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FERMION-MESON VERTICES IN DUAL THEORIES

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ABSTRACT

We construct the general dual vertex describing the transition of a Ramond-fermion into a Neveu-Schwarz meson by the emission of a general excited fermion state. Our vertex thus generalizes the previous results of Thorn and Schwarz and is now put into a relatively compact and manageable form which enables us to check that certain of the general gauge conditions are correctly converted between the meson and fermion legs by the vertex. The particular construction of the vertex can be applied to the other older dual theories and leads to some new insights into them.

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1. INTRODUCTION

Apart from the linear trajectories, one of the most interesting features of the dual resonance models[1] has been the connection with an underlying quark structure, as indicated by the duality diagrams. This has been particularly emphasized and exploited by Mandelstam[2]. Yet only recently have Ramond[3], Neveu and Schwarz[4], and Thorn[5] shown how to incorporate half-integral spin particles, whilst retaining, hopefully, the gauge structures which should eliminate the ghosts due to time components. It is tantalizing to think of the Ramond fermion as a quark[6] or a baryon, but in fact it is probably neither[7], but just an important clue on the way to more physically realistic theories.

So far amplitudes describing two interacting fermions have not been constructed, and it is necessary to do this to see whether the theory has a consistent physical interpretation [see Schwarz[6]]. The necessary building block is the 3-Reggeon vertex involving two fermions and a meson. This has to be constructed to satisfy duality and ought to respect the gauge conditions which eliminate the ghost states. The first steps towards this vertex were made by Thorn[5] and Schwarz[6]. We carry their procedure a step further by constructing a more general vertex than theirs and presenting it in a simpler form, which enables us to investigate its group transformation and gauge properties.

Another line of thought has motivated us to consider the Ramond theory, apart from its intrinsic interest. Owing to the underlying group structure, possible dual theories can be enumerated and classified according to the available representations of the Möbius group.

For a certain subclass of these possibilities, Corrigan and Montonen[8] have constructed the N-Reggeon vertex, but found that if, in addition to the duality, one requires ghost-killing mechanisms of the familiar type, then only two possibilities survive: the ordinary dual resonance theory and the one due to Neveu and Schwarz[9]. The latter theory is tantalizingly close to meson reality [and links up with the Ramond theory[4-6]], but has the unsatisfactory feature that certain trajectories lie one half unit too high. The problem is to find some latitude in the group theory which might enable the construction of more realistic theories. The class of theories for which N-Reggeon vertices could easily be constructed had the characteristic feature that

$$L^{+}|0\rangle = 0 ,$$

where L^{+} is the Möbius "raising operator". Thus the vacuum state was always Möbius invariant. This is not the case for the Ramond theory, because there L^{+} contains a term $\gamma \cdot d_1^{\dagger}$, where γ is the usual Dirac gamma matrix which is responsible for the half-integral spin content of the theory. Thus we are regarding the Ramond theory as the prototype of a more general class of dual theory[10] for which we wish to construct the vertices.

The amplitude for a Ramond fermion emitting mesons involves the non-invariant vacuum state and our first step in Section 2 is to find a way of overcoming this difficulty by writing the amplitude in a general Möbius frame of reference. Then it is easy to write down the duality equation which should determine the required vertex. These equations are solved in Section 3, the solution involving an exponential of a contour integral. We find that the typical manipulations involve Cauchy's theorem. Another technical trick involves the introduction of an auxiliary set of oscillators, and this facilitates the investigation of the group transformation properties of the vertex in Section 4, where we check the desired transformations with respect to the Virasoro generators L_N for $N \geq -1$. This limitation is purely technical since we (and Dr. P. Goddard) have found another method of proof which extends to all N if only ground state emission is considered. (This argument is not included in the present paper). The restricted result means that the Möbius gauges convert both ways but the higher Virasoro gauges only convert from meson lines to fermion lines, but not vice versa.

In Section 5 we see how our form of "3-Reggeon vertex" can be applied to more conventional theories and related to familiar results.

In the Appendix we present some general theory which is used in the main part of this paper and make some comments about redundant dependence of legs in N-Reggeon vertices on adjacent variables.

2. THE VERTEX AND ITS DEFINING EQUATIONS

The fundamental constructs in the dual theories we shall consider are the Fubini-Veneziano fields[11)]

$$Q^\mu(z) = q^\mu - i p^\mu \ln z - i \sum_{n=1}^{\infty} \left[a_n^{\mu +} z^n - a_n^\mu z^{-n} \right] / \sqrt{n} \tag{2.1}$$

$$P^\mu(z) = i z \frac{d Q^\mu(z)}{dz} = p^\mu + \sum_{n=1}^{\infty} \sqrt{n} \left[a_n^{\mu +} z^n + a_n^\mu z^{-n} \right] \tag{2.2}$$

and, respectively, the Neveu-Schwarz[9)] and Ramond[3)] fields:

$$H^\mu(z) = \sum_{n=0}^{\infty} \left[b_n^{\mu +} z^{n+\frac{1}{2}} + b_n^\mu z^{-n-\frac{1}{2}} \right] \tag{2.3}$$

$$\Gamma^\mu(z) = \gamma^\mu + i\sqrt{2}\,\gamma_5 \sum_{n=1}^{\infty} \left[d_n^{\mu +} z^n + d_n^\mu z^{-n} \right] \tag{2.4}$$

where a_n^μ, b_n^μ, d_n^μ are the usual annihilation operators. Virasoro generators L_n satisfying the algebra

$$[L_n, L_m] = (n - m)\, L_{n+m} + \delta_{n+m,0}\, c_n \tag{2.5}$$

can be constructed out of these fields:

$$L_n^a = \frac{1}{2\pi i} \oint \frac{dz}{z}\, z^n\, \frac{1}{2} :P^2(z): \tag{2.6}$$

$$L_n^b = \frac{1}{4\pi i} \oint dz\, z^{n+1} : \frac{H(z)}{\sqrt{z}} \cdot \frac{d}{dz} \frac{H(z)}{\sqrt{z}} : \tag{2.7}$$

$$L_n^d = \frac{1}{8\pi i} \oint dz\, z^{n+1} : \frac{\Gamma(z)}{\sqrt{z}} \cdot \frac{d}{dz} \frac{\Gamma(z)}{\sqrt{z}} : \tag{2.8}$$

The contours of integration enclose the origin, which is an isolated essential singularity of the integrand in each case. We take special care that the c-number c_n in Eq. (2.5) vanishes when $n = \pm 1$ by redefining

$$L_0^d = \frac{1}{4} - \sum_{n=1}^{\infty} n\, d_n^\dagger \cdot d_n \tag{2.9}$$

The figure $\frac{1}{4}$ is thus responsible for ensuring that the Möbius algebras are satisfied precisely and will be seen to affect the value of the fermion mass.

Figure 1 depicts a Ramond fermion (solid line) emitting successive mesons (dashed line):

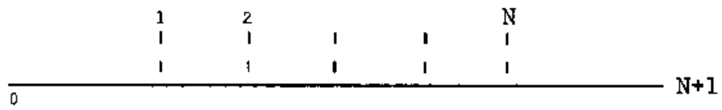

Fig. 1

and the corresponding amplitude is:

$$\bar{u}(k_0) \langle B | U_1(1) \frac{1}{L_0 - 1} U_2(1) \frac{1}{L_0 - 1} \cdots U_N(1) | A \rangle u(k_{N+1}) \tag{2.10}$$

where the emission vertex is defined by

$$\mathcal{U}_i(z) = k_i \cdot \Gamma(z)\,\Gamma_5 :e^{-ik_i\cdot Q(z)}: /z \tag{2.11}$$

and $\Gamma_5 = \gamma_5(-1)^{\sum_1^\infty d_n^\dagger d_n}$. $|A\rangle$ and $|B\rangle$ are excited physical fermion states satisfying

$$(L_0 - 1)|A\rangle = (L_0^a + L_0^d - 1)|A\rangle = 0 \tag{2.12}$$

$$(L_n^d + L_n^a)|A\rangle \quad = 0\ , \qquad n \geq 1 \tag{2.13}$$

and may be also Dirac equation conditions. Note that the ground state fermion automatically has mass squared $-3/2$ as a consequence of Eqs. (2.9) and (2.12).

In order to understand how to dualize expression (2.10) in order to exhibit propagators in meson channels, we must first write the expression in an arbitrary Möbius frame of reference as:

$$\int \frac{\prod_{i=0}^{N+1} dz_i}{dV_{abc}} \quad \bar{u}(k_0) \langle B| \frac{e^{L^-/z_0}}{z_0^2} \mathcal{U}_1(z_1)\mathcal{U}_2(z_2)\cdots \mathcal{U}_N(z_N) e^{z_{N+1}L^+} |A\rangle\, u(k_{N+1}) \tag{2.14}$$

That this is indeed Möbius invariant follows from the fact that if $O(\gamma)$ is the representative in the combined a,d Fock space of the Möbius transformation $\gamma(z) = (az + b)/(cz + d)$ with $ad - bc = 1$, then

$$O(\gamma)\,\mathcal{U}(z)\,O(\gamma)^{-1} = \mathcal{U}(\gamma(z))\ (cz+d)^{-2} \tag{2.15}$$

$$O(\gamma)\,e^{zL^+}|A\rangle = e^{\gamma(z)L^+}|A\rangle\ (cz+d)^{-2} \tag{2.16}$$

$$\langle B| \frac{e^{L^-/z}}{z^2}\, O(\gamma)^{-1} = \langle B| \frac{e^{L^-/\gamma(z)}}{\gamma(z)^2}\ (cz+d)^{-2} \tag{2.17}$$

where, in order to prove Eqs. (2.16) and (2.17), one uses Eqs. (2.12) and (2.13) for $n = 1$ (and $L^+ = L_{-1}$).

The fixing of a Möbius frame is achieved by division by the standard differential dV_{abc}, which effectively fixes any three points z_a, z_b, z_c at any values. If one chooses to fix $z_0 = \infty$, $z_1 = 1$, and $z_{N+1} = 0$, then the remaining integrations in expression (2.14) can be evaluated to recover expression (2.10). Notice that the states $|A\rangle$ and $|B\rangle$ are not Möbius invariant even if they are ground states, and need not be. This is the difference with previous theories.

For example, in both the conventional theory with the Fubini-Veneziano operator momentum p (which gives zero on $|0\rangle$), and in the Neveu-Schwarz case (2.7)

$$e^{zL^+}\,|0\rangle = |0\rangle \;.$$

in the conventional theory with a c-number momentum π

$$e^{zL^+}|0\rangle = e^{-\pi\cdot\sum_{n=1}^{\infty}\frac{z^n a_n^+}{\sqrt{n}}}|0\rangle$$

is a "coherent state", and this fact has been exploited considerably in the formalism of this theory[12]. On the other hand, in the Ramond case (2.8), it can be shown that

$$e^{zL_d^+} = e^{\sum_{n=1}^{\infty}(-z)^n\binom{-1/2}{n}\frac{i}{\sqrt{2}}\gamma_5\gamma\cdot d_n^+ + \frac{1}{4}\sum_{n,m=1}^{\infty}\frac{n-m}{n+m}\binom{-1/2}{n}\binom{-1/2}{m}(-z)^{n+m}d_n^+\cdot d_m^+} \cdot e^{-z\sum_{n=1}^{\infty}(n+1/2)\,d_{n+1}^+\cdot d_n} \;. \tag{2.18}$$

This means that

$$e^{zL_d^+}|0_d\rangle = e^{\sum_{n=1}^{\infty}(-z)^n\binom{-1/2}{n}\frac{i}{\sqrt{2}}\gamma_5\gamma\cdot d_n^+ + \frac{1}{4}\sum_{n,m=1}^{\infty}\frac{n-m}{n+m}\binom{-1/2}{n}\binom{-1/2}{m}(-z)^{m+n}d_n^+ d_m^+}|0_d\rangle \tag{2.19}$$

has an extremely complicated structure, owing to the non-commutativity of the Dirac gamma-matrices. We omit proofs of Eqs. (2.18) and (2.19), because it turns out that we shall not need these results. Equation (2.16), which follows from the group theory and the subsidiary conditions, is the all-important property of these states. Another important property, following from Eq. (2.16), is

$$\langle A|\,\frac{e^{L^-/x}}{x^2}\,e^{yL^+}\,|B\rangle = (x-y)^{-2}\langle A|B\rangle \tag{2.20}$$

These states can thus be regarded as generalized coherent states.

Now we wish to dualize Fig. 1 into the form in Fig. 2.

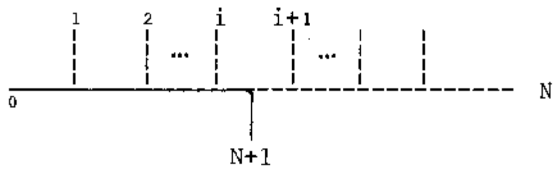

Fig. 2

by the steps in Fig. 3:

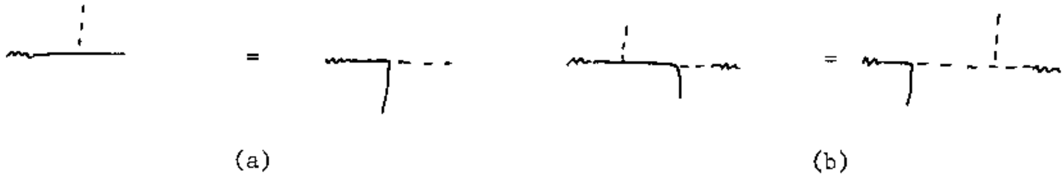

Fig. 3

That is we want to construct a vertex $V_A(z)$ corresponding to the emission of particle N + 1 in Fig. 2, which maps the b Fock space into the d Fock space and enjoys the operator relations corresponding to Figs. 3a and 3b.

$$V_A(z)\,|0_{a,b}\rangle = e^{zL^+}\,e^{-ik_A\cdot z}\,|A\rangle \tag{2.21}$$

and

$$\mathcal{U}(y)\;V_A(z) = V_A(z)\;\hat{\mathcal{U}}(y)/\beta \qquad z \neq y \tag{2.22}$$

where $\hat{U}(y)$ is the vertex for a meson leg emitting a meson. From the work of Neveu and Schwarz[4)] and Thorn[5)] we expect it to be

$$\hat{U}(y) = k \cdot H(y) \; : e^{-ik\cdot Q(y)} : /y \;. \tag{2.23}$$

Equation (2.22) corresponds to Fig. 3b and corresponds to the possibility of dualizing a fermion line to get a meson line. The quantity β is a number or operator to be determined as a sort of duality eigenvalue. Equation (2.22) has to be satisfied whatever state $|A\rangle$ is emitted and therefore has many solutions. Equation (2.21) is the "boundary condition" which expresses which particular state is actually emitted. If we can find V_A, satisfying these conditions, then the amplitude (2.10) or (2.14) can be written in a form corresponding to Fig. 2 as

$$\frac{\int \prod_{i=0}^{N+1} dz_i}{dV_{abc}} \; \bar{u}(k_0) < B \left| \frac{e^{L_{-}k_0}}{z_0^2} U_1(z_1) \cdots U_i(z_i) V_A(z_{N+1}) \hat{U}_{i+1}(z_{i+1}) \cdots \hat{U}_N(z_N) \right| 0 > \; . \; u(k_{N+1}) \tag{2.24}$$

Notice that the integration variables z_0, z_1, ... z_N are always ordered around the Koba-Nielsen circle in their numbered order so that there is never any possibilities of z_{N+1} coinciding with z_{i+1} to z_N.

In order to simplify the subsequent discussion we shall make an assumption (which will be removed at the end of the paper) namely that $|A>$ consists of the "a" vacuum times a state $|A_d>$ in the "d" Fock space. Then we expect $V_A(z)$ to factorize into

$$V_A(z) = W_A(z) \; : e^{-ik\cdot Q(z)} : z^{k^2/2} \; , \tag{2.25}$$

where $W_A(z)$ is independent of "a" oscillators and maps the "b" Fock space into the "d" Fock space, and will satisfy

$$W_A(z) \; H^\mu(y)\beta = \Gamma^\mu(y) \Gamma_5 \; W_A(z) \tag{2.26}$$

and

$$W_A(z) | 0_b > = e^{z L_d^+} | A_d > \tag{2.27}$$

It is convenient to "renormalize" $H^\mu(y)$ by defining

$$\hat{H}^\mu(y) = i\sqrt{2}\,\gamma_5 \; H^\mu(y)$$

The significance of this is that on the unit circle $y = e^{i\theta}$, we have formally

$$\left\{ \hat{H}^\mu(e^{i\theta}), \hat{H}^\nu(e^{i\theta'}) \right\} = 4\pi g^{\mu\nu}\delta(\theta-\theta') = \left\{ \Gamma^\mu(e^{i\theta}), \Gamma^\nu(e^{i\theta'}) \right\} \tag{2.28}$$

and that $\hat{H}^\mu$, unlike H^μ anticommutes with Γ^μ (if the b and d's relatively anticommute). Then Eq. (2.26) becomes

$$W_A(z) \; \hat{H}^\mu(y) = \lambda \; \Gamma^\mu(y) \; W_A(z) \tag{2.29}$$

and, we see, applying $W_A(z)$ to Eq. (2.28) by using Eq. (2.29) that $\lambda^2 = 1$, so that our "eigenvalue" is determined.

If the solution to Eqs. (2.26) and (2.27) is unique we find, considering these equations in two Möbius frames related by $\gamma(z)$, that

$$O_d(\gamma)\, W_A(z)\, O_b(\gamma)^{-1} = W_A(\gamma(z)) / (cz+d)^{2c_d} \tag{2.30}$$

where $$L^0_d \,|A_d\rangle = c_d\, |A_d\rangle$$

and $-\frac{1}{2}M_A^2 + c_d = 1$ from Eq. (2.12). If in Eq. (2.16) $O(\gamma)$ is replaced by $O_d(\gamma)$ then $(cz + d)^{-2c}$ is replaced by $(cz + d)^{-2c_d}$.

Later on when $W_A(z)$ is constructed we shall take pains to verify expression (2.30) explicitly, since we need it to get the correct Möbius gauge properties.

Because of Eq. (2.18) we can expect that the right-hand side of Eq. (2.27) is very complicated. We can therefore simplify this condition by writing

$$W_A(z) = e^{zL^+_d}\, \tilde{W}_A(z) \tag{2.31}$$

so that $\tilde{W}_A(z)$ satisfies, instead of Eqs. (2.27) and (2.29),

$$\tilde{W}_A(z)\,|0_b\rangle = |A_d\rangle \tag{2.32}$$

$$\tilde{W}_A(z)\, \frac{\hat{H}^\mu(y)}{\sqrt{y}} = \frac{\Gamma^\mu(y-z)}{\sqrt{y-z}}\, \tilde{W}_A(z) \tag{2.33}$$

where we have used the known Möbius transformation properties of Γ. In the next section we shall construct a $\tilde{W}_A(z)$ satisfying these equations.

As explained in the Appendix, we are now in a position to write down the amplitude corresponding to Fig. 4.

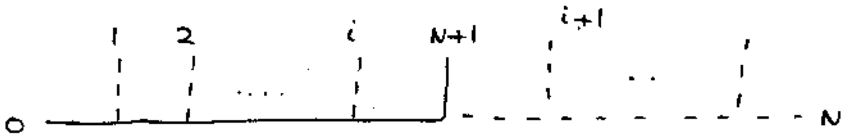

Fig. 4

The integrand is the same as that in expression (2.14), but the integration variables are held in a different order, namely in the same cyclic order as in Fig. 4. Thus if we now take the Möbius frame in which $z_0 = \infty$, $z_1 = 1$, $z_N = 0$, we have

$$\bar{u}(k_0) \langle B | U_1(1) \frac{1}{L_0-1} \cdots U_i(1) \frac{1}{L_0-1} V_A(1) \frac{1}{L_0-1} \hat{U}_{i+1}(1) \frac{1}{L_0-1} \cdots \hat{U}_{N-1}(1) k_N \cdot b_0^{+} | 0 \rangle \cdot u(k_{N+1})$$

It is understood that the appropriate propagators appear on the appropriate lines.

3. CONSTRUCTION OF THE VERTEX

Our main result is that the Fubini-Veneziano vertex $W_A(z)$ for converting a fermion line into a meson line, by emitting a Fermi occupation number state $|A\rangle$, is given by Eqs. (2.25) and (2.31) in terms of

$$\tilde{W}_A(z) = \langle 0_{bb'} | e^{I(z) + i J(\epsilon)} | 0_{b'} A_d \rangle \tag{3.1}$$

where

$$I(z) = \oint \frac{dx}{4\pi i} \frac{\Gamma(x-z)}{\sqrt{x-z}} \cdot \frac{\hat{H}(x)}{\sqrt{x}} \tag{3.2}$$

$$J(\epsilon) = \oint \frac{dx}{4\pi i} \frac{\hat{H}'(x-\epsilon)}{\sqrt{x-\epsilon}} \cdot \frac{\hat{H}(x)}{\sqrt{x}} \tag{3.3}$$

In each of these integrals $I(z)$ and $J(\varepsilon)$ the contour of integration encircles the isolated singularity of the integrand at the origin in a positive sense, but excludes the other singularities [a branch point at $x = z$ in $I(z)$ and an isolated essential singularity at ε in $J(\varepsilon)$].

The dashes indicate an auxiliary set of oscillators b'_n similar to the b_n, but anticommuting with these and the d_n (just as the b_n anticommute with the d_n). The oscillators play a purely internal role, since their vacuum expectation value is taken. As will be seen later the introduction of these oscillators is a convenient device to achieve a generalized sort of normal ordering in the d oscillators. We shall also see that $\tilde{W}_A(z)$ is in fact independent of the ε which can be thought of as small, but non-zero, so that the contour described above is well defined.

Our procedure will be to solve Eqs. (2.32) and (2.33) with the trial expression

$$\chi(z) = \langle 0_{bb'} | e^{\lambda I(z) + \mu J(\epsilon)} | 0_{b'} A_d \rangle \tag{3.4}$$

If, in addition, the exponential can be normal ordered in the d and b$'$ oscillators to give a non-singular expression, then we find that

$$\lambda = \pm 1 \ , \qquad \mu = \pm i \ ,$$

as quoted in Eq. (3.1). Normal ordering of the exponential with respect to the b oscillators is unnecessary since by the residue theorem the contour integral eliminates the positive powers of x occuring in $\hat{H}(x)/\sqrt{x}$ and hence all the "$b^\dagger$'s" occurring in the exponent. In view of this, expression (3.4) automatically satisfies the "boundary condition" (2.32) and we need only study the "duality equation" (2.33) by means of the identity

$$\chi(z)\,\frac{\hat{H}_\nu(y)}{\sqrt{y}} = \langle 0_{bb'} | e^{\lambda I + \mu J}\, \frac{\hat{H}_\nu(y)}{\sqrt{y}} | 0_{b'} A_d \rangle$$

$$= \langle 0_{bb'} | \left\{ \frac{\hat{H}_\nu(y)}{\sqrt{y}} + \left[\lambda I + \mu J, \frac{\hat{H}_\nu(y)}{\sqrt{y}}\right] + \frac{1}{2}\left[\lambda I + \mu J, \left[\lambda I + \mu J, \frac{\hat{H}_\nu(y)}{\sqrt{y}}\right]\right] + \ \cdots\cdots \right\} e^{\lambda I + \mu J} | 0_{b'} A_d \rangle \tag{3.5}$$

We shall evaluate the first commutators

$$\left[I(z)\,,\ \frac{\hat{H}_\nu(y)}{\sqrt{y}} \right] = \frac{\Gamma_\nu(y-z)}{\sqrt{y-z}} \tag{3.6}$$

$$\left[J(\epsilon)\,,\ \frac{\hat{H}_\nu(y)}{\sqrt{y}} \right] = \frac{\hat{H}'_\nu(y-\epsilon)}{\sqrt{y-\epsilon}} \tag{3.7}$$

and we find that the second and therefore subsequent commutators vanish.

Only $\hat{H}^+(y)$ fails to commute with I(z), and so Eq. (3.6) follows by the residue theorem, if we choose the contour of integration to enclose y and use the commutator

$$\left\{ \frac{\hat{H}_\mu(x)}{\sqrt{x}} , \frac{\hat{H}^+_\nu(y)}{\sqrt{y}} \right\} = \frac{2 g_{\mu\nu}}{x-y} \qquad \text{if} \quad |y| < |x| \tag{3.8}$$

Equation (3.7) follows similarly. Equations (3.6) and (3.7) can be continued to all values of y. To evaluate the second commutator we need $\left[I(z), \Gamma_\nu(y-z)/\sqrt{y-z}\right]$ and we must split $\Gamma(y-z)$ into parts $\Gamma^+ + \Gamma^0$ and Γ^- (involving powers of $z \geq 0$ and < 0, respectively) so that we can use the commutators

$$\begin{aligned} \left\{ \Gamma_\mu(x_1), \Gamma^+_\nu(x_2) + \Gamma^0_\nu(x_2) \right\} &= \frac{2 g_{\mu\nu} x_1}{x_1 - x_2} \qquad \text{if} \quad |x_2| < |x_1| \\ \left\{ \Gamma_\mu(x_1), \Gamma^-_\nu(x_2) \right\} &= \frac{2 g_{\mu\nu} x_1}{x_2 - x_1} \qquad \text{if} \quad |x_1| < |x_2| \end{aligned} \tag{3.9}$$

to see that

$$\left[I(z), \frac{\Gamma^+_\nu(y-z) + \Gamma^0_\nu(y-z)}{\sqrt{y-z}} \right] = -\oint \frac{dx}{4\pi i} \frac{2(x-z)}{\sqrt{(x-z)(y-z)}\,(x-y)} \hat{H}_\nu(x)$$

providing $|y-z| < |x-z|$

and
$$\left[I(z), \frac{\Gamma^-_\nu(y-z)}{\sqrt{y-z}} \right] = -\oint \frac{dx}{4\pi i} \frac{2(x-z)}{\sqrt{(x-z)(y-z)}\,(y-x)} \hat{H}_\nu(x)$$

providing $|x-z| < |y-z|$

In either case, in order to satisfy the inequalities for all x on the contour of integration, the point y must lie outside the contour in both cases, that is in the shaded regions illustrated in Fig. 5 in the respective cases.

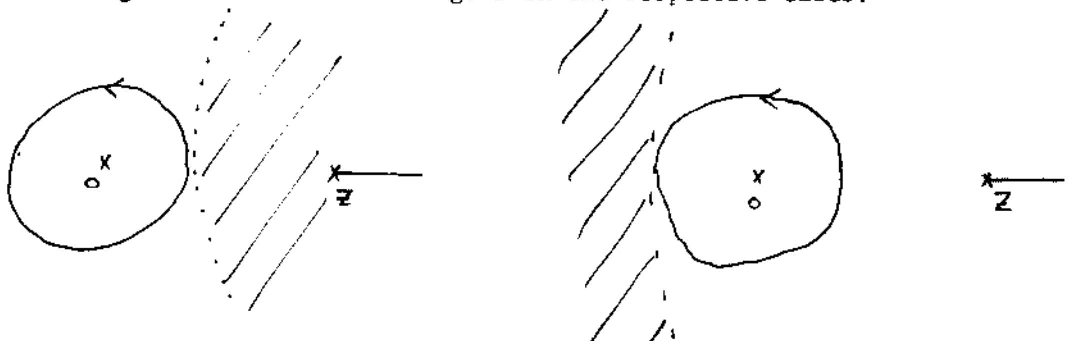

Fig. 5

When we continue the last two equations into a common region of y, so that we can add them together, y must still lie outside the contour of integration in each case. Then we get a cancellation and the commutator $[I(z), \Gamma_\nu(y-z)]$ vanishes. Similarly $[J(\varepsilon), \hat{H}'_\nu(y-\varepsilon)]$ is zero and we have now deduced that

$$\chi(z)\frac{\hat{H}_\nu(y)}{\sqrt{y}} = \langle 0_{bb'}| \left\{ \frac{\hat{H}_\nu(y)}{\sqrt{y}} + \lambda\frac{\Gamma_\nu(y-z)}{\sqrt{y-z}} + \mu\frac{\hat{H}'_\nu(y-\epsilon)}{\sqrt{y-\epsilon}} \right\} e^{\lambda I(z)+\mu J(\epsilon)} |0_{b'} A_d\rangle \tag{3.10}$$

Before showing that for a suitable choice of μ, the first and third terms, involving $\hat{H}$ and $\hat{H}'$ will cancel, we now examine the conditions under which $\exp[\lambda I(z) + \mu J(\varepsilon)]$ can be rearranged into a form which looks finite and in which the $d\, d^\dagger$ operators are normal ordered. Corresponding to the splittings of Γ and $\hat{H}'$ we split

$$I(z) = I^+(z) + I^0(z) + I^-(z) \quad ; \quad J(\epsilon) = J^+(\epsilon) + J^-(\epsilon)$$

and we have formally that

$$e^{\lambda I(z)+\mu J(\epsilon)} = e^{\lambda I^+(z)+\mu J^+(\epsilon)}\, e^{\lambda I^0(z)}\, e^{\lambda I^-(z)+\mu J^-(\epsilon)} \cdot e^{-\frac{1}{2}[\lambda I^+(z)+\mu J^+(\epsilon),\, \lambda I^-(z)+\mu J^-(\epsilon)]} \tag{3.11}$$

and the question is whether the exponent of the final factor is finite. According to Eqs. (3.8) and (3.9) it can be evaluated as

$$\frac{1}{2}\oint\frac{dx}{4\pi i}\oint\frac{dy}{4\pi i}\left[\lambda^2\sqrt{\frac{(x-z)}{(y-z)}}\;\frac{1}{y-x} + \frac{\mu^2}{y-x}\right]\frac{\hat{H}(x)}{\sqrt{x}}\cdot\frac{\hat{H}(y)}{\sqrt{y}} \tag{3.12}$$

and we can only define the contours to enclose 0 and exclude z if the square bracket is regular at x = y, which appears to be a pole unless

$$\lambda^2 + \mu^2 = 0 \;. \tag{3.13}$$

We shall, therefore, require this condition. Notice that when we take vacuum expectation values of expression (3.11) with respect to the b′ Fock space, $J^\pm(\varepsilon)$ disappears and the only remnant of the b′ oscillators is the second term in the square brackets of expression (3.12) and this is independent of ε, as claimed earlier.

Finally, we want to get the desired cancellation of the $\hat{H}$ and $\hat{H}'$ terms in Eq. (3.10). In fact, proceeding along the lines of the previous arguments we can show that, for any μ,

$$\frac{\hat{H}_\nu^{-\prime}(y-\epsilon)}{\sqrt{y-\epsilon}}\, e^{\mu J^+(\epsilon)} |0_{b'}\rangle = \mu\, \frac{\hat{H}_\nu^-(y)}{\sqrt{y}}\, e^{\mu J^+(\epsilon)} |0_{b'}\rangle \tag{3.14}$$

This leads to the desired cancellation providing $\mu^2 = -1$. Putting this together with Eqs. (3.13) and (3.10) we have indeed put X (3.4) into the form (3.1) and verified Eqs. (2.32) and (2.33).

Notice that since $\Gamma_5(-1)^{\sum_{n=1}^\infty (b_n^\dagger b_n + b_n^{\dagger\prime} b_n')}$ commutes with $I(z)$ and $J(\epsilon)$, Eq. (2.33) can be written as

$$\tilde{W}_A(z)\, i\sqrt{2}\, \frac{H^\mu(y)}{\sqrt{y}} (-1)^{\sum_{n=1}^\infty b_n^\dagger \cdot b_n} = \frac{\Gamma^\mu(y-z)}{\sqrt{y-z}} \Gamma_5 \tilde{W}_A(z)$$

which leads to Eqs. (2.26) and (2.22) with $\beta = i\sqrt{2}(-1)^{\sum_{n=1}^\infty b_n^\dagger b_n}$.

4. PROPERTIES OF THE VERTEX OPERATOR $\tilde{W}_A(z)$

According to Eqs. (3.1) and (3.11) we can write our vertex operator in a form in which the b′ oscillators are eliminated

$$\tilde{W}_A(z) = \langle 0_b |\, e^{I^+(z)+I^0(z)}\, e^{I^-(z)}\, e^{E(z)} |A_d\rangle$$

where E(z) denotes the expression (3.12). All the terms in the exponents are integrals which can be explicitly evaluated by the residue theorem:

$$\begin{aligned} I^+(z) &= -\gamma_5 \sum_{\substack{m=0\\n=1}}^{\infty} d_n^{\mu\,\dagger} (-z)^{n-m-1/2} \binom{n-1/2}{m} b_{m\,\mu} \\ I^0(z) &= \frac{i}{\sqrt{2}} \gamma^\mu \gamma_5 \sum_{m=0}^{\infty} (-z)^{-m-1/2} \binom{-1/2}{m} b_{m\,\mu} \\ I^-(z) &= -\gamma_5 \sum_{\substack{m=0\\n=1}}^{\infty} d_n^{\mu} (-z)^{-n-m-1/2} \binom{-n-1/2}{m} b_m^\mu \\ E(z) &= \frac{1}{2} \sum_{n,m=0}^{\infty} b_n^\mu A_{nm} b_{m\,\mu} \end{aligned} \tag{4.1}$$

where the matrix A_{nm} has been explicitly antisymmetrized and is defined by the equation

$$\sum_{n,m=0}^{\infty} x^n A_{nm} y^m = \frac{1}{2(y-x)}\left\{2 - \sqrt{\frac{y-z}{x-z}} - \sqrt{\frac{x-z}{y-z}}\right\}$$

Because

$$\frac{1}{x-y}\left\{\sqrt{\frac{x-z}{y-z}} - 1\right\} = -\frac{1}{2}\int_z^{\infty}\frac{d\xi}{\sqrt{x-\xi}\,(y-\xi)^{3/2}}$$

and because the latter can be expanded in powers of x and y and integrated term by term, we soon find that

$$A_{nm} = \frac{1}{2}\,\frac{n-m}{n+m+1}\,(-z)^{-n-m-1}\binom{-\frac{1}{2}}{n}\binom{-\frac{1}{2}}{m}$$

Notice how similar this is to the coefficient of $d_n^{\dagger}\, d_m^{\dagger}$ in expression (2.18) (but not the same). We can now compare our vertex with the results of previous authors who were concerned with the case with $|A_d\rangle = |0_d\rangle$. Then I^- disappears and we have an expression $\tilde{W}_0(z)$ like Schwarz'[6] (but not the same, since his was a different quantity).

If we consider $\langle 0_d|W_0(z) = \langle 0_d|\tilde{W}_0(z)$, I^+ also disappears and we are left with precisely Thorn's expression[5], describing the coupling of a meson Reggeon to fermion antifermion ground states.

We have seen that e^{zL^+} and $\tilde{W}(z)$ are both finite when written in normal ordered form. This means that they are "good" operators in the sense that their matrix elements between occupation number states are finite. This need not be true of the product of the two operators, and as we shall now see, there is indeed a problem. For, consider

$$e^{(z-\eta)L^+}\,\tilde{W}_A(z) = \langle 0_{bb'}|\, e^{I(\eta) + iJ(\epsilon)}\, e^{(z-\eta)L^+}\,|0_{b'}A_d\rangle \qquad (4.2)$$

As we take the limit $\eta \to 0$ to get $W_A(z)$ we see that the integral $I(\eta)$ in Eq. (3.3) is singular, since its contour of integration becomes pinched between the essential singularities at 0 and η. This is related to the fact that the coefficients I^+, I^0, and I^- (4.1) diverge as $z \to 0$. Thus, properly speaking $W_A(z)$ does not exist as an operator and this is not surprising perhaps since, according to Eq. (2.29), it does the peculiar job of converting integral powers of y into $\frac{1}{2}$ integral powers.

In the applications we must remember that we do not take general matrix elements of $W_A(z)$ but matrix elements between physical states. Then we get sensible results since the L^+ can be absorbed on the left, leaving $\tilde{W}_A(z)$, which as we have seen, has finite matrix elements. For example,

$$\langle \text{phys} | \Gamma^\mu(z_1)\, W_A(z_2) = \langle \text{phys} | \Gamma^\mu(z_1 - z_2)\, \tilde{W}_A(z_2)$$

since the L^+ has been taken through $\Gamma(z_1)$ and vanishes when acting on the physical state on the left. Since Γ is linear in annihilation and creation operators $\langle \text{phys}|\Gamma(z_1-z_2)$ is a linear combination of occupation number states. Notice that it would not be correct to omit the e^{zL^+} and work with $\tilde{W}$ rather than W from the beginning.

According to the way it was constructed $W_A(z)$ should have straightforward Möbius transformation properties (2.30). In addition, we would like it to have simple transformation properties with respect to the Virasoro generators of the type:

$$L_n^d\, W_A(z) - W_A(z)\, L_n^b = z^{n+1}\frac{d}{dz} W_A(z) + c_d (n+1) z^n W_A(z) \tag{4.3}$$

So far as the manipulations of Section 3 were concerned the state $|A_d\rangle$ could be any occupation number state in the fermion Fock space. In effect this means that we have really calculated the 3-Reggeon vertex for a fermion emitting a meson. In order to prove an equation like that above (4.3), we shall have further to assume that $|A_d\rangle$ is a physical state in the sense that

$$\begin{aligned} L_n^d\, |A_d\rangle &= 0 \qquad n \geqslant 1 \\ L_0^d\, |A_d\rangle &= c_d\, |A_d\rangle \end{aligned} \tag{4.4}$$

where c_d is a number (e.g. $\frac{1}{4}$ for the fermion vacuum state).

Since $W_A(z)$ is not really meaningful, as we have discussed above, the above equation is not meaningful either, and so we shall in fact work with the corresponding equation for $\widetilde{W}_A(z)$. It is not difficult to prove, considering e^{-zL^+} $L_n\, e^{zL^+}$, that

$$L_n\, e^{zL^+} - e^{zL^+} \sum_{s=0}^{n} z^{n-s} \binom{n+1}{s+1} L_s = z^{n+1} \frac{d}{dz} e^{zL^+} \tag{4.5}$$

at least for $n \geq -1$. Combining this with Eq. (4.3) we find that $\widetilde{W}_A(z)$ must satisfy

$$\sum_{s=0}^{n} z^{n-s} \binom{n+1}{s+1} L_s^d\, \widetilde{W}_A(z) - \widetilde{W}_A(z) L_n^b = z^n \left(z \frac{d}{dz} + c_d (n+1) \right) \widetilde{W}_A(z) \tag{4.6}$$

and this is the equation we shall prove at least for $n \geq -1$.

Note that for $n \geq -1$ this equation (4.6) reduces to a trivial identity if we let it act on the meson vacuum state and use the boundary condition (2.32), since we get

$$\sum_{s=0}^{N} z^{N-s} \binom{N+1}{s+1} L_s^d\, |A_d\rangle = c_d (N+1) |A_d\rangle$$

which is correct by Eq. (4.4). This illustrates the relevance of the conditions (4.4).

Using Eqs. (3.2), (3.3) and the commutators

$$\left[L_n^b\,,\ \frac{H_\nu(y)}{\sqrt{y}} \right] = y^n \left(y \frac{d}{dy} + \frac{n+1}{2} \right) \frac{H_\nu(y)}{\sqrt{y}}$$

and a similar one with Γ replacing H, we can easily find that

$$\left[\sum_{s=-1}^{N} z^{N-s} \binom{N+1}{s+1} L_s^d + L_N^b\,,\ I(z) \right] = 0$$

$$\left[\sum_{r=-1}^{N} \epsilon^{N-r} \binom{N+1}{r+1} L_r^{b'} + L_N^b\,,\ J(\epsilon) \right] = 0$$

since on evaluation we obtain integrals of total derivatives about closed paths. The $s = -1$ term is not really wanted in the above equation so that we calculate separately that

$$\left[L_{-1}^d\,,\ I(z) \right] = -\frac{dI(z)}{dz}$$

Putting together these equations we have

$$\left[\sum_{s=0}^{N} z^{N-s}\binom{N+1}{s+1}L_s^d + \sum_{r=-1}^{N} \epsilon^{N-r}\binom{N+1}{r+1}L_r^{b'} + L_N^b,\ e^{I(z)+iJ(\epsilon)}\right]$$

$$= -z^{N+1}\left[L_{-1}^d,\ e^{I(z)+iJ(\epsilon)}\right] \qquad (4.7)$$

$$= z^{N+1}\frac{d}{dz}\, e^{I(z)+iJ(\epsilon)}$$

If we now take matrix elements of this with respect to the states $\langle 0_b, 0_b|$ and $|0_b, A_d\rangle$ and use Eqs. (4.4) and

$$L_r^{b'}\,|0_{b'}\rangle = 0 \qquad r \geq -1$$

we find the desired equation (4.6) except for an extra term

$$\langle 0_{bb'}|\left[L_N^b + \sum_{s=-1}^{N}\epsilon^{N-s}\binom{N+1}{s+1}L_s^{b'}\right]e^{I(z)+iJ(\epsilon)}\,|0_{b'}A_d\rangle \qquad (4.8)$$

This is precisely analogous to the unwanted extra terms in Eq. (3.10) and we shall now show that they will cancel for the same reason. The parts of the L_n which contribute to Eq. (4.8) are those quadratic in H^- (i.e. containing no creation operators). Thus what we need to show is that

$$0 = \langle 0_{b'}|\left\{\oint dz\, z^{N+1}\frac{\hat{H}^-(z)}{\sqrt{z}}\frac{d}{dz}\frac{\hat{H}^-(z)}{\sqrt{z}} + \sum_{s=-1}^{N}\epsilon^{N-s}\binom{N+1}{s+1}\oint dx\, x^{s+1}\frac{\hat{H}'(x)}{\sqrt{x}}\frac{d}{dx}\frac{\hat{H}'(x)}{\sqrt{x}}\right\}$$

$$\cdot\, e^{iJ^+(\epsilon)}\,|0_{b'}\rangle$$

$$\equiv \langle 0_{b'}|\oint dz\, z^{N+1}\left[\frac{\hat{H}^-(z)}{\sqrt{z}}\frac{d}{dz}\frac{\hat{H}^-(z)}{\sqrt{z}} + \frac{\hat{H}'^-(z-\epsilon)}{\sqrt{z-\epsilon}}\frac{d}{dz}\frac{\hat{H}'^-(z-\epsilon)}{\sqrt{z-\epsilon}}\right]e^{iJ^+(\epsilon)}|0_{b'}\rangle$$

and this is indeed zero by Eq. (3.14) (since $i^2 = -1$).

This completes the proof of Eq. (4.6) and hence for Eq. (4.3) for $n \geq -1$. This includes the Möbius invariance which involves $n = -1$, 0 and 1. As mentioned earlier it seems that a different method is needed to extend the argument to $N < -1$.

5. APPLICATION OF THE FORMALISM TO OTHER THEORIES

We have now presented the fermion vertex and its main properties. It has some weird features, for example the auxiliary oscillators and the singularities discussed above. We now want to see to what extent the features depend upon the presence of fermions and to what extent they are general features of dual theories. We shall do this by applying the formalism developed in this paper to more familiar theories and shall, as a by-product, gain some new insight into these older theories.

Let us suppose that the fermion lines are replaced by Neveu-Schwarz meson lines, so that the vertex now describes the emission of a meson with occupation number state $|B''\rangle$ by a meson (that it is almost a 3-meson-Reggeon vertex). To do this we replace $\Gamma^{\mu}(x)$ by $\hat{H}^{\mu''}(x)$, depending on a third set of oscillators b'' and all our previous arguments still apply. Thus the Fubini-Veneziano vertex for the process in Fig. 6 is

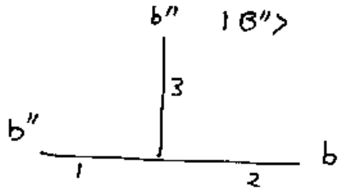

Fig. 6

$$e^{z L^{+}_{b''}} \langle 0_b 0_{b'} | e^{I''(z) + i J''(\epsilon)} | B'' 0_{b'} \rangle \tag{5.1}$$

where I'' and J'' are I and J [Eqs. (3.2) and (3.3)] with Γ replaced by $\hat{H}''$. Now when we rearrange so as to normal order the b'' we see that

$$\left[I''^{+} + i J''^{+} , I^{-''} + i J''^{-} \right] = 0$$

since the $H'H'$ anticommutator cancels with the $H''H''$ anticommutator completely, and not just in the singular part as happened in expression (3.12). The effect of the auxiliary b' oscillators is simply to normal order with respect to the b'' oscillators and the vertex (5.1) can therefore simply be written as

$$e^{z L^{+}_{b''}} \langle 0_b | : e^{I''(z)} : | B'' \rangle \tag{5.2}$$

in which the normal ordering refers to the b'' oscillators and the b' auxiliary oscillators have disappeared. This is why we regarded the auxiliary oscillators as a trick to provide a generalized sort of normal ordering. The interested reader who is sufficiently careful with contour integrals can evaluate expression (5.2) in the case where

$$|B''\rangle = k \cdot b_0^{+''} |0_{b''}\rangle$$

by use of the techniques of Section 3 and obtain

$$k \cdot \frac{H^{+''}(z)}{\sqrt{z}} \langle q_b| e^{-\Sigma b^{+''} \cdot b} |0_{b''}\rangle + \langle q_b| e^{-\Sigma b^{+''} \cdot b} |0_{b''}\rangle \frac{k \cdot H^{-}(z)}{\sqrt{z}}$$

$$= \langle 0_b | e^{-\Sigma b^{+''} \cdot b} |0_{b''}\rangle \frac{k \cdot H(z)}{\sqrt{z}}$$

as it should be for a Neveu-Schwarz pion[9]. Notice that now $I''^{+}(0)$ exists and is simply

$$I''^{+}(0) = -\Sigma b^{+''} \cdot b$$

while

$$I''^{-}(0) = -\Sigma b_n^{+''} \Gamma_{nm} b_m$$

where $\Gamma_{nm} x^m = 1/x^n$ and so Γ_{nm} is the familiar non-existent matrix which always comes into N-Reggeon vertices[13]. This is the effect of the pinch singularity (4.2) noted earlier. Let us now consider the formal expression

$$\langle 0_b | : e^{I''(0)} : \tag{5.3}$$

In view of the above remarks it is the Möbius 3-vertex[13,8] evaluated in the Neveu-Schwarz representation in which

$$\gamma_{12}(z) = z \; , \; \gamma_{13}(z) = z \; , \; \gamma_{23}(z) = 1/z \tag{5.4}$$

and in which the oscillators of legs 1 and 3 are identified. Notice that Eqs. (5.4) are symmetric under the interchange of 2 and 3. Thus, if instead, we identified the oscillators of legs 1 and 2 and called them b, and left the oscillator of leg 3 as b′, which is distinct, as indicated in Fig. 7, then the following would be the same mathematical object as expression (5.3)

$$\langle 0_b | : e^{I(0)} :$$

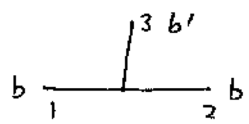

Fig. 7

If we were now to construct the same mathematical object with "a" rather than "b" oscillators, so as to get conventional dual theory we would replace

$$\langle 0_{b'}| \longrightarrow \langle 0_{a'}|$$

$$I(0) \longrightarrow -\frac{1}{2\pi}\oint \frac{dx}{x}\, Q(x)\cdot P'(x)$$

where $Q(x)$ and $P'(x)$ are the fields (2.1) and (2.2) formed with a and a$'$ oscillators, respectively. The integration contour formally includes the singularity of P' and excludes that of Q, in accordance with the procedure described previously. Notice that this exponent is the one given by the Ramond correspondence principle[13)]

$$-i\, p\cdot q \;=\; -i\,\langle P\rangle\cdot\langle Q\rangle \;\rightarrow\; -i\,\langle P\cdot Q\rangle$$

The object constructed is the Möbius 3-vertex[13)] given by Eqs. (5.4) with oscillators on legs 1 and 2 identified.

Now let us comment that the Sciuto vertex[14)] is the Mobius 3-vertex with

$$\gamma_{13}(x) = \Omega\,\Omega^+(x) = 1/1-x$$

$$\gamma_{12}(x) = I(x) = x$$

$$\gamma_{32}(x) = \Omega^+(x) = -x/1-x$$

and hence is

$$\Omega\,\Omega^+\langle 0_{a'}| : e^{-\frac{1}{2\pi}\oint\frac{dx}{x}Q(x)P'(x)} : \Omega\,\Omega^+$$

$$= \langle 0_{a'}| : e^{-\frac{1}{2\pi}\oint\frac{dx}{x}Q\left(\frac{1}{1-x}\right)\cdot P'(x)} :$$

and this is the form first found by Della Selva and Saito[15)]. Notice that the contour of integration is now well defined.

It also follows that the Fubini-Veneziano vertex for the emission of a state $e^{-ik\cdot q}|\psi\rangle$ on leg 3 is

$$V_\psi(z) = z^{K^2/2}\langle 0_{a'}|e^{ik\cdot q'} : e^{-\frac{1}{2\pi}\oint\frac{dx}{x}Q(x)\cdot P'(x)} : e^{zL'_+}\, e^{-ik\cdot q'}|\psi\rangle$$

$$= z^{K^2/2}\langle 0_{a'}|e^{ik\cdot q'} : e^{-\frac{1}{2\pi}\oint\frac{dx}{x}Q(x+z)\cdot P'(x)} : e^{-ik\cdot q'}|\psi\rangle \qquad (5.5)$$

on absorbing the L_+ in $\langle 0_{a'}|$ and making a change of variables. If $|\psi\rangle = \mathcal{P}(a_n^\dagger)|0\rangle$, where $\mathcal{P}$ denotes some polynomial we can evaluate Eq. (5.5) to give

$$V_\psi(z) = z^{k^2/2} : e^{-ik\cdot Q(z)} \mathcal{P}\left(\frac{i\sqrt{n}}{n!} \frac{d^n}{dz^n} Q(z)\right):$$

as expected. Using the methods of the previous section we could prove that

$$\left[L_N, V_\psi(z)\right] = z^{N+1}\frac{d}{dz} V_\psi(z) + c_\psi (N+1) z^N V_\psi(z) \qquad (5.6)$$

providing

$$L_N\, e^{-ik\cdot q'}|\psi\rangle = 0 \quad N \geq 1 \quad , \quad (L_0 - c_\psi) e^{-ik\cdot q'}|\psi\rangle = 0 \qquad (5.7)$$

Once again our argument would only be valid for $N \geq -1$, but our result could immediately be extended to all N by virtue of the complex conjugation symmetry of the vertex $V_\psi(z)$ (which is not valid for the fermion vertex). Notice that we have a natural converse to the theorem of Di Vecchia and Fubini[16], who, given a vertex satisfying Eq. (5.6), construct a state $|\psi\rangle$ satisfying Eqs. (5.7).

It is now clear that the most general vertex $V_A(z)$ for emitting a fermion state $e^{-ik\cdot q}|A\rangle$, with A a state in the combined a,d Fock space, as sought in Section 2, is

$$V_A(z) = z^{k^2/2} e^{zL_d^+} \langle 0_b 0_{b'} 0_{a'}| e^{ik\cdot q'} e^{I(z)+iJ(z)} : e^{-\frac{1}{2\pi}\oint \frac{dx}{x} Q(x+z)\cdot P'(x)} :$$

$$\cdot\, e^{-ik\cdot q'}|0_{b'} A_{a'd}\rangle \qquad (5.8)$$

and that if $|A\rangle$ satisfies Eqs. (2.12) and (2.13),

$$\left(L_N^a + L_N^d\right)V_A(z) - V_A(z)\left(L_N^a + L_N^b\right) = z^N\left[z\frac{d}{dz} V_A(z) + (N+1)V_A(z)\right] \qquad (5.9)$$

at least for $N > -1$. From this we deduce that if

$$W_N = L_0 - L_N - 1$$

is the N^{th} Virasoro gauge operator acting to the right, then

$$\left(W_N^a + W_N^d\right)V_A(1) - V_A(1)\left(W_N^a + W_N^b\right) = -N\, V_A(1)$$

for all $N \geq 1$. On the other hand, because of the restriction $N \geq -1$ on L_N the only result for the gauge operators W_N^+ acting to the left concerns the value $N = 1$:

$$\left(W_N^{a\,+} + W_N^{d\,+}\right) V_A(1) - V_A(1)\left(W_N^{a\,+} + W_N^{d\,+}\right) = N\; V_A(1) \tag{5.10}$$

Thus there are no spurious states in a fermion leg coupled to a system consisting of one physical fermion together with M physical mesons but as far as our results go, there may be states spurious with respect to the higher ($N > 1$) gauges in a meson leg coupled to a system consisting of two physical fermions together with an indefinite number of physical mesons.

As mentioned in the introduction we and Dr. P. Goddard have found separate proofs of Eq. (5.9) and hence (5.10) valid for all appropriate N in the special case that the vertex describes fermion ground state emission. The result is not proved here but it leads us to believe that our inability to prove the full result is due to technical difficulties. Because of the results we do have we believe that our vertex provides the best candidate so far for constructing amplitudes with two fermion lines e.g. that corresponding to the process shown in Fig. 8

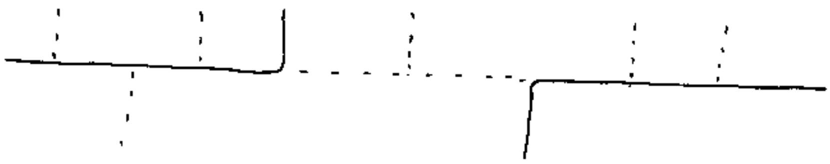

Fig. 8

but we cannot be totally sure that there are no spurious states propagating in the meson lines joining the two fermion lines until we have extended the proof of Eq. (5.9) to account for all N and A, and, what is a more serious problem, until we have understood how the so-called "G-gauges" behave with respect to our vertex[17].

Let us recall that the "full" 3-Reggeon vertex is obtained from the Möbius 3-vertex given by Eq. (5.4) by adjoining to legs 1, 2 and 3 the operator factors X_1, Y_2 and Y_3 (in the notation of Refs. 13 and 18). We therefore expect that the "full" 3-Reggeon vertex for two fermions and a meson is the usual "a" oscillator 3-Reggeon vertex times an expression involving b and d oscillators:

$$\lim_{\eta \to 0} \; X_1^d \, \langle 0_b, 0_b | e^{I(\eta) + i J(\epsilon)} | 0_{b'} \rangle \, Y_2^b \, Y_3^d$$

Acknowledgements

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APPENDIX

This Appendix concerns generalizations of the arguments of Section 2 which we think reveal important structural features of dual theories.

Suppose we have vertex type operators $V_i(z_i)$ with Möbius transformation properties

$$O(\gamma)\, V_i(z_i)\, O(\gamma)^{-1} = V_i(\gamma(z_i)) \,/ (c z_i + d)^{2c_i}$$

The V_i could stand for many possible objects: $: e^{-iQ(z_i)\cdot k} : /z_i$, $H^\mu(z_i)/\sqrt{z_i}$ or $\Gamma^\mu(z_i)/\sqrt{z_i}$ for example. Then since

$$\begin{pmatrix}\alpha & \beta & z\\ \infty & 0 & 1\end{pmatrix} V(z) \begin{pmatrix}\infty & 0 & 1\\ \alpha & \beta & z\end{pmatrix} = V(1)\, \left[(z-\alpha)(z-\beta)\right]^{-2c}$$

where α and β can be any numbers, we find for a product of such vertices that

$$\begin{pmatrix}\alpha_1 & \beta_1 & z_1\\ \infty & 0 & 1\end{pmatrix} V_1(z_1)\, V_2(z_2) \ldots V_N(z_N) \begin{pmatrix}\infty & 0 & 1\\ \alpha_N & \beta_N & z_N\end{pmatrix} = V_1(1)\, P_{12}\, V_2(1)\, P_{23}\, V_3(1) \ldots V_N(1) \tag{A.1}$$

where

$$P_{i,i+1} = \begin{pmatrix}\alpha_i & \beta_i & z_i\\ \infty & 0 & 1\end{pmatrix} \begin{pmatrix}\infty & 0 & 1\\ \alpha_{i+1} & \beta_{i+1} & z_{i+1}\end{pmatrix} \tag{A.2}$$

and we have omitted the c-number factors $(cz + d)^{-2c_i}$ which are not relevant to the points we want to make.

If the variables $z_1 \ldots z_N$ are ordered consecutively on the Koba-Nielsen circle the products in expression (A.1) can be associated with the multiperipheral configuration in Fig. A1

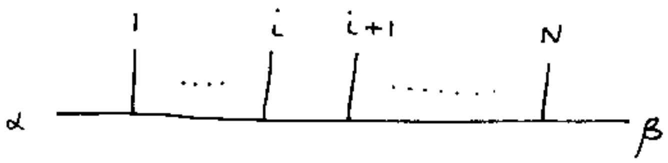

Fig. A1

If we choose each $\alpha_i = \alpha$ and each $\beta_i = \beta$ with α and β the Koba-Nielsen variables of the end particles on Fig. A1, then we find that $P_{i,i+1}$ is the expected "untwisted" propagator

$$P_{i\;i+1} = \left(x_{i,i+1} \right)^{L_0}$$

where

$$x_{i\;i+1} = \left(z_{i+1}, \beta, z_i, \alpha \right)$$

is the Chan variable for the internal line joining external lines i and i+1. So we have found the relation between the Fubini-Veneziano form and the old-fashioned multiperipheral operator formalism.

If now we consider an ordering of Koba-Nielsen variables on the circle appropriate to the semiperipheral diagram (i.e. $z_1 \ldots z_n$, $z_{N+1} \ldots z_{n+1}$) in Fig. A2,

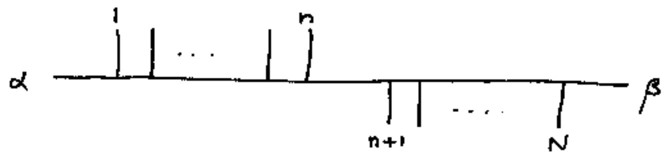

Fig. A2

we see from the previous results that in order to get the correct propagator $(x_{i,i+1})^{L_0}$ on the untwisted lines i = 1 ... n - 1; n + 1 ... N - 1, we must choose

$$\alpha_i = \alpha \qquad \beta_i = z_{n+1} \qquad i = 1 \ldots n$$

$$\alpha_i = z_n \qquad \beta_i = \beta \qquad i = n+1 \ldots N$$

This determines

$$P_{n,n+1} = \begin{pmatrix} \alpha & z_{n+1} & z_n \\ \infty & 0 & 1 \end{pmatrix} \begin{pmatrix} \infty & 0 & 1 \\ z_n & \beta & z_{n+1} \end{pmatrix} = (y)^{L_0}\, \Theta(y)$$

which is the conventional twisted propagator[12)] in terms of the appropriate Chan variable $y = (z_n, \alpha, \beta, z_{n+1})$. Since typically $\left[V(z_n), V(z_{n+1}) \right]_{\pm} = 0 \;\; z_n \neq z_{n+1}$ the twisted line duality relation is trivial.

The conclusion of our argument is that products of Fubini-Veneziano vertices describe both twisted or untwisted configurations, the particular configuration being determined by the ordering of the integration variables z_i on the Koba-Nielsen circle. This result is implied by the literature but we wanted to spell it out since we have used it in the text of this paper.

If we take matrix elements of Eq. (A.1) with respect to physical states $\langle P_0|$ and $|P_{N+1}\rangle$, we see that in the general Möbius frame the (N+1) state is defined by

$$\begin{pmatrix} \infty & 0 & 1 \\ \alpha_N & \beta_N & z_N \end{pmatrix} |P_{N+1}\rangle \tag{A.3}$$

where α_N, β_N, z_N are the Koba-Nielsen variables of respectively the next leg in a clockwise sense, the leg itself, and the next leg in an anticlockwise sense. Thus the operator in expression (A.3) is like that normally associated with the leg of an N-Reggeon vertex[18,13]. Because of the lemma:

$$\begin{pmatrix} \infty & 0 & 1 \\ \alpha & \beta & \gamma \end{pmatrix} = e^{\beta L_+} e^{\frac{L^-}{\beta-\alpha}} \left[\frac{(\gamma-\beta)(\beta-\alpha)}{\gamma-\alpha}\right]^{L_0}$$

we see that due to the physical state and mass shell conditions (2.12) and (2.13) the expression (A.3) reduces to

$$e^{\beta_N L_+} |P_{N+1}\rangle \frac{(z_N-\beta_N)(\beta_N-\alpha_N)}{(z_N-\alpha_N)} \tag{A.4}$$

This is important for several reasons:

i) It leads to an alternative derivation of our general Möbius form (2.14) used in the text.

ii) It clarifies a peculiar feature of the N-Reggeon[13,18] vertex: that the operator associated with a particular leg should depend also on the variables of the adjacent legs as well as the leg itself. We now see that this unwanted extra dependence simplifies when a physical mass shell state is adjoined.

Finally notice that in order to have a cyclically symmetric formalism like that usually applied to mesons one would want

$$V_{N+1}(z_{N+1})|0\rangle = e^{z_{N+1} L^+} |P_{N+1}\rangle$$

(another example of the boundary condition used in this paper). The reader can easily verify examples of this

$$\frac{k \cdot H(z)}{\sqrt{z}} |0\rangle = e^{z L^+_b} k \cdot b_0^+ |0\rangle$$

$$:e^{-ik\cdot Q(z)}:/z|0\rangle = e^{z L^+_a} e^{-ik\cdot q} |0\rangle$$

REFERENCES AND FOOTNOTES

1) The following are reviews providing more references:
V. Alessandrini, D. Amati, M. Le Bellac and D. Olive, Physics Reports, Vol. 1C, No. 6 (1971).
We use the convention by the above review and particularly the convention for metric $g^{\mu\nu} = \mathrm{diag}\ (1,-1,-1,-1)$.
V. Alessandrini and D. Amati, Dual models: Their group theoretic structure, CERN preprint TH 1425 (1971).

2) S. Mandelstam, Phys. Rev. D1, 1720 (1970); D1, 1734 (1970).

3) P. Ramond, Phys. Rev. D3, 2415 (1971).

4) A. Neveu and J. Schwarz, Phys. Rev. D4, 1109 (1971).

5) C.B. Thorn, Phys. Rev. D4, 1112 (1971).

6) J. Schwarz, Phys. Letters 37 B, 315 (1971).

7) P.G.O. Freund, Chicago preprint EFI 71-36 (1971).

8) E. Corrigan and C. Montonen, Nuclear Phys. B36, 58 (1972).

9) A. Neveu and J. Schwarz, Nuclear Phys. B31, 86 (1971).

10) D. Fairlie and D.Olive (unpublished).

11) S. Fubini and G. Veneziano, Nuovo Cimento 67 A, 29 (1970).

12) V. Alessandrini, D. Amati, M. Le Bellac and D. Olive, Physics Reports, Vol. 1C, No. 6 (1971), Chapter 5.

13) V. Alessandrini, D. Amati, M. Le Bellac and D. Olive, Physics Reports, Vol. 1C, No. 6 (1971), Chapter 9.

14) S. Sciuto, Nuovo Cimento Letters 2, 411 (1969).

15) A. Della Selva and S. Saito, Nuovo Cimento Letters 4, 689 (1970).

16) P. Di Vecchia and S. Fubini, Nuovo Cimento Letters 1, 823 (1971).

17) A. Neveu, J. Schwarz and C.B. Thorn, Phys. Letters 35 B, 529 (1971).

18) C. Lovelace, Phys. Letters 32 B, 589 (1970).
D. Olive, Nuovo Cimento 3 A, 399 (1971).