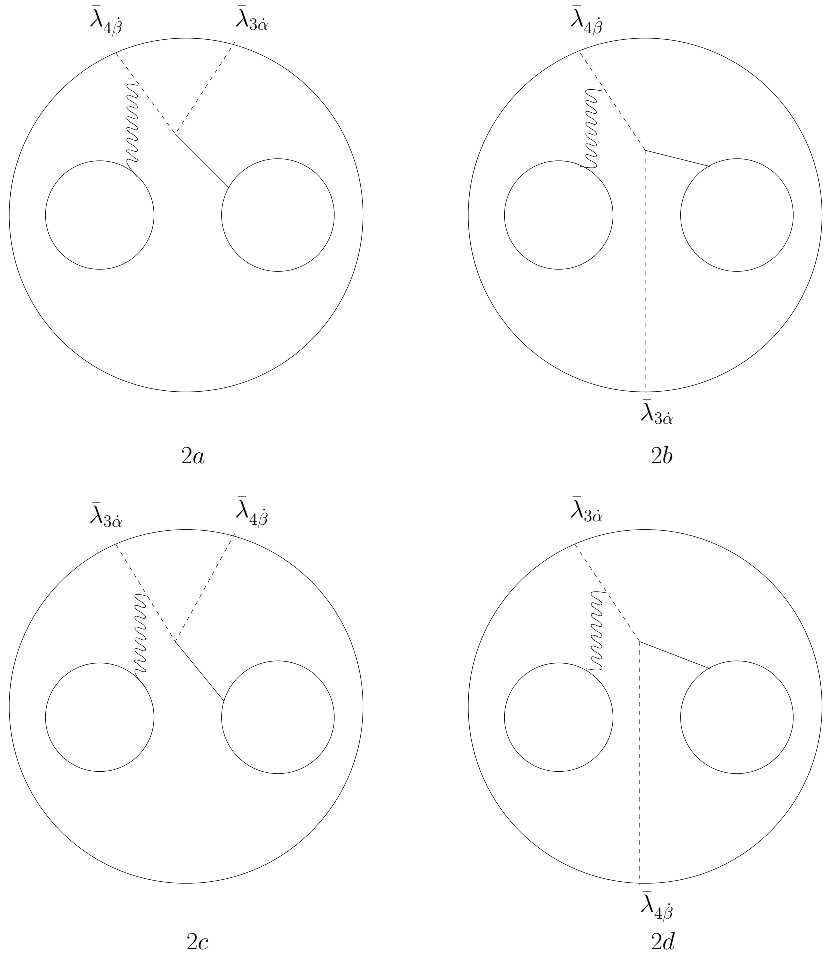

Figure 2: *Gluon emission diagrams originating from the compensating term in the internal operator. The gluon is absorbed by the fermion field. There are also mirror diagrams, not drawn in this figure.*

Now we perform the inversion for the fermions, thus arriving at:

$$I^{\alpha\beta}_{\hat{\alpha}\hat{\gamma}} \;=\; 8\sqrt{2}\frac{1}{2^8\pi^6}\log x_{12}^2\,(\delta^{\alpha}_{\hat{\alpha}}\delta^{\beta}_{\hat{\gamma}} \;+\; \delta^{\alpha}_{\hat{\gamma}}\delta^{\beta}_{\hat{\alpha}})\;. \tag{5.32}$$

The calculation for the diagram in Figure $2c$ (where the other fermion absorbs the gluon) proceeds in a similar fashion to that of Figure $2a$, giving the same result as in (5.32) but with α and β swapped. Since the expression for $I^{\alpha\beta}_{\hat{\alpha}\hat{\gamma}}$ is symmetric under this exchange, the results is the same as for the diagram in Figure $2a$. We should also not forget to multiply our result by $1/4$ due to the normalisation of the operators ($1/2$ for each operator), and by a factor of $-\sqrt{2}/4$ coming from the compensating term on the right hand side of (2.8). Taking also account the powers of $g^2/2$ associated with vertices and propagators, we finally get the result for the sum of diagrams $2a$ and $2c$:

$$-\,2\left(\frac{g^2}{2}\right)^3\frac{1}{2^8\pi^6}\log x_{12}^2\,(\delta^{\alpha}_{\hat{\alpha}}\delta^{\beta}_{\hat{\gamma}} \;+\; \delta^{\alpha}_{\hat{\gamma}}\delta^{\beta}_{\hat{\alpha}})\;. \tag{5.33}$$

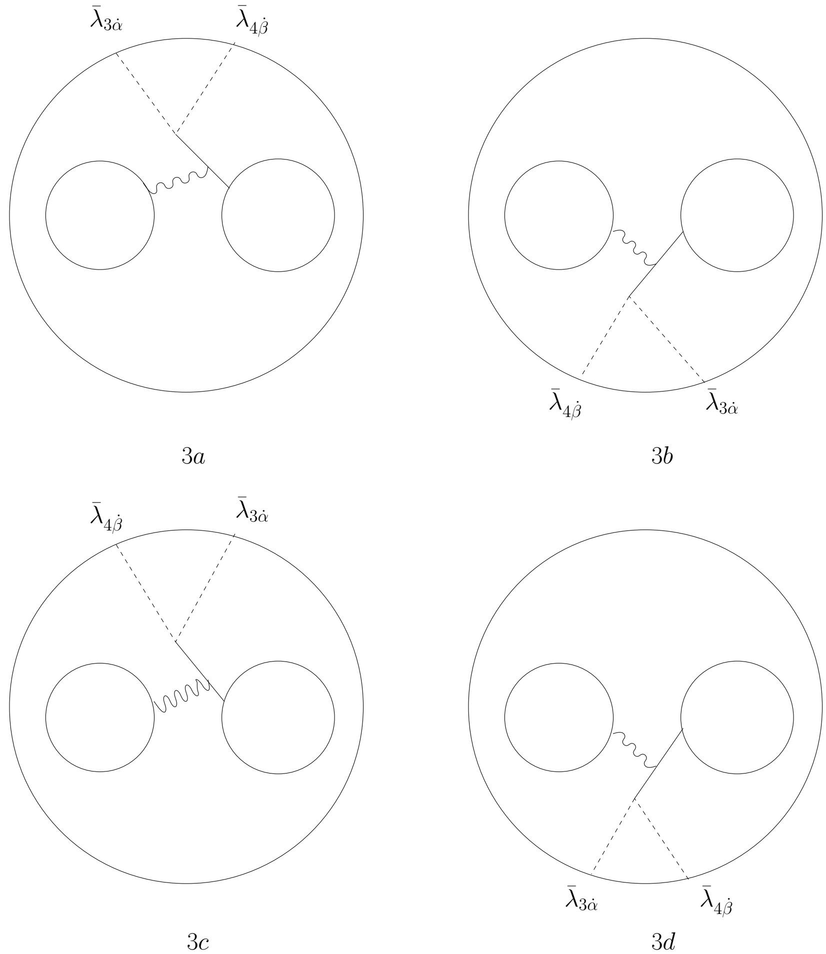

Figure 3: *Gluon emission diagrams originating from the compensating term in the internal operator. The gluon is absorbed by the Z field.*

The diagrams in Figure $2b$ and $2d$ can be computed in a similar way. Notice that they have a relative minus sign compared to $2a$ and $2c$, and a different BMN factor $\bar{q}^{J_1+1}$.

For $\alpha \neq \beta$, the result of (5.33) is exactly the same as the result obtained from the compensating diagrams for the case of BMN operators with *mixed* impurities (one vector and one scalar impurity). We addressed this case in section 4 of [10]. Furthermore, for $\alpha = \beta$ we get the same result as that of the compensating diagrams of BMN operators with *vector* impurities, i.e. twice that of the mixed case (see [10, 20]). From (5.14)-(5.16), one can easily work out the contributions of the compensating terms alone for the mixed and vector case,

$$[b_{m,ny}]^{\rm c.t.}_{\rm mixed} = [b_{m,ny}]_{\rm mixed} - [b_{m,ny}]_{\rm scalar} = -\frac{1}{2}\Big(m - \frac{n}{y}\Big)^2 , \tag{5.34}$$

$$\begin{aligned} [b_{m,ny}]^{\rm c.t.}_{\rm vector} &= [b_{m,ny}]_{\rm vector} - [b_{m,ny}]_{\rm scalar} = -\Big(m - \frac{n}{y}\Big)^2 \\ &= 2\ [b_{m,ny}]^{\rm c.t.}_{\rm mixed} \ . \end{aligned} \tag{5.35}$$

Finally, let us note that at the order we are working, the diagrams of Figure 3 are also present. However, the diagram in $3a$ cancels against that in $3b$ because of the relative minus sign associated with the vertex where the gluon is absorbed; and, similarly, $3c$ cancels against $3d$. Therefore the net contribution of the diagrams in Figure 3 is zero.

We conclude by summarising the results of this section.
a. The contribution of the pure BMN part of the operators involved in the three-point function (5.17) is precisely the same obtained for *scalar* BMN operators. Hence the corresponding coefficient $b_{m,ny}$ is given by (5.14).
b. The previous remark also implies that, when no compensating term is present in the expression for the BMN operator considered, the result (5.14) gives the full answer.
c. When a compensating term appears, it contributes precisely as the compensating term of the mixed (scalar-vector) case when $\alpha \neq \beta$, of the vector case for $\alpha = \beta$. The corresponding expressions for $b_{m,ny}$ are then (5.15) and (5.16), respectively.

6 Testing the BMN correspondence in the fermion sector

We can now apply the results derived in the previous section to test the pp-wave/SYM correspondence in the fermionic sector. In particular, we will reproduce in the gauge theory the three-string amplitudes for the following flavour-conserving processes:

$$(\lambda_{31} \dots \lambda_{32})_m \longrightarrow (\lambda_{31} \dots \lambda_{32})_n + \text{vac.} \ , \tag{6.1}$$

$$(\lambda_{31} \dots \lambda_{31})_m \longrightarrow (\lambda_{31} \dots \lambda_{31})_n + \text{vac.} \ , \tag{6.2}$$

$$(\lambda_{31} \dots \lambda_{42})_m \longrightarrow (\lambda_{31} \dots \lambda_{42})_n + \text{vac.} \ , \tag{6.3}$$

$$(\lambda_{31} \dots \lambda_{41})_m \longrightarrow (\lambda_{31} \dots \lambda_{41})_n + \text{vac.} \ . \tag{6.4}$$

A few comments are in order.

1. First, it does not take long to realise that these cases actually cover all the irreducible representations of the two-impurity fermion BMN operators, where the two impurities are Weyl spinor of the same chirality (see (2.16) and (2.17)).

2. The case where the two impurities have opposite chirality, $\lambda_{r\alpha}$ and $\bar{\lambda}_{\dot{r}\dot{\beta}}$ (see (2.18)), can of course be treated with a similar technique. Notice that, in that case, a compensating term containing $(D_\mu \phi^i)\sigma^\mu_{\alpha\dot{\beta}}$ will always be present in the precise expression of the BMN operator containing the $\lambda_{r\alpha}$, $\bar{\lambda}_{\dot{r}\dot{\beta}}$ impurities, in agreement with the fact that the right hand side of (2.18) contains only one irreducible representation.

3. Finally, notice that the operators taking part in the first two processes (6.1) and (6.2) do not have compensating terms, i.e. they do not have a projection onto the $(\mathbf{1},\mathbf{3}^+)$ representation of $SO(4)\times SO(4)$, in contradistinction with the operators taking part in the remaining last two, (6.3) and (6.4).

Let us write down the string amplitudes corresponding to the processes of (6.1)-(6.4), taking into account that in the three-string vertex of [11,12,14,23] all the external states are written as ket states:

$$
\begin{aligned}
\mu^{-1}\langle \mathrm{v}|\, \beta^{12}_{-m(3)}\, \beta^{11}_{m(3)}\, \beta_{11,n(1)}\, \beta_{12,-n(1)}\, |H_3\rangle &\equiv A_1 \ , \\
\mu^{-1}\langle \mathrm{v}|\, \beta^{11}_{-m(3)}\, \beta^{11}_{m(3)}\, \beta_{11,n(1)}\, \beta_{11,-n(1)}\, |H_3\rangle &\equiv A_2 \ , \\
\mu^{-1}\langle \mathrm{v}|\, \beta^{22}_{-m(3)}\, \beta^{11}_{m(3)}\, \beta_{11,n(1)}\, \beta_{22,-n(1)}\, |H_3\rangle &\equiv A_3 \ , \\
\mu^{-1}\langle \mathrm{v}|\, \beta^{21}_{-m(3)}\, \beta^{11}_{m(3)}\, \beta_{11,n(1)}\, \beta_{21,-n(1)}\, |H_3\rangle &\equiv A_4 \ .
\end{aligned}
\tag{6.5}
$$

We have already computed the string amplitudes in (6.5) in (4.6)-(4.9), finding

$$
A_1 = -\lambda' C_{\mathrm{norm}} \frac{\beta+1}{\pi^2} \sin^2 \pi m\beta \ , \tag{6.6}
$$

$$
A_2 = 0 \ , \tag{6.7}
$$

$$
A_3 = 0 \ , \tag{6.8}
$$

$$
A_4 = \lambda' C_{\mathrm{norm}} \frac{\beta+1}{\pi^2} \sin^2 \pi m\beta \ . \tag{6.9}
$$

We start our analysis from the first process (6.1). For this case, we have found in the previous section that the corresponding coefficient $b_{m,ny}$ of the field theory three-point function is exactly the same as that obtained in the case of BMN operators with two scalar impurities of different flavours. This is due to the absence of compensating terms in the operators participating in the process (6.1), so that only the diagrams of Figure 1 contribute (with λ_4 replaced by the second λ_3). As explained in (5.13), supersymmetry guarantees that the anomalous dimension of the two fermion BMN operators is the same as that of two scalars; therefore, the coefficient $a_{m,ny}$ for the fermions is identical to that of two different scalars. The consequence of this is that the gauge theory prediction for the string amplitude of (6.1) is exactly the same as the prediction obtained in the case of BMN operators with two different scalar impurities; and it was shown in [10] that there is precise agreement between the field theory and string theory prediction in the case of two scalar impurities of different flavours. The result obtained in string field theory for the process (6.1) is given in (6.6). Indeed, this result (6.6) precisely coincides with the string amplitude for the case of two scalar impurities, Eq. (4.10). This is our first test.

Two identical fermions, both in the unbarred and barred operators, take part in the process in (6.2). The corresponding gauge theory calculation is therefore slightly more complicated, since there are twice as many contractions as in the previous case, and thus twice as many Feynman diagrams. Taking these diagrams into account, the S and T matrices take the following form:

$$
\begin{aligned}
S &= \begin{pmatrix} \delta_{mn} & g_2\,(C_{m,qz} - C_{m,-qz}) \\ g_2\,(C_{py,n} - C_{-py,n}) & \delta_{pq} \end{pmatrix} + \mathcal{O}(g_2^2) \qquad (6.10)\\
&= \mathbb{1} + g_2 s + \mathcal{O}(g_2^2)\ ,
\end{aligned}
$$

$$
\begin{aligned}
T &= \lambda' \begin{pmatrix} m^2\,\delta_{mn} & \begin{matrix} g_2\big[C_{m,qz}\,(a+b)_{m,qz} \\ -C_{m,-qz}\,(a+b)_{m,-qz}\big] \end{matrix} \\ \begin{matrix} g_2\big[C_{py,n}\,(a+b)_{py,n} \\ -C_{-py,n}\,(a+b)_{-py,n}\big] \end{matrix} & (p^2/y^2)\,\delta_{pq}\delta_{yz} \end{pmatrix} + \mathcal{O}(g_2^2) \qquad (6.11)\\
&= d \ +\ g_2 t\ +\ \mathcal{O}(g_2^2)\ .
\end{aligned}
$$

In (6.11) the coefficient $a_{m,ny}$ is as in (5.13), and $b_{m,ny}$ is given in (5.14), as explained in section 5. We should note the crucial minus sign between the two terms appearing in the non-diagonal matrix elements of T. This comes from the anticommuting nature of the fermion impurities. We can now work out the expression for the matrix Γ of the SYM dilatation operator in the field theory basis which is isomorphic to the natural string basis. Using (5.12), we get immediately

$$
\Gamma = d + g_2\Big[t - (1/2)\{s,d\}\Big] = \lambda' \begin{pmatrix} m^2\,\delta_{mn} & 0 \\ 0 & (p^2/y^2)\,\delta_{pq}\delta_{yz} \end{pmatrix}\ . \qquad (6.12)
$$

Thus, we conclude that the field theory prediction for the second process (6.2) is 0. This is in agreement with the vanishing of the corresponding string amplitude of (6.7).

Next, we consider the third process, (6.3). In this case the compensating diagrams of Figure 2 should be taken into account. As we have noticed in the previous section (see the discussion after (5.33)), the contribution of the compensating term is the same as that arising from compensating terms of BMN operators with mixed (one scalar-one vector) impurities. Furthermore, the contribution of the diagrams where only the pure BMN parts of the operators is taken into account, is the same for all the cases (i.e. scalar, mixed, vector and fermion impurities). Therefore, we conclude that the coefficient $b_{m,ny}$ appearing in the the three-point function, and thus the whole calculation, are identical to the mixed case studied in [10].[14] It was found in [10] that the matrix elements of the dilatation operator in the isomorphic to string basis for the case of BMN operators with mixed impurities is given by

$$\Gamma_{\rm mixed} \;=\; d+g_2\Big[t_{\rm mixed}-(1/2)\{s,d\}\Big] \;=\; \lambda'\left(\begin{array}{cc} m^2\,\delta_{mn} & 0 \\ & \\ 0 & (p^2/y^2)\,\delta_{pq}\delta_{yz}\end{array}\right)\;. \qquad (6.13)$$

Therefore, the previous result (6.13) precisely reproduces, in the gauge theory, the vanishing three-string amplitude of (6.8).

Finally, we focus on the last process of (6.4). We noticed in section 5 that the diagrams from the compensating terms contribute, in this case, exactly as the diagrams from the compensating term for two vector operators. Following similar arguments as before, we conclude that the field theory prediction for this process is the same as that for the vector case. The result for the vector case was found in [10] to be equal to the negative of the result for the process for the scalars, which in turns is equal to the result for (6.1). This is again in perfect agreement with the string amplitude obtained in (6.9). This is our last test.

We close this section with a comment about how the $\mathbb{Z}_2$ symmetry of the pp-wave background is realised in the string amplitudes of (6.5). It is known that under the $\mathbb{Z}_2$ symmetry the two indices of a fermion creation or annihilation operator are exchanged,

$$\mathbb{Z}_2: \qquad \beta_{\alpha\beta}\longrightarrow\beta_{\beta\alpha}\;. \qquad (6.14)$$

However, whereas the string vertex $|H_3\rangle$ is invariant under $\mathbb{Z}_2$, the true vacuum $|\mathrm{v}\rangle$ corresponds to a combination of the trace of the metric and the five-form field on one of the $\mathbb{R}_4$'s [55], and thus one has to assign negative $\mathbb{Z}_2$ parity to it [18, 23]. The correctness of this assignment was also verified from the field theory perspective in [10] (see also the discussion in the Introduction). If we apply (6.14) to the string amplitudes in (6.5) we obtain:

$$\begin{aligned}\hat{\mathbb{Z}}_2 A_1 &\equiv -A_4 = A_1\;,\\ \hat{\mathbb{Z}}_2 A_2 &\equiv -A_2 = A_2\;,\\ \hat{\mathbb{Z}}_2 A_3 &\equiv -A_3 = A_3\;,\\ \hat{\mathbb{Z}}_2 A_4 &\equiv -A_1 = A_4\;.\end{aligned} \qquad (6.15)$$

Therefore, we conclude that the $\mathbb{Z}_2$ symmetry leaves the value of the string amplitudes of (6.1)-(6.4) invariant.

[14] As before, in order to reach this conclusion we also used the fact that the anomalous dimension of all two-impurity operators, i.e. is the $a_{m,ny}$ coefficient, is the same for any kind of impurity.

Acknowledgements

It is a pleasure to thank Carlo Angelantonj, Andreas Brandhuber, Sanjaye Ramgoolam, Rodolfo Russo, Bill Spence and in particular Valya Khoze for stimulating conversations. GG acknowledges a grant from the State Scholarship Foundation of Greece (I.K.Y.), and the hospitality of Queen Mary College, London, where part of this work was carried out. The work of GT was supported by PPARC.

Appendix A: Notation and conventions in gauge theory

We write the Euclidean $\mathcal{N} = 4$ Lagrangian as

$$\mathcal{L} = \mathcal{L}_{\mathcal{B}} + \mathcal{L}_{\mathcal{F}} \ , \tag{A.1}$$

where

$$\mathcal{L}_{\mathcal{B}} = \frac{2}{g^2}\mathrm{Tr}\left(\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + (D^\mu \bar{Z}^i)(D_\mu Z_i) - [Z_i, Z_j][\bar{Z}^i, \bar{Z}^j] + \frac{1}{2}[Z_i, \bar{Z}^i][Z_j, \bar{Z}^j]\right) \ , \tag{A.2}$$

and

$$\mathcal{L}_{\mathcal{F}} = \frac{2}{g^2}\mathrm{Tr}\left(\lambda_A \sigma^\mu D_\mu \bar{\lambda}^A - \sqrt{2}i([\lambda_4, \lambda_i]\bar{Z}^i + [\bar{\lambda}^4, \bar{\lambda}^i]Z_i) + \frac{i}{\sqrt{2}}(\epsilon^{ijk}[\lambda_i, \lambda_j]Z_k + \epsilon_{ijk}[\bar{\lambda}^i, \bar{\lambda}^j]\bar{Z}^k)\right) \ . \tag{A.3}$$

In the above equation $A = 1, \ldots, 4$ and $i, j, k = 1, \ldots, 3$. Z_i are the the three complex scalars defined by

$$Z_1 = \frac{\phi_1 + i\phi_2}{\sqrt{2}} \ , \quad Z_2 = \frac{\phi_3 + i\phi_4}{\sqrt{2}} \ , \quad Z_3 = \frac{\phi_5 + i\phi_6}{\sqrt{2}} \ , \tag{A.4}$$

where ϕ_i, $i = 1, \ldots, 6$ are the real scalar fields transforming under the $SO(6)$ R-symmetry group. We will also set $Z_3 := Z$.

We define the covariant derivative is $D_\mu \phi_i = \partial_\mu \phi_i - i[A_\mu, \phi_i]$, where $A_\mu = A^a_\mu T^a$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$.

Our $SU(N)$ generators are normalised as

$$\mathrm{Tr}\left(T^a T^b\right) = \delta^{ab} \ , \tag{A.5}$$

so that, for example,

$$\left\langle Z^i_j(x)\bar{Z}^l_m(0)\right\rangle = \frac{g^2}{2}\,\delta^i_m \delta^l_j\,\Delta(x) \quad , \quad \Delta(x) = \frac{1}{4\pi^2 x^2} \ . \tag{A.6}$$

Our Euclidean sigma matrices satisfy

$$\sigma_\mu \bar{\sigma}_\nu + \sigma_\nu \bar{\sigma}_\mu = 2\delta_{\mu\nu} \ , \qquad \bar{\sigma}_\mu \sigma_\nu + \bar{\sigma}_\nu \sigma_\mu = 2\delta_{\mu\nu} \ . \tag{A.7}$$

The completeness relation reads:

$$\sigma^\mu_{\alpha\dot{\beta}}\,\bar{\sigma}^{\dot{\gamma}\delta}_\mu \ = \ 2\,\delta^\delta_\alpha\,\delta^{\dot{\gamma}}_{\dot{\beta}} \ . \tag{A.8}$$

Another useful identity is:

$$\sigma_\nu \,\bar\sigma_\rho \,\sigma_\mu \;=\; \delta_{\nu\rho}\sigma_\mu \,+\, \delta_{\mu\rho}\sigma_\nu \,-\, \delta_{\mu\nu}\sigma_\rho \,+\, \epsilon_{\nu\rho\mu\tau}\sigma_\tau \tag{A.9}$$

We also define $\sigma_{\mu\nu}$ and $\bar\sigma_{\mu\nu}$ by:

$$\sigma_{\mu\nu} \;=\; \frac{1}{2}(\sigma_\mu\bar\sigma_\nu - \sigma_\nu\bar\sigma_\mu) = i\eta^a_{\mu\nu}\sigma^a \ , \tag{A.10}$$

$$\bar\sigma_{\mu\nu} \;=\; \frac{1}{2}(\bar\sigma_\mu\sigma_\nu - \bar\sigma_\nu\sigma_\mu) = i\bar\eta^a_{\mu\nu}\sigma^a \ , \tag{A.11}$$

where $\eta^a_{\mu\nu}$ ($\bar\eta^a_{\mu\nu}$) are the (anti-)self-dual 't Hooft symbols [56].

Finally, we will use the definitions $J := J_1 + J_2$ and $J_1 = y \cdot J$, where $y \in (0,1)$.

Appendix B: The three-string vertex

We begin by specifying the notation and conventions used in pp-wave string field theory. The combination $\alpha' p^+$ for the r-th string is denoted $\alpha_{\rm r}$ and $\sum_{\rm r=1}^3 \alpha_{\rm r} = 0$. As is standard in the literature, we will choose a frame in which $\alpha_3 = -1$,

$$\alpha_{\rm r} = \alpha' p^+_{(\rm r)} \ : \qquad \alpha_3 = -1, \qquad \alpha_1 = y, \qquad \alpha_2 = 1-y. \tag{B.1}$$

In terms of the $U(1)$ R-charges of the BMN operators in the gauge theory three-point function, $\langle \mathcal{O}_1^{J_1}\, \mathcal{O}_2^{J_2}\, \bar{\mathcal{O}}_3^{J}\rangle$ we have

$$y = \frac{J_1}{J} \ , \qquad 1-y = \frac{J_2}{J}, \qquad y \in (0,1) \ , \tag{B.2}$$

and $J = J_1 + J_2$.

The effective SYM coupling constant (1.2) in the frame (B.1) takes the simple form

$$\lambda' = \frac{1}{(\mu p^+ \alpha')^2} \ \equiv \frac{1}{(\mu\alpha_3)^2} \ = \frac{1}{\mu^2}. \tag{B.3}$$

Here μ is the mass parameter which appears in the pp-wave metric, in the chosen frame it is dimensionless[15] and the expansion in powers of $1/\mu^2$ is equivalent to the perturbative expansion in λ'. Finally, the frequencies are defined via

$$\omega_{{\rm r}m} = \sqrt{m^2 + (\mu\alpha_{\rm r})^2} \ . \tag{B.4}$$

The three-string vertex $|H_3\rangle$ can be represented as a ket-state in the tensor product of three single-string Fock spaces. It has the form [11, 12]

$$\frac{1}{\mu}|H_3\rangle = \mathsf{P}|V_F\rangle|V_B\rangle\, \delta\Big(\sum_{\rm r=1}^3 \alpha_{\rm r}\Big) \ , \tag{B.5}$$

[15]It is $p^+\mu$ which is invariant under longitudinal boosts and is frame-independent.

where the kets $|V_B\rangle$ and $|V_F\rangle$ are constructed by requiring they satisfy the bosonic and fermionic kinematical symmetries, and $\alpha_{\rm r}$ are defined in (B.1). $|V_B\rangle$ is given by

$$|V_B\rangle = \exp\left(\frac{1}{2}\sum_{\rm r,s=1}^{3}\sum_{m,n=-\infty}^{\infty}\sum_{I=1}^{8} a_m^{({\rm r})\,I\dagger}\overline{N}^{({\rm rs})}_{mn}a_n^{({\rm s})\,I\dagger}\right)|{\rm v}\rangle_1|{\rm v}\rangle_2|{\rm v}\rangle_3\ , \tag{B.6}$$

where the $\overline{N}^{({\rm rs})}_{mn}$ are the Neumann matrices in the number operator basis. The fermionic ket state $|V_F\rangle$, which is relevant for this paper, is given in the $SO(4)\times SO(4)$ formalism by [23, 58]

$$|V_F\rangle = \exp\left(\sum_{\rm r,s=1}^{3}\sum_{m,n\geq 0}(b^{\alpha\beta\,\dagger}_{-m({\rm r})}\,b^{\dagger}_{n({\rm s})\,\alpha\beta}\ +\ b^{\dot\alpha\dot\beta\,\dagger}_{m({\rm r})}\,b^{\dagger}_{-n({\rm s})\,\dot\alpha\dot\beta})\overline{Q}^{({\rm rs})}_{mn}\right)|{\rm v}\rangle_1|{\rm v}\rangle_2|{\rm v}\rangle_3\ , \tag{B.7}$$

where $\overline{Q}^{({\rm rs})}_{mn}$ are the fermionic Neumann matrices. The complete perturbative expansion of the Neumann matrices in the pp-wave background in the vicinity of $\mu=\infty$, was constructed in [57][16]. The vacuum state $|{\rm v}\rangle\equiv|{\rm v}\rangle_1|{\rm v}\rangle_2|{\rm v}\rangle_3$ is defined as the state which is annihilated by all a's and b's,

$$a_{n({\rm r})}\,|{\rm v}\rangle_{\rm r}\ =\ 0\ ,\qquad b_{n({\rm r})}\,|{\rm v}\rangle_{\rm r}\ =\ 0\ ,\qquad \forall n\ . \tag{B.8}$$

The prefactor P is determined by imposing the dynamical symmetries of the pp-wave superalgebra, and was derived in [23]. Its expressions reads:

$$\begin{aligned}\mathsf{P}\ &=\ \left[\left(\mathcal{K}^i\widetilde{\mathcal{K}}^j\ +\ \frac{\mu\beta(\beta+1)}{2}\alpha_3^3\,\delta^{ij}\right)V_{ij}\ -\ \left(\mathcal{K}^a\widetilde{\mathcal{K}}^b\ +\ \frac{\mu\beta(\beta+1)}{2}\alpha_3^3\,\delta^{ab}\right)V_{ab}\right.\\ &-\ \left.\mathcal{K}_1^{\dot\alpha\alpha}\widetilde{\mathcal{K}}_2^{\dot\beta\beta}S^+_{\alpha\beta}(Y)S^-_{\dot\alpha\dot\beta}(Z)\ -\ \widetilde{\mathcal{K}}_1^{\dot\alpha\alpha}\mathcal{K}_2^{\dot\beta\beta}S^-_{\alpha\beta}(Y)S^+_{\dot\alpha\dot\beta}(Z)\right]C_{\rm norm}\ ,\end{aligned} \tag{B.9}$$

where $i=1\ldots4$ and $a=1\ldots4$ label the first and second group of four bosonic directions of the pp-wave geometry, respectively. Full details about the expressions appearing in (B.9) can be found in the original paper [23] or, for instance, in the review [35]. We will only need the following expressions:

$$\begin{aligned}V_{ij}\ &=\ \delta_{ij}\left[1+\frac{1}{12}(Y^4+Z^4)+\frac{1}{144}Y^4Z^4\right]-\frac{i}{2}\left[Y^2_{ij}(1+\frac{1}{12}Z^4)-Z^2_{ij}(1+\frac{1}{12}Y^4)\right]\\ &+\ \frac{1}{4}(Y^2Z^2)_{ij}\ ,\end{aligned} \tag{B.10}$$

$$\begin{aligned}V_{ab}\ &=\ \delta_{ab}\left[1-\frac{1}{12}(Y^4+Z^4)+\frac{1}{144}Y^4Z^4\right]-\frac{i}{2}\left[Y^2_{ab}(1-\frac{1}{12}Z^4)-Z^2_{ab}(1-\frac{1}{12}Y^4)\right]\\ &+\ \frac{1}{4}(Y^2Z^2)_{ab}\ ,\end{aligned} \tag{B.11}$$

where

$$Y^{\alpha\beta}\ =\ \sum_{\rm r=1}^{3}\sum_{n\geq0}\bar{G}_{n({\rm r})}\,b^{\dagger\alpha\beta}_{n({\rm r})}\ ,\qquad Z^{\dot\alpha\dot\beta}\ =\ \sum_{\rm r=1}^{3}\sum_{n\geq0}\bar{G}_{n({\rm r})}\,b^{\dagger\dot\alpha\dot\beta}_{-n({\rm r})}\ , \tag{B.12}$$

[16]See also [59], and Appendix of [60] for some useful properties of the Neumann matrices.

$$Y^2_{\alpha\beta} := Y_{\alpha\gamma} Y_\beta{}^\gamma , \qquad Y^4 := Y^2_{\alpha\beta} (Y^2)^{\alpha\beta} . \tag{B.13}$$

Similar expressions hold for the Z's. The matrices $\bar{G}_{n(\mathrm{r})}$ are given in (3.12) of [23]. Finally, the overall normalisation C_{norm} cannot be fixed by imposing the dynamical constraints, and is determined (once and for all) by requiring agreement with a single field theory calculation. Its value will be taken to be:

$$C_{\mathrm{norm}} = -\frac{g_2}{2} \frac{1}{\sqrt{J}} \frac{1}{\sqrt{y(1-y)}} . \tag{B.14}$$

Appendix C: Summing over the BMN phase factors

We report here the expressions for the coefficients P_I and P_{II} which arise after summing over the BMN phase factors in the interacting diagrams derived in section 5.1. Defining

$$q = e^{2\pi i m/J} , \qquad q_1 = e^{2\pi i n/J_1} , \tag{C.1}$$

the expressions for P_I and P_{II} are given by

$$P_I = \sum_{l=0}^{J_1} (\bar{q} q_1)^l \, \bar{q} , \qquad P_{II} = \sum_{l=0}^{J_1} (\bar{q} q_1)^l . \tag{C.2}$$

We also need to evaluate the quantity $2(P_I + \bar{P}_I) - 2(P_{II} + \bar{P}_{II})$, which in the BMN limit is

$$2(P_I + \bar{P}_I) - 2(P_{II} + \bar{P}_{II}) = -\frac{8m}{m - n/y} \sin^2 \pi m y . \tag{C.3}$$

Appendix D: The functions X, Y and H

The expressions for three-point functions of BMN operators with scalar, vector, mixed or fermion impurities involve the integral

$$X_{1234} = \int d^4 z \; \Delta(x_1 - z) \Delta(x_2 - z) \Delta(x_3 - z) \Delta(x_4 - z) . \tag{D.1}$$

X_{1234} develops a $\log x_{12}^2$ term X as x_1 approaches x_2, which repeatedly appears in section 5.1. The expression for X is [20]

$$X := X_{1234}|_{x_3 = x_4} = \frac{\log (x_{12}\Lambda)^{-1}}{8\pi^2 \, (4\pi^2 x_{31}^2)^2} . \tag{D.2}$$

Another important function ubiquitously appearing in the calculations is

$$Y_{123} = \int d^4z \, \Delta(x_1 - z)\Delta(x_2 - z)\Delta(x_3 - z) \ . \tag{D.3}$$

It is easy to realise that, as $x_{12} \to 0$, the function Y_{123} contains a logarithmic term given by

$$Y_{123}|_{x_{12}\to 0} = -\frac{1}{2^4\pi^2}\Delta(x_{13}) \log x_{12}^2 \ . \tag{D.4}$$

One also needs the following expression for the $\log x_{12}^2$ term in the first derivative of Y:

$$(\partial_{1\alpha} Y_{123})_{x_{12}\to 0} = \frac{1}{2^5\pi^2} \log x_{12}^2 \ \partial_{3\alpha}\Delta(x_1 3) \ . \tag{D.5}$$

Notice that (D.5) should be derived directly from (D.3) rather than by differentiating (D.4).

In the calculation, we also encounter the function H defined by

$$H_{14,23} = (\partial_\mu^{x_1} - \partial_\mu^{x_4})(\partial_\mu^{x_2} - \partial_\mu^{x_3}) \int d^4z \ d^4t \ \ \Delta(x_1 - z)\Delta(x_4 - z)\Delta(x_2 - t)\Delta(x_3 - t)\Delta(z - t) \ , \tag{D.6}$$

which can be evaluated with the useful relation proved in [31]

$$\frac{H_{14,23}}{\Delta_{14}\Delta_{23}} = X_{1234}\left(\frac{1}{\Delta_{12}\Delta_{43}} - \frac{1}{\Delta_{13}\Delta_{24}}\right) + G_{1,23} - G_{4,23} + G_{2,14} - G_{3,14} \ \ , \tag{D.7}$$

where $\Delta_{ij} = \Delta(x_i - x_j)$ and

$$G_{i,jk} = Y_{ijk}\left(\frac{1}{\Delta_{ik}} - \frac{1}{\Delta_{ij}}\right) \ . \tag{D.8}$$

We can recast (D.7) as

$$\begin{aligned} H_{14,23} &= -X_{1234}\frac{\Delta_{14}\Delta_{23}}{\Delta_{13}\Delta_{24}} + \left(\frac{Y_{123}}{\Delta_{13}} + \frac{Y_{124}}{\Delta_{24}}\right)\Delta_{14}\Delta_{23} + \cdots \\ &= H_I + H_{II} + \cdots \ , \end{aligned} \tag{D.9}$$

where the dots stand for terms which either vanish or do not contain the $\log x_{12}^2$.

Appendix E: More detailed calculations for the evaluation of the Feynman diagrams

The three-point functions with fermion BMN operators discussed in section 5 are expressed in terms of J_A, J_B and J_C defined in (5.21)-(5.23). Here we sketch the calculation of the $\log(x_{12}^2)$ parts of these quantities. Let us start by calculating the following integral:

$$A = \partial_k^1 \partial_k^4 H_{1432} = \int d^4z \, d^4t \, \partial_k^z \Delta_{1z} \, \partial_k^z \Delta_{4z} \Delta_{zt} \, \Delta_{2t} \Delta_{3t} = \tag{E.1}$$

$$- \int d^4z \, d^4t \, \Delta_{1z} \Box_z \Delta_{4z} \, \Delta_{zt} \Delta_{2t} \Delta_{3t} - \int d^4z \, d^4t \, \Delta_{1z} \partial_k^z \Delta_{4z} \, \partial_k^z \Delta_{zt} \, \Delta_{2t} \Delta_{3t} \; .$$

The box acting on the propagator gives a delta function which eliminates the z integration, giving a result proportional to Y_{234}. Y_{234}, however, does not contain any $\log x_{12}^2$ term so for our purposes this term can safely be ignored. Therefore we are left with:

$$\begin{aligned} A &= \int d^4z \, d^4t \, \partial_k^z \Delta_{1z} \, \Delta_{4z} \partial_k^z \Delta_{zt} \, \Delta_{2t} \Delta_{3t} + \int d^4z \, d^4t \, \Delta_{1z} \Delta_{4z} \Box_z \Delta_{zt} \, \Delta_{2t} \Delta_{3t} \\ &= -\int d^4z \, d^4t \, \Box_z \Delta_{1z} \, \Delta_{4z} \Delta_{zt} \Delta_{2t} \Delta_{3t} - \int d^4z \, d^4t \, \partial_k^z \Delta_{1z} \, \partial_k^z \Delta_{4z} \Delta_{zt} \, \Delta_{2t} \Delta_{3t} - X_{1234} \\ &= \Delta_{14} Y_{123} - A - X_{1234} \; . \end{aligned} \tag{E.2}$$

From the last expression one can obtain A:

$$A = \frac{1}{2} \big(-X_{1234} + \Delta_{14} Y_{123} \big) \; . \tag{E.3}$$

In the above derivation , we have integrated by parts with respect to z several times, and we used $\Box_x \Delta(x) = -\delta(x)$. Since the $\log x_{12}$ terms of X_{1234} and Y_{123} are well known (see Appendix D), the same is also true for A.

Upon using the useful identity $(\partial_\mu^1 + \partial_\mu^2 + \partial_\mu^3 + \partial_\mu^4) H_{1423} = 0$, and the expression for A derived above, one can evaluate $(\partial^3 \cdot \partial^4 + \partial^2 \cdot \partial^4) H_{1423}$:

$$(\partial^2 + \partial^3) \cdot \partial^4 H_{1423} = -(\partial^1 + \partial^4) \cdot \partial^4 H_{1423} = -A - \Box_4 H_{1423} \rightarrow = -A \; , \tag{E.4}$$

since again $\Box_4$ acting on H_{1423} does not give rise to a $\log x_{12}^2$ term. One can also evaluate the difference $(\partial^3 \cdot \partial^4 - \partial^2 \cdot \partial^4) H_{1423}$ using

$$\partial^i \cdot \partial^j H_{1423} = \frac{1}{2} (\Box_k + \Box_l - \Box_i - \Box_j) H_{1423} + \partial^k \cdot \partial^l H_{1423} \; , \tag{E.5}$$

where (E.5) holds for $i \neq j \neq k \neq l$.

Starting from

$$H_{14,23} = (\partial^1 - \partial^4) \cdot (\partial^2 - \partial^3) H_{1423} \; , \tag{E.6}$$

substituting for $\partial^1 \cdot \partial^2$ and $\partial^1 \cdot \partial^3$ the corresponding expressions from (E.5), and solving for $(\partial^3 \cdot \partial^4 - \partial^2 \cdot \partial^4) H_{1423}$, we obtain:

$$(\partial^3 \cdot \partial^4 - \partial^2 \cdot \partial^4) H_{1423} = \frac{1}{2} \Big[H_{14,23} + (\Box_2 - \Box_3) H_{1423} \Big] \; . \tag{E.7}$$

Now, since the divergences of the right hand side of (E.7) are known [10], the divergence of $(\partial^3\cdot\partial^4-\partial^2\cdot\partial^4)H_{1423}$ is also known. In conclusion, we have computed the $\log x_{12}^2$ parts of (E.7) and (E.4). That means we can evaluate the $\log x_{12}^2$ parts of $\partial^3\cdot\partial^4\, H_{1423}$ and $\partial^2\cdot\partial^4\, H_{1423}$ separately.

We are now in position to write down the expressions for the J's as functions of X_{1234}, Y_{123} and Y_{124}. These are the following:

$$J_A = -\frac{1}{2}(X_{1234}+\Delta_{41}Y_{123}) + \cdots , \tag{E.8}$$

$$J_B = -\frac{1}{2}(-X_{1234}+\Delta_{23}Y_{124}) + \cdots , \tag{E.9}$$

$$J_C = \frac{1}{4}(-\Delta_{41}Y_{123}-X_{1234}+\Delta_{23}Y_{124}+H_{14,23}) + \cdots , \tag{E.10}$$

where the dots stand for terms which do not contain the $\log x_{12}^2$. $H_{14,23}$ is given in (D.7). In the evaluation of the diagrams involving the compensating term, we also made use of the following relations:

$$\partial_\nu^4\partial_\mu^3 X_{1234}\,|_{x_3\,=\,x_4,x_{12}\to 0} = -\frac{1}{(4\pi^2)^3}\frac{x_{3\mu}x_{3\nu}}{(x_3^2)^4}\log x_{12}^2 , \tag{E.11}$$

$$\begin{aligned}\partial_\nu^4\partial_\mu^4 X_{1234}\,|_{x_3=x_4,x_{12}\to 0} &= \Delta_{23}\partial_\nu^4\partial_\mu^4 Y_{124}\,|_{x_3=x_4,x_{12}\to 0}\\ &= \frac{\log x_{12}^2}{2(4\pi^2)^3(x_3^2)^3}\left(\delta_{\mu\nu}-4\frac{x_{3\mu}x_{3\nu}}{x_3^2}\right) ,\end{aligned} \tag{E.12}$$

$$\partial_\nu^1 Y_{123}\,|_{x_3=x_4,x_{12}\to 0} = \frac{-x_{3\nu}}{2^6\pi^4(x_3^2)^2}\log x_{12}^2 , \tag{E.13}$$

$$\partial_\nu^3 Y_{123}\,|_{x_3=x_4,x_{12}\to 0} = \frac{x_{3\nu}}{2^5\pi^4(x_3^2)^2}\log x_{12}^2 . \tag{E.14}$$

References

[1] D. Berenstein, J. M. Maldacena and H. Nastase, *"Strings in flat space and pp waves from $N = 4$ super Yang Mills,"* JHEP **0204** (2002) 013, hep-th/0202021.

[2] R. R. Metsaev, *"Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background,"* Nucl. Phys. B **625** (2002) 70, hep-th/0112044.

[3] A. Santambrogio and D. Zanon, *"Exact anomalous dimensions of N=4 Yang-Mills operators with large R charge,"* Phys. Lett. B **545** (2002) 425, hep-th/0206079.

[4] H. Verlinde, *"Bits, matrices and 1/N,"* JHEP **0312** (2003) 052, hep-th/0206059.

[5] D. J. Gross, A. Mikhailov and R. Roiban, *"A calculation of the plane wave string Hamiltonian from $N = 4$ super-Yang-Mills theory,"* JHEP **0305** (2003) 025, hep-th/0208231.

[6] C. Kristjansen, J. Plefka, G. W. Semenoff and M. Staudacher, *"A new double-scaling limit of $N = 4$ super Yang-Mills theory and PP-wave strings,"* Nucl. Phys. B **643** (2002) 3, hep-th/0205033.

[7] N. R. Constable, D. Z. Freedman, M. Headrick, S. Minwalla, L. Motl, A. Postnikov and W. Skiba, *"PP-wave string interactions from perturbative Yang-Mills theory,"* JHEP **0207** (2002) 017, hep-th/0205089.

[8] J. Gomis, S. Moriyama and J. w. Park, *"SYM description of SFT Hamiltonian in a pp-wave background,"* Nucl. Phys. B **659** (2003) 179, hep-th/0210153.

[9] J. Gomis, S. Moriyama and J. w. Park, *"SYM description of pp-wave string interactions: Singlet sector and arbitrary impurities,"* Nucl. Phys. B **665** (2003) 49, hep-th/0301250.

[10] G. Georgiou, V. V. Khoze and G. Travaglini, *"New tests of the pp-wave correspondence,"* JHEP **0310** (2003) 049, hep-th/0306234.

[11] M. Spradlin and A. Volovich, *"Superstring interactions in a pp-wave background,"* Phys. Rev. D **66** (2002) 086004, hep-th/0204146.

[12] M. Spradlin and A. Volovich, *"Superstring interactions in a pp-wave background. II,"* JHEP **0301** (2003) 036, hep-th/0206073.

[13] A. Pankiewicz, *"More comments on superstring interactions in the pp-wave background,"* JHEP **0209** (2002) 056, hep-th/0208209.

[14] A. Pankiewicz and B. Stefanski, *"pp-wave light-cone superstring field theory,"* Nucl. Phys. B **657** (2003) 79, hep-th/0210246.

[15] Y. j. Kiem, Y. b. Kim, S. m. Lee and J. m. Park, *"pp-wave / Yang-Mills correspondence: An explicit check,"* Nucl. Phys. B **642** (2002) 389, hep-th/0205279.

[16] P. Lee, S. Moriyama and J. w. Park, *"Cubic interactions in pp-wave light cone string field theory,"* Phys. Rev. D **66** (2002) 085021, hep-th/0206065.

[17] C. S. Chu, V. V. Khoze, M. Petrini, R. Russo and A. Tanzini, *"A note on string interaction on the pp-wave background,"* hep-th/0208148.

[18] Y. j. Kiem, Y. b. Kim, J. Park and C. Ryou, *"Chiral primary cubic interactions from pp-wave supergravity,"* JHEP **0301** (2003) 026, hep-th/0211217.

[19] J. Pearson, M. Spradlin, D. Vaman, H. Verlinde and A. Volovich, *"Tracing the string: BMN correspondence at finite J^2/N,"* JHEP **0305** (2003) 022, hep-th/0210102.

[20] C. S. Chu, V. V. Khoze and G. Travaglini, *"BMN operators with vector impurities, $\mathbb{Z}_2$ symmetry and pp-waves,"* JHEP **0306** (2003) 050, hep-th/0303107.

[21] B. Eden, *"On two fermion BMN operators,"* Nucl. Phys. B **681** (2004) 195, hep-th/0307081.

[22] M. Bianchi, G. Rossi and Y. S. Stanev, *"Surprises from the resolution of operator mixing in $N = 4$ SYM,"* hep-th/0312228.

[23] A. Pankiewicz, *"An alternative formulation of light-cone string field theory on the plane wave,"* JHEP **0306** (2003) 047, hep-th/0304232.

[24] A. Pankiewicz and B. J. Stefanski, *"On the uniqueness of plane-wave string field theory,"* hep-th/0308062.

[25] J. Gomis, S. Moriyama and J. w. Park, *"Open + closed string field theory from gauge fields,"*, Nucl. Phys. B **678** (2004) 101, hep-th/0305264.

[26] P. Di Vecchia, J. L. Petersen, M. Petrini, R. Russo and A. Tanzini, *"The 3-string vertex and the AdS/CFT duality in the pp-wave limit,"* hep-th/0304025.

[27] R. Slansky, *"Group Theory For Unified Model Building,"* Phys. Rept. **79** (1981) 1.

[28] A. Parnachev and A. V. Ryzhov, "Strings in the near plane wave background and AdS/CFT," JHEP **0210** (2002) 066, hep-th/0208010.

[29] U. Gursoy, *"Vector operators in the BMN correspondence,"* JHEP **0307** (2003) 048, hep-th/0208041.

[30] N. Beisert, *"BMN operators and superconformal symmetry,"* Nucl. Phys. B **659** (2003) 79, hep-th/0211032.

[31] N. Beisert, C. Kristjansen, J. Plefka, G. W. Semenoff and M. Staudacher, *"BMN correlators and operator mixing in $N = 4$ super Yang-Mills theory,"* Nucl. Phys. B **650** (2003) 125, hep-th/0208178.

[32] C. S. Chu, V. V. Khoze and G. Travaglini, *"Three-point functions in $N = 4$ Yang-Mills theory and pp-waves,"* JHEP **0206** (2002) 011, hep-th/0206005.

[33] C. S. Chu, V. V. Khoze and G. Travaglini, *"pp-wave string interactions from n-point correlators of BMN operators,"* JHEP **0209** (2002) 054, `hep-th/0206167`.

[34] N. R. Constable, D. Z. Freedman, M. Headrick and S. Minwalla, *"Operator mixing and the BMN correspondence,"* JHEP **0210** (2002) 068, `hep-th/0209002`.

[35] D. Sadri and M. M. Sheikh-Jabbari, *"The plane-wave / super Yang-Mills duality,"* `hep-th/0310119`.

[36] E. S. Fradkin and M. Y. Palchik, *"Conformal Quantum Field Theory In D-Dimensions,"* Mathematics and its applications, **376**, Kluver, Dordrecht 1996.

[37] R. A. Janik, *"BMN operators and string field theory,"* Phys. Lett. B **549** (2002) 237, `hep-th/0209263`.

[38] J. A. Minahan and K. Zarembo, *"The Bethe-ansatz for N = 4 super Yang-Mills,"* JHEP **0303** (2003) 013, `hep-th/0212208`.

[39] N. Beisert, C. Kristjansen, J. Plefka and M. Staudacher, *"BMN gauge theory as a quantum mechanical system,"* Phys. Lett. B **558** (2003) 229, `hep-th/0212269`.

[40] M. Spradlin and A. Volovich, *"Note on plane wave quantum mechanics,"* Phys. Lett. B **565** (2003) 253, `hep-th/0303220`.

[41] N. Beisert, C. Kristjansen and M. Staudacher, *"The dilatation operator of N = 4 super Yang-Mills theory,"* `hep-th/0303060`.

[42] N. Beisert, *"The complete one-loop dilatation operator of N = 4 super Yang-Mills theory,"* Nucl. Phys. B **676** (2004) 3, `hep-th/0307015`.

[43] N. Beisert and M. Staudacher, *"The N = 4 SYM integrable super spin chain,"* Nucl. Phys. B **670** (2003) 439, `hep-th/0307042`.

[44] N. Beisert, *"The su(2|3) dynamic spin chain,"* `hep-th/0310252`.

[45] C. Kristjansen, *"Three-spin strings on AdS(5) x S**5 from N = 4 SYM,"* `hep-th/0402033`.

[46] G. Arutyunov and M. Staudacher, *"Two-loop commuting charges and the string / gauge duality,"* `hep-th/0403077`.

[47] S. Mandelstam, *"Interacting String Picture Of The Neveu-Schwarz-Ramond Model,"* Nucl. Phys. B **69** (1974) 77.

[48] M. B. Green and J. H. Schwarz, *"Superstring Interactions,"* Nucl. Phys. B **218** (1983) 43.

[49] M. B. Green, J. H. Schwarz and L. Brink, *"Superfield Theory Of Type Ii Superstrings,"* Nucl. Phys. B **219** (1983) 437.

[50] M. B. Green and J. H. Schwarz, *"Superstring Field Theory,"* Nucl. Phys. B **243** (1984) 475.

[51] U. Gursoy, *"Predictions for pp-wave string amplitudes from perturbative SYM,"* JHEP **0310** (2003) 027, `hep-th/0212118`.

[52] T. Klose, *"Conformal dimensions of two-derivative BMN operators,"* JHEP **0303** (2003) 012, `hep-th/0301150`.

[53] G. Georgiou and V. V. Khoze, *"BMN operators with three scalar impurities and the vertex-correlator duality in pp-wave,"* JHEP **0304** (2003) 015, `hep-th/0302064`.

[54] M. Bianchi, B. Eden, G. Rossi and Y. S. Stanev, *"On operator mixing in $N = 4$ SYM,"* Nucl. Phys. B **646** (2002) 69, `hep-th/0205321`.

[55] R. R. Metsaev and A. A. Tseytlin, *"Exactly solvable model of superstring in plane wave Ramond-Ramond background,"* Phys. Rev. D **65** (2002) 126004, `hep-th/0202109`.

[56] G. 't Hooft, *"Computation Of The Quantum Effects Due To A Four-Dimensional Pseudoparticle,"* Phys. Rev. D **14** (1976) 3432 [Erratum-ibid. D **18** (1978) 2199].

[57] Y. H. He, J. H. Schwarz, M. Spradlin and A. Volovich, *"Explicit formulas for Neumann coefficients in the plane-wave geometry,"* Phys. Rev. D **67** (2003) 086005, `hep-th/0211198`.

[58] C. S. Chu, M. Petrini, R. Russo and A. Tanzini, *"String interactions and discrete symmetries of the pp-wave background,"* Class. Quant. Grav. **20** (2003) S457, `hep-th/0211188`.

[59] J. Lucietti, S. Schafer-Nameki and A. Sinha, *"On the plane-wave cubic vertex,"* `hep-th/0402185`.

[60] C. S. Chu and V. V. Khoze, *"Correspondence between the 3-point BMN correlators and the 3-string vertex on the pp-wave,"* JHEP **0304** (2003) 014, `hep-th/0301036`.