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FERMION - FIELD NONTOPOLOGICAL SOLITONS

II. MODELS FOR HADRONS

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Abstract

We examine the possibility, and its consequences, that in a relativistic local field theory, consisting of color quarks q, scalar gluon σ, color gauge field V_μ and color Higgs field ϕ, the mass of the soliton solution may be much lower than any mass of the plane wave solutions; i.e., m_q the quark mass, m_σ the gluon mass, etc. There appears a rather clean separation between the physics of these low mass solitons and that of the high energy excitations, in the range of m_q and m_σ, provided that the parameters $\xi \equiv (\mu/m_q)^2$ and $\eta \equiv \mu/m_\sigma$ are both $\ll 1$, where μ is an overall low energy scale appropriate for the solitons (but the ratio η/ξ is assumed to be $O(1)$, though otherwise arbitrary).

Under very general assumptions, we show that independently of the number of parameters in the original Lagrangian, the mathematical problem of finding the quasi-classical soliton solutions reduces, through scaling, to that of a simple set of two coupled first-order differential equations, neither of which contains any explicit free parameters. The general properties and the numerical solutions of this reduced set of differential equations are given. The resulting solitons exhibit physical characteristics very similar to those of a "gas bubble" immersed in a "medium": there is a constant surface tension and a constant pressure exerted by the medium on the gas; in addition, there are the "thermodynamical" energy of the gas and the related gas pressure, which are determined by the solutions of the reduced equations. Both a SLAC-like bag and the Creutz-Soh version of the MIT bag may appear, but only as special limiting cases.

These soliton solutions are applied to the physical hadrons; their static properties are calculated and, within a 10–15 % accuracy, agree with observations.

Contents

I. Introduction

In a previous paper[1] (hereafter called I), we have made a systematic comparison between the quasiclassical soliton results and the exact answer in a quantum field theory, whenever the exact answer is available. In a fully relativistic renormalizable theory of a Fermion field interacting with a scalar gluon field, the exact answer is known only in the weak coupling region. There, it is found that the quasiclassical result becomes exact when the Fermion number N is large. Even when $N = 2$, the quasiclassical result remains a fair approximation. For example, the ratio between the exact two-body binding energy and the corresponding quasiclassical soliton result is $\cong .77$ in the weak coupling limit. When the Fermions are nonrelativistic (like electrons in a crystal), but the scalar field remains relativistic, exact answers are also known in the strong coupling limit. We find that the quasiclassical soliton result becomes exact for arbitrary N, provided that the coupling is sufficiently strong; it is also exact in any coupling range, when N is sufficiently large. It is not difficult to trace the underlying reason for the validity of the quasiclassical description. When N is $\gg 1$, there is a large number of real particles in the system. Similarly, when the coupling is strong, the number of virtual particles becomes large. In either case, the system possesses some large coherent modes of field quanta, which are accessible to quasiclassical descriptions. It is quite remarkable that even in the worst case, $N = 2$ and weak coupling, the quasiclassical binding energy derived from the soliton solution remains a fairly reasonable approximation to the exact quantum value. [The same conclusion can be reached if the conserved quantum number, say N, is carried by a Boson field, instead of a Fermion field.]

From these comparisons, we infer that strong coupling is by no means detrimental to a quasiclassical approximation.[1,2] Rather, because of the large number of virtual quanta involved, and because of the strong potential energy which may develop against fluctuations, one expects the quasiclassical approximation to be more reliable in the strong coupling region. With this assumption, we shall in this paper extend our studies of quasiclassical soliton solutions to quark models for hadrons, where strong coupling is clearly required. Our starting point is identical to that of Bardeen, Chanowitz, Drell, Weinstein and Yan;[3] it is also similar to the work of many others.[4-6] On the other hand, as we shall see, the details are different; our analysis of the quasiclassical soliton solutions will be more systematic. Both a SLAC-like bag[3] and the Creutz-Soh version[5] of the MIT bag[7] will appear only as special limiting cases.

The specific system that we wish to study contains a quark field ψ, which has nine components representing the (3,3) representation of the color[8] SU(3) times the usual[9] SU(3) symmetry. [The generalization to SU(4) is straightforward.] Instead of a permanent confinement, we assume a very large mass m_q for the free quark, which accounts for its escape from detection so far. A scalar gluon field σ is introduced to bind the quarks into observed hadrons. By applying the same mechanism as that used in the discussions on abnormal nuclear states,[10] we can reduce the effective mass of a bound quark to almost zero inside the hadron, and thereby realize some of the well-known features of a relativistic quark model, such as SU(6) symmetry[11] and the related electromagnetic properties. In addition, we follow the suggestion of Nambu[12] to introduce a color-gauge vector field V_μ to unglue the color-nonsinglet states; this necessitates that the vector forces be strong and long range[13] inside the hadron. Consequently, the vector field must also be of a very small effective mass inside the hadron, though its physical mass m_V in a free state has to be rather large since it has escaped detection so far. A color Higgs[14] field ϕ is then introduced to achieve this purpose.

The general Lagrangian density $\mathcal{L}$ of these four fields Ψ, σ, V_μ and ϕ is given by Eq. (3.8) in Sec. III. In the Lagrangian density, the potential function $U(\sigma, \phi)$ between the scalar gluon field σ and the color Higgs field ϕ is assumed to have an absolute minimum at the vacuum value

$$\sigma = \sigma_{vac} \neq 0 \qquad \text{and} \qquad \phi = \phi_{vac} \neq 0 \tag{1.1}$$

with the convention

$$U(\sigma_{vac}, \phi_{vac}) = 0 \quad . \tag{1.2}$$

The free quark mass m_q and the free vector mass m_V are

$$m_q = g\,\sigma_{vac} \qquad \text{and} \qquad m_V = f\,\phi_{vac} \quad , \tag{1.3}$$

where g and f are the appropriate coupling constants in the theory. These two masses are both heavy, $\gg 1$ GeV. In addition, the potential function $U(\sigma, \phi)$ is assumed to have a local minimum at the origin

$$\sigma = \phi = 0 \quad ,$$

where the effective masses of the quark and the vector field are both zero. We define

$$p \equiv U(0, 0) > 0 \quad . \tag{1.4}$$

For color-singlet states, the average value of the color gauge field V_μ is zero; therefore, we can simply ignore V_μ in a quasiclassical calculation for observed hadrons, since these are all color-singlets.

As we shall see, in accordance with the aforementioned description inside the hadron, we expect the interior of our soliton solution to be in the neighborhood of $\sigma = \phi = 0$. Consequently, the energy scale of the low-lying solitons is expected to be at least partly determined by p. Since, in this paper, we are interested only in soliton models for hadrons, which are supposed to be much lighter than the quark and the gluon, we shall always assume

$$m_q \gg p^{\frac{1}{4}} \qquad \text{and} \qquad m_\sigma \gg p^{\frac{1}{4}} \tag{1.5}$$

where m_σ is the mass of the gluon field σ. [In the case of $\sigma - \phi$ coupling, m_σ must be more carefully defined. See (3.29) in Sec. III.]

Near the surface of the soliton, as we shall also see, there is a rapid transition of the scalar fields, σ and ϕ, changing from values near $(\sigma, \phi) = (0, 0)$ to $(\sigma_{vac}, \phi_{vac})$. The simplest way to calculate this transition is to solve the corresponding mechanical analog problem of a point particle, whose "coordinates" are (σ, ϕ), moving in a "potential" $- U(\sigma, \phi)$, starting from the origin $(0, 0)$ at a finite "time" and reaching the point $(\sigma_{vac}, \phi_{vac})$ at an "infinite time". Such a transition of σ and ϕ gives rise to a surface energy density s, which will be denoted by

$$s = \text{surface energy/area} \equiv \frac{1}{6} \mu^3 \equiv \frac{1}{6} \bar{m}_\sigma \sigma_{vac}^2 , \tag{1.6}$$

where μ, thus defined, has the dimensionality of a mass. It can be readily verified that if there is only the σ-field, without the Higgs field ϕ, then $\bar{m}_\sigma = m_\sigma$, the free σ-mass; thus, if one wishes, one may regard $\bar{m}_\sigma$, defined by (1.6), to be an "effective" σ-mass, relevant for the description of the soliton surface. [See (2.44) and (3.27) below.] In parallel with (1.5), we assume

$$m_q \gg \mu \qquad \text{and} \qquad m_\sigma \gg \mu \tag{1.7}$$

where, in accordance with (1.6),

$$\mu = (\bar{m}_\sigma \sigma_{vac}^2)^{\frac{1}{3}} . \tag{1.8}$$

Under the assumptions (1.5) and (1.7), the low-lying solitons are characterized by the energy scales $p^{\frac{1}{4}}$ and μ (or $s^{\frac{1}{3}}$). For convenience in order of magnitude estimations, the dimensionless ratio between p and μ^4,

$$\lambda \cong p/\mu^4 , \tag{1.9}$$

though arbitrary, will be regarded as $O(1)$; i.e., $\lambda^{\frac{1}{4}}$ is considered to be much smaller than either (m_q/μ) or (m_σ/μ), so that (1.7) implies (1.5). Hence, in a soliton model of hadrons, we expect[15]

$$\mu = O(m_B) \tag{1.10}$$

where $m_B \cong 1316$ MeV is the average baryon mass of the observed lowest SU(6) 56-multiplet. It is useful to define

$$\xi \cong (\mu/m_q)^2 \quad \text{and} \quad \eta \cong (\mu/m_\sigma) . \tag{1.11}$$

Both dimensionless parameters are assumed to be quite small.

In the limit ξ and η both $\to 0$, at a fixed but arbitrary ratio η/ξ, a rather remarkable simplification arises. As we shall see, the low-lying soliton solutions can be analysed independently of the high energy excitations (which may involve free quarks, free gluons, etc.). Furthermore, through scaling, the mathematical problem can be reduced to a simple system of two coupled first order differential equations neither of which contains any explicit free parameters:

$$\frac{d\hat{u}}{d\rho} = (-1+\hat{u}^2-\hat{v}^2)\,\hat{v}$$

and (1.12)

$$\frac{d\hat{v}}{d\rho} + \frac{2}{\rho}\,\hat{v} = (1+\hat{u}^2-\hat{v}^2)\,\hat{u} \quad .$$

This reduction is established first in Sec. II for a simple system of only color quarks and scalar gluons, and then in Sec. III for a more general system including vector color gauge fields and color Higgs fields. The general properties of the reduced equations (1.12) together with the numerical solutions are given in Sec. II.4.

In Sec. IV, it is shown that the resulting low-lying states exhibit physical characteristics very similar to those of a "gas bubble" (i.e., the soliton) immersed in a "medium" (i.e., the vacuum): there is a constant pressure p exerted by the medium on the gas and a constant surface tension s. In addition, there are the "thermodynamical" energy of the gas and the related gas pressure; both are determined by the solutions of the reduced equations. Also in Sec. IV, we apply these soliton solutions to the known hadrons. The static properties agree with observations to within 10–15% accuracy.

Because of the rather clean separation of physics of the low energy hadrons states from the physics at a much higher energy ($\sim$ quark mass), identical results can be derived for these low-lying solutions, whether we assume the quarks are integer-charged[16] or fractionally charged,[17] whether they are stable or unstable (provided that the interaction causing the instability does not play a major role in the binding). What emerges is the possibility of a relatively self-contained description of hadron physics in the GeV range that is based on the quasiclassical soliton solutions of a relativistic local field theory. The fact that these low-lying states form almost a closed system indicates that the theory can at least be regarded as a phenomenological one, somewhat analogous to Fermi's theory of β-decay. The familiar "current X current" description of the weak interaction, though not fundamental, seems to be quite adequate up to the present energy range; it can be formulated without any specific reference to the precise nature of the underlying structure of the weak interaction. Likewise, the Lagrangian density used in our derivation of the soliton solutions may not be fundamental. Even some of the "local" fields used in our description, such as gluon, quark, etc., may turn out to be approximate concepts, valid only at relatively large distances, $\sim 10^{-13} - 10^{-15}$ cm.

II. Systems of Quarks and Scalar Gluons

1. Hamiltonian

From the discussions given in the previous section, we see that for color-singlets, the system can be reduced to that of a spin $\frac{1}{2}$ quark field and some scalar fields. For clarity of presentation, in this section we examine a simpler system, consisting of only the quark field and the scalar gluon field σ, without the Higgs field. [The complete Lagrangian, which contains the vector and Higgs fields as well, is given in Sec. III.] The Hamiltonian density $\mathcal{H}$ of this simpler system may be written as

$$\mathcal{H} = \tfrac{1}{2}\Pi^2 + \tfrac{1}{2}(\nabla\sigma)^2 + U(\sigma) + \sum_{j,k} \psi_j^{k\dagger}\,(-i\vec{\alpha}\cdot\vec{\nabla} + g\beta\sigma)\psi_j^k + \text{counterterms} \tag{2.1}$$

where $\vec{\alpha}$ and β are the standard Dirac matrices, σ is the gluon field, Π its conjugate momentum, and ψ_j^k is the quark field, with the subscript j and the superscript k varying independently from 1 to 3 representing, respectively, the "color" SU(3) index and the usual "flavor" SU(3) index. In this section, for definiteness, we assume $U(\sigma)$ to be a fourth order polynomial of σ. Since, on account of (1.4), $\sigma = 0$ is assumed to be a local minimum, we have

$$U(\sigma) = \tfrac{1}{2}a\sigma^2 + \tfrac{1}{3!}b\sigma^3 + \tfrac{1}{4!}c\sigma^4 + p \tag{2.2}$$

where

$$b^2 > 3ac \tag{2.3}$$

so that the absolute minimum of $U(\sigma)$ is at $\sigma = \sigma_{vac} \neq 0$. In accordance with (1.2) and (1.4), the constant p is introduced in order that

$$U(\sigma_{vac}) = 0 \quad \text{and} \quad U(0) = p \quad . \tag{2.4}$$

Without any loss of generality, we may choose $b < 0$, and therefore $\sigma_{vac} > 0$:

$$\sigma_{vac} = \frac{3}{2c}\left[-b + \left(b^2 - \frac{8}{3}ac\right)^{\frac{1}{2}}\right] \quad . \tag{2.5}$$

The free gluon mass m_σ and the free quark mass m_q are given, respectively, by

$$m_\sigma^2 = d^2U/d\sigma^2 \quad \text{at} \quad \sigma = \sigma_{vac} \quad ,$$

and (2.6)

$$m_q = g\,\sigma_{vac} \quad .$$

The parameters a, b, c and p in $U(\sigma)$ and $\mathcal{H}$ all refer to the appropriate renormalized constants, and the counterterms in (2.1) are for renormalization purposes.

By following exactly the same steps used in Sec. I of I, leading from Eq. (1.1) to Eq. (1.16) in that paper, we can decompose our total Hamiltonian $H = \int \mathcal{H}\, d^3r$ into a sum of two terms: a quasiclassical part H_{qcl} and a quantum correction H_{corr}

$$H = H_{qcl} + H_{corr} \quad . \tag{2.7}$$

In the present paper, we are interested only in states with quark number $N \leqq 3$. For these states, just as in Eq. (1.19) of I, the lowest eigenvalue E of H_{qcl} is given by the minimum of the functional

$$E(\sigma) \equiv N\epsilon + \int\left[\tfrac{1}{2}(\vec{\nabla}\sigma)^2 + U(\sigma)\right] d^3r \tag{2.8}$$

where $\sigma(\vec{r})$ is a c. number function of $\vec{r}$ and ϵ is defined to be the lowest <u>positive</u> eigenvalue of the c. number Dirac equation

$$(-i\vec{a}\cdot\vec{\nabla} + g\beta\sigma)\;\psi = \epsilon\psi \quad . \tag{2.9}$$

It has been shown elsewhere[1, 18] that the eigenvalue ϵ of (2.9) is never zero (in contrast to the topological soliton[19]). Furthermore, because of charge-conjugation symmetry, ϵ always appears in pairs: $\pm|\epsilon_1|$, $\pm|\epsilon_2|$, $\cdots$. From (2.8) and (2.9), one sees that the minimum of $E(\sigma)$ occurs when σ is the solution of

$$-\nabla^2\sigma + U'(\sigma) = -gN\psi^\dagger\beta\psi \quad , \tag{2.10}$$

where $U'(\sigma) = dU/d\sigma$ and $\int\psi^\dagger\psi\, d^3r = 1$.

It is useful to define

$$\Delta \equiv \text{maximum of } U(\sigma) \text{ between } \sigma = 0 \text{ and } \sigma_{vac} \quad ,$$

and

$$\zeta \equiv p/\Delta \quad .$$

As already mentioned in the introduction [and as we shall also show later in (2.44)], in the present simple case, the mass $\bar{m}_\sigma$ defined by (1.6) is the same as m_σ; thus, (1.8) becomes simply

$$\mu = (m_\sigma \sigma_{vac}^2)^{\frac{1}{3}} \quad . \tag{2.11}$$

From (1.9) and (1.11), we see that $p/(m_\sigma \sigma_{vac})^2 = \lambda\eta$. Thus, when $\eta \to 0$, so does ζ, since in this limit $\Delta = m_\sigma^2 \sigma_{vac}^2/32$, and therefore

$$\lambda\eta = \zeta/32 \quad .$$

It is convenient to express the parameters a, b, c and p in terms of ζ, σ_{vac} and m_σ. For $\zeta \ll 1$, we find

$$\begin{aligned} a &= m_\sigma^2\left[1 - \frac{3}{8}\zeta + O(\zeta^2)\right] \quad , \\ b &= -6(m_\sigma^2/\sigma_{vac})\left[1 - \frac{1}{4}\zeta + O(\zeta^2)\right] \quad , \\ c &= 12(m_\sigma/\sigma_{vac})^2\left[1 - \frac{3}{16}\zeta + O(\zeta^2)\right] \end{aligned} \tag{2.12}$$

and

$$p = \frac{1}{32} m_\sigma^2 \sigma_{vac}^2 \left[\zeta + O(\zeta^2)\right] \quad .$$

Through (1.11) and (2.11), m_σ, σ_{vac} and m_q may in turn be expressed in terms of μ, η and ξ. Thus the problem defined by (2.9) and (2.10) contains a mass μ and four dimensionless parameters ξ, η, λ and N (or ξ, η, ζ and N).

2. Reduction of differential equations

In this section, we discuss the simplification of the differential equations (2.9) and (2.10), when the parameters $\xi = (\mu/m_q)^2$ and $\eta = \mu/m_\sigma$, defined by (1.11) and (2.11), are both small.

It is convenient to make the standard separation of angular variables for the lowest positive energy solution of (2.9). We write

$$\psi = \begin{pmatrix} u \\ i(\vec{\sigma}\cdot\vec{r}/r)\,v \end{pmatrix} S \qquad (2.13)$$

where $\vec{\sigma}$ is the Pauli matrix, $u = u(r)$, $v = v(r)$ and

$$S = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad .$$

Equations (2.9) and (2.10) take on the radial form

$$\frac{du}{dr} = (-\epsilon - g\sigma)\,v \quad ,$$

$$\frac{dv}{dr} + \frac{2}{r} v = (\epsilon - g\sigma)\,u \qquad (2.14)$$

and

$$\frac{d^2\sigma}{dr^2} + \frac{2}{r}\frac{d\sigma}{dr} - U'(\sigma) = Ng(u^2 - v^2) \quad ,$$

with $\int (u^2 + v^2)\, d^3r = 1$. From (2.14), we see that

$$\frac{d}{dr}(u^2 - v^2) = -4v\left[\epsilon u - (v/r)\right] \quad . \tag{2.15}$$

We define the dimensionless variable

$$\rho \equiv \epsilon r \quad . \tag{2.16}$$

As we shall see, for $N = 2$ or 3, in the limit ξ and η both $\to 0$ at a fixed but arbitrary ratio η/ξ, through scaling the above rather complicated set of coupled equations in r can be reduced to the following simple set of two coupled first order differential equations in ρ :

$$\frac{d\hat{u}}{d\rho} = (-1 + \hat{u}^2 - \hat{v}^2)\,\hat{v}$$

and

$$\frac{d\hat{v}}{d\rho} + \frac{2}{\rho}\hat{v} = (1 + \hat{u}^2 - \hat{v}^2)\,\hat{u} \quad . \tag{2.17}$$

The relation between these two sets of equations, (2.14) and (2.17), will be given below. It is quite remarkable that (2.17) does not explicitly contain any free parameter, while in the original set of equations (2.14) there are five independent parameters a, b, c, g and N (or μ, ξ, η, λ and N).

To see how the solutions of (2.14) can be expressed in terms of those of (2.17), we first comment on some simple properties of the reduced equations (2.17). At $\rho = 0$, the initial value $\hat{u}(0)$ can be arbitrary, while $\hat{v}(0) = 0$ because of the term $2\hat{v}/\rho$ in the second equation of (2.17). By assigning an initial value $\hat{u}(0)$, we can integrate (2.17) from $\rho = 0$ to the point when $\hat{u}(\rho) = \hat{v}(\rho)$, say at $\rho = \rho_1$. Let us define

$$\hat{u}_1 \equiv \hat{u}(\rho_1) = \hat{v}(\rho_1) \quad , \tag{2.18}$$

$$n \equiv 4\pi \int_0^{\rho_1} \rho^2 (\hat{u}^2 + \hat{v}^2)\, d\rho \tag{2.19}$$

and

$$q \equiv 4\pi \int_0^{\rho_1} \rho^2 (\hat{u}^2 - \hat{v}^2)^2 \, d\rho \quad . \tag{2.20}$$

Each initial value $\hat{u}(0)$ leads to a given set of $\hat{u}_1$, ρ_1, n and q. It is just as convenient to choose n to be the independent parameter, and regard $\hat{u}(0)$, ρ_1, $\hat{u}_1$ and q as functions of n. The following theorem (proved in the next section) establishes the relation between the solutions of (2.14) and (2.17):

Theorem I In the limit ξ and η both $\to 0$ at a fixed, though arbitrary, ratio η/ξ, for $N = 2$ or 3, the lowest soliton energy $E(\sigma)$, which is determined by (2.8) and (2.14), is given by

$$\frac{E}{\mu} = \left(\frac{N}{n}\right)^{\frac{1}{2}} \left(\frac{\xi}{\eta}\right)^{\frac{1}{2}} (n + \tfrac{1}{2} q) + \tfrac{2}{3}\pi \left(\frac{N}{n}\right) \left(\frac{\eta}{\xi}\right) \rho_1^{\,2} + \frac{4}{3}\pi \left(\frac{N}{n}\right)^{\frac{3}{2}} \left(\frac{\eta}{\xi}\right)^{\frac{3}{2}} \lambda \rho_1^{\,3} \tag{2.21}$$

and

$$0 = -2 \left(\frac{N}{n}\right)^{\frac{1}{2}} \left(\frac{\xi}{\eta}\right)^{\frac{1}{2}} (\rho_1 - 1) \, \hat{u}_1^{\,2} + \tfrac{1}{3} \left(\frac{N}{n}\right) \left(\frac{\eta}{\xi}\right) + \left(\frac{N}{n}\right)^{\frac{3}{2}} \left(\frac{\eta}{\xi}\right)^{\frac{3}{2}} \lambda \rho_1 \tag{2.22}$$

where μ and λ are defined by (2.11) and (1.9) respectively.

Before giving the proof of the theorem, it may be useful first to discuss its content. For definiteness, let us consider in (2.14) a given set of parameters a, b, c, p, g and N. The other parameters such as μ, η/ξ and λ are then all determined. On the other hand, from the solution of (2.17), one has $q = q(n)$, $\rho_1 = \rho_1(n)$ and $\hat{u}_1 = \hat{u}_1(n)$. We may then use (2.22) to determine n, and (2.21) to determine E.

The physical meaning of the theorem becomes clearer if we express (2.21) and (2.22) in the following alternative (but equivalent) form, (2.23) and (2.24), also proved in the next section:

$$E = N\epsilon \left[1 + \tfrac{1}{2}(q/n) \right] + \tfrac{2}{3}\pi R^2 \mu^3 + \tfrac{4}{3}\pi R^3 p \quad , \tag{2.23}$$

and

$$-2N\upsilon_1^2(\epsilon - R^{-1}) + \tfrac{1}{3}(\mu^3/R) + p = 0 \quad , \tag{2.24}$$

where

$$n = (g\epsilon/m_\sigma)^2 N = \frac{\epsilon^2 N\eta}{\mu^2\xi} \quad , \tag{2.25}$$

$$R = \rho_1/\epsilon \tag{2.26}$$

and

$$\upsilon_1 = (\epsilon^3/n)^{\frac{1}{2}} \hat{\upsilon}_1 \quad . \tag{2.27}$$

In this new form, we may first derive the functions $q = q(n)$, $\rho_1 = \rho_1(n)$ and $\hat{\upsilon}_1 = \hat{\upsilon}_1(n)$ from the solution of (2.17), just as in the preceding paragraph. Next, we use (2.25)-(2.27) to obtain $\epsilon = \epsilon(n)$, $R = R(n)$ and $\upsilon_1 = \upsilon_1(n)$. We then choose R to be the independent parameter instead of n; i.e., we regard $n = n(R)$, $\epsilon = \epsilon(R)$, $\upsilon_1 = \upsilon_1(R)$, $q = q(R)$, etc. Equation (2.23) can now be used to derive $E = E(R)$, and (2.24) to determine R. The parameter R will turn out to be essentially the radial extension of the soliton solution. The physical origin of the various terms in (2.23) for $E(R)$ can be traced rather directly. As we shall see, the Fermions contribute an energy $N\epsilon$. The Boson field gives a surface energy $\frac{2}{3}\pi R^2\mu^3$; in addition, it has a volume energy $\frac{4}{3}\pi R^3 p + \frac{1}{2}N\epsilon(q/n)$, in which the first term is due to the integral of $U(0) = p$ over the volume $\frac{4}{3}\pi R^3$, and the second term is due to the deviation $\sigma \neq 0$, and therefore $U(\sigma) \neq p$, in the same volume. As will be shown in Appendix A, Eq. (2.24) is simply the condition $dE(R)/dR = 0$.

Since (2.21) and (2.22) depend on η and ξ only through their ratio η/ξ, one sees that when the parameters ξ and η are both $<< 1$, the physics of these low-lying states becomes separated from that of high energy excitations which may consist of free quarks and free gluons.

As we shall see in Sec. III, Theorem 1 is equally applicable to the general case, which includes not only the quark and the scalar gluon fields but also the gauge and the Higgs fields. The applications of Theorem 1 to the observed hadrons will be discussed in Sec. IV. Because of Theorem 1, the resulting soliton admits of a phenomenological description very similar to that of a gas bubble immersed in a medium: there is a constant surface tension $s = \frac{1}{6}\mu^3$, and a constant pressure p exerted by the medium on the gas bubble; in addition, there is the "thermodynamical" energy $N\epsilon\left[1 + \frac{1}{2}(q/n)\right]$ of the gas bubble itself. The details are given in Sec. IV.1.

3. Proof of Theorem 1

In this proof, we shall assume ξ and η both to be infinitesimal, but regard their ratio η/ξ to be $O(1)$. It is convenient to divide the space into three regions:

$$\begin{array}{lll} & \text{the inside region} & r \leqq R_1 = R - O(m_q^{-1}) \ , \\ & \text{the outside region} & r \geqq R_2 = R - O(m_q^{-1}) \qquad (2.28) \\ \text{and} & \text{the transition region} & R_1 \leqq r \leqq R_2 \end{array}$$

where R is defined by (2.26), and R_1 and R_2 will be determined below.

(i) inside region $r \leqq R_1$

According to (2.12), when $\zeta \equiv p/\Delta$ is $\ll 1$, the local minimum $\sigma = 0$ of $U(\sigma)$ is almost degenerate with the absolute minimum $\sigma = \sigma_{vac}$. Thus, we expect the classical scalar field σ to be near $\sigma = 0$ over a large region of space, which is defined to be the inside region $r \leqq R_1$. As we shall see, $R_1 < R$, although their difference is small. Let $\bar{\sigma}$ be the average value of σ in the inside region. The volume energy due to the integral of $U(\sigma)$ is $\sim \frac{4}{3}\pi R^3\left[p + \frac{1}{2}(m_\sigma \bar{\sigma})^2\right]$, which should be $\lesssim$ the total energy $E = O(\mu)$. [The justification of $E = O(\mu)$, of course, comes from (2.21) and (2.22), which are yet to be proved. To facilitate our order of magnitude estimations, we shall first assume it to be true.] As already mentioned in the introduction, $p = O(\mu^4)$. Since $\eta \equiv \mu/m_\sigma$ and R will turn out to be $O(\mu^{-1})$, it follows then that

$$\bar{\sigma} = O(\mu\eta) \quad . \tag{2.29}$$

From (2.11), we see that $\sigma_{vac}^2 = \mu^3/m_\sigma = \mu^2\eta$; i.e.,

$$\sigma_{vac} = (\mu\eta^{\frac{1}{2}}) \quad . \tag{2.30}$$

By using (2.12), we obtain

$$c = O(\eta^{-3}) \qquad \text{and} \qquad b\bar{\sigma}/a = O(\eta^{\frac{1}{2}}) \quad . \tag{2.31}$$

Since $g\sigma_{vac} = m_q$ and $\xi \equiv (\mu/m_q)^2$, we also have

$$g = (\xi\eta)^{-\frac{1}{2}} = O(\eta^{-1}) \quad . \tag{2.32}$$

Thus, in the inside region, since $\sigma = O(\bar{\sigma})$ and $d\sigma/dr = O(\bar{\sigma}/R)$, we can approximate

$$U(\sigma) = p + \tfrac{1}{2}m_\sigma^2\sigma^2\left[1 + O(\eta^{\frac{1}{2}})\right] \quad , \tag{2.33}$$

and neglect the derivatives of σ in the last equation in (2.14). This leads to

$$\sigma \cong -(Ng/m_\sigma^2)(u^2 - v^2) \quad . \tag{2.34}$$

As a result, (2.14) becomes

$$\frac{du}{dr} = \left[-\epsilon + (Ng^2/m_\sigma^2)(u^2 - v^2)\right] v$$

and (2.35)

$$\frac{dv}{dr} + \frac{2}{r} v = \left[\epsilon + (Ng^2/m_\sigma^2)(u^2 - v^2) \right] u$$

with $\int (u^2 + v^2)\, d^3r = 1$. By defining

$$u = (\epsilon^3/n)^{\frac{1}{2}} \hat{u} \qquad \text{and} \qquad v = (\epsilon^3/n)^{\frac{1}{2}} \hat{v} \quad , \tag{2.36}$$

where n is given by (2.25), we see that (2.35) becomes simply (2.17) on account of (2.16), and that n is expressed in terms of $\hat{u}$ and $\hat{v}$ by (2.19).

(ii) outside region $r \geqq R_2$

In the outside region, we assume σ rises from near zero to its asymptotic value σ_{vac} at $r = \infty$. As we shall see, although $R_2 < R$, which is given by (2.26), R_2 is also $= R - O(m_q^{-1})$, like R_1 . From the definition (2.26) of R, we see that the extrapolation of the inside solution gives $u^2 - v^2 = 0$ at $r = R$ (which is in the outside region, but quite near $r = R_2$) . Therefore, we expect $u^2 - v^2$ to be small in the entire outside region; i.e.

$$\mu^{-3}(u^2 - v^2) \ll 1 \quad . \tag{2.37}$$

Thus, we may neglect $u^2 - v^2$ in the equation for σ in (2.14). Because $r \gtrsim R = O(\mu^{-1})$, we may also neglect the curvature term $(2/r)\, d\sigma/dr$. Since as shown in Sec. II.1, $\zeta \equiv p/\Delta \ll 1$, we may regard $U(\sigma)$ as approximate degenerate at $\sigma = 0$ and $\sigma = \sigma_{vac}$. To the zeroth order in the small parameter (2.37), we find in the outside region

$$\sigma(r) \cong \tfrac{1}{2}\sigma_{vac}\left[1 + \tanh \tfrac{1}{2} m_\sigma (r - R_0)\right] \tag{2.38}$$

and

$$u(r) \cong v(r) \cong \exp\left[-\int g\, \sigma(r)\, dr\right] \tag{2.39}$$

where R_0 is a constant, and $r = R_0$ lies <u>within</u> the outside region. The indefinite integral in (2.39) carries an integration constant, which will be determined by the connection to the inside solution. By using (2.38), we can simplify (2.39), and derive

$$u \cong v \cong u_0 \left[1 + e^{m_\sigma (r - R_0)} \right]^{-m_q/m_\sigma} \tag{2.40}$$

where u_0 is a constant. Since both $\xi \equiv (\mu/m_q)^2$ and $\eta = (\mu/m_\sigma)$ are $\ll 1$, for $\xi = O(\eta)$ we have $m_\sigma \gg m_q$. Thus, while σ changes rapidly from near 0 to σ_{vac} in the region $r = R_0 + O(m_\sigma^{-1})$, u and v change much more slowly. The expression (2.40) can be further approximated:

$$u \cong v \cong \begin{cases} u_0 & \text{for } r \leqq R_0 \\ u_0 \exp\left[-m_q (r - R_0)\right] & \text{for } r > R_0 \end{cases} . \tag{2.41}$$

To first order in the small parameter (2.37), we may substitute (2.41) into the righthand side of (2.15), and approximate $r^{-1} \cong R_0^{-1}$. We obtain, for $r = R_0 - O(m_q^{-1})$, but $\leqq R_0$,

$$u^2 - v^2 \cong 2u_0^2 (\epsilon - R_0^{-1}) \left[m_q^{-1} + 2(R_0 - r) \right] , \tag{2.42}$$

and for $r \geqq R_0$

$$u^2 - v^2 \cong 2u_0^2 (\epsilon - R_0^{-1})\, m_q^{-1} \exp\left[-2m_q (r - R_0)\right] . \tag{2.43}$$

In passing, we note that, by using (2.38), the energy $\int \left[\frac{1}{2} (\nabla \sigma)^2 + U(\sigma) \right] d^3 r$ integrated over the outside region is given by

$$\tfrac{4}{3} \pi R^2 m_\sigma \sigma_{vac}^2 = \tfrac{4}{3} \pi R^2 \mu^3 \equiv 4\pi R^2 s \tag{2.44}$$

where μ is given by (2.11) and s is, as defined before in (1.6), the surface energy per unit area. By comparing (2.44) to (1.6), we see that $\bar{m}_\sigma = m_\sigma$, and (1.8) is the same as (2.11).

(iii) transition region $R_1 \leq r \leq R_2$

In this region σ changes sign, so that neither $\nabla^2\sigma$ nor $u^2 - v^2$ can be neglected in the last line of (2.14). However, it is easily seen from the first two lines of (2.14) that u and v do not change appreciably in this region, so that (2.42) continues to hold. We discuss first the connection between the Fermion wave function u and v in the inside solution and that in the outside solution. As before, let R be given by (2.26). Although the boundary of the inside region is within the surface $r = R$, we may extend the inside solution of (2.35), which we shall denote by u_i, v_i, up to $r = R$. At $r = R$, by definition, we have $u_i(R) = v_i(R) = u_1$. Thus, by using (2.15), we find $d(u_i^2 - v_i^2)/dr = -4u_1^2(\epsilon - R^{-1})$ at $r = R$; i.e., in the region $r = R - O(m_q^{-1})$ we have

$$u_i^2 - v_i^2 \cong 4u_1^2(\epsilon - R^{-1})(R - r) \quad . \tag{2.45}$$

By matching (2.42) and (2.45) as well as their derivatives, at R_1, one finds

$$R_0 = R - (2m_q)^{-1} \tag{2.46}$$

and

$$u_0 = u_1 \tag{2.47}$$

where u_0 is given by (2.40) and u_1 by (2.27). So far, the values of R_1 and R_2 are arbitrary, provided both are $R - O(m_q^{-1})$, and

$$R_1 < R_2 < R_0 \quad . \tag{2.48}$$

Next, we consider the joining of the scalar field σ. Let us choose the boundary $r = R_2$ of the outside region such that

$$\exp\left[m_\sigma(R_0 - R_2)\right] \gg 1 \quad . \tag{2.49}$$

The condition (2.49) is totally consistent with $R_2 = R - O(m_q^{-1})$, since for $\eta = O(\xi)$, m_σ is $\gg m_q$. From (2.38), one sees that

$$\sigma(R_2) \cong \sigma_{vac} \exp\left[m_\sigma(R_2 - R_0)\right] \ll \sigma_{vac} \quad . \tag{2.50}$$

From (2.29) and (2.30), it follows that $|\bar{\sigma}| \ll \sigma_{vac}$. Thus, in both the transition and the inside region

$$|\sigma| \ll \sigma_{vac} \quad . \tag{2.51}$$

In the transition region, $u^2 - v^2$ is given by (2.42); in addition, $r^{-1} \ll m_\sigma$. Therefore, the third equation of (2.14) takes on the approximate form

$$\left(\frac{d^2}{dr^2} - m_\sigma^2\right)\sigma = Ng(u^2 - v^2) = \Lambda(R - r) \tag{2.52}$$

where

$$\Lambda = 4u_1^2(\epsilon - R^{-1})\,Ng \quad . \tag{2.53}$$

The desired solution is

$$\sigma = -(\Lambda/m_\sigma^2)(R - r) + \sigma_0 \exp\left[m_\sigma(r - R_2)\right] \tag{2.54}$$

where σ_0 is a constant to be determined. By assuming $\xi = O(\eta)$, and by using (2.27), (2.32) and $\epsilon = O(\mu)$, we find $\Lambda = O(\mu^4/\eta)$. In the transition region, since $r = R - O(m_q^{-1})$, the first term $(\Lambda/m_\sigma^2)(R - r)$ in (2.54) is $O(\mu\,\eta^{\frac{3}{2}})$. According to (2.31), σ_{vac} is $O(\mu\,\eta^{\frac{1}{2}})$. We shall choose σ_0 and R_1 such that

$$\sigma_{vac} \gg \sigma_0 \gg O(\mu\,\eta^{\frac{3}{2}})$$

and

$$\sigma_0 \exp\left[-m_\sigma(R_2 - R_1)\right] \ll O(\mu\,\eta^{\frac{3}{2}}) \quad . \tag{2.55}$$

As $r \to R_2$, (2.54) becomes $\sigma \cong \sigma_0 \exp\left[m_\sigma(r - R_2)\right]$, which approaches the same outside solution (2.38), provided

$$\sigma_0 = \sigma_{vac} \exp\left[m_\sigma(R_2 - R_0)\right] \quad ; \tag{2.56}$$

at $r = R_1$, (2.54) becomes

$$\sigma \cong -(Ng/m_\sigma^2)(u^2 - v^2) \tag{2.57}$$

which is the same inside solution (2.34).

We note that (2.56) is consistent with $\sigma_{vac} \gg \sigma_0$ from (2.55) because of (2.49). Moreover, the two parts of (2.55) are consistent provided that $m_\sigma(R_2 - R_1) \gg 1$. This in turn can be satisfied with $R_1 = R - O(m_q^{-1})$ since $m_\sigma \gg m_q$. The above discussion completes the joining of both the Dirac wave function, u and v, and the scalar field σ between the inside solution and the outside solution. The total energy E of the system is given by (2.8). In the inside region $(\nabla\sigma)^2$ is $\sim(\bar{\sigma}/R)^2 \sim \mu^4\eta^2$ which is much smaller than $U(\sigma) \sim p + \frac{1}{2}m_\sigma^2\sigma^2 \sim \mu^4$. Therefore, the integral $\int\left[\frac{1}{2}(\nabla\sigma)^2 + U(\sigma)\right]d^3r$ over the inside region becomes, because of (2.34),

$$\frac{4}{3}\pi R^3 p + \frac{1}{2}(Ng/m_\sigma)^2 \int (u^2 - v^2)^2 d^3r \quad . \tag{2.58}$$

The same integral over the transition region can be neglected, and that over the outside region is given by (2.44). By using (2.16), (2.20), (2.25), (2.26) and (2.36), one sees that the second term in (2.58) is $\frac{1}{2}N\epsilon q/n$. Thus, the total energy E is given by (2.23).

To derive (2.24), the simplest way is to multiply the last equation of (2.14) on both sides by $\frac{d\sigma}{dr}$, and then integrate from r to ∞. We find

$$\frac{1}{2}\left(\frac{d\sigma}{dr}\right)^2 - U(\sigma) = \int_r^\infty dr\left[\frac{2}{r}\left(\frac{d\sigma}{dr}\right)^2 - Ng(u^2 - v^2)\frac{d\sigma}{dr}\right] \quad . \tag{2.59}$$

According to (2.54) and (2.57), at $r = R_1$, $\frac{1}{2}(d\sigma/dr)^2 \cong \frac{1}{2}(\Lambda^2/m_\sigma^4) = O(\mu^4\eta^2)$, $\sigma \cong -(\Lambda/m_\sigma^2)(R - R_1) = O(\mu\eta^{\frac{3}{2}})$ and therefore $U(\sigma) \cong p + \frac{1}{2}m_\sigma^2\sigma^2 = p + O(\mu^4\eta)$, where $p = O(\mu^4)$. The righthand side of (2.59) is dominated by the integration over the region when σ changes rapidly from near zero to σ_{vac}. After neglecting $O(\eta)$ as compared to 1, and by using (2.38), (2.42) and (2.47), we find that, at $r = R_1$, (2.59) becomes

$$-p = \frac{1}{3}(\mu^3/R) - 2Nu_1^2(\epsilon - R^{-1}) ,$$

which is (2.24). From (2.23) and (2.24), and by using (1.11), (2.11) and (2.25), one derives (2.21) and (2.22). This completes the proof of Theorem 1.

There is an alternative way to derive (2.24), which will be given in Appendix A. We recall that from the solution of (2.17), we can obtain the functions $\rho_1 = \rho_1(n)$, $q = q(n)$ and $\hat{u}_1 = \hat{u}_1(n)$. Consequently, at a given set of parameters N, g, p, m_σ and m_q, (2.25) and (2.26) may be used to define $\epsilon = \epsilon(n)$ and $R = R(n)$. Of course, we may equally well choose R to be the independent variable, and regard $\epsilon = \epsilon(R)$ and $n = n(R)$. Equation (2.23) then gives $E = E(R)$. As will be shown in Appendix A, (2.24) can also be established by setting

$$\frac{dE}{dR} = 0 \quad . \tag{2.60}$$

From the discussions given in Appendix A, one sees that (2.59) implies $dE/dR = 0$; thereby, one gains a further insight into the interrelation between these equations.

We note that the discussion of the inside region shows that the "reduced" functions $\hat{u}$ and $\hat{v}$ are proportional to the actual quark wave function u, v. Hence all physical averages with respect to the quark density can be calculated from $\hat{u}$, $\hat{v}$, the contribution from $r > R_1$ being negligible.

4. Solutions of the reduced equations

Our starting point in this section is the pair of differential equations (2.17). As explained before, in the paragraph preceding (2.18)-(2.20), the solutions of these equations form a one-parameter family since the functions $\hat{u}(\rho)$, $\hat{v}(\rho)$ are completely determined when $\hat{u}(0)$ is given. Without loss of generality we assume $\hat{u}(0) > 0$.

There is a critical value $\hat{u}_{crit}$ such that if $\hat{u}(0) > \hat{u}_{crit}$, the functions $\hat{u}$, $\hat{v}$ become infinite at some value $\rho_2 < 1$, with $\hat{u} > \hat{v}$ for all $0 < \rho < \rho_2$. Such solutions are of no interest to us, since they do not correspond to any solution of (2.14). Therefore we restrict ourselves to the range

$$\hat{u}(0) \;<\; \hat{u}_{crit} \;\cong\; 1.7419 \quad . \tag{2.61}$$

The parameter n can take values from 0 to ∞. When $n \to 0$, $\hat{u}(0) \to 0$; when $n \to \infty$, $\hat{u}(0) \to \hat{u}_{crit}$. In Figure 1, $\hat{u}^2 - \hat{v}^2$ is plotted vs. ρ for two initial values of $\hat{u}(0)$, one near 0 and the other near $\hat{u}_{crit}$. One sees that the solution is volume-dominated for small n $(\hat{u}(0) \sim 0)$ and surface-dominated for large n $(\hat{u}(0) \sim \hat{u}_{crit})$.

We shall first discuss the two limits $n \to 0$ (MIT-like) and $n \to \infty$ (SLAC-like).

(i) When $n \ll 4\pi$, both $\hat{u}$ and $\hat{v}$ remain small for $0 < \rho < \rho_1$. Thus we may neglect the nonlinear terms in (2.17), obtaining

$$\begin{aligned} \frac{d\hat{u}}{d\rho} &= -\hat{v} \quad , \\ \frac{d\hat{v}}{d\rho} + \frac{2}{\rho}\hat{v} &= \hat{u} \quad . \end{aligned} \tag{2.62}$$

The solutions to (2.62) are elementary and well-known:

$$\begin{aligned} \hat{u} &= \hat{u}(0)\, j_0(\rho) = \hat{u}(0)\, \rho^{-1} \sin\rho \quad , \\ \hat{v} &= \hat{u}(0)\, j_1(\rho) = \hat{u}(0)\, \rho^{-2}(\sin\rho - \rho\cos\rho) \quad . \end{aligned} \tag{2.63}$$

We then have $j_0(\rho_1) = j_1(\rho_1)$ or

$$\rho_1 = 2.0428 \quad ,$$

$$(4\pi)^{-1} n = \hat{u}(0)^2 \rho_1 \, | \sin(2\rho_1) | = 1.6545 \, \hat{u}(0)^2 \quad ,$$

$$\hat{u}_1^2 = \left[\hat{u}(0) \rho_1^{-1} \sin \rho_1 \right]^2 = 0.1149 \left(\frac{n}{4\pi} \right) \quad ,$$

$$4\pi \int_0^{\rho_1} \rho^2 (\hat{u}^2 - \tfrac{1}{3} \hat{v}^2) \, d\rho = \frac{\rho_1 n}{3(\rho_1 - 1)} = 0.6530 n \quad , \qquad (2.64)$$

$$4\pi \int_0^{\rho_1} \rho^3 \hat{u} \hat{v} \, d\rho = \frac{(4\rho_1 - 3) n}{8(\rho_1 - 1)} = 0.6199 n \quad ,$$

$$4\pi \int_0^{\rho_1} \rho^4 (\hat{u}^2 + \hat{v}^2) \, d\rho = \frac{2\rho_1^3 - 2\rho_1^2 + 4\rho_1 - 3}{6(\rho_1 - 1)} n = 2.2175 n$$

and

$$q = O(n^2) \quad .$$

In Fig. 2 we plot $\hat{u}$, $\hat{v}$, $\hat{u}^2 + \hat{v}^2$ and $\hat{u}^2 - \hat{v}^2$ against ρ for this case.

(ii) The case of large n can best be understood by considering first the limiting solution for $\hat{u}(0) = u_{crit}$. This initial value yields a definite pair of curves for $\hat{u}$, $\hat{v}$ which are graphed in Fig. 3. As $\rho \to 1-$, $\hat{u}$, $\hat{v}$, and $\hat{u}^2 - \hat{v}^2$ all become large. The manner in which this happens can be found by letting

$$x = \hat{u}\hat{v} \quad ,$$

$$y = \hat{u}^2 - \hat{v}^2 \quad , \qquad (2.65)$$

$$\tau = 2(\rho - 1) \quad ,$$

and neglecting terms of relative order τ, $1/y$, or $y/(x\tau)$. One thus obtains from (2.17) the approximate equations

$$\frac{dx}{d\tau} = yx \quad ,$$

$$\frac{dy}{d\tau} = -\tau x \quad . \tag{2.66}$$

The solution that becomes infinite as $\tau \to 0-$ is

$$\begin{aligned} x &= 3|\tau|^{-3} , \\ y &= 3|\tau|^{-1} \end{aligned} \tag{2.67}$$

which explains why we regard τ, $1/y$ and $y/(x\tau)$ to be the same order.

For finite $n \gg 4\pi$, the functions $\hat{u}$ and $\hat{v}$ lie very close to the "critical" curves except in a region $\rho = 1 \pm O\left[n^{-\frac{1}{2}}\right]$. In this region the approximation (2.66) still holds, but instead of obeying (2.67), x and y remain finite at $\tau = 0$ and y decreases to zero at $\tau = \tau_1 = 2(\rho_1 - 1)$.

The finite solutions of (2.66) with which we must deal can be reduced to a single universal solution by the transformation

$$x \to \hat{x} = x\tau_1^3 \quad , \quad y \to \hat{y} = y\tau_1 \quad , \quad \tau \to \hat{\tau} = \tau/\tau_1 \quad , \tag{2.68}$$

which leaves (2.66) invariant. The functions $\hat{x}(\hat{\tau})$, $\hat{y}(\hat{\tau})$ are now completely determined by

$$\begin{aligned} \frac{d\hat{x}}{d\hat{\tau}} &= \hat{y}\hat{x} \quad , \\ \frac{d\hat{y}}{d\hat{\tau}} &= -\hat{\tau}\hat{x} \end{aligned} \tag{2.69}$$

with the boundary conditions $\hat{y} = 0$ at $\hat{\tau} = +1$, and $\hat{x}, \hat{y} \to 0$ at $\hat{\tau} \to -\infty$. The first condition sets the scale for $\hat{\tau}$, which would otherwise be adjustable through a transformation like (2.68). The second condition makes x, y obey (2.67) in the region

$1 \gg 1 - \rho \gg n^{-\frac{1}{2}}$. Thus (2.67) provides the transition from the peak region described by (2.69) to the region $0 < \rho < 1 - O\left[n^{-\frac{1}{2}}\right]$ where $\hat{u}$ and $\hat{v}$ are almost the same as their limiting values for $n \to \infty$. The results corresponding to (2.64) in the limit $n \to \infty$ is:

$$
\begin{aligned}
\rho_1 &= 1 + (3.531\,\pi/n)^{\frac{1}{2}} \quad , \\
n &= 3.531\,\pi(\rho_1 - 1)^{-2} \quad , \\
\hat{u}_1^{\,2} &= \frac{3.531}{8}\,(\rho_1 - 1)^{-3} \quad , \\
4\pi \int_0^{\rho_1} \rho^2(\hat{u}^2 - \tfrac{1}{3}\hat{v}^2)\,d\rho &= \tfrac{1}{3}\,n \quad , \qquad\qquad (2.70) \\
4\pi \int_0^{\rho_1} \rho^3\,\hat{u}\,\hat{v}\,d\rho &= \tfrac{1}{2}\,n \quad , \\
4\pi \int_0^{\rho_1} \rho^4(\hat{u}^2 + \hat{v}^2)\,d\rho &= n
\end{aligned}
$$

and

$$q = O(n^{\frac{1}{2}}) \quad .$$

The solutions of (2.69) are plotted in Fig. 4, with the asymptotic forms (2.67) shown for comparison. The relation $n = 8\pi\hat{u}_1^{\,2}(\rho_1 - 1)$ is exact in this limit, as seen from the equation $\int \hat{x}\,d\hat{\tau} = \hat{x}\hat{\tau} + \frac{1}{2}\hat{y}^2$ which follows from (2.69).

We note that it is possible to eliminate τ in (2.66), or $\hat{\tau}$ in (2.69). Let us define $a = \tau^3 x = \hat{\tau}^3\hat{x}$ and $b = \tau y = \hat{\tau}\hat{y}$. From (2.66), we see that

$$\frac{db}{da} = \frac{b - a}{a(b + 3)} \quad .$$

(iii) For intermediate values of n, the equations (2.17) have been integrated numerically. The quantities $\hat{u}$, $\hat{v}$, $\hat{u}^2 + \hat{v}^2$, and $\hat{u}^2 - \hat{v}^2$ are graphed against ρ for several values of n in Fig. 5.

From the arguments of the previous section, we see that $\hat{u}^2 + \hat{v}^2$ is proportional to the quark density, while $\hat{u}^2 - \hat{v}^2$ is proportional to the gluon field inside the hadron, $\sigma \cong -(\epsilon/g)(\hat{u}^2 - \hat{v}^2)$. The following results on $\hat{u}^2 - \hat{v}^2$, $\hat{u}$ and $\hat{u}^2 + \hat{v}^2$ are rigorously true:

<u>Theorem 2</u> If $\hat{u}(0) < \sqrt{2}$ (i.e., $n < 74.84$), the quantity $\hat{u}^2 - \hat{v}^2$ decreases monotonically from $\rho = 0$ to $\rho = \rho_1$. If $\hat{u}(0) > \sqrt{2}$ (i.e., $n > 74.84$), the quantity $\hat{u}^2 - \hat{v}^2$ increases monotonically from $\rho = 0$ to a maximum at $\rho = \bar{\rho} < \rho_1$, and decreases monotonically from $\rho = \bar{\rho}$ to $\rho = \rho_1$.

<u>Proof</u> Let $y = \hat{u}^2 - \hat{v}^2$, and $z = (\hat{v}/\hat{u}) - \rho$. Then from (2.17) we obtain

$$\frac{dy}{d\rho} = 4\rho^{-1}\hat{u}\hat{v}z \tag{2.71}$$

and

$$\frac{dz}{d\rho} = \hat{u}^{-2}(y-1)y - 2\rho^{-1}z - 1 \quad . \tag{2.72}$$

For $\rho \to 0$, $y \to \hat{u}(0)^2$, $\rho^{-1}z \to dz/d\rho$ and (2.72) becomes

$$3\frac{dz}{d\rho} = \hat{u}(0)^2 - 2 \quad . \tag{2.73}$$

Let $\hat{u}(0) < \sqrt{2}$, then z is initially negative.

Suppose that $z(\rho) = 0$ has a root between 0 and ρ_1. Let $\bar{\rho}$ be the smallest such root. Then z must be increasing at $\bar{\rho}$, and so from (2.72) we have

$$0 < \left.\frac{dz}{d\rho}\right|_{\bar{\rho}} = \left[\hat{u}^{-2}(y-1)y\right]_{\bar{\rho}} - 1 \tag{2.74}$$

where the subscript $\bar{\rho}$ denotes $\rho = \bar{\rho}$. Now, by definition, ρ_1 is the (smallest) root of $y(\rho) = 0$. Hence,

$$0 < y < \hat{u}^2 \quad \text{for} \quad \rho < \rho_1 \quad . \tag{2.75}$$

Since $\bar{\rho} < \rho_1$, (2.74) and (2.75) imply

$$1 < \left[\hat{u}^{-2}(y-1)\,y\right]_{\bar{\rho}} < y(\bar{\rho}) - 1 ,$$

which, on account of (2.71), leads to

$$2 < y(\bar{\rho}) = y(0) + \int_0^{\bar{\rho}} 4\rho^{-1}\,\hat{u}\hat{v}z < y(0) = \hat{u}(0)^2 \qquad (2.76)$$

contrary to hypothesis. [We know that $z < 0$ for $0 < \rho < \bar{\rho}$; in addition, from (2.75), $\hat{u} > 0$ for $\rho < \rho_1$, and since from (2.17), $d(\rho^2 v)/d\rho = \rho^2(1+y)\,\hat{u} > 0$, $\hat{v}$ is also > 0 for $0 < \rho < \rho_1$.]

The contradiction shows that z has no root between 0 and ρ_1. Therefore it remains negative, and the first part of Theorem 2 follows from (2.71).

Now let $\hat{u}(0) > \sqrt{2}$. Then z is initially positive, as seen from (2.73). By integrating (2.71) from 0 to ρ_1, we see that z cannot remain positive throughout; therefore it has a root. Let $\bar{\rho}$ be the smallest positive root of $z(\rho) = 0$.

If $z(\rho) = 0$ possesses a second root between $\bar{\rho}$ and ρ_1, let $\bar{\rho}'$ be the smallest such root. Then $\frac{dz}{d\rho}$ must be negative at $\bar{\rho}$ and positive at $\bar{\rho}'$, so that from (2.72) we find

$$\left[\hat{u}^{-2}y(y-1)\right]_{\bar{\rho}'} > 1 > \left[\hat{u}^{-2}y(y-1)\right]_{\bar{\rho}}$$

which, because of $\hat{u}^{-2}y = 1 - (\hat{v}/\hat{u})^2 = 1 - (\rho + z)^2$, may be rewritten as

$$\left[y(\bar{\rho}') - 1\right](1 - \bar{\rho}'^2) > 1 > \left[y(\bar{\rho}) - 1\right](1 - \bar{\rho}^2) \quad . \qquad (2.77)$$

Now, $y(\bar{\rho}) > y(0) = \hat{u}(0)^2 > 2$ since z is positive between 0 and $\bar{\rho}$. Therefore $y(\bar{\rho}) - 1 > 0$, so that

$$\left[y(\bar{\rho}) - 1\right](1 - \bar{\rho}^2) > \left[y(\bar{\rho}) - 1\right](1 - \bar{\rho}'^2) \quad . \qquad (2.78)$$

On the other hand, $1 - \bar{\rho}'^2 = (y\hat{u}^{-2})_{\bar{\rho}'} > 0$ and $y(\bar{\rho}) > y(\bar{\rho}')$ since z is negative (and therefore $dy/d\rho$ is negative) between $\bar{\rho}$ and $\bar{\rho}'$. Thus

$$\left[y(\bar{\rho}) - 1\right] (1 - \bar{\rho}'^2) \;>\; \left[y(\bar{\rho}') - 1\right] (1 - \bar{\rho}'^2) \quad . \tag{2.79}$$

Combining (2.77), (2.78) and (2.79), we have a contradiction. Therefore there is no second root of $z(\rho) = 0$. The second part of Theorem 2 now follows from (2.71).

Theorem 3 If $\hat{u}(0) < 1$ (i.e., $n < 20.47$), the function $\hat{u}(\rho)$ decreases monotonically from $\rho = 0$ to $\rho = \rho_1$. If $\hat{u}(0) > 1$ (i.e., $n > 20.47$), then $\hat{u}(\rho)$ increases monotonically from $\rho = 0$ to a maximum at $\rho = \rho_0 < \rho_1$, and decreases monotonically from $\rho = \rho_0$ to $\rho = \rho_1$.

Proof The first equation in (2.17) may be written as

$$\frac{d\hat{u}}{d\rho} = \hat{v}\, w \tag{2.80}$$

where

$$w(\rho) = y(\rho) - 1 = \hat{u}^2 - \hat{v}^2 - 1 \quad . \tag{2.81}$$

As $\rho \to 0$, $w(\rho) \to \hat{u}(0)^2 - 1$. Thus, when $\hat{u}(0) < 1$, $w(0) < 0$. Furthermore, when $\hat{u}(0) < 1 < \sqrt{2}$, we know from Theorem 2, $dw/d\rho = dy/d\rho < 0$ for $0 < \rho \leqq \rho_1$; consequently $w(\rho) < 0$, and therefore $\hat{u}(\rho)$ decreases monotonically.

Next, we consider the case $\hat{u}(0) > 1$. Since $w(0) > 1$ and $w(\rho_1) = -1$, in the interval from $\rho = 0$ to $\rho = \rho_1$, there must be a root of $w(\rho) = y(\rho) - 1 = 0$. From Theorem 2, one can show readily that there is only one such root. By using (2.80) we establish Theorem 3.

<u>Theorem 4</u> If $\hat{u}(0) < \frac{1}{\sqrt{2}}$ (i.e., $n < 9.618$), the quantity $\hat{u}^2 + \hat{v}^2$ decreases monotonically from $\rho = 0$ to $\rho = \rho_1$.

<u>Proof</u> Let

$$Y \equiv \hat{u}^2 + \hat{v}^2$$

and (2.82)

$$Z \equiv (\hat{u}y/\hat{v}) - \rho^{-1}$$

where, as before, $y = \hat{u}^2 - \hat{v}^2$. From (2.17), we find

$$\frac{dY}{d\rho} = 4\hat{v}^2 Z$$

and (2.83)

$$\frac{dZ}{d\rho} = (\hat{u}/\hat{v})\frac{dy}{d\rho} + (y/\hat{v}^2)\left[-Y - y^2 + 2(\hat{u}\hat{v}/\rho)\right] + \rho^{-2} .$$

As $\rho \to 0$, since $\hat{v}/\rho \to \frac{1}{3}\hat{u}(0)\left[1 + \hat{u}(0)^2\right]$, we have

$$\rho Z \to 2\left[1 + \hat{u}(0)^2\right]^{-1}\left[\hat{u}(0)^2 - \tfrac{1}{2}\right] . \qquad (2.84)$$

Hence, for $\hat{u}(0) < 1/\sqrt{2}$, $\rho Z < 0$ as $\rho \to 0$.

Suppose that, when $\hat{u}(0) < 1/\sqrt{2}$, $Z(\rho) = 0$ has a root between $\rho = 0$ and $\rho = \rho_1$. Let $\rho = \ell$ be the smallest such root. Thus, $Z(\rho)$ must increase at $\rho = \ell$; i.e.,

$$\left(\frac{dZ}{d\rho}\right)_\ell > 0 ,$$

where the subscript ℓ denotes $\rho = \ell$. Since $Z(\ell) = 0$, by using (2.83) we find

$$\left(\frac{dZ}{d\rho}\right)_\ell = (\hat{u}/\hat{v})_\ell \left(\frac{dy}{d\rho}\right)_\ell - (y/\hat{v}^2)_\ell\left[\hat{v}^2(\hat{v}^2 + \hat{u}^2 + 1) + \hat{u}^2(1 - 2\hat{u}^2)\right]_\ell . \qquad (2.85)$$

For $\hat{u}(0) < 1/\sqrt{2}$, which is less than both 1 and $\sqrt{2}$, we have $(dy/d\rho)_\ell < 0$ by Theorem 2, and $(1 - 2\hat{u}^2)_\ell > 0$ by Theorem 3. Hence, (2.85) leads to $(dZ/d\rho)_\ell < 0$, which contradicts the hypothesis that $Z(\rho) = 0$ has a root between 0 and ρ_1. From (2.82) and (2.84), we also see that when $\hat{u}(0) < 1/\sqrt{2}$, $dY/d\rho < 0$ as $\rho \to 0$. Theorem 4 is then proved.

Remarks

1. From our numerical solutions, we find that for $\hat{u}(0) > 1/\sqrt{2}$ (i.e., $n > 9.618$), the quantity $\hat{u}^2 + \hat{v}^2$ increases monotonically from $\rho = 0$ to a maximum at $\rho = \ell < \rho_1$, and then decreases monotonically from $\rho = \ell$ to $\rho = \rho_1$.

2. From (2.17), one sees that as $\rho \to 0$, $\hat{v}/\rho \to \frac{1}{3}\hat{u}(0)\left[1 + \hat{u}(0)^2\right] > 0$, and when $\rho = \rho_1$, $d\hat{v}/d\rho = \left[1 - (2/\rho_1)\right]\hat{u}(\rho_1)$, which is positive if $\rho_1 > 2$ (i.e., $\hat{u}(0) < .3066$, $n < 1.901$) and negative if $\rho_1 < 2$. From our numerical solutions, we find that $\hat{v}(\rho)$ has at most one maximum between $\rho = 0$ and $\rho = \rho_1$. Thus, if $\hat{u}(0) < .3066$, $\hat{v}(\rho)$ increases monotonically from $\rho = 0$ to $\rho = \rho_1$. If $\hat{u}(0) > .3066$, $\hat{v}(\rho)$ increases monotonically from $\rho = 0$ to a maximum at, say, $\rho = \rho_0' < \rho_1$, and then decreases monotonically from $\rho = \rho_0'$ to $\rho = \rho_1$.

3. An exact relation among n, q, ρ_1, and $\hat{u}_1$ may be derived by noting that (2.17) has the consequence

$$\frac{d}{d\rho}\left[\rho^3(\hat{u}^2 - \hat{v}^2)^2 + 2\rho^3(\hat{u}^2 + \hat{v}^2) - 4\rho^2\hat{u}\hat{v}\right] = 2\rho^2(\hat{u}^2 + \hat{v}^2) - \rho^2(\hat{u}^2 - \hat{v}^2)^2 . \tag{2.86}$$

Multiplying by 2π and integrating from 0 to ρ_1, we have

$$8\pi\hat{u}_1^2\rho_1^2(\rho_1 - 1) = n - \tfrac{1}{2}q . \tag{2.87}$$

III. Inclusion of Vector and Higgs Fields

In this section we consider the general case in which, in addition to the spin $\frac{1}{2}$ quark field ψ and the scalar gluon field σ introduced before, there are also the color SU(3) gauge field V_μ and the color Higgs field ϕ. Through the spontaneous symmetry-breaking mechanism,[14] the eight vector-field components of V_μ are all going to be massive; the number of scalar-field components of ϕ must, therefore, be more than eight. Since the color SU(3) is expected to remain a good (or, at least, approximately good) symmetry after the spontaneous symmetry-breaking mechanism, the Lagrangian density that one starts from should be invariant under a larger group $\mathcal{G}$ which includes the color SU(3) as a subgroup. There is a certain arbitrariness in choosing the group $\mathcal{G}$ and the representation of ϕ. For definiteness, we adopt the specific example discussed by Sirlin and ourselves in an earlier paper.[20] We assume $\mathcal{G}$ to be SU(3) X SU(3) and ϕ to form the (3, 3) representation of $\mathcal{G}$. [In addition to $\mathcal{G}$, there is the usual "flavor" SU(3), or SU(4).] Thus, ϕ consists of nine complex scalar fields ϕ_0 and ϕ_a where, as well as throughout the paper, the subscripts

$$\begin{aligned} &a, b, c \text{ vary from } 1 \text{ to } 8 \ , \\ &\mu, \nu, \lambda \text{ vary from } 1 \text{ to } 4 \\ \text{and} \quad &i, j, k \text{ vary from } 1 \text{ to } 3 \ . \end{aligned} \tag{3.1}$$

It is convenient to represent the gauge field and the Higgs field by 3 X 3 matrices:

$$\begin{aligned} V_\mu &\equiv \tfrac{1}{2}\lambda_a (V_\mu)_a \ , \\ \phi &\equiv \tfrac{1}{2}\lambda_0 \phi_0 + \tfrac{1}{2}\lambda_a \phi_a \end{aligned} \tag{3.2}$$

where $\lambda_0 = (\frac{2}{3})^{\frac{1}{2}}$ times the 3 X 3 unit matrix, and λ_a's are the 3 X 3 Gell-Mann matrices which satisfy the usual relations

$$\mathrm{tr}\,(\lambda_a \lambda_b) = 2\delta_{ab} \quad , \quad [\lambda_a , \lambda_b] = 2i\, F_{abc}\, \lambda_c \quad ,$$

and (3.3)

$$\{\lambda_a , \lambda_b\} = 2\, D_{abc}\, \lambda_c + \tfrac{4}{3}\, \delta_{ab} \quad .$$

All repeated indices are to be summed over. The gauge field forms a (8, 1) representation of $\mathcal{G}$, the gluon field σ is invariant under $\mathcal{G}$, and each of the "flavor"-components of the quark field ψ^k forms a (3, 1) representation of $\mathcal{G}$. In terms of the components ψ_j^k introduced in Eq. (2.1), we may write

$$\psi^k = \begin{pmatrix} \psi_1^k \\ \psi_2^k \\ \psi_3^k \end{pmatrix} \quad . \qquad (3.4)$$

The group $\mathcal{G}$ = SU(3) X SU(3) consists of the transformations

$$V_\mu \rightarrow u\, V_\mu\, u^\dagger \quad , \qquad \phi \rightarrow u\, \phi\, v^\dagger \quad ,$$

$$\psi^k \rightarrow u\, \psi^k \qquad \text{and} \qquad \sigma \rightarrow \sigma \qquad (3.5)$$

where u and v are two arbitrary x-independent 3 X 3 unitary matrices with det = 1 .

The Lagrangian density $\mathcal{L}$ is assumed to be invariant under a local SU(3) gauge transformation

$$V_\mu \rightarrow u(x)\, V_\mu\, u(x)^\dagger - \frac{i}{f}\left(\frac{\partial u(x)}{\partial x_\mu}\right) u(x)^\dagger \quad ,$$

$$\phi \rightarrow u(x)\, \phi \quad , \qquad \psi^k \rightarrow u(x)\, \psi^k \qquad (3.6)$$

and $\qquad \sigma \rightarrow \sigma$

where $u(x)^\dagger u(x) = 1$ and $\det u(x) = 1$; in addition, $\mathcal{L}$ is invariant under the global $\mathcal{G} \times SU(3)$ transformations, where $\mathcal{G}$ is given by (3.5), and the extra SU(3) group denotes the usual "flavor" transformations, under which V_μ , ϕ and σ are all invariant, but

$$\psi^k \rightarrow w_j^k \, \psi^j \tag{3.7}$$

where $w \equiv (w_j^k)$ is another x-independent 3×3 unitary matrix with $\det = 1$.

[The generalization of the "flavor" transformation group to SU(4) is straightforward.]

The general renormalizable form of $\mathcal{L}$ can be readily found:

$$\mathcal{L} = -\operatorname{tr}\left[\tfrac{1}{2} V_{\mu\nu}^2 + (\bar{D}_\mu \phi^\dagger)(D_\mu \phi)\right] - \psi^{k\dagger} \gamma_4 (\gamma_\mu D_\mu + g\sigma)\psi^k - \tfrac{1}{2}\left(\frac{\partial \sigma}{\partial x_\mu}\right)^2 - U(\sigma, \phi) \tag{3.8}$$

where

$$\begin{aligned}
V_{\mu\nu} &\equiv \tfrac{1}{2}\lambda_a (V_{\mu\nu})_a = \frac{\partial}{\partial x_\mu} V_\nu - \frac{\partial}{\partial x_\nu} V_\mu - if\left[V_\mu, V_\nu\right] , \\
D_\mu \phi &= \frac{\partial}{\partial x_\mu} \phi - if V_\mu \phi , \\
\bar{D}_\mu \phi^\dagger &= \frac{\partial}{\partial x_\mu} \phi^\dagger + if \phi^\dagger V_\mu , \\
D_\mu \psi^k &= \frac{\partial}{\partial x_\mu} \psi^k - if V_\mu \psi^k
\end{aligned} \tag{3.9}$$

and $U(\sigma, \phi)$ is a fourth order polynomial in σ and ϕ . Because of our convention $x_\mu = (\vec{r}, it)$, we have

$$\bar{D}_j \phi^\dagger = (D_j \phi)^\dagger \quad \text{and} \quad \bar{D}_4 \phi^\dagger = -(D_4 \phi)^\dagger \quad . \tag{3.10}$$

As already explained in the introduction, the function $U(\sigma, \phi)$ satisfies (1.2) and (1.4); i.e., it has an absolute minimum at $(\sigma, \phi) = (\sigma_{vac}, \phi_{vac})$ and a local minimum at the origin $(\sigma, \phi) = (0, 0)$, with

$$U(\sigma_{vac}, \phi_{vac}) = 0$$

and (3.11)

$$U(0, 0) \equiv p > 0 \ .$$

Both σ_{vac}, ϕ_{vac} are assumed to be $\neq 0$. The general form of $U(\sigma, \phi)$ that satisfies these properties still contains a rather large number of constants $a, b, \cdots,$ $a', b', \cdots, a'', b'', \cdots,$ defined as follows:

$$\begin{aligned} U(\sigma, \phi) = & \tfrac{1}{2} a \sigma^2 + \tfrac{1}{3!} b \sigma^3 + \tfrac{1}{4!} c \sigma^4 + p \\ & + a' \, tr(\phi^\dagger \phi) + \tfrac{1}{2}\left[b' \det \phi + (b' \det \phi)^\dagger\right] \\ & + c' \, tr\left[(\phi^\dagger \phi)^2\right] + d' \left[tr(\phi^\dagger \phi)\right]^2 + (a'' \sigma + c'' \sigma^2) \, tr(\phi^\dagger \phi) \\ & + \tfrac{1}{2} \sigma \left[b'' \det \phi + (b'' \det \phi)^\dagger\right] \ . \end{aligned} \tag{3.12}$$

At first sight, it seems almost unmanageable to discuss such a general case with so many independent constants. As we shall see, the problem is actually quite simple, provided that the parameters ξ and η, introduced in (1.11), are both small, $\ll 1$. Of course, in the present general case because σ is coupled to ϕ, there are many scalar masses. The definition of m_σ used in (1.11) has to be made precise. [See (3.30) below.]

To begin with, we may adopt the unitary gauge by choosing the transformation $u(x)$ in (3.6) so that for $a = 1, 2, \cdots 8$,

$$tr\left[\lambda_a(\phi - \phi^\dagger)\right] = 0 \tag{3.13}$$

everywhere.[20] We may then write

$$\sigma = \sigma_{vac} + R' \ ,$$

$$\phi = \phi_{vac} + \tfrac{1}{2}\lambda_0(R + i\,I) + H \qquad (3.14)$$

where

$$H = \tfrac{1}{2}\lambda_a H_a \quad ,$$

and R, R', I and H_a are all Hermitian fields. For simplicity, we assume the constants b' and b'' in (3.12) to be real, and therefore ϕ_{vac} is real. Because $(\sigma, \phi) = (\sigma_{vac}, \phi_{vac})$ is the absolute minimum of U, $\frac{\partial U}{\partial \sigma} = \frac{\partial U}{\partial R} = \frac{\partial U}{\partial I} = \frac{\partial U}{\partial H_a} = 0$ at $(\sigma, \phi) = (\sigma_{vac}, \phi_{vac})$. Near $(\sigma, \phi) = (\sigma_{vac}, \phi_{vac})$, we may expand

$$U(\sigma, \phi) = \tfrac{1}{2}\left[M_{RR}R^2 + 2\,M_{RR'}\,RR' + M_{R'R'}\,R'^2\right] + \tfrac{1}{2}\,m_I^2\,I^2 + \tfrac{1}{2}\,m_H^2\,H_a^2 + \cdots \qquad (3.15)$$

where $\cdots$ denotes cubic and higher order terms in the fields R, R', I and H_a. The mass-squares m_I^2, m_H^2 and the eigenvalues m_1^2 and m_2^2 of the matrix

$$M \equiv \begin{pmatrix} M_{RR} & M_{RR'} \\ M_{RR'} & M_{R'R'} \end{pmatrix} \qquad (3.16)$$

are all positive; these parameters are related to the constants a, b, c, a', b', $\cdots$ by

$$m_I^2 = a' - b'\phi_{vac} + 2(c' + 3d')\phi_{vac}^2 + (a'' - b''\phi_{vac} + c''\sigma_{vac})\sigma_{vac} \quad ,$$

$$m_H^2 = a' - \tfrac{1}{2}b'\phi_{vac} + 6(c' + d')\phi_{vac}^2 + (a'' - \tfrac{1}{2}b''\phi_{vac} + c''\sigma_{vac})\sigma_{vac} \quad ,$$

$$M_{RR} = a' + b'\phi_{vac} + 6(c' + 3d')\phi_{vac}^2 + (a'' + b''\phi_{vac} + c''\sigma_{vac})\sigma_{vac} \quad , \qquad (3.17)$$

$$M_{R'R'} = a + b\,\sigma_{vac} + \tfrac{1}{2}\,c\,\sigma_{vac}^2 + 6c''\phi_{vac}^2$$

and

$$M_{RR'} = \sqrt{6}\,(a'' + \tfrac{1}{2}b''\phi_{vac} + 2\,c''\sigma_{vac})\,\phi_{vac} \quad .$$

It can be readily verified that after the spontaneous symmetry-breaking, the system remains symmetric under a global (i.e., x-independent) "color" SU(3) transformation $\{u\}$:

$$V_\mu \rightarrow u V_\mu u^\dagger \quad , \quad H \rightarrow u H u^\dagger \quad ,$$
$$\psi^k \rightarrow u \psi^k \tag{3.18}$$

and R, R' and I are all invariant; of course, the "flavor" SU(3) symmetry (3.7) also remains valid.

When $(\sigma, \phi) = (\sigma_{vac}, \phi_{vac})$, the masses of the vector field V_μ and the quark field ψ^k are given respectively by

$$m_V = f \phi_{vac} \qquad \text{and} \qquad m_q = g \sigma_{vac} \quad ; \tag{3.19}$$

they are both also assumed to be large, $\gg 1$ GeV. When $(\sigma, \phi) = (0, 0)$, both fields V_μ and ψ^k are of zero mass.

Near the origin $(\sigma, \phi) = (0, 0)$, we have

$$U(\sigma, \phi) = p + \tfrac{1}{2} a \sigma^2 + a' \, \mathrm{tr}(\phi^\dagger \phi) + \cdots \tag{3.20}$$

where $\cdots$ denotes cubic and higher order terms in σ and ϕ. Clearly, both constants a and a' are > 0, in order that the origin be a local minimum of U. In the present case, there are many scalar masses. For simplicity, we assume all scalar masses in the theory m_I, m_H, m_1, m_2, $a^{\frac{1}{2}}$ and $a'^{\frac{1}{2}}$ to be large [where m_1^2 and m_2^2 are the eigenvalues of the matrix (3.16)], $\gg$ the lowest soliton mass ~ 1 GeV. Furthermore, for simplicity we assume them to be all of the same order of magnitude. It is appropriate to call

$$a^{\frac{1}{2}} = \text{"}\sigma\text{-mass" near the origin.} \tag{3.21}$$

As we shall see, $a^{\frac{1}{2}}$ is relevant for the description of the interior of the soliton. For the surface of the soliton a different definition of "σ - mass" will be introduced. In order to do that, let us consider the following (hypothetical) problem of a topological soliton solution in one space-dimension.

In this (hypothetical) problem, $x_\mu = (x, it)$ and the Lagrangian density is

$$\mathcal{L}_0 \equiv -\tfrac{1}{2}\left(\frac{\partial\sigma}{\partial x_\mu}\right)^2 - \mathrm{tr}\left(\frac{\partial\phi_0^\dagger}{\partial x_\mu}\ \frac{\partial\phi_0}{\partial x_\mu}\right) - U_0(\sigma, \phi_0) \tag{3.22}$$

where

$$\phi_0 \equiv \phi_{vac} + \tfrac{1}{2}\lambda_0(R + iI)$$

and U_0 is related to the same U in (3.8) by

$$U_0(\sigma, \phi_0) \equiv \lim_{p\to 0} U(\sigma, \phi_0) \ , \tag{3.23}$$

such that the limiting function $U_0(\sigma, \phi_0)$ has two absolute minima, at $(\sigma, \phi_0) = (\sigma_{vac}, \phi_{vac})$ and $(\sigma, \phi_0) = (0, 0)$, with

$$U_0(0, 0) = U_0(\sigma_{vac}, \phi_{vac}) = 0 \ . \tag{3.24}$$

It is straightforward to see that there is a t-independent topological soliton solution, which satisfies

$$\tfrac{1}{2}\left(\frac{d\sigma}{dx}\right)^2 + \tfrac{1}{2}\left(\frac{dR}{dx}\right)^2 + \tfrac{1}{2}\left(\frac{dI}{dx}\right)^2 - U_0 = 0 \ . \tag{3.25}$$

A convenient way to visualize the solution is to consider the mechanical analog problem of a point particle of unit mass, whose "position" coordinate is (σ, ϕ_0) [i.e., (σ, R, I)] and whose "time" coordinate is x, moving in a "potential" $-U_0$. Equation (3.25), then, denotes simply the law of conservation of "energy" of the particle.

According to (3.24), the "potential" $-U_0$ has two peaks at $(\sigma, \phi_0) = (0, 0)$ and $(\sigma, \phi_0) = (\sigma_{vac}, \phi_{vac})$. There is clearly a solution, described by a path P, in which at "time" $x = -\infty$, the particle is on one of the two peaks, but when $x \to +\infty$, it moves onto the other peak. The corresponding 1 space-dimensional soliton solution is

$$\begin{aligned} &\sigma_P(x) \equiv \sigma(x) \\ \text{and} \quad & \\ &\phi_P(x) \equiv \phi_0(x) \qquad \text{along } P . \end{aligned} \tag{3.26}$$

Its energy is given by the path integral along P:

$$2\int_{-\infty}^{\infty} U_0\, dx = \frac{1}{6} a^{\frac{1}{2}} \sigma_{vac}^2 \gamma \equiv \frac{1}{6} \bar{m}_\sigma \sigma_{vac}^2 \tag{3.27}$$

where $a^{\frac{1}{2}}$ is as introduced in (3.21) and γ is a dimensionless number. In accordance with (1.6) and (1.8),

$$\begin{aligned} &\bar{m}_\sigma = a^{\frac{1}{2}} \gamma \\ \text{and} \quad & \\ &\mu = (\bar{m}_\sigma \sigma_{vac}^2)^{\frac{1}{3}} . \end{aligned} \tag{3.28}$$

We now define

$$m_\sigma \equiv \bar{m}_\sigma / \gamma^2 = a^{\frac{1}{2}} / \gamma \tag{3.29}$$

and, as before in (1.11),

$$\xi = (\mu / m_q)^2 \qquad \text{and} \qquad \eta = (\mu / m_\sigma) . \tag{3.30}$$

The purpose of these definitions is to make the quantity $\mu^2 \xi / \eta$ independent of γ, so as to justify the second line of (3.35) below. Then γ will not appear in the final equations (3.41) and (3.42).

We recall again that if the system consists of only the quark field and a single scalar field σ, without the Higgs field ϕ, then as in Sec. II, by solving the corresponding one-dimensional problem for $U_0 = \frac{1}{2}a(\sigma - \sigma_{vac})^2 (\sigma/\sigma_{vac})^2$, we would obtain

$$\gamma = 1 \qquad \text{and} \qquad m_\sigma = \bar{m}_\sigma = a^{\frac{1}{2}} .$$

The definition of μ given above by (3.28) then becomes identical to that of (2.11) in Sec. II; the same applies to the definitions of ξ and η. In the following, for convenience of order of magnitude estimations, we regard

$$\gamma = O(1) . \qquad (3.31)$$

We now return to the original Lagrangian (3.8). For color singlet states, we may set in the quasi-classical solution

$$V_\mu = H = 0 ,$$

and σ, R, I to be all c.number functions. Just as in (2.8), for color singlet states with a quark number $N = 2$ or 3, the soliton energy is given by

$$E = N\epsilon + \int \left[\tfrac{1}{2}(\vec{\nabla}\sigma)^2 + \tfrac{1}{2}(\vec{\nabla}R)^2 + \tfrac{1}{2}(\vec{\nabla}I)^2 + U(\sigma, \phi) \right] d^3r \qquad (3.32)$$

where $\phi = \phi_{vac} + \frac{1}{2}\lambda_0 (R + iI)$, ϵ is the lowest <u>positive</u> eigenvalue of the c.number Dirac equation

$$(-i\vec{\alpha}\cdot\vec{\nabla} + g\beta\sigma)\psi = \epsilon\psi , \qquad (3.33)$$

and σ, R and I satisfy

$$-\nabla^2\sigma + \frac{\partial}{\partial\sigma} U = -gN\psi^\dagger\beta\psi ,$$

$$-\nabla^2 R + \frac{\partial}{\partial R} U = 0 \tag{3.34}$$

and

$$-\nabla^2 I + \frac{\partial}{\partial I} U = 0 \quad .$$

Assuming that the two parameters ξ and η , defined above, are both small, $\ll 1$, we may now go through exactly the same argument used in Sec. II.3 . We first divide the space into three regions: inside, outside and transition, in accordance with (2.28).

In the inside region $r \leqq R_1$, we have

$$\phi = 0 \quad .$$

So far as the solution σ and Ψ is concerned, the entire discussion given in Sec. II.3, from (2.33)-(2.36) can be carried over to the present case, without any change except that m_σ is replaced by $a^{\frac{1}{2}}$; therefore, just as in (2.16), (2.25), (2.26), (2.34) and (2.36), we have, for the present case, also

$$\rho = \epsilon r \ , \qquad \rho_1 = \epsilon R \ ,$$

$$n = (g\epsilon)^2 N/a = \epsilon^2 N \eta/(\mu^2 \xi) \ ,$$

$$u = (\epsilon^3/n)^{\frac{1}{2}} \hat{u} \ , \qquad v = (\epsilon^3/n)^{\frac{1}{2}} \hat{v} \tag{3.35}$$

and

$$\sigma \cong -(Ng/a)(u^2 - v^2)$$

where μ , ξ and η are defined by (3.28) and (3.30). Equations (3.33) and (3.34) can now be again reduced to (2.17), with $\hat{u}$ and $\hat{v}$ related to Ψ through (2.13) and (3.35).

In the outside region $r \geqq R_2$, the present case is slightly more complicated than the simple system discussed in Sec. II. Both ϕ and σ rise from zero, or near zero, to their respective vacuum values σ_{vac} and ϕ_{vac} . This results in the replacement of (2.38) by

$$\sigma(R) \cong \sigma_p(r-R_0)$$

and (3.36)

$$\phi(R) \cong \phi_p(r-R_0)$$

where σ_p and ϕ_p are the appropriate 1 space-dimensional solutions given by (3.26). Equations (2.39) and (2.41)-(2.43) remain valid. Just as in (2.44), in the present general case, the energy integrated over the outside region is, because of (3.27),

$$\tfrac{2}{3}\pi R^2 \bar{m}_\sigma \sigma_{vac}^2 = \tfrac{2}{3}\pi R^2 \mu^3 = 4\pi R^2 s \quad ,$$

in accordance with (1.6).

In the transition region $R_1 \leq r \leq R_2$, the entire argument in Sec. II.3, leading from (2.45) to (2.58), is applicable, except that m_σ is replaced by $a^{\frac{1}{2}}$. Thus, the soliton energy E, defined by (3.32), is given by

$$E = N\epsilon\left[1+\tfrac{1}{2}(q/n)\right] + \tfrac{2}{3}\pi R^2\mu^3 + \tfrac{4}{3}\pi R^3 p \qquad (3.37)$$

which is identical to (2.23). Next, we multiply the three equations in (3.34) by $d\sigma/dr$, dR/dr and dI/dr respectively; after integrating from r to ∞, we obtain the generalization of (2.59):

$$\tfrac{1}{2}\left(\frac{d\sigma}{dr}\right)^2 + \tfrac{1}{2}\left(\frac{dR}{dr}\right)^2 + \tfrac{1}{2}\left(\frac{dI}{dr}\right)^2 - U$$

$$= \int_r^\infty dr \left\{\frac{2}{r}\left[\left(\frac{d\sigma}{dr}\right)^2 + \left(\frac{dR}{dr}\right)^2 + \left(\frac{dI}{dr}\right)^2 - Ng\,(u^2-v^2)\frac{d\sigma}{dr}\right]\right\} . \qquad (3.38)$$

By going through the same argument, which is given immediately after (2.59) in Sec. II.3, but with (2.38) replaced by (3.36) and m_σ by $a^{\frac{1}{2}}$, we find that, at $r = R_1$, after neglecting $O(\eta)$ as compared to 1, (3.38) becomes

$$-p = \tfrac{1}{3}(\mu^3/R) - 2N\upsilon_1^2(\epsilon - R^{-1}) , \tag{3.39}$$

which is again identical to (2.24). By using the third equation in (3.35), one sees that

$$\epsilon = \mu(n/N)^{\frac{1}{2}} (\xi/\eta)^{\frac{1}{2}} . \tag{3.40}$$

Consequently, (3.37) and (3.39) can also be written in a form identical to (2.21) and (2.22):

$$\frac{E}{\mu} = \left(\frac{N}{n}\right)^{\frac{1}{2}} \left(\frac{\xi}{\eta}\right)^{\frac{1}{2}} (n + \tfrac{1}{2}q) + \tfrac{8}{3}\pi\left(\frac{N}{n}\right)\left(\frac{\eta}{\xi}\right)\rho_1^2 + \tfrac{4}{3}\pi\left(\frac{N}{n}\right)^{\frac{3}{2}}\left(\frac{\eta}{\xi}\right)^{\frac{3}{2}} \lambda\rho_1^3 \tag{3.41}$$

and

$$0 = -2\left(\frac{n}{N}\right)^{\frac{1}{2}}\left(\frac{\xi}{\eta}\right)^{\frac{1}{2}}(\rho_1 - 1)\,\hat{\upsilon}_1^2 + \tfrac{1}{3}\left(\frac{\eta}{\xi}\right) + \left(\frac{N}{n}\right)^{\frac{1}{2}}\left(\frac{\eta}{\xi}\right)^{\frac{3}{2}} \lambda\rho_1 \tag{3.42}$$

where $\lambda = p/\mu^4$ is defined by (1.9). Thus, the theorem stated in Sec. II.2 is applicable to the general case as well, provided that μ, ξ and η are defined by (3.28)-(3.30).

Through (3.35) we may also use (3.37) to determine the function $E = E(R)$. By following the same argument given in Appendix A, but replacing m_σ by $a^{\frac{1}{2}}$, we can show that (3.39) is equivalent to the condition $dE(R)/dR = 0$, just as in the simple case, discussed in Sec. II.

IV. Static Properties of Hadrons

We start from the general system considered in Sec. III, and assume, as before, that the parameters $\xi \equiv (\mu/m_q)^2$ and $\eta \equiv (\mu/m_\sigma)$ are both small, $\ll 1$. As we have seen, independently of the number of parameters in the original Lagrangian (3.8), in the limit when ξ and $\eta \to 0$, at a fixed though arbitrary ratio η/ξ, the low-lying soliton states, at a given $N = 2$ or 3, depend only on an overall energy scale μ and two dimensionless parameters $\lambda = p/\mu^4$ and η/ξ. The application of these soliton solutions to the observed hadrons will be discussed in this section.

1. Phenomenological description

For the moment, let us leave aside the soliton problem, and discuss instead a hypothetical <u>analog</u> system, consisting of a "gas bubble" of radius R immersed in a "medium". We define

$$\begin{aligned} E_g &\equiv \text{"thermodynamical" energy of the gas,} \\ s &\equiv \text{surface tension,} \\ \text{and} \quad p &\equiv \text{pressure exerted on the gas by the medium.} \end{aligned} \tag{4.1}$$

Each of these terms contributes a part to the (total) energy of the system, which may be written as a sum

$$E = E_g + E_s + E_p \tag{4.2}$$

where, under the assumption that s and p are both constants,

$$E_s = 4\pi R^2 s \qquad \text{and} \qquad E_p = \frac{4\pi}{3} R^3 p \quad . \tag{4.3}$$

The radius R is determined by

$$dE/dR = 0 \quad . \tag{4.4}$$

The appropriate thermodynamical energy E_g to be used depends on, among other things, the heat transfer condition (e.g., isothermal or adiabatic); its dependence on R can be rather complicated. However, so far as the equilibrium configuration and its immediate neighborhood are concerned, we may assume a simple power law

$$E_g = K/R^k \tag{4.5}$$

where k and K are both positive constants. Equation (4.4) gives

$$k\,E_g = 2\,E_s + 3\,E_p \quad . \tag{4.6}$$

It is convenient to introduce

$$\ell \equiv \frac{E_p}{E_p + E_s} \quad . \tag{4.7}$$

Hence,

$$\frac{E_g}{E} = \frac{2+\ell}{2+\ell+k} \quad . \tag{4.8}$$

This simple system carries four constants: s , p , k and K , or, the equivalent set

$$E\ ,\quad R\ ,\quad k \quad \text{and} \quad \ell \quad . \tag{4.9}$$

Returning now to the field-theoretic problem we see that, by comparing (4.2) with (3.37), the phenomenological description used above can be directly transferred to the soliton solution. The "thermodynamical energy of the gas" is

$$E_g = N\epsilon\left[1 + \tfrac{1}{2}(q/n)\right] \quad . \tag{4.10}$$

In addition, there is a surface energy $E_s = 4\pi R^2 s = \frac{2}{3}\pi R^2 \mu^3$ due to the "surface tension" $s = \frac{1}{6}\mu^3$ and a volume energy $E_p = \frac{4\pi}{3} R^3 p$ due to the "pressure" p of the surrounding "medium", which is really the vacuum, since according to (1.2) and (1.4), $p = U(0,0) - U(\sigma_{vac}, \phi_{vac})$. The resulting sum of these three energies is exactly (3.37). A more general definition of the exponent k introduced in (4.5) is

$$k = -\frac{d \ln E_g}{d \ln R} \quad . \tag{4.11}$$

By comparing (3.39) with (4.6) [or by directly differentiating (4.10), as done explicitly in Appendix A], and by using (2.87), we find that for the soliton problem, k is a function only of n, given by

$$k(n) = 8\pi \hat{u}_1^2 \rho_1^2 (\rho_1 - 1)/(n + \tfrac{1}{2}q) = \frac{n - \frac{1}{2}q}{n + \frac{1}{2}q} \tag{4.12}$$

where $\hat{u}_1(n)$, $\rho_1(n)$ and $q(n)$ are all defined in Sec. II; these functions are determined by the solutions of the reduced equation (2.17). In Figure 6a, n is plotted vs. the initial value $\hat{u}(0)$ of the solution $\hat{u}(\rho)$ of (2.17); likewise, in Figure 6b, ρ_1, k and q/n are also plotted vs. $\hat{u}(0)$. The functions $\rho_1(n)$, $k(n)$ and $q(n)$ can then be deduced from these two figures by eliminating $\hat{u}(0)$.

As noted in (4.9), the "gas bubble" problem is characterized by four phenomenological constants. On the other hand, the soliton solution (at a given $N = 2$ or 3) depends only on three parameters: μ, $\lambda = p/\mu^4$ and η/ξ. By using (2.26), (4.8) and (4.10), we find

$$RE = \frac{2 + \ell + k}{n(2 + \ell)} \, (n + \tfrac{1}{2}q) \, N\rho_1 \quad , \tag{4.13}$$

which together with (4.12) introduces a constraint on the four parameters in (4.9).

If the complete Lagrangian is known, then η/ξ , $\mu = (6s)^{\frac{1}{3}}$ and p are all determined; among these, μ and p can be directly used in a phenomenological description, while the physical interpretation of η/ξ is a less direct one. For phenomenological descriptions, a different choice of the three independent parameters can be either k , s and p , or, since $k = k(n)$,

$$n \ , \ s \ \text{ and } \ p \ . \tag{4.14}$$

Of course, since η/ξ , $\lambda = p/\mu^4$ and n satisfy (3.42), all these sets of parameters are equivalent. We note that from Figures 6a and 6b, the function $k(n)$ is single-valued, while its inverse $n(k)$ is double-valued. Hence, the set (4.14) may well be the most convenient one to use.

From Figure 6b, one sees that $k = 1$ and $q/n = 0$ at both limits $n \to 0$ and $n \to \infty$. At $n = 79$ $(\hat{u}(0) = 1.42)$, k has a minimum and q/n a maximum; the bounds thus set are

$$k \geqq .7895 \qquad \text{and} \qquad q/n \leqq .2352 \ . \tag{4.15}$$

As the ratio $\lambda = p/\mu^4$ varies from 0 to ∞ , one sees that by using (4.3) and (4.7) ℓ also varies from 0 to 1 . Thus, from (4.8), it follows that at any given $k = k(n)$

$$\frac{2}{2+k} \leqq \frac{E_g}{E} \leqq \frac{3}{3+k} \ , \tag{4.16}$$

which together with (4.15) leads to

$$\frac{2}{3} \leqq \frac{E_g}{E} \leqq 0.7902 \ . \tag{4.17}$$

Also, from (4 .10) and (4.15) we obtain

$$0.8905 \leqq \frac{N\epsilon}{E_g} \leqq 1 \ . \tag{4.18}$$

Similarly, we can set bounds on $N\epsilon/E$ and RE/N. At a given n, we have

$$\frac{2n}{(n+\frac{1}{2}q)(2+k)} \leq \frac{N\epsilon}{E} \leq \frac{3n}{(n+\frac{1}{2}q)(3+k)} \qquad (4.19)$$

and, since $R\epsilon = \rho_1$,

$$\left[1+\tfrac{1}{2}(q/n)\right](1+\tfrac{1}{3}k)\,\rho_1 \leq \frac{RE}{N} \leq \left[1+\tfrac{1}{2}(q/n)\right](1+\tfrac{1}{2}k)\,\rho_1 \,. \qquad (4.20)$$

By using (4.12) and (4.15), we find

$$.641 \leq \frac{N\epsilon}{E} \leq \frac{3}{4}$$

and

$$\frac{4}{3} \leq \frac{RE}{N} \leq 3.0642 \,. \qquad (4.21)$$

The upper bound on RE/N is reached as $n \to 0$.

2. Baryon and meson masses

In our model, the low-lying solitons are color singlets; the color nonsinglets have all been unglued by the strongly interacting vector gauge field. These low-lying solitons will be identified as the observed hadrons. Within our approximation, the energy levels exhibit a typical SU(6) degeneracy.[11] (For the present discussion, we assume the quarks have only three "flavors".) The baryons are the color singlets of the three-quark system; the lowest energy state belongs to the 56 representation of SU(6), which consists of the usual spin $\frac{3}{2}$ SU(3)-decuplet and the usual spin $\frac{1}{2}$ SU(3)-octet. The mesons are the color singlets of the quark-antiquark system. The lowest energy meson states have a 36-fold degeneracy, consisting of two SU(6) representations, 35 and 1; alternatively, these states can also be resolved into the usual vector and pseudoscalar nonets. The mass of these soliton solutions is given by (3.37) and (3.39). We have

and
$$E = m_B \qquad \text{for} \quad N = 3$$
$$E = m_M \qquad \text{for} \quad N = 2 \tag{4.22}$$

where m_B denotes the lowest baryon mass averaged over the 56 representation, and m_M the corresponding lowest meson mass averaged over the vector and pseudoscalar nonets.

Of course, we may also adopt the phenomenological description developed in the previous section. For definiteness, we may choose, as in (4.14), n, $s = \frac{1}{6}\mu^3$ and p to be the independent phenomenological constants in the theory. It is instructive to first examine some limiting cases:

(i) $n \to 0$

From (2.64), (4.10) and Figure 6b, we see that in this limit,

$$\rho_1 = 2.0428 \;, \qquad k = 1 \;,$$
$$q/n = 0 \qquad \text{and} \qquad E_g = N\rho_1/R \;. \tag{4.23}$$

Hence, (4.2) and (4.6) become

$$E = N\rho_1 R^{-1} + 4\pi R^2 s + \tfrac{4}{3}\pi R^3 p$$
and
$$N\rho_1 = 8\pi R^3 s + 4\pi R^4 p \;. \tag{4.24}$$

The problem is then completely determined by the two remaining constants $s = \frac{1}{6}\mu^3$ and p. By using (2.13), (2.63) and (3.35), we know that in this limit the charge density $\psi^\dagger\psi$ and the scalar density $\psi^\dagger\beta\psi$ of the quark wave functions are distributed entirely <u>within the soliton volume.</u> [See especially Theorems 2 and 4 in Sec. II.4.] Furthermore, in this limit, since $E_g = N\epsilon$, the scalar fields (gluon σ and Higgs ϕ) only contribute directly to the volume energy $E_p = \frac{4}{3}\pi R^3 p$ and the surface energy $E_s = 4\pi R^2 s = \frac{2}{3}\pi R^2 \mu^3$. The following are two extreme cases:

Case (ia). In addition to $n \to 0$, we may take the limit $s \to 0$. Thus, $E_s = 0$, and we find

$$N = 4\pi(p/2.0428)\, R^4 \quad ,$$

$$E = \tfrac{4}{3} N\epsilon = \tfrac{4}{3}\sqrt{2}\, (\pi p)^{\frac{1}{4}}\, (2.0428)^{\frac{3}{4}}\, N^{\frac{3}{4}} \tag{4.25}$$

and

$$\frac{m_M}{m_B} = (\tfrac{2}{3})^{\frac{3}{4}} \quad .$$

This double limit $n \to 0$ and $s \to 0$ gives the Creutz-Soh version[5] of the MIT bag.[7] [We note that the description of the vector gauge field in our model is quite different from that in the MIT bag. Also, our model does not give permanent quark confinement, except in the limit when $m_q = \infty$.]

Case (ib). In the double limit $n \to 0$ and $p \to 0$, then $E_p = 0$, and because $s = \frac{1}{6}\mu^3$, we have in place of (4.26)

$$N = \tfrac{4}{3}\pi(\mu R)^3/2.0428 \quad ,$$

$$E = \tfrac{3}{2} N\epsilon = (9\pi/2)^{\frac{1}{3}}\, (2.0428)^{\frac{2}{3}}\, \mu\, N^{\frac{2}{3}} \tag{4.26}$$

and

$$\frac{m_M}{m_B} = (\tfrac{2}{3})^{\frac{2}{3}} \quad .$$

(ii) $n \to \infty$

From (2.70), (4.10) and Figure 1b, we see that in this limit

$$\rho_1 = 1 \quad , \qquad k = 1 \quad ,$$

and (4.27)

$$q/n = 0 \qquad \text{and} \qquad E_g = N/R \ .$$

Hence, (4.2) and (4.6) become

$$E = NR^{-1} + 4\pi R^2 s + \tfrac{4}{3}\pi R^3 p$$

and (4.28)

$$N = 8\pi R^3 s + 4\pi R^4 p \qquad .$$

By using (2.67)-(2.69), we find that the charge density of the quark wave function $\psi^\dagger \psi \propto \hat{u}^2 + \hat{v}^2 \cong 2x$ of (2.65) now concentrates entirely <u>on the surface</u> $r = R$ of the soliton solution. The corresponding scalar density $\psi^\dagger \beta \psi \propto \hat{u}^2 - \hat{v}^2 = y$ of (2.65) also peaks near the surface at $r = R\left[1 - O(n^{-\frac{1}{2}})\right]$, but then drops quickly to zero at $r = R$. While the quark wave function in these two limiting cases, $n \to 0$ and ∞, behaves totally differently, the gluon and the Higgs fields exhibit the same characteristics. Since $E_g = N\epsilon$ in both limits, the scalar fields contribute only directly to E_s and E_p. Again, we examine two extreme cases:

Case (iia). In the double limit $n \to \infty$ and $s \to 0$, we have $E_s = 0$,

$$N = 4\pi R^4 p \ ,$$

$$E = \tfrac{4}{3} N\epsilon = \tfrac{4}{3}\sqrt{2}\ (\pi p)^{\frac{1}{4}}\ N^{\frac{3}{4}} \tag{4.29}$$

and

$$\frac{m_M}{m_B} = (\tfrac{2}{3})^{\frac{3}{4}} \qquad .$$

Case (iib). In the double limit $n \to \infty$ and $p \to 0$, we have $E_p = 0$, and since $s = \frac{1}{6}\mu^3$,

$$N = \tfrac{4}{3}\pi(\mu R)^3 \ ,$$

$$E = \tfrac{3}{2} N\epsilon = (9\pi/2)^{\frac{1}{3}} \mu N^{\frac{2}{3}} \tag{4.30}$$

and

$$\frac{m_M}{m_B} = (\tfrac{2}{3})^{\frac{2}{3}} .$$

In case (iib), both the quark wave function and the energy density of the gluon and the Higgs field concentrate on the surface of the soliton, similar to the SLAC bag.[3] [Note, however, in our field-theoretic model the symmetric point $(\sigma, \phi) = (0, 0)$ is a local minimum of $U(\sigma, \phi)$, while in the SLAC version, it corresponds to a local maximum. In order to have the vector gluon be effective in ungluing the color non-singlets, we must have $m_V = 0$ inside the soliton solution,[13] which makes it desirable to have the symmetric point $(\sigma, \phi) = (0, 0)$ be a local minimum of U.]

<u>Remarks.</u> A. At any finite $n \neq 0$, inside the soliton the gluon field σ may deviate appreciably from being a constant $\sigma = 0$. Hence, in accordance with (4.10), E_g contains an additional part $\frac{1}{2} N\epsilon (q/n)$, besides the total quark energy $N\epsilon$. In addition, $k \equiv - d \ln E_g / d \ln R$ becomes different from 1. Only in the limit $n \to 0$, or ∞, is $k = 1$ and $N\epsilon = E_g$.

B. We may choose, instead of n, $s = \frac{1}{6}\mu^3$ and p,

$$\mu , \quad \lambda = p/\mu^4 \quad \text{and} \quad \eta/\xi \tag{4.31}$$

as the set of independent parameters, where η and ξ are defined by (1.11), as before. Then,

$$n = n(\lambda, \eta/\xi)$$

is given by (3.42). Both λ and η/ξ vary from 0 to ∞. At any finite fixed value of λ,

$$n = O(\eta/\xi) \quad \text{as} \quad \eta/\xi \to \text{ either } 0 \text{ or } \infty \; . \tag{4.32}$$

At any finite fixed $\eta/\xi \neq 0$, as $\lambda \to 0$ (3.42) reduces to

$$\frac{\eta}{\xi} = F(n) \equiv \left(\frac{n}{N}\right)^{\frac{1}{3}} \left[6(\rho_1 - 1)\,\hat{u}_1^2\right]^{\frac{2}{3}} \tag{4.33}$$

which gives a finite nonzero n ; as $\lambda \to \infty$ (3.42) leads to, because of (2.70),

$$n = (4\pi N\lambda)^{\frac{1}{2}}\,\eta/\xi \to \infty \; . \tag{4.34}$$

Thus, $n \to 0$ only when $\eta/\xi \to 0$, while $n \to \infty$ when either $\eta/\xi \to \infty$, or $\lambda \to \infty$, or both.

3. Charge radius, magnetic moment and g_A/g_V of the nucleon

Let r_N , μ_N and g_A/g_V be, respectively, the root mean-squared charge radius, the magnetic moment and the ratio between the axial vector and the vector β-decay coupling constants of the nucleon, where the subscript N denotes either the neutron n or the proton p . In our model, we have

$$r_p^2 = \langle \rho^2 \rangle / \epsilon^2 \; , \qquad r_n^2 = 0$$

where

$$\langle \rho^2 \rangle = n^{-1} \int_0^{\rho_1} 4\pi \rho^4 (\hat{u}^2 + \hat{v}^2)\, d\rho \; ,$$

$$\mu_p = \hat{\mu}_p / \epsilon \; , \qquad \mu_n = -\tfrac{2}{3}\mu_p$$

where

$$\hat{\mu}_p = \tfrac{2}{3}\, n^{-1} \int 4\pi \rho^3\, \hat{u}\hat{v}\, d\rho \tag{4.35}$$

and

$$g_A/g_V = \frac{5}{3n} \int_0^{\rho_1} 4\pi \rho^2 (\hat{u}^2 - \tfrac{1}{3}\hat{v}^2)\, d\rho$$

where, as before, $n = 4\pi \int_0^{\rho_1} \rho^2 (\hat{u}^2 + \hat{v}^2)\, d\rho$. Thus, $\langle \rho^2 \rangle$, $\hat{\mu}_p$ and g_A/g_V are functions only of n ; their values are plotted in Fig. 7. Because of (2.26) and (2.25), each quark carries an energy

$$\epsilon = \rho_1/R = \mu(n/N)^{\frac{1}{2}}(\eta/\xi)^{-\frac{1}{2}} \quad . \tag{4.36}$$

The derivation of (4.35) follows the standard route:[3,7] Let Ψ denote the quark wave function whose total z-component angular moment is $\frac{1}{2}$; i.e., Ψ is given by (2.13) with

$$S = \binom{1}{0} \quad .$$

In either the Gell-Mann-Zweig quark model, or the Han-Nambu model, one can readily show that

$$r_p^2 = \int \psi^\dagger \psi\, r^2 d^3r / \int \psi^\dagger \psi\, d^3r \quad ,$$

$$\mu_p = \tfrac{1}{2}\left[\int \vec{r} \times \psi^\dagger \vec{a}\, \psi\, d^3r\right]_z / \int \psi^\dagger \psi\, d^3r \tag{4.37}$$

and

$$g_A/g_V = \frac{5}{3} \int \psi^\dagger \sigma_z \psi\, d^3r / \int \psi^\dagger \psi\, d^3r \quad .$$

Hence, the expressions for r_p , μ_p and g_A/g_V given in (4.35) follow. The corresponding expression for r_n is obvious, and that for μ_n is due to the relevant SU(6) Clebsch-Gordon coefficients. We list below the values of these quantities for the limiting cases that have been examined in the previous section.

(i) $n \to 0$

In this limit, $\rho_1 = 2.0428$, $\hat{u}$ and $\hat{v}$ are given by (2.63). By using (2.64), we find (4.35) becomes

$$r_p = \epsilon^{-1}\left[\frac{2\rho_1^3 - 2\rho_1^2 + 4\rho_1 - 3}{6(\rho - 1)}\right]^{\frac{1}{2}} = 1.4891/\epsilon \quad ,$$

$$r_n = 0 \quad ,$$

$$\mu_p = \frac{(4\rho_1 - 3)}{12\,\epsilon\,(\rho_1 - 1)} = 0.4133/\epsilon \quad ; \tag{4.38}$$

$$\mu_n = -\tfrac{2}{3}\,\mu_p$$

and

$$g_A/g_V = \frac{5\rho_1}{9\,(\rho_1 - 1)} = 1.088 \quad .$$

(ia) If in addition to $n \to 0$, we assume $s \to 0$, then by using (4.25), since $N = 3$ we have

$$\epsilon = \tfrac{1}{4}\,m_B \tag{4.39}$$

and therefore, from (4.38)

$$r_p = 5.956/m_B \quad \text{and} \quad \mu_p = 1.653/m_B \quad . \tag{4.40}$$

(ib) If in addition to $n \to 0$, we assume $p \to 0$, then by using (4.26), we have

$$\epsilon = \frac{2}{9}\,m_B \quad , \tag{4.41}$$

and therefore, from (4.38),

$$r_p = 6.701/m_B \quad \text{and} \quad \mu_p = 1.860/m_B \quad . \tag{4.42}$$

(ii) $n \to \infty$

In this limit, $\rho_1 = 1$ and the Fermion wave functions $\hat{u}$ and $\hat{v}$ both concentrate on the surface $r = R$. Hence,

$$r_p = \epsilon^{-1} \;, \qquad r_n = 0 \;,$$

$$\mu_p = (3\,\epsilon)^{-1} \;, \qquad \mu_n = -\tfrac{2}{3}\,\mu_p \tag{4.43}$$

and

$$g_A/g_V = \frac{5}{9} \quad .$$

(iia) If in addition to $n \to \infty$, we assume $s \to 0$, then we have, just as in (4.39),

$$\epsilon = \frac{1}{4} m_B \ ,$$

and therefore, from (4.43),

$$r_p = 4/m_B \qquad \text{and} \qquad \mu_p = 4/(3m_B) \ . \qquad (4.44)$$

(iib) If in addition to $n \to \infty$, we assume $p \to 0$, then, just as in (4.41),

$$\epsilon = \frac{2}{9} m_B$$

and therefore, from (4.43),

$$r_p = 9/(2m_B) \qquad \text{and} \qquad \mu_p = 3/(2m_B) \ . \qquad (4.45)$$

These limiting values are also summarized in Table 1. For comparison with experimental results, it is more convenient to use the average nucleon mass $m_N \cong 939$ MeV as the basic energy scale, rather than $m_B \cong 1316$ MeV, the baryon mass averaged over the 56 representation of SU(6). From Table 1, we conclude that for applications to hadrons, the parameter n could be either O(1) or smaller. In any case, it should be away from the $n \to \infty$ limit. Otherwise, g_A/g_V would be 5/9 and the charge density would be distributed only on the surface of the soliton; both features seem to be quite different from those of the physical nucleon.

Physical observable	Experimental value	Theoretical value in some limiting cases			
		$n \to 0$		$n \to \infty$	
		$\lambda \to \infty$	$\lambda \to 0$	$\lambda \to \infty$	$\lambda \to 0$
r_p	$3.86/m_N$	$4.25/m_N$	$4.78/m_N$	$2.86/m_N$	$3.21/m_N$
μ_p	$2.79/(2m_N)$	$2.36/(2m_N)$	$2.66/(2m_N)$	$1.90/(2m_N)$	$2.14/(2m_N)$
μ_n	$-0.685\,\mu_p$	$-\frac{2}{3}\mu_p$	$-\frac{2}{3}\mu_p$	$-\frac{2}{3}\mu_p$	$-\frac{2}{3}\mu_p$
g_A/g_V	1.25	1.09	1.09	5/9	5/9
scalar-field energy density		volume	surface	volume	surface
charge density		volume		surface	

Table 1. Root mean-squared charge radius r_N, magnetic moment μ_N and g_A/g_V of the nucleon N. The parameters $\lambda = p/\mu^4 = p/(6s)^{\frac{4}{3}}$ and n are defined by (1.9) and (2.19). In the last two rows, "volume" and "surface" mean respectively "within the volume" and "on the surface". See Sec. IV.3 for further details.

V. Remarks

In this paper, we have presented a new formulation of the relativistic quark model of hadrons, based on the quasiclassical soliton solutions of local field theories. We have shown that, once the low-lying soliton mass is assumed to be much smaller than the masses of the plane wave solutions (i.e., quarks, gluons, etc.), then under very general conditions, independently of the number of constants in the original Lagrangian, the description of the solitons depends only on three phenomenological parameters: n, s and p, as given by (4.14). There is a direct physical interpretation of these parameters. The soliton (i.e., the hadron) resembles a "gas bubble" immersed in a medium (i.e., the vacuum); p is the pressure exerted by the medium on the gas bubble, s is the surface tension and n determines the thermodynamic functions of the gas. In the double limit $n \to 0$ and $s \to 0$, one obtains an MIT-like bag, while in the opposite extreme $n \to \infty$ and $p \to 0$, a SLAC-like bag.

Such reductions occur frequently in physics, whenever the system under consideration contains two or more very different scales of length (or energy). As examples, one may mention Fermi's β-decay theory of weak interactions, the usual scattering length and effective range approximation of nuclear forces, etc. In all these cases, if one starts from the underlying Lagrangian, it may be difficult to give a rigorous proof of the validity of the approximations used. Quite often, this difficulty is compounded by lack of knowledge of the correct basic theory. The same is true here. In our case, one of the important questions is the validity of quasiclassical soliton solutions in the strong coupling region. For a fully relativistic local quantum field theory, this question is not resolved. However, in the case of nonrelativistic Fermions interacting with Bosons (which can be relativistic), the answer is known: the quasiclassical solution does give an accurate description when the coupling is sufficiently strong.

Because the solitons are solutions of a local field theory, it should be possible to calculate matrix elements of operators between different soliton states, e.g., nuclear charge form factors with large momentum transfer, π-decay rate, etc. Some of these calculations are under investigation.

Appendix A

In this appendix, we give an alternative proof of (2.24). In accordance with (2.60) and the discussion preceding it, in this alternative proof one should first start from the expression $E = E(R)$, given by (2.23), and then derive (2.24) by setting

$$\frac{dE}{dR} = 0 \quad . \tag{A.1}$$

By using (2.58), we may rewrite (2.23) as

$$E(R) = N\epsilon + 2\pi(Ng/m_\sigma)^2 \int_0^R r^2 dr\,(u^2 - v^2)^2$$
$$+ \tfrac{2}{3}\pi R^2 \mu^3 + \tfrac{4}{3}\pi R^3 p \tag{A.2}$$

where u and v are solutions of (2.35), and $u = v$ at $r = R$. Thus, the variation of E is

$$\delta E = N\delta\epsilon + 4\pi(Ng/m_\sigma)^2 \int_0^R r^2 dr\,(u^2 - v^2)\,\delta(u^2 - v^2)$$
$$+ 4\pi R^2 \delta R \left[\tfrac{1}{3}(\mu^3/R) + p\right] \quad . \tag{A.3}$$

Throughout this appendix, we keep the parameters N, g, p, m_σ and μ fixed. Since in (2.17), each solution determines a definite value of n, defined by (2.19), we may regard the solution of (2.17) as a function of ρ and n; i.e.,

$$\hat{u} = \hat{u}(\rho, n) \qquad \text{and} \qquad \hat{v} = \hat{v}(\rho, n) \tag{A.4}$$

where ρ varies from 0 to $\rho_T(n)$. We may then use (2.25) and (2.26) to define $\epsilon = \epsilon(n)$ and $R = R(n)$, or its inverse function $n = n(R)$. Through (2.16), (2.36) and (A.4), we may regard the solution of (2.35) as a function of r and n; i.e.,

$$u = u(r, n) \qquad \text{and} \qquad v = v(r, n) \tag{A.5}$$

with
$$4\pi \int_0^R (u^2 + v^2)\, r^2\, dr = 1 \quad , \tag{A.6}$$

where at $r = R(n)$,

$$u(R, n) = v(R, n) \equiv u_1(n) \quad . \tag{A.7}$$

Equation (2.35) can be written in its original form (2.9):

$$H_F \psi = \epsilon \psi \quad , \tag{A.8}$$

where
$$H_F = -i\vec{\alpha}\cdot\vec{\nabla} + g\beta\sigma \quad , \tag{A.9}$$

$$\sigma = \sigma(\vec{r}, n) = -(Ng/m_\sigma^2)(u^2 - v^2) \tag{A.10}$$

and $\psi = \psi(\vec{r}, n)$ is related to $u(r, n)$ and $v(r, n)$ by (2.13). From (A.8), one has

$$(\delta H_F)\psi + H_F\,\delta\psi = (\delta\epsilon)\psi + \epsilon(\delta\psi) \tag{A.11}$$

where
$$\delta H_F = g\beta\delta\sigma = -(Ng/m_\sigma^2)\,\delta(u^2 - v^2) \quad . \tag{A.12}$$

In this variation, $\vec{r}$ is kept fixed, but $n \to n + \delta n$. Since $4\pi \int_0^R \psi^\dagger \psi\, r^2\, dr = 1$, on account of (A.6), we find, upon multiplying (A.11) by $\psi^\dagger$ and integrating from $r = 0$ to R,

$$\delta\epsilon = 4\pi \int_0^R r^2\, dr \left[\psi^\dagger (H_F - \epsilon)\,\delta\psi + g\psi^\dagger \beta \psi\, \delta\sigma\right] ,$$

which, through partial integration and because of (A.8)-(A.10), may be written as

$$\delta\epsilon = 4\pi R^2 (u\,\delta v - v\,\delta u)_R - 4\pi \int_0^R (m_\sigma^2/N)\,\sigma\,\delta\sigma\, r^2\, dr \tag{A.13}$$

where the subscript R denotes $r = R$,

$$\delta u = \left[\frac{\partial}{\partial n} u(r, n)\right] \delta n \quad \text{and} \quad \delta v = \left[\frac{\partial}{\partial n} v(r, n)\right] \delta n \quad . \tag{A.14}$$

Because

$$u(r, n) = (\epsilon^3/n)^{\frac{1}{2}} \hat{u}(\rho, n) \quad , \quad v(r, n) = (\epsilon^3/n)^{\frac{1}{2}} \hat{v}(\rho, n)$$

$$\rho = r\epsilon \qquad \text{and} \qquad \epsilon = \epsilon(n) \quad , \tag{A.15}$$

we have

$$u\delta v - v\delta u = \frac{\epsilon^3}{n}\left[r\delta\epsilon\left(\hat{u}\frac{\partial}{\partial\rho}\hat{v} - \hat{v}\frac{\partial}{\partial\rho}\hat{u}\right) + \delta n\left(\hat{u}\frac{\partial}{\partial n}\hat{v} - \hat{v}\frac{\partial}{\partial n}\hat{u}\right)\right] \quad . \tag{A.16}$$

By using (2.17), and noting that the derivative $d/d\rho$ there is the partial derivative $\partial/\partial\rho$ above, we obtain

$$\hat{u}\frac{\partial}{\partial\rho}\hat{v} - \hat{v}\frac{\partial}{\partial\rho}\hat{u} = -\frac{2}{\rho}\hat{u}\hat{v} + \hat{u}^2 + \hat{v}^2 + (\hat{u}^2 - \hat{v}^2)^2 \tag{A.17}$$

and

$$\frac{\partial}{\partial\rho}(\hat{u}^2 - \hat{v}^2) = -4\hat{v}\left[\hat{u} - (\hat{v}/\rho)\right] \quad . \tag{A.18}$$

At $r = R$, $\rho = \rho_1$, $\hat{u} = \hat{v} \equiv \hat{u}_1$, and therefore

$$\left(\hat{u}\frac{\partial}{\partial n}\hat{v} - \hat{v}\frac{\partial}{\partial n}\hat{u}\right) = -\tfrac{1}{2}\left[\frac{\partial}{\partial n}(\hat{u}^2 - \hat{v}^2)\right] \quad . \tag{A.19}$$

Let us define

$$X(\rho, n) \equiv \hat{u}^2(\rho, n) - \hat{v}^2(\rho, n) \quad . \tag{A.20}$$

Since $X(\rho, n) = 0$ at $\rho = \rho_1(n)$, it follows then that $X(\rho, n + \delta n) = 0$ at

$\rho = \rho_1(n) + \delta\rho_1$ where $\delta\rho_1 = (d\rho_1/dn)\,\delta n$. In the (X, ρ) plane, we may consider an infinitesimal right angle triangle ABC, whose vertices are $A = (0, \rho_1)$, $B = (0, \rho_1 + \delta\rho_1)$ and $C = (\delta X, \rho_1)$ where $\delta X \equiv [\partial X(\rho, n)/\partial n]\,\delta n$. Hence, the point A lies on the curve $X(\rho, n)$ vs. ρ, and the points B and C on the curve $X(\rho, n + \delta n)$ vs. ρ; CA is $\perp$ AB, and their lengths are, respectively, $\overline{CA} = \delta X$ and $\overline{AB} = \delta\rho_1$. The ratio $-\overline{CA}/\overline{AB}$ is the slope of CB. By using (A.18) and setting $\rho = \rho_1$, we find that this slope is

$$-4\hat{u}_1^2(1 - \rho_1^{-1}) \;.$$

Thus,

$$\frac{\partial X}{\partial n}\,\delta n = \delta X = 4\hat{u}_1^2(1 - \rho_1^{-1})\,\delta\rho_1 \;. \tag{A.21}$$

Because of (A.17)-(A.21), at $r = R$ (i.e., at $\rho = \rho_1$), (A.16) becomes

$$(u\,\delta v - v\,\delta u)_R = 2\,\frac{\epsilon^3}{n}\,\hat{u}_1^2(1 - \rho_1^{-1})\,(R\,\delta\epsilon - \delta\rho_1) \;;$$

therefore, (A.13) reduces to

$$\delta\epsilon + 4\pi\int_0^R (m_\sigma^{\ 2}/N)\,\sigma\,\delta\sigma\,r^2\,dr = -8\pi R^2 u_1^2(\epsilon - R^{-1})\,\delta R \;. \tag{A.22}$$

By using (A.3), (A.10) and (A.22), we obtain

$$\delta E = 4\pi R^2\,\delta R\left[-2N\,u_1^2(\epsilon - R^{-1}) + p + \tfrac{1}{3}(\mu^3/R)\right] \;. \tag{A.23}$$

Thus, $dE/dR = 0$ gives (2.24).

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Figure Captions

Figure 1. $\hat{u}^2 - \hat{v}^2$ vs. ρ for $(4\pi)^{-1} n \ll 1$ (with an arbitrary scale for $\hat{u}^2 - \hat{v}^2$) and for $(4\pi)^{-1} n = 3.53 \times 10^6$ (with the exact scale for $\hat{u}^2 - \hat{v}^2$).

Figure 2. $\hat{u}(\rho)/\hat{u}(0)$, $\hat{v}(\rho)/\hat{v}(0)$ and $\left[\hat{u}^2(\rho) \pm \hat{v}^2(\rho)\right]/\hat{u}^2(0)$ vs. ρ from $\rho = 0$ to $\rho = \rho_1 = 2.0428$ when $n \to 0+$.

Figure 3. $\hat{u}$, $\hat{v}$ (solid curves) and $\hat{u}^2 \pm \hat{v}^2$ (dashed curves) vs. ρ from $\rho = 0$ to $\rho = \rho_1 = 1$ when $n \to \infty$. For small ρ, one uses the left-hand scale for the ordinate; for large ρ, the right-hand scale.

Figure 4. Solutions $\hat{x}(\hat{\tau})$ and $\hat{y}(\hat{\tau})$ of (2.69).

Figure 5. $\hat{u}(\rho)$, $\hat{v}(\rho)$ (solid curves) and $\hat{u}^2(\rho) \pm \hat{v}^2(\rho)$ (dashed curves) vs. ρ from $\rho = 0$ to $\rho = \rho_1$ for $n = 9.43$ (in a), 20.5 (in b), 47.8 (in c), 117 (in d), 259 (in e) and 1631 (in f). The right-hand scale for the ordinate refers in c and d to $\hat{u}^2 + \hat{v}^2$ and $\hat{u}^2 - \hat{v}^2$, in e to $\hat{u}^2 + \hat{v}^2$ alone, and in f to $\hat{u}^2 + \hat{v}^2$ only for $\rho > 0.75$. The left-hand scale refers to everything else.

Figure 6. (a) n (solid curve) and $\log_{10} n$ (dotted curve) vs. $\hat{u}(0)$. As $\hat{u}(0) \to \hat{u}_c \cong 1.7419$, $n \to \infty$. (b) ρ_1 (dotted curve), k (solid curve) and q/n (dashed curve) vs. $\hat{u}(0)$. See (2.18)-(2.20) and (4.12) for their definitions.

Figure 7. The integrals $\langle \rho^2 \rangle^{\frac{1}{2}}$, $\hat{\mu}_p = \epsilon \mu_p$ and g_A/g_V vs. $\hat{u}(0)$ which ranges from 0 to $\hat{u}_c \cong 1.7419$. See (4.35) for their definitions.

FIGURE 1

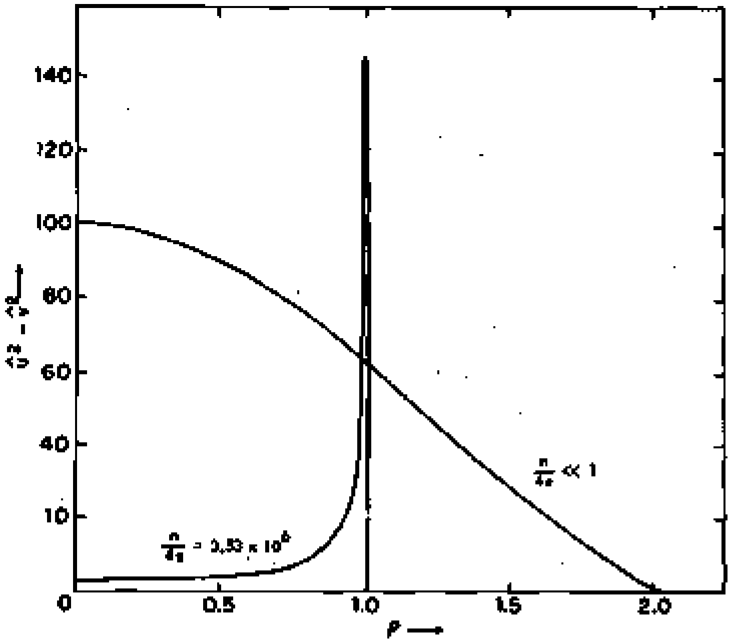
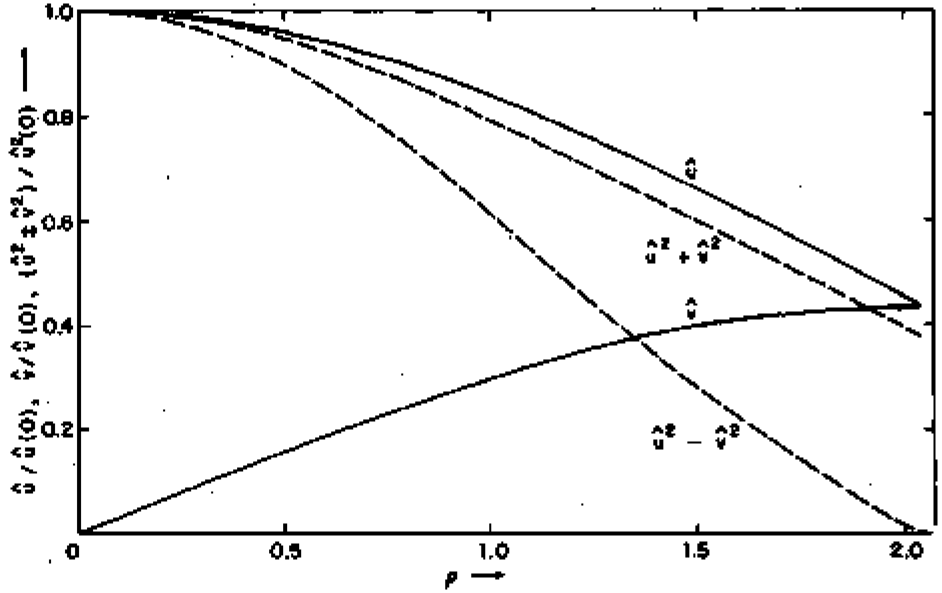


FIGURE 2



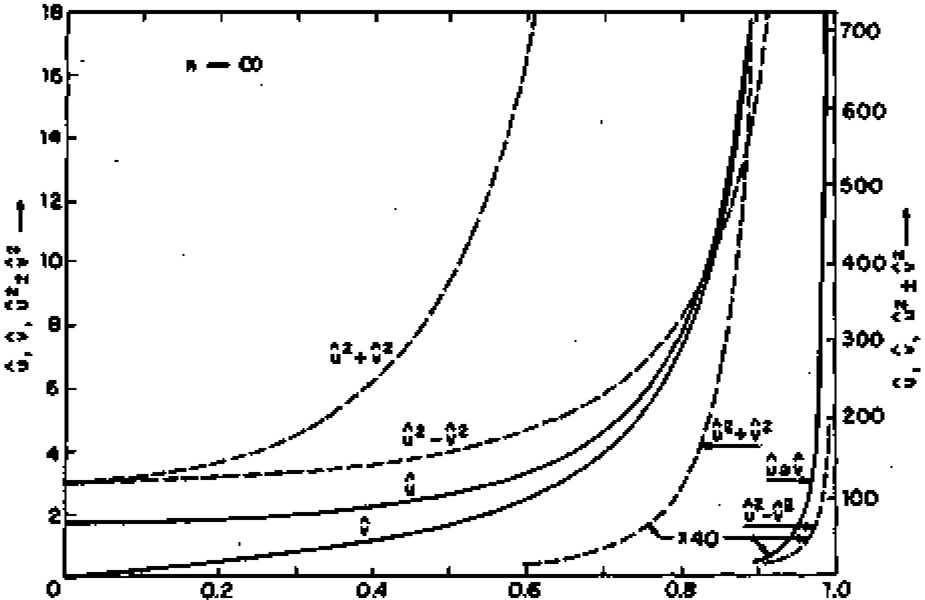


FIGURE 3

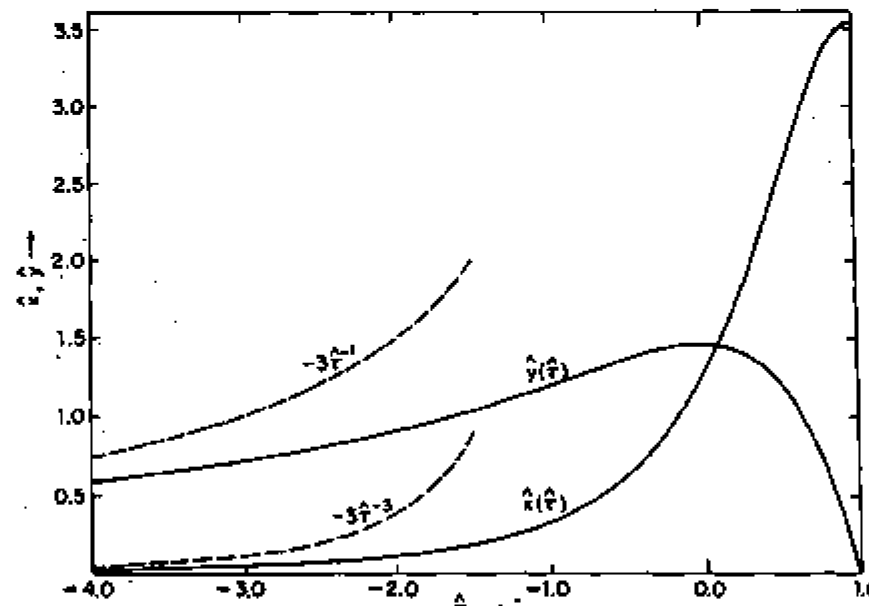


FIGURE 4

FIGURE 5a

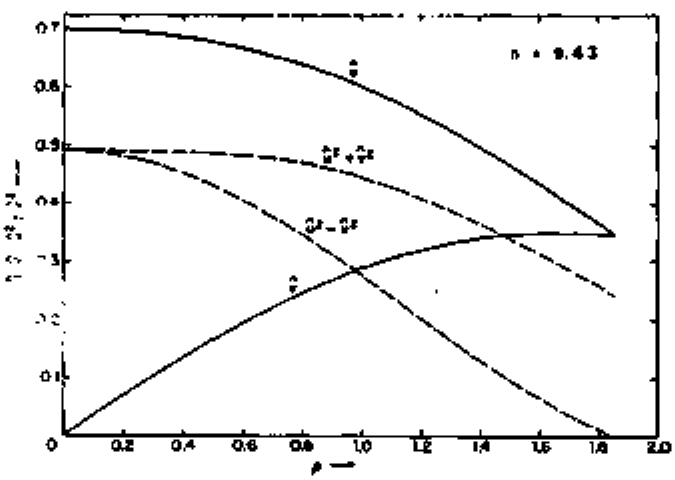


FIGURE 5b

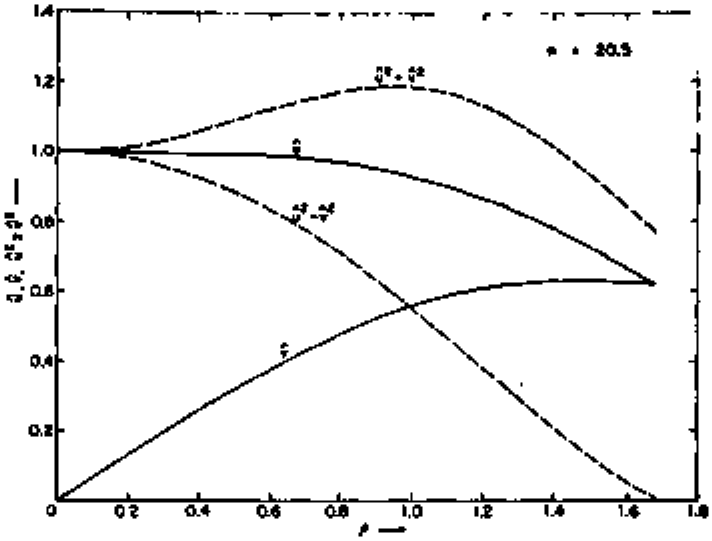


FIGURE 5c

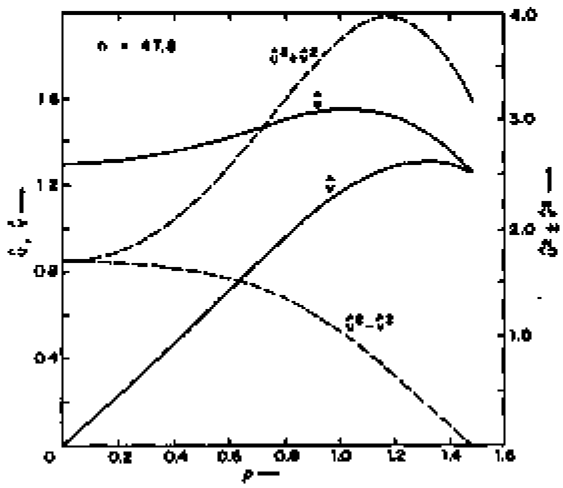


FIGURE 5d

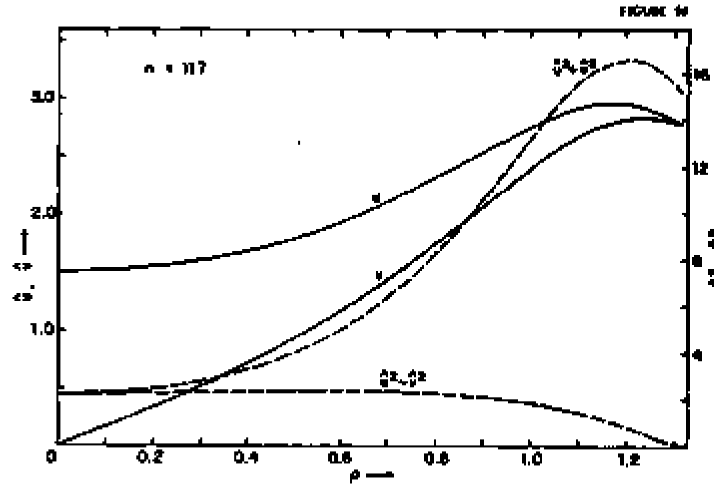


FIGURE 5e

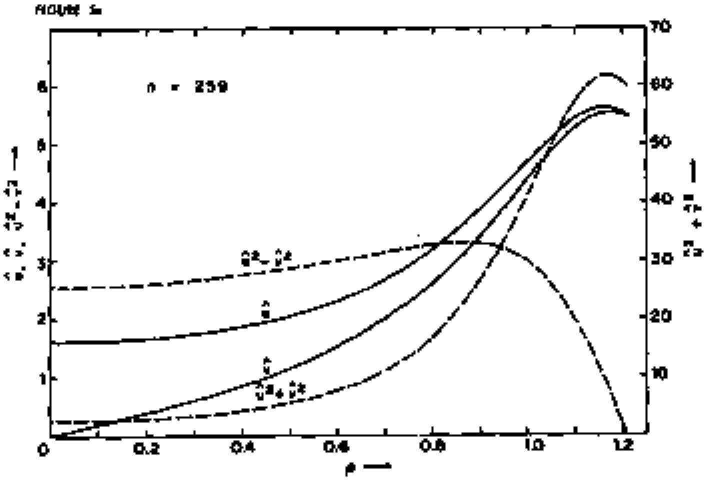
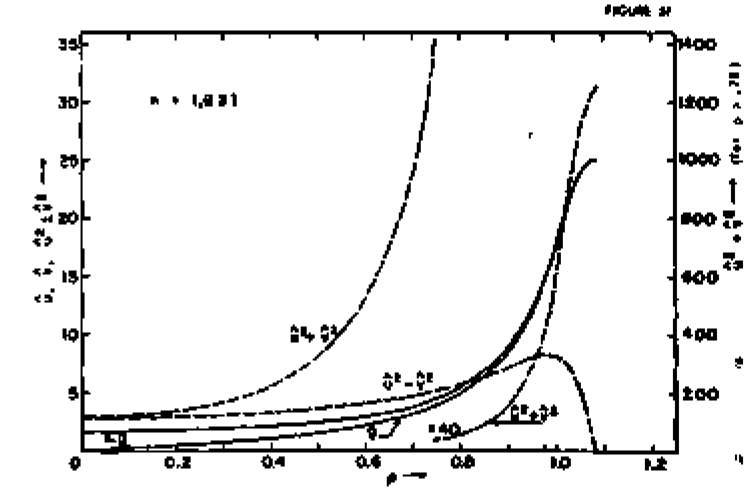


FIGURE 5f



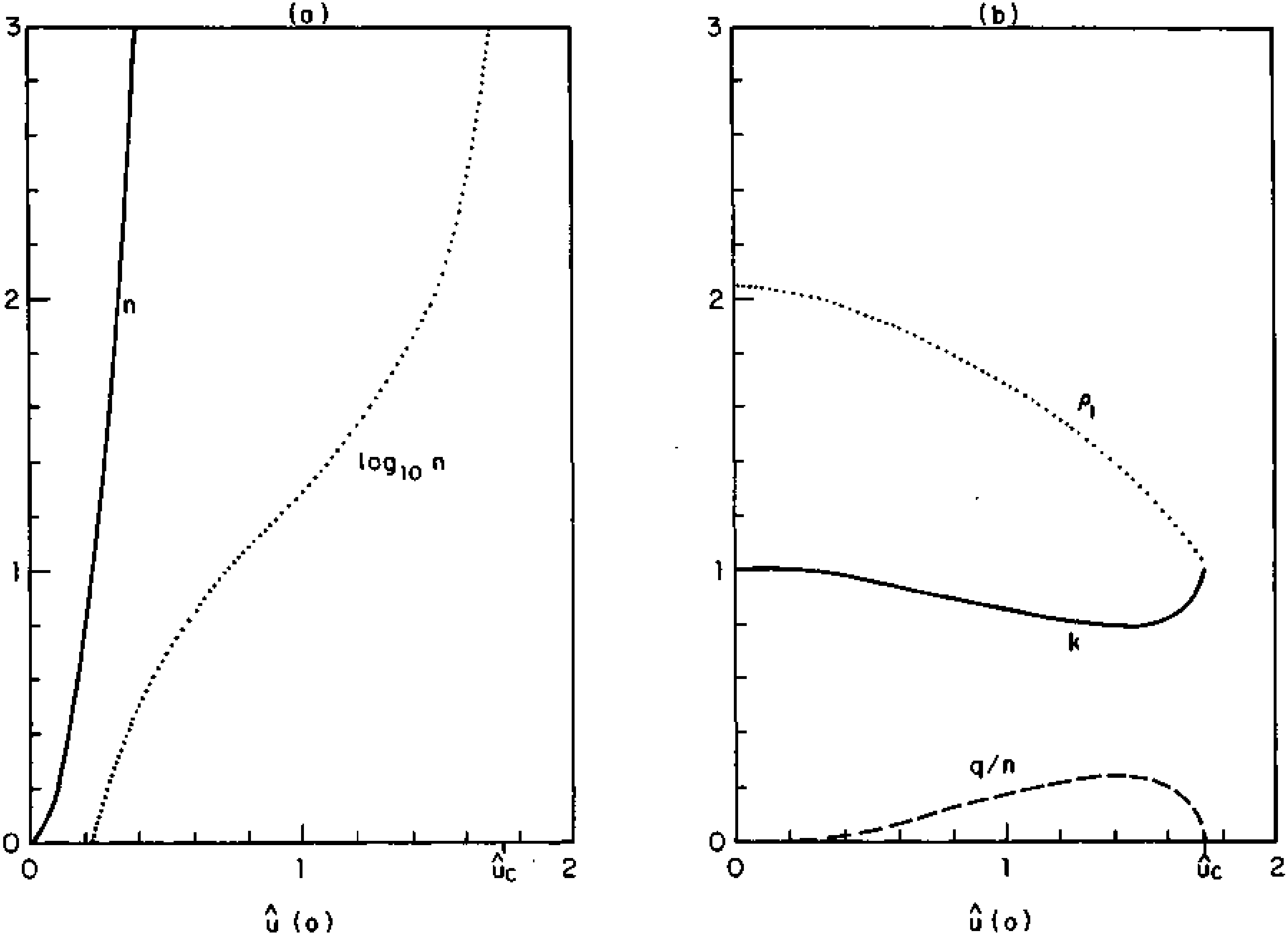


FIGURE 6

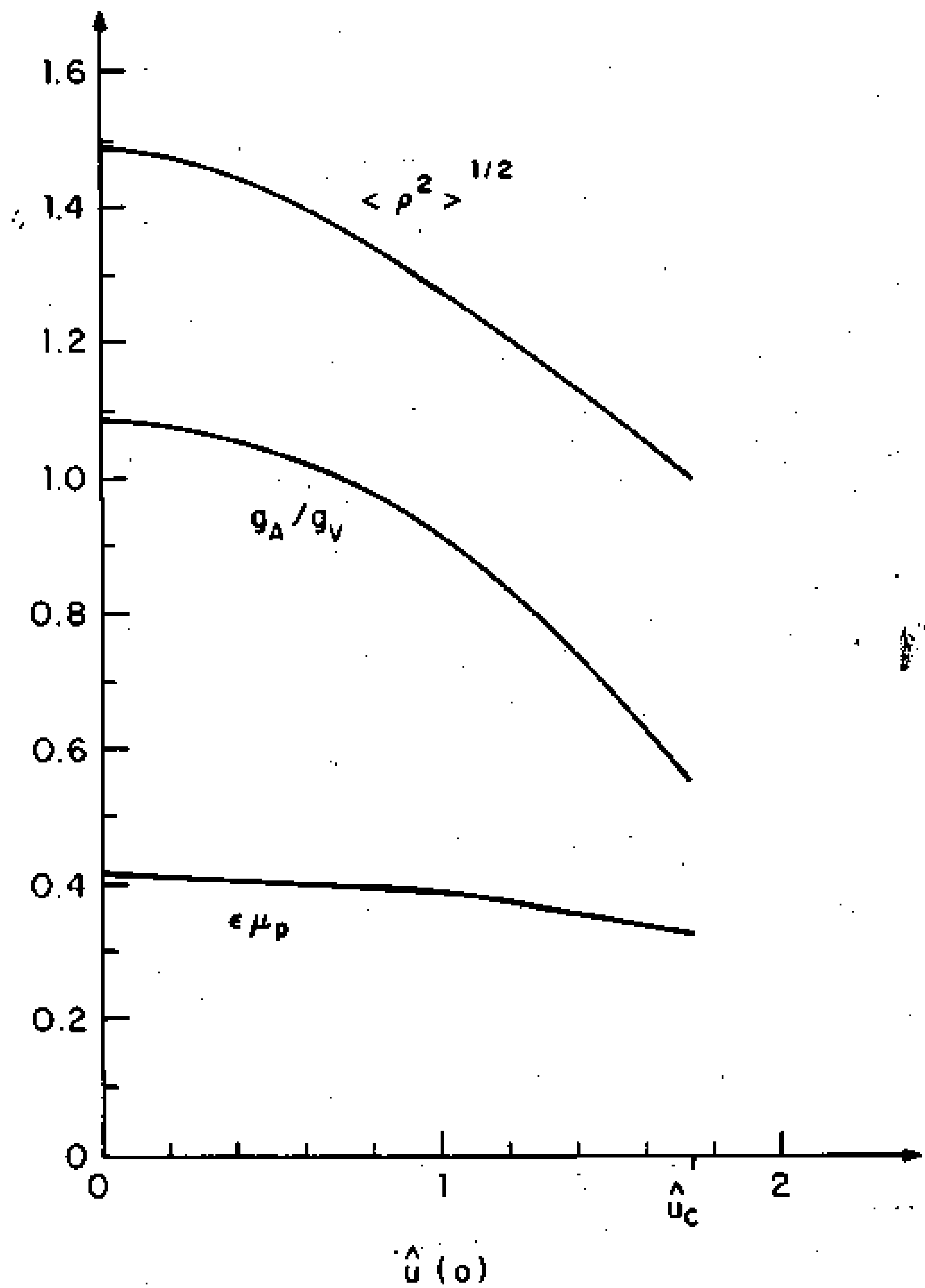


FIGURE 7

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