Fermion realization of the nuclear Sp(6,R) model

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A fermion realization of the nuclear Sp(6,R) model, which complements the traditional bosonic representation, is developed. A recursive process is presented in which symplectic matrix elements of arbitrary one-body fermion operators between states of excitation $N\hbar\omega$ and $N'\hbar\omega$ in the same or in different symplectic bands are related back to valence shell matrix elements, which can be evaluated by standard shell model techniques. Matrix elements so determined may be used to calculate observables such as electron scattering form factors which carry detailed structural information on nuclear wave functions. © *1998 American Institute of Physics.* [S0022-2488(98)02110-0]

I. INTRODUCTION

Extensive effort has been devoted to developing the relevant mathematical and computational tools for a fully microscopic treatment of collective nuclear phenomena. In particular, ever since the noncompact symplectic group in three dimensions, Sp(6,R), was recognized as the appropriate dynamical group for a many-body theory of nuclear collective motion,[1–3] it has received careful and detailed attention. This group is of special interest because it is also the dynamical group of the three-dimensional harmonic oscillator and thus it establishes an important link between the nuclear shell model and the collective model. Generalized vector coherent state theory and boson realizations of the symplectic algebra have been employed to construct the discrete infinite-dimensional unitary irreducible representations of Sp(6,R).[4–13] Methods have been introduced for calculating the necessary orthonormalization factors for symplectic basis states and the requisite matrix elements of the symplectic generators.[7,14–16] In addition, a procedure has been developed for evaluating matrix elements of general two-body interactions of the type used in standard microscopic treatments of nuclear many-body systems.[17–19]

Despite practical limitations due to the large sizes of the Hilbert spaces involved, various applications have proven the symplectic approach to be successful in reproducing collective nuclear properties, such as excitation energies, quadrupole moments, and electromagnetic transition probabilities.[20–24] Unfortunately, until now it has not been possible to evaluate matrix elements of arbitrary one-body operators between symplectic basis states. These matrix elements are of particular interest since they are required for the calculation of observables, such as nuclear form factors, which carry very detailed structural information on nuclear wave functions. It is our purpose in this article to provide a new, fermionic, realization of the sp(6,R) algebra, which complements the traditional bosonic representation and leads to a method for determining matrix elements of a general one-body operator in a Sp(6,R)$\supset$U(3) basis. Specifically, a recursive process is presented in which symplectic matrix elements of arbitrary one-body fermion operators between states of excitation $N\hbar\omega$ and $N'\hbar\omega$ in the same or in different symplectic bands are related back to valence shell matrix elements, which can be evaluated by standard shell model techniques. The fermionic realization of the symplectic algebra is particularly well suited for this approach, since the requisite valence shell matrix elements are readily available.[25]

The article is organized as follows: To establish the notation, in Sec. II we give a brief summary of the generators of the symplectic algebra, their bosonic representation, commutation relations, and matrix elements. In Sec. III the symplectic generators are recast in terms of fermi-

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onic creation and annihilation operators. The symplectic model space is reviewed in Sec. IV. In Sec. V we present the derivation of the matrix element formula for a general one-body operator acting in the symplectic space. Concluding remarks are given in Sec. VI. The derivations in Secs. III and V make use of both oscillator boson operators and fermion creation and annihilation operators. Phase conventions, conjugation properties, commutation relations, and matrix elements of the former are given in Appendix A, and conjugation properties and commutation relations of various SU(3)-coupled products of the latter are derived in Appendix B. In addition, Appendix C provides a compilation of various important SU(3) recoupling rules.

II. GENERATORS OF THE SYMPLECTIC ALGEBRA

The generators of the symplectic algebra $sp(6,R)$ can be realized in terms of bilinear products of harmonic oscillator bosons in a Cartesian scheme:[26]

$$C_{ij}=\sum_s\left(b_i^\dagger(s)b_j(s)+\frac{1}{2}\delta_{ij}\right)-\frac{1}{A}\sum_s b_i^\dagger(s)\sum_t b_j(t)-\frac{1}{2}\delta_{ij},$$

$$B_{ij}^\dagger=\frac{1}{2}\sum_s b_i^\dagger(s)b_j^\dagger(s)-\frac{1}{2A}\sum_s b_i^\dagger(s)\sum_t b_j^\dagger(t), \tag{1}$$

$$B_{ij}=\frac{1}{2}\sum_s b_i(s)b_j(s)-\frac{1}{2A}\sum_s b_i(s)\sum_t b_j(t),$$

where the sums run over all A particles in the system and the two-body $1/A$ terms effect the removal of spurious center-of-mass excitations from the A-particle system. (Refer to Appendix A for details on the boson creation and annihilation operators.) The commutation relations for the symplectic algebra in this basis are easily inferred from the commutation relations of its building blocks and are listed by Hecht.[13]

Making use of the spherical components $b_{1q}^{\dagger(10)}(s)$ and $\tilde{b}_{1q}^{(01)}(s)$ of the boson creation and annihilation operators, the symplectic generators can be written as SU(3) irreducible tensor operators:

$$C_{lm}^{(11)}=\sqrt{2}\sum_s\{b^\dagger(s)\times\tilde{b}(s)\}_{lm}^{(11)}-\frac{\sqrt{2}}{A}\sum_{st}\{b^\dagger(s)\times\tilde{b}(t)\}_{lm}^{(11)},$$

$$A_{lm}^{(20)}=\frac{1}{\sqrt{2}}\sum_s\{b^\dagger(s)\times b^\dagger(s)\}_{lm}^{(20)}-\frac{1}{\sqrt{2}A}\sum_{st}\{b^\dagger(s)\times b^\dagger(t)\}_{lm}^{(20)}, \tag{2}$$

$$B_{lm}^{(02)}=\frac{1}{\sqrt{2}}\sum_s\{\tilde{b}(s)\times\tilde{b}(s)\}_{lm}^{(02)}-\frac{1}{\sqrt{2}A}\sum_{st}\{\tilde{b}(s)\times\tilde{b}(t)\}_{lm}^{(02)},$$

transforming according to the SU(3) irreps $(\lambda\mu)=(11)$, (20), and (02), respectively. The overall normalization factor is chosen in agreement with the convention used by Rosensteel.[27] The symplectic $2\hbar\omega$ raising and lowering operators, $A_{lm}^{(20)}$ and $B_{lm}^{(02)}$, respectively, are related to each other via Hermitian conjugation: $B_{lm}^{(02)}=(-1)^{l-m}(A_{l-m}^{(20)})^\dagger$. Both $A_{lm}^{(20)}$ and $B_{lm}^{(02)}$ have $l=0$ and 2 components, whereas $C_{lm}^{(11)}$ has $l=1$ and 2 components. The relevant commutation relations between the spherical components of the symplectic raising and lowering operators are given by Rosensteel:[27]

$$[A^{(20)}_{l_1m_1},A^{(20)}_{l_2m_2}]=[B^{(02)}_{l_1m_1},B^{(02)}_{l_2m_2}]=0,$$

$$[B^{(02)}_{l_1m_1},A^{(20)}_{l_2m_2}]=\frac{1}{2}\sqrt{10}\sum_{lm}\langle(02)l_1m_1;(20)l_2m_2|(11)lm\rangle C^{(11)}_{lm}$$
$$+2\sqrt{\frac{2}{3}}(-1)^{l_2+m_2}\delta_{l_1l_2}\delta_{m_1(-m_2)}H_0. \tag{3}$$

Here $\langle -;-|-\rangle$ denotes a Wigner SU(3) coupling coefficient (see Appendix C) and $H_0=\hat{N}_b+\frac{3}{2}$, where $\hat{N}_b=\Sigma_{i=1}^{3}C_{ii}$ counts the number of harmonic oscillator bosons in the system.

The eight operators $C^{(11)}_{lm}$ generate the $su(3)$ subalgebra of $sp(3,R)$ and are related to the angular momentum operator L_q and the Elliott (algebraic) quadrupole operator $Q^a_{2m}=\sqrt{4\pi/5}\Sigma_s(r_s^2Y_{2m}(\hat{r}_s)+p_s^2Y_{2m}(\hat{p}_s))$, here given in units of $\hbar=\omega=m=1$, as follows:

$$C^{(11)}_{1q}=L_q,\quad q=0,\pm1$$
$$C^{(11)}_{2m}=\frac{1}{\sqrt{3}}Q^a_{2m},\quad m=0,\pm1,\pm2. \tag{4}$$

We can also express the collective quadrupole operator $Q^c_{2m}=\sqrt{16\pi/5}\Sigma_s r_s^2Y_{2m}(\hat{r}_s)$ as a linear combination of SU(3) irreducible tensor operators:

$$Q^c_{2m}=Q^a_{2m}+\sqrt{3}(A^{(20)}_{2m}+B^{(02)}_{2m}). \tag{5}$$

Matrix elements for $C^{(11)}$ in the standard SU(3) basis are given by[28,29]

$$\langle(\lambda'\mu')|||C^{(11)}|||(\lambda\mu)\rangle=(-1)^{\phi}\sqrt{2C_2(\lambda\mu)}\,\delta_{(\lambda'\mu')(\lambda\mu)},$$
$$\phi=\begin{cases}1, & \text{for }\ \mu\neq0\\ 0, & \text{for }\ \mu=0,\end{cases} \tag{6}$$
$$C_2(\lambda\mu)=\frac{2}{3}(\lambda^2+\lambda\mu+\mu^2+3\lambda+3\mu).$$

The reduced matrix element $\langle(\lambda'\mu')|||C^{(11)}|||(\lambda\mu)\rangle$ is related to the full SU(3) matrix element via the Wigner–Eckart theorem for SU(3) (see Appendix C). The symbol $C_2(\lambda\mu)$ denotes the second-order Casimir invariant of SU(3), and the choice of the phase is consistent with that of reference 29.

Several strategies for calculating matrix elements of the symplectic generators $A^{(20)}$ and $B^{(02)}$ have been explored. A direct way is to use the Sp$(6,R)$ commutation relations to derive recursion formulas, as shown by Rosensteel.[27] Another approach is to start from approximate matrix elements and proceed by successive approximations, adjusting the matrix elements until the commutation relations are precisely satisfied.[3] Deenen and Quesne[16] have employed a boson mapping to obtain the generator matrix elements, and Castaños *et al.*[15] have derived simple analytical functions for some special irreps. The most elegant method, outlined by Rowe in reference 7, involves vector-valued coherent state representation theory and evaluates matrix elements of the symplectic raising and lowering operators by relating them to the matrix elements of a much simpler $u(3)\otimes$*Weyl* algebra.

III. FERMION REALIZATION OF THE SYMPLECTIC GENERATORS

In a fermion second quantization formulation, the one-body part of $C^{(11)}$ takes the form

$$\mathscr{F}(C^{(11)s=0}_{LM\sigma=0})=\left(1-\frac{1}{A}\right)\sqrt{2}\sum_{\rho\rho'}\left\langle\underbrace{(\nu0)lm\frac{1}{2}\sigma''}_{\rho}\right|\{b^{\dagger}\times\tilde{b}\}^{(11)s=0}_{LM\sigma=0}\left|\underbrace{(\nu'0)l'm'\ \frac{1}{2}\sigma'}_{\rho'}\right\rangle$$
$$\times a^{\dagger}_{(\nu0)lm(1/2)\sigma''}a_{(\nu'0)l'm'(1/2)\sigma'},\tag{7}$$

where $a^{\dagger}_{\rho}$ and a_{ρ} are fermion creation and annihilation operators (see Appendix B). Note that $b^{\dagger}$, $\tilde{b}$, and $C^{(11)}_{LM}$ do not act on the spin part of the wave functions; thus one needs to treat these operators as $s=\sigma=0$ objects, as has been explicitly expressed in the above equation.

Utilizing the symmetry properties of the Clebsch–Gordan and SU(3) Wigner coefficients and the definition of the proper SU(3) irreducible tensor operator $\tilde{a}_{(0\nu')l'(-m')(1/2)(-\sigma')}\equiv(-1)^{\nu'+l'-m'+1/2-\sigma'}a_{(\nu'0)l'm'(1/2)\sigma'}$, we obtain

$$\mathscr{F}(C^{(11)s=0}_{LM\sigma=0})=\left(1-\frac{1}{A}\right)\sqrt{2}\sum_{\nu\nu'}\left\langle(\nu0)\frac{1}{2}\right\|\left|\{b^{\dagger}\times\tilde{b}\}^{(11)s=0}\right\|\left|(\nu'0)\frac{1}{2}\right\rangle\sqrt{d(\nu0)}\,\frac{1}{2}$$
$$\times\{a^{\dagger}_{(\nu0)(1/2)}\times\tilde{a}_{(0\nu')(1/2)}\}^{(11)s=0}_{LM\sigma=0}$$
$$=\left(1-\frac{1}{A}\right)\sum_{\nu}\sqrt{\frac{1}{6}\nu(\nu+1)(\nu+2)(\nu+3)}$$
$$\times\{a^{\dagger}_{(\nu0)(1/2)}\times\tilde{a}_{(0\nu)(1/2)}\}^{(11)s=0}_{LM\sigma=0},\tag{8}$$

where $d(\lambda\mu)=\frac{1}{2}(\lambda+1)(\mu+1)(\lambda+\mu+2)$ denotes the dimension of the su(3) irrep $(\lambda\mu)$ and use has been made of Eq. (A6), which expresses the triple-reduced matrix element in terms of ν and ν' only (see Appendix A). Analogously one derives the fermion realization for the one-body part of the symplectic raising and lowering operators.

The two-body parts of the symplectic generators can be obtained in the same manner. A general symmetric two-body operator for A identical particles, $\mathscr{G}=\Sigma^{A}_{s<t=1}g(\mathbf{x}_s,\boldsymbol{\sigma}_s,\mathbf{x}_t,\boldsymbol{\sigma}_t)$, where s and t refer to the s-th and t-th particles, respectively, takes the following form in a fermion second quantized formulation:

$$\mathscr{G}=\frac{1}{4}\sum_{\substack{\rho_1,\rho_1'\\ \rho_2,\rho_2'}}\langle\rho_1\rho_2|g(\mathbf{x}_1,\boldsymbol{\sigma}_1,\mathbf{x}_2,\boldsymbol{\sigma}_2)|\rho_1'\rho_2'\rangle a^{\dagger}_{\rho_1}a^{\dagger}_{\rho_2}a_{\rho_2'}a_{\rho_1'}.\tag{9}$$

Here $|\rho_1\rho_2\rangle$ denotes the direct product of the single-particle wave functions $|\rho_1\rangle$ and $|\rho_2\rangle$. For the cases that are of interest here, the function $g(\mathbf{x}_1,\boldsymbol{\sigma}_1,\mathbf{x}_2,\boldsymbol{\sigma}_2)$ can be written as a product $g(1)g(2)\equiv g(\mathbf{x}_1,\boldsymbol{\sigma}_1)g(\mathbf{x}_2,\boldsymbol{\sigma}_2)$, where $g(t)$, with $t=1$ or 2, acts solely on the single-particle wave function with subscript t. Thus the two-body part of the symplectic generator $C^{(11)}$ is written as

$$
\begin{aligned}
\mathscr{G}(C^{(11)s=0}_{LM\sigma=0}) = &-\frac{1}{\sqrt{8}A}\sum_{\substack{\rho_1,\rho_1'\\ \rho_2,\rho_2'}} \Big\langle \underbrace{(\nu_1 0)l_1 m_1 \tfrac{1}{2}\sigma_1}_{\rho_1};\underbrace{(\nu_2 0)l_2 m_2 \tfrac{1}{2}\sigma_2}_{\rho_2}\Big| \\
&\times\{b^\dagger(1)\times\tilde{b}(2)\}^{(11)s=0}_{LM\sigma=0}\Big|\underbrace{(\nu_1' 0)l_1' m_1' \tfrac{1}{2}\sigma_1'}_{\rho_1'};\underbrace{(\nu_2' 0)l_2' m_2' \tfrac{1}{2}\sigma_2'}_{\rho_2'}\Big\rangle \\
&\times a^\dagger_{(\nu_1 0)l_1 m_1(1/2)\sigma_1} a^\dagger_{(\nu_2 0)l_2 m_2(1/2)\sigma_2} a_{(\nu_2' 0)l_2' m_2'(1/2)\sigma_2'} a_{(\nu_1' 0)l_1' m_1'(1/2)\sigma_1'} \\
= &-\frac{1}{\sqrt{8}A}\sum_{\substack{(\lambda\mu)S\\ (\lambda'\mu')S'}} \sqrt{\frac{1}{8}\,d(\lambda\mu)(2S+1)}\Big\langle \{(\nu_1 0)\times(\nu_2 0)\}(\lambda\mu);\Big\{\frac{1}{2}\times\frac{1}{2}\Big\}S\Big|\Big|\Big| \\
&\times\{b^\dagger(1)\times\tilde{b}(2)\}^{(11)s=0}\Big|\Big|\Big|\{(\nu_1' 0)\times(\nu_2' 0)\}(\lambda'\mu');\Big\{\frac{1}{2}\times\frac{1}{2}\Big\}S'\Big\rangle \\
&\times\sum_{\substack{\nu_1\nu_2\\ \nu_1'\nu_2'}} \{\{a^\dagger_{(\nu_1 0)(1/2)}\times a^\dagger_{(\nu_2 0)(1/2)}\}^{(\lambda\mu)S}\times\{\tilde{a}_{(0\nu_1')(1/2)}\times\tilde{a}_{(0\nu_2')(1/2)}\}^{(\mu'\lambda')S'}\}^{(11)s=0}_{LM\sigma=0},
\end{aligned}
\tag{10}
$$

and analogously for $\mathscr{G}(A^{(20)s=0}_{LM\sigma=0})$ and $\mathscr{G}(B^{(02)s=0}_{LM\sigma=0})$. Here use has been made of the symmetry properties of the Clebsch–Gordan and Wigner coefficients and of the Wigner–Eckart theorem for both SU(2) and SU(3). Upon evaluation of the triple-reduced matrix elements of $\{b^\dagger(1)\times\tilde{b}(2)\}^{(11)}$, $\{b^\dagger(1)\times b^\dagger(2)\}^{(20)}$, and $\{\tilde{b}(1)\times\tilde{b}(2)\}^{(02)}$ (see Appendix A), we find the following fermionic expressions for the symplectic generators $C^{(11)}$, $A^{(20)}$, and $B^{(02)}$, respectively:

$$
\begin{aligned}
C^{(11)}_{LM} = &\left(1-\frac{1}{A}\right)\sum_\nu \sqrt{\frac{1}{6}\nu(\nu+1)(\nu+2)(\nu+3)}\{a^\dagger_{(\nu 0)(1/2)}\times\tilde{a}_{(0\nu)(1/2)}\}^{(11)s=0}_{LM\sigma=0} \\
&-\frac{1}{4A}\sum_{\substack{\nu\nu' S\\ (\lambda\mu)(\lambda'\mu')}} \sqrt{\frac{1}{4}\,d(\lambda\mu)\nu(\nu'+3)(2S+1)}\left\{\begin{matrix} (\nu-1,0) & (10) & (\nu 0) & - \\ (\nu'+1,0) & (01) & (\nu' 0) & - \\ (\lambda'\mu') & (11) & (\lambda\mu) & - \\ - & - & - & \end{matrix}\right\} \\
&\times\{\{a^\dagger_{(\nu 0)(1/2)}\times a^\dagger_{(\nu' 0)(1/2)}\}^{(\lambda\mu)S}\times\{\tilde{a}_{(0,\nu-1)(1/2)}\times\tilde{a}_{(0,\nu'+1)(1/2)}\}^{(\mu'\lambda')S}\}^{(11)s=0}_{LM\sigma=0},
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
A^{(20)}_{LM} = &\left(1-\frac{1}{A}\right)\sum_\nu \sqrt{\frac{1}{12}(\nu+1)(\nu+2)(\nu+3)(\nu+4)}\{a^\dagger_{(\nu+2,0)(1/2)}\times\tilde{a}_{(0\nu)(1/2)}\}^{(20)s=0}_{LM\sigma=0} \\
&-\frac{1}{4A}\sum_{\substack{\nu\nu' S\\ (\lambda\mu)(\lambda'\mu')}} \sqrt{\frac{1}{12}\,d(\lambda\mu)\nu\nu'(2S+1)}\left\{\begin{matrix} (\nu-1,0) & (10) & (\nu 0) & - \\ (\nu'-1,0) & (10) & (\nu' 0) & - \\ (\lambda'\mu') & (20) & (\lambda\mu) & - \\ - & - & - & \end{matrix}\right\} \\
&\times\{\{a^\dagger_{(\nu 0)(1/2)}\times a^\dagger_{(\nu' 0)(1/2)}\}^{(\lambda\mu)S}\times\{\tilde{a}_{(0,\nu-1)(1/2)}\times\tilde{a}_{(0,\nu'-1)(1/2)}\}^{(\mu'\lambda')S}\}^{(20)s=0}_{LM\sigma=0},
\end{aligned}
\tag{12}
$$

and

$$B_{LM}^{(02)}=\left(1-\frac{1}{A}\right)\sum_{\nu}\sqrt{\frac{1}{12}(\nu+1)(\nu+2)(\nu+3)(\nu+4)}\{a^{\dagger}_{(\nu 0)(1/2)}\times\tilde{a}_{(0,\nu+2)(1/2)}\}^{(02)s=0}_{LM\sigma=0}$$

$$-\frac{1}{4A}\sum_{\substack{\nu\nu' S\\(\lambda\mu)(\lambda'\mu')}}\sqrt{\frac{1}{12}d(\lambda\mu)(\nu+3)(\nu'+3)(2S+1)}\begin{Bmatrix}(\nu+1,0) & (01) & (\nu 0) & -\\ (\nu'+1,0) & (01) & (\nu' 0) & -\\ (\lambda'\mu') & (02) & (\lambda\mu) & -\\ - & - & - & \end{Bmatrix}$$

$$\times\{\{a^{\dagger}_{(\nu 0)(1/2)}\times a^{\dagger}_{(\nu' 0)(1/2)}\}^{(\lambda\mu)S}\times\{\tilde{a}_{(0,\nu+1)(1/2)}\times\tilde{a}_{(0,\nu'+1)(1/2)}\}^{(\mu'\lambda')S}\}^{(02)s=0}_{LM\sigma=0}. \tag{13}$$

Making use of the properties of the coupling coefficients and of the Hermitian conjugation properties of the generators, one can verify that $B_{LM}^{(02)}=(-1)^{L-M}(A_{L-M}^{(20)})^{\dagger}$ holds, as expected. This relation serves as a stringent test for the fermionic expressions of $A^{(20)}$ and $B^{(02)}$.

IV. SYMPLECTIC MODEL SPACE

A basis for the Hilbert space is generated by applying symmetrically coupled products of the $2\hbar\omega$ raising operator $A^{(20)}$ with itself to the usual $0\hbar\omega$ shell-model states. The $0\hbar\omega$ starting configurations are labeled by the Elliott SU(3) quantum numbers $(\lambda_\sigma\mu_\sigma)$[30,31] and by N_σ, the eigenvalue of the oscillator boson number operator which takes the minimum value consistent with the Pauli Exclusion Principle. The product of $N_n/2$ raising tensors $A^{(20)}$, each of which promotes a particle from a given shell into a higher-lying shell $2\hbar\omega$ above, generates $N_n\hbar\omega$ excitations for each starting irrep $N_\sigma(\lambda_\sigma\mu_\sigma)$. Each such product operator $\mathscr{P}^{N_n(\lambda_n\mu_n)}_{\alpha_n}$, labeled according to its SU(3) content, $(\lambda_n\mu_n)$, is then coupled with $|N_\sigma(\lambda_\sigma\mu_\sigma)\rangle$ to good SU(3) symmetry $\rho(\lambda_\omega\mu_\omega)$, with ρ denoting the multiplicity of the coupling $(\lambda_n\mu_n)\times(\lambda_\sigma\mu_\sigma)$.

It will be convenient to use the general shorthand notation, Γ, for a U(3) or SU(3) representation label, and α for an appropriate set of U(3) subgroup labels. We thus introduce, following the notation of Hecht,[32]

$$\begin{aligned}\Gamma_\sigma&\equiv[\sigma_1\sigma_2\sigma_3]\equiv N_\sigma(\lambda_\sigma\mu_\sigma)=N_\sigma(\sigma_1-\sigma_2,\sigma_2-\sigma_3),\\ \Gamma_n&\equiv[n_1n_2n_3]\equiv N_n(\lambda_n\mu_n)=N_n(n_1-n_2,n_2-n_3),\\ \Gamma_\omega&\equiv[\omega_1\omega_2\omega_3]\equiv N_\omega(\lambda_\omega\mu_\omega)=N_\omega(\omega_1-\omega_2,\omega_2-\omega_3),\end{aligned} \tag{14}$$

where $(\lambda\mu)$ are SU(3) labels and the $N_\sigma=\sigma_1+\sigma_2+\sigma_3$, $N_n=n_1+n_2+n_3$, and $N_\omega=\omega_1+\omega_2+\omega_3$ give the number of squares in the U(3) Young tableaux. With this convention, the product operators, which are defined recursively, can be written as

$$\mathscr{P}^{\Gamma_n}_{\alpha_n}(A^{(20)})=\sum_{\Gamma_{n'}\beta\alpha_{n'}}\langle(20)\beta;\Gamma_{n'}\alpha_{n'}|\Gamma_n\alpha_n\rangle X^{\Gamma_n}(\Gamma_{n'})A^{(20)}_{\beta}\mathscr{P}^{\Gamma_{n'}}_{\alpha_{n'}}(A^{(20)}), \tag{15}$$

with $A^{(20)}$ adding a $2\hbar\omega$ excitation to the $N_{n'}$ excitation $(N_{n'}=n_1'+n_2'+n_3')$ that is created by the action of the operator $\mathscr{P}^{\Gamma_{n'}}_{\alpha_{n'}}(A^{(20)})$; and the factor

$$X^{\Gamma_n}(\Gamma_{n'})=\frac{1}{n_1+n_2+n_3}(n_1n_2n_3|||\mathscr{A}^{(20)}|||n_1'n_2'n_3') \tag{16}$$

is required to properly normalize the raising polynomials (for details see reference 32). The n_i in the above equation denote the number of oscillator bosons in the i-th direction, and we have $N_n=N_{n'}+2$. The operator $\mathscr{A}^{(20)}$ is a generator of the $u(3)\otimes Weyl$ algebra; the evaluation of its matrix elements and its relation to the symplectic generator $A^{(20)}$ are discussed in references 33, 7.

We thus obtain a basis of $\mathrm{Sp}(3,R)$ states that are reduced according to the subgroup chain

$$\begin{array}{ccccccc}\mathrm{Sp}(3,R) & \supset & \mathrm{U}(3) & \supset & \mathrm{SO}(3) & \supset & \mathrm{SO}(2)\\ \Gamma_\sigma & \Gamma_n\rho & \Gamma_\omega & \kappa & L & & M.\end{array} \tag{17}$$

For each $0\hbar\omega$ SU(3) starting irrep $\Gamma_\sigma = N_\sigma(\lambda_\sigma\mu_\sigma)$ a basis for a symplectic representation is given by states of the form

$$|\Phi[\Gamma_\sigma\Gamma_n\rho\Gamma_\omega\alpha_\omega]\rangle \equiv \sum_{\alpha_n\alpha_\sigma} \langle\Gamma_n\alpha_n;\Gamma_\sigma\alpha_\sigma|\Gamma_\omega\alpha_\omega\rangle_\rho \mathscr{P}^{\Gamma_n}_{\alpha_n}(A^{(20)})|\Gamma_\sigma\alpha_\sigma\rangle, \tag{18}$$

where $N_n/2=0,1,2,\ldots,$ counts the number of boson excitations, $N_\omega = N_\sigma + N_n$, $(\lambda_n\mu_n)$ ranges over the set $\Omega = \{(n_1-n_2, n_2-n_3)|n_1 \geqslant n_2 \geqslant n_3 \geqslant 0; N_n = n_1+n_2+n_3; n_1,n_2,n_3$ even integers$\}$, $\rho\Gamma_\omega$ includes all SU(3) irreps resulting from the coupling $\Gamma_n \times \Gamma_\sigma$, and $\alpha_\omega = \kappa LM$ denotes quantum numbers associated with the group chain SU(3)$\supset$SO(3)$\supset$SO(2). Alternatively, one can also choose subgroup labels $\alpha_\omega = \varepsilon\Lambda M_\Lambda$, which are associated with the chain SU(3)$\supset$SU(2)$\times$U(1)$\supset$SO(2). The states of the $\mathrm{Sp}(3,R)\supset\mathrm{SU}(3)$ basis are thus labeled by three types of U(3) quantum numbers: $\Gamma_\sigma = N_\sigma(\lambda_\sigma\mu_\sigma)$, the symplectic bandhead or Sp(3,R) lowest weight U(3) symmetry, which specifies the Sp(3,R) irreducible representation; $\Gamma_n = N_n(\lambda_n\mu_n)$, the U(3) symmetry of the raising polynomial; and $\Gamma_\omega = N_\omega(\lambda_\omega\mu_\omega)$, the U(3) symmetry of the final state. Any given symplectic representation space $N_\sigma(\lambda_\sigma\mu_\sigma)$ is infinite dimensional, since $N_n/2$, the number of boson excitations, can take any positive integer value. In practical applications, one must therefore either truncate the symplectic Hilbert space, or restrict oneself to interactions and observables for which the matrix elements depend solely on the symplectic irrep and can be calculated analytically.

The states of Equation (18) are eigenstates of the harmonic oscillator Hamiltonian, $H_0|\Phi\rangle = E_0|\Phi\rangle$, with eigenvalues $E_0 = (N_\omega + \frac{3}{2})\hbar\omega$. Two such states with different U(3) content $\Gamma_\omega = N_\omega(\lambda_\omega\mu_\omega)$ are orthogonal, whereas two states with identical U(3) symmetry Γ_ω, but different $\rho\Gamma_n = \rho N_n(\lambda_n\mu_n)$ quantum numbers, are generally not orthogonal. The states $|\Phi(\Gamma_\sigma\Gamma_n\rho\Gamma_\omega\alpha_\omega)\rangle$ of Equation (18) can be related to the orthonormal basis states $|\Gamma_\sigma\Gamma_n\rho\Gamma_\omega\alpha_\omega\rangle$ of the unitary irreducible representation of $\mathrm{Sp}(3,R)$, by

$$|\Phi[\Gamma_\sigma\Gamma_{n_i}\rho_i\Gamma_\omega\alpha_\omega]\rangle = \sum_j [\mathscr{K}(\Gamma_\sigma,\Gamma_\omega)]_{\Gamma_{n_i}\rho_i,\Gamma_{n_j}\rho_j}|\Gamma_\sigma\Gamma_{n_j}\rho_j\Gamma_\omega\alpha_\omega\rangle. \tag{19}$$

Here $|\Gamma_\sigma\Gamma_n\rho\Gamma_\omega\alpha_\omega\rangle$, without the letter Φ stands for the orthonormal basis states, and the symbol $[\mathscr{K}(\Gamma_\sigma,\Gamma_\omega)]_{n_i\rho_i,n_j\rho_j}$ denotes the matrix elements of $\mathscr{K}$, the positive Hermitian square root of the overlap matrix $\mathscr{K}^2$, which has matrix elements

$$[\mathscr{K}^2(\Gamma_\sigma,\Gamma_\omega)]_{\Gamma_{n'}\rho',\Gamma_n\rho} \equiv \langle\Phi[\Gamma_\sigma\Gamma_{n'}\rho'\Gamma_\omega\alpha_\omega]|\Phi[\Gamma_\sigma\Gamma_n\rho\Gamma_\omega\alpha_\omega]\rangle. \tag{20}$$

The matrix $\mathscr{K}^2$ is diagonal in Γ_σ and Γ_ω and independent of U(3) subgroup labels α_ω, and its rows and columns are labeled by Γ_n and ρ. Due to the smallness of the off-diagonal matrix elements of $\mathscr{K}^2$ the orthonormal basis states in Equation (19) can also be tagged by those labels $\Gamma_{n_i}\rho_i$ which correspond to the dominant values of $\Gamma_n\rho$ in these states. The method for calculating the matrix elements of $\mathscr{K}^2$, and therefore of $\mathscr{K}$, is given in reference 7. An approximation formula for the matrix elements of $\mathscr{K}$ has been worked out by Hecht.[32]

V. MATRIX ELEMENTS OF ARBITRARY ONE-BODY OPERATORS

In this section we derive a recursion formula in which symplectic matrix elements of arbitrary one-body operators between states of excitation $N_{n_1}\hbar\omega$ and $N_{n_2}\hbar\omega$ in the same or in different symplectic bands are related back to valence shell matrix elements, which can be evaluated by standard shell model techniques. The derivation of the desired recursion formula makes use of the fact that the symplectic basis states are constructed by applying polynomials $\mathscr{P}^{\Gamma_n}_{\alpha_n}(A^{(20)})$ of symmetrically coupled products of the symplectic raising operator $A^{(20)}$ with itself to $0\hbar\omega$ shell model configurations $|\Gamma_\sigma\alpha_\sigma\rangle$. (As before, we will denote the orthonormal basis states by $|\Gamma_\sigma\Gamma_n\rho\Gamma_\omega\alpha_\omega\rangle$, without the letter Φ.) Using Equations (18) and (15) and the definition of the SU(3) Racah coefficients U (see Appendix C), we can now express a symplectic basis state $|\Phi[\Gamma_\sigma\Gamma_n\rho\Gamma_\omega\alpha_\omega]\rangle$ of excitation $N_n\hbar\omega$ in terms of basis states $|\Phi[\Gamma_\sigma\Gamma_{n'}\rho'\Gamma_{\omega'}\alpha_{\omega'}]\rangle$ of excitation $N_{n'}\hbar\omega$ with $N_{n'} = N_n - 2$:

$$|\Phi[\Gamma_\sigma\Gamma_n\rho\Gamma_\omega\alpha_\omega]\rangle=\sum_{\Gamma_{n'}\rho'\Gamma_{\omega'}\beta\alpha_{\omega'}} U[(20)\Gamma_{n'}\Gamma_\omega\Gamma_\sigma;\Gamma_{n-}\rho\Gamma_{\omega'}\rho'_-]$$
$$\times\langle(20)\beta;\Gamma_{\omega'}\alpha_{\omega'}|\Gamma_\omega\alpha_\omega\rangle X^{\Gamma_n}(\Gamma_{n'})A^{(20)}_\beta|\Phi[\Gamma_\sigma\Gamma_{n'}\rho'\Gamma_{\omega'}\alpha_{\omega'}]\rangle. \quad (21)$$

[This can be proved by decoupling the raising polynomial $\mathscr{P}^{\Gamma_n}_{\alpha_n}(A^{(20)})$ from the lowest weight state $|\Gamma_\sigma\alpha_\sigma\rangle$, inserting the definition of $\mathscr{P}^{\Gamma_n}_{\alpha_n}(A^{(20)})$ given in Equation (15), coupling the raising polynomial $\mathscr{P}^{\Gamma'_n}_{\alpha'_n}(A^{(20)})$ to $|\Gamma_\sigma\alpha_\sigma\rangle$, and making use of the property (C22) of the SU(3) Racah coefficient U.] An analogous expression can be obtained for the bra state $\langle\Phi[\Gamma_\sigma\Gamma_n\rho\Gamma_\omega\alpha_\omega]|$.

The task at hand is to evaluate matrix elements of arbitrary one-body operators. Since any one-body operator can be expanded in terms of the fermion unit operators $\{a^\dagger_{(\eta 0)(1/2)}\times\tilde{a}_{(0\eta')(1/2)}\}^{\Gamma S}_{\alpha\Sigma}$ by employing the formalism of second quantization and SU(3) recoupling techniques, it suffices to evaluate the following matrix element:

$$\langle\Gamma_{\sigma_2}\Gamma_{n_2}\rho_2\Gamma_{\omega_2}\alpha_{\omega_2};S_2\Sigma_2|\{a^\dagger_\eta\times\tilde{a}_{\eta'}\}^{\Gamma S}_{\alpha\Sigma}|\Gamma_{\sigma_1}\Gamma_{n_1}\rho_1\Gamma_{\omega_1}\alpha_{\omega_1};S_1\Sigma_1\rangle. \quad (22)$$

Here we have introduced a simplified, but unambiguous, notation for the fermion creation and annihilation operators: $a^\dagger_{(\eta 0)(1/2)}\to a^\dagger_\eta$, $\tilde{a}_{(0\eta)(1/2)}\to\tilde{a}_\eta$, and $S_1\Sigma_1$ and $S_2\Sigma_2$ denote the spin and spin projection of the ket and bra states, respectively. We need only consider the case of $\eta\geqslant\eta'$, since the matrix element of $\{a^\dagger_\eta\times\tilde{a}_{\eta'}\}^{\Gamma S}_{\alpha\Sigma}$ for $\eta<\eta'$ can be obtained through complex conjugation from the matrix element of $\{a^\dagger_{\eta'}\times\tilde{a}_\eta\}^{\tilde{\Gamma}S}_{\bar{\alpha}-\Sigma}$, where $\tilde{\Gamma}=(\mu\lambda)$ is the irrep conjugate to $\Gamma=(\lambda\mu)$ and $\overline{\alpha}=\kappa l(-m)$ for $\alpha=\kappa lm$.

Using the $\mathscr{K}$-matrix and applying the step-down procedure outlined above to the ket state $|\Phi[\Gamma_\sigma\Gamma_n\rho\Gamma_\omega\alpha_\omega]\rangle$, the matrix element can be written as

$$\begin{aligned}
&\langle\Gamma_{\sigma_2}\Gamma_{n_2}\rho_2\Gamma_{\omega_2}\alpha_{\omega_2};S_2\Sigma_2|\{a^\dagger_\eta\times\tilde{a}_{\eta'}\}^{\Gamma S}_{\alpha\Sigma}|\Gamma_{\sigma_1}\Gamma_{\hat{n}_1}\hat{\rho}_1\Gamma_{\omega_1}\alpha_{\omega_1};S_1\Sigma_1\rangle\\
&=\sum_{\Gamma_{n_1}\rho_1}[\mathscr{K}^{-1}(\Gamma_{\sigma_1},\Gamma_{\omega_1})]_{\Gamma_{\hat{n}_1}\hat{\rho}_1,\Gamma_{n_1}\rho_1}\\
&\quad\times\langle\Gamma_{\sigma_2}\Gamma_{n_2}\rho_2\Gamma_{\omega_2}\alpha_{\omega_2};S_2\Sigma_2|\{a^\dagger_\eta\times\tilde{a}_{\eta'}\}^{\Gamma S}_{\alpha\Sigma}|\Phi[\Gamma_{\sigma_1}\Gamma_{n_1}\rho_1\Gamma_{\omega_1}\alpha_{\omega_1}];S_1\Sigma_1\rangle\\
&=\sum_{\Gamma_{n_1}\rho_1}[\mathscr{K}^{-1}(\Gamma_{\sigma_1},\Gamma_{\omega_1})]_{\Gamma_{\hat{n}_1}\hat{\rho}_1,\Gamma_{n_1}\rho_1}\\
&\quad\times\sum_{\Gamma_{\omega'_1}\Gamma_{\hat{n}'_1}\hat{\rho}'_1}U[(20)\Gamma_{\hat{n}'_1}\Gamma_{\omega_1}\Gamma_{\sigma_1};\Gamma_{n_1-}\rho_1\Gamma_{\omega'_1}\hat{\rho}'_{1-}]X^{\Gamma_{n_1}}(\Gamma_{\hat{n}'_1})\\
&\quad\times\sum_{\Gamma_{n'_1}\rho'_1}[\mathscr{K}(\Gamma_{\sigma_1},\Gamma_{\omega'_1})]_{\Gamma_{\hat{n}'_1}\hat{\rho}'_1,\Gamma_{n'_1}\rho'_1}\Bigg\{\sum_{\alpha_{\omega'_1}\beta_1}\langle(20)\beta_1;\Gamma_{\omega'_1}\alpha_{\omega'_1}|\Gamma_{\omega_1}\alpha_{\omega_2}\rangle\\
&\quad\times\langle\Gamma_{\sigma_2}\Gamma_{n_2}\rho_2\Gamma_{\omega_2}\alpha_{\omega_2};S_2\Sigma_2|\{a^\dagger_\eta\times\tilde{a}_{\eta'}\}^{\Gamma S}_{\alpha\Sigma}A^{(20)}_{\beta_1}|\Gamma_{\sigma_1}\Gamma_{n'_1}\rho'_1\Gamma_{\omega'_1}\alpha_{\omega'_1};S_1\Sigma_1\rangle\Bigg\}. \quad (23)
\end{aligned}$$

Note that the symplectic raising and lowering operators do not act on the spin part of the wave functions.

We now consider the term in parentheses and express the operator $\{a^\dagger_\eta\times\tilde{a}_{\eta'}\}^{\Gamma S}_{\alpha\Sigma}A^{(20)}_{\beta_1}$ as

$$\{a^\dagger_\eta\times\tilde{a}_{\eta'}\}^{\Gamma S}_{\alpha\Sigma}A^{(20)}_{\beta_1}=A^{(20)}_{\beta_1}\{a^\dagger_\eta\times\tilde{a}_{\eta'}\}^{\Gamma S}_{\alpha\Sigma}+[\{a^\dagger_\eta\times\tilde{a}_{\eta'}\}^{\Gamma S}_{\alpha\Sigma},A^{(20)}_{\beta_1}]. \quad (24)$$

Recalling that the fermion realization of $A^{(20)}$ is given by Equation (12), we obtain

$$
\begin{aligned}
\{a^\dagger_\eta \times \tilde{a}_{\eta'}\}^{\Gamma S}_{\alpha\Sigma} A^{(20)}_{\beta_1} &= A^{(20)}_{\beta_1} \{a^\dagger_\eta \times \tilde{a}_{\eta'}\}^{\Gamma S}_{\alpha\Sigma} \\
&+ \left(1-\frac{1}{A}\right) \sum_\nu \sqrt{\frac{1}{12}(\nu+1)(\nu+2)(\nu+3)(\nu+4)} [\{a^\dagger_\eta \times \tilde{a}_{\eta'}\}^{\Gamma S}_{\alpha\Sigma}, \{a^\dagger_{\nu+2} \times \tilde{a}_\nu\}^{(20)s=0}_{\beta_1 \sigma=0}] \\
&- \frac{1}{4A} \sum_{\substack{\nu\nu' S' \\ \Gamma_a \Gamma_b}} \sqrt{\frac{d(\Gamma_a)}{12} \nu\nu'(2S'+1)} \left\{ \begin{array}{cccc} (\nu-1,0) & (10) & (\nu 0) & - \\ (\nu'-1,0) & (10) & (\nu' 0) & - \\ \Gamma_b & (20) & \Gamma_a & - \\ - & - & - & \end{array} \right\} \\
&\times [\{a^\dagger_\eta \times \tilde{a}_{\eta'}\}^{\Gamma S}_{\alpha\Sigma}, \{\{a^\dagger_\nu \times a^\dagger_{\nu'}\}^{\Gamma_a S'} \times \{\tilde{a}_{\nu-1} \times \tilde{a}_{\nu'-1}\}^{\tilde{\Gamma}_b S'}\}^{(20)s=0}_{\beta_1 \sigma=0}].
\end{aligned}
\tag{25}
$$

Therefore, the expression in the parentheses of Equation (23) is comprised of three terms: $\{\cdots\} = C_1 + (1-1/A)C_2 + (1/4A)C_3$, where A denotes the number of nucleons in the system. The first of these is given by

$$
\begin{aligned}
C_1 \equiv \sum_{\alpha_{\omega_1'}\beta_1} & \langle (20)\beta_1; \Gamma_{\omega_1'}\alpha'_{\omega_1} | \Gamma_{\omega_1}\alpha_{\omega_1}\rangle \\
& \times \langle \Gamma_{\sigma_2}\Gamma_{n_2}\rho_2\Gamma_{\omega_2}\alpha_{\omega_2}; S_2\Sigma_2 | A^{(20)}_{\beta_1} \{a^\dagger_\eta \times \tilde{a}_{\eta'}\}^{\Gamma S}_{\alpha\Sigma} | \Gamma_{\sigma_1}\Gamma_{n_1'}\rho_1'\Gamma_{\omega_1'}\alpha_{\omega_1'}; S_1\Sigma_1\rangle.
\end{aligned}
\tag{26}
$$

To evaluate C_1, we insert a complete set of states:

$$
1 = \sum_{\substack{\Gamma_{\sigma''}\Gamma_{n''}\rho'' \\ \Gamma_{\omega''}\alpha_{\omega''}S''\Sigma''}} |\Gamma_{\sigma''}\Gamma_{n''}\rho''\Gamma_{\omega''}\alpha_{\omega''}; S''\Sigma''\rangle\langle \Gamma_{\sigma''}\Gamma_{n''}\rho''\Gamma_{\omega''}\alpha_{\omega''}; S''\Sigma''|,
\tag{27}
$$

between $A^{(20)}_{\beta_1}$ and $\{a^\dagger_\eta \times \tilde{a}_{\eta'}\}^{\Gamma S}_{\alpha\Sigma}$ and make use of the following relation:

$$
\begin{aligned}
& \langle \Gamma_{\sigma_2}\Gamma_{n_2}\rho_2\Gamma_{\omega_2}\alpha_{\omega_2}; S_2\Sigma_2 | A^{(20)}_{\beta_1} | \Gamma_{\sigma''}\Gamma_{n''}\rho''\Gamma_{\omega''}\alpha_{\omega''}; S''\Sigma''\rangle \\
&\quad = \delta_{\Gamma_{\sigma''}\Gamma_{\sigma_2}} \delta_{N_{n''}N_2-2} \delta_{S''S_2} \delta_{\Sigma''\Sigma_2} \langle \Gamma_{\sigma_2}\Gamma_{n_2}\rho_2\Gamma_{\omega_2}\alpha_{\omega_2}; S_2\Sigma_2 | A^{(20)}_{\beta_1} | \Gamma_{\sigma_2}\Gamma_{n''}\rho''\Gamma_{\omega''}\alpha_{\omega''}; S_2\Sigma_2\rangle.
\end{aligned}
\tag{28}
$$

The delta functions in this expression reflect the fact that the symplectic generator $A^{(20)}$ only connects states within a symplectic irrep which have the same spin and differ by $2\hbar\omega$ in their excitation. Note that these delta functions significantly restrict the sum over $\Gamma_{\sigma''}\Gamma_{n''}\rho''\Gamma_{\omega''}\alpha_{\omega''}S''\Sigma''$ of the complete set of states. Only states which are constructed from the lowest weight irrep $\Gamma_{\sigma''} = \Gamma_{\sigma_2}$ by applying a raising polynomial $\mathscr{P}^{\Gamma_{n''}}_{\alpha_{n''}}(A^{(20)})$ which is characterized by $\Gamma_{n''} = (n_1'', n_2'', n_3'')$ with $N_{n''} = n_1'' + n_2'' + n_3'' = n_1^{(2)} + n_2^{(2)} + n_3^{(2)} - 2$ [where $\Gamma_{n_2} = (n_1^{(2)}, n_2^{(2)}, n_3^{(2)})$] can yield nonvanishing contributions. Employing the (generalized) Wigner–Eckart theorem, and making use of the symmetry properties of the SU(3) coupling and recoupling coefficients, we obtain the following expression for C_1:

$$C_1=(-1)^{\Gamma+\Gamma_{\omega_1}+\Gamma_{\omega_2}}\sqrt{\frac{d(\Gamma_{\omega_1})}{d(20)d(\Gamma)}}$$
$$\times\sum_{\Gamma_{n''}\rho''\Gamma_{\omega''}}\sqrt{d(\Gamma_{\omega''})}\langle\Gamma_{\sigma_2}\Gamma_{n_2}\rho_2\Gamma_{\omega_2};S_2|||A^{(20)}|||\Gamma_{\sigma_2}\Gamma_{n''}\rho''\Gamma_{\omega''};S_2\rangle$$
$$\times\sum_{\rho_3}\langle\Gamma_{\sigma_2}\Gamma_{n''}\rho''\Gamma_{\omega''};S_2|||\{a^\dagger_\eta\times\tilde{a}_{\eta'}\}^{\Gamma S}|||\Gamma_{\sigma_1}\Gamma_{n_1'}\rho_1'\Gamma_{\omega_1'};S_1\rangle_{\rho_3}$$
$$\times\sum_{\rho_4\rho_5}\Phi_{\rho_3\rho_4}[\Gamma_{\omega''}\tilde{\Gamma};\Gamma_{\omega_1'}]\Phi_{\rho_4\rho_5}[\Gamma_{\omega_1'}\tilde{\Gamma}_{\omega''};\tilde{\Gamma}]$$
$$\times\sum_{\rho_6}U[\Gamma_{\omega_2}\tilde{\Gamma}_{\omega''}\Gamma_{\omega_1}\Gamma_{\omega_1'};(20)_{-}{}_{-}\tilde{\Gamma}\rho_5\rho_6]\langle\Gamma_{\omega_1}\alpha_{\omega_1};\Gamma\alpha|\Gamma_{\omega_2}\alpha_{\omega_2}\rangle_{\rho_6}\langle S_1\Sigma_1S\Sigma|S_2\Sigma_2\rangle, \tag{29}$$

where we have introduced the abbreviation $(-1)^{\Gamma_i}\equiv(-1)^{(\lambda_i+\mu_i)}$ for $\Gamma_i=(\lambda_i\mu_i)$.

The matrix element of the second term in Equation (25) takes the form

$$C_2\equiv\sum_\nu\sqrt{\frac{1}{12}(\nu+1)(\nu+2)(\nu+3)(\nu+4)}\sum_{\alpha_{\omega_1'}\beta_1}\langle(20)\beta_1;\Gamma_{\omega_1'}\alpha_{\omega_1'}|\Gamma_{\omega_1}\alpha_{\omega_1}\rangle$$
$$\times\langle\Gamma_{\sigma_2}\Gamma_{n_2}\rho_2\Gamma_{\omega_2}\alpha_{\omega_2};S_2\Sigma_2|[\{a^\dagger_\eta\times\tilde{a}_{\eta'}\}^{\Gamma S}_{\alpha\Sigma},\{a^\dagger_{\nu+2}\times\tilde{a}_\nu\}^{(20)s=0}_{\beta_1\sigma=0}]|\Gamma_{\sigma_1}\Gamma_{n_1'}\rho_1'\Gamma_{\omega_1'}\alpha_{\omega_1'};S_1\Sigma_1\rangle, \tag{30}$$

and can be evaluated by using the commutator of Equation (B8) (see Appendix B). The delta functions in Equation (B8) restrict the sum over ν to two terms only: one term with $\nu=\eta$ and one with $\nu=\eta'-2$. Again making use of the Wigner–Eckart theorem and the symmetry properties of the coupling coefficients, we obtain

$$C_2=-(-1)^\Gamma\sqrt{\frac{1}{2}(\eta+1)(\eta+2)}\sqrt{\frac{d(\Gamma_{\omega_1})}{d(\Gamma)d(\Gamma_{\omega_1'})}}$$
$$\times\sum_{\Gamma''}(-1)^{\Gamma''}\sqrt{d(\Gamma'')}U[(20)(\eta0)\Gamma''(0\eta');(\eta+2,0)\Gamma]$$
$$\times\sum_{\rho_3}\langle\Gamma_{\sigma_2}\Gamma_{n_2}\rho_2\Gamma_{\omega_2};S_2|||\{a^\dagger_{\eta+2}\times\tilde{a}_{\eta'}\}^{\Gamma''S}|||\Gamma_{\sigma_1}\Gamma_{n_1'}\rho_1'\Gamma_{\omega_1'};S_1\rangle_{\rho_3}$$
$$\times\sum_{\rho_4}U[\Gamma_{\omega_2}\tilde{\Gamma}''\Gamma_{\omega_1}(20);\Gamma_{\omega_1'}\rho_3{}_{-}\tilde{\Gamma}{}_{-}\rho_4]\langle\Gamma_{\omega_1}\alpha_{\omega_1};\Gamma\alpha|\Gamma_{\omega_2}\alpha_{\omega_2}\rangle_{\rho_4}$$
$$\times\langle S_1\Sigma_1S\Sigma|S_2\Sigma_2\rangle$$
$$+(-1)^{\eta+\eta'-\Gamma}\sqrt{\frac{1}{24}(\eta'-1)\eta'(\eta'+1)(\eta'+2)}\sqrt{\frac{d(\Gamma_{\omega_1})}{d(\eta0)d(\Gamma_{\omega_1'})}}$$
$$\times\sum_{\Gamma''}\sqrt{d(\Gamma'')}U[\Gamma(\eta'0)\Gamma''(0,\eta'-2);(\eta0)(20)]$$
$$\times\sum_{\rho_3}\langle\Gamma_{\sigma_2}\Gamma_{n_2}\rho_2\Gamma_{\omega_2};S_2|||\{a^\dagger_\eta\times\tilde{a}_{\eta'-2}\}^{\Gamma''S}|||\Gamma_{\sigma_1}\Gamma_{n_1'}\rho_1'\Gamma_{\omega_1'};S_1\rangle_{\rho_3}$$
$$\times\sum_{\rho_4}U[\Gamma_{\omega_2}\tilde{\Gamma}''\Gamma_{\omega_1}(20);\Gamma_{\omega_1'}\rho_3{}_{-}\tilde{\Gamma}{}_{-}\rho_4]\langle\Gamma_{\omega_1}\alpha_{\omega_1};\Gamma\alpha|\Gamma_{\omega_2}\alpha_{\omega_2}\rangle_{\rho_4}\langle S_1\Sigma_1S\Sigma|S_2\Sigma_2\rangle. \tag{31}$$

Note that the sum over Γ'' is restricted by coupling requirements $(\eta+2,0)\times(0\eta')\to\Gamma''$, $(20)\times\Gamma\to\Gamma''$, and $\Gamma_{\omega_1'}\times\Gamma''\to\Gamma_{\omega_2}$ in the first term and $(\eta0)\times(0,\eta'-2)\to\Gamma''$, $\Gamma\times(20)\to\Gamma''$, and $\Gamma_{\omega_1'}\times\Gamma''\to\Gamma_{\omega_2}$ in the second term.

The matrix element of the third term in Equation (25) originates from the two-body center-of-mass correction in the fermion realization of $A^{(20)}$ [Equation (13)] and can be evaluated once the commutator,

$$[\{a^\dagger_\eta\times\tilde a_{\eta'}\}^{\Gamma S}_{\alpha\Sigma},\{\{a^\dagger_\nu\times a^\dagger_{\nu'}\}^{\Gamma_a S'}\times\{\tilde a_{\nu-1}\times\tilde a_{\nu'-1}\}^{\tilde\Gamma_b S'}\}^{(20)s=0}_{\beta_1\sigma=0}],\tag{32}$$

is worked out. Upon doing so and employing the Wigner–Eckart theorem, recoupling coefficients, their symmetry and orthogonalization properties, it turns out that (while the commutator is nonzero) the contribution to the matrix element of Equation (22) vanishes identically: $C_3=0$.

We can now combine Equations (23), (29), (31), apply the Wigner–Eckart theorem to the matrix element under consideration, and utilize the orthonormality of the Clebsch–Gordan and Wigner coupling coefficients to obtain the final expression, a recursion formula for triple-reduced matrix elements of the one-body unit operator $\{a^\dagger_\eta\times\tilde a_{\eta'}\}^{\Gamma S}_{\alpha\Sigma}$:

$$\begin{aligned}
&\langle\Gamma_{\sigma_2}\Gamma_{n_2}\rho_2\Gamma_{\omega_2};S_2|||\{a^\dagger_{(\eta 0)(1/2)}\times\tilde a_{(0\eta')(1/2)}\}^{\Gamma S}|||\Gamma_{\sigma_1}\Gamma_{\hat n_1}\hat\rho_1\Gamma_{\omega_1};S_1\rangle_\rho\\
&=\sum_{\Gamma_{n_1}\rho_1}[\mathscr{K}^{-1}(\Gamma_{\sigma_1}\Gamma_{\omega_1})]_{\Gamma_{\hat n_1}\hat\rho_1,\Gamma_{n_1}\rho_1}\\
&\times\sum_{\Gamma_{\hat n_1'}\hat\rho_1'\Gamma_{\omega_1'}}U[(20)\Gamma_{\hat n_1'}\Gamma_{\omega_1}\Gamma_{\sigma_1};\Gamma_{n_1}{}_-\rho_1\Gamma_{\omega_1'}\hat\rho_1'{}_-]X^{\Gamma_{n_1}}(\Gamma_{\hat n_1'})\\
&\times\sum_{\Gamma_{n_1'}\rho_1'}[\mathscr{K}(\Gamma_{\sigma_1}\Gamma_{\omega_1'})]_{\Gamma_{\hat n_1'}\hat\rho_1',\Gamma_{n_1'}\rho_1'}\\
&\times\Bigg\{(-1)^{\Gamma+\Gamma_{\omega_1}+\Gamma_{\omega_2}}\sqrt{\frac{d(\Gamma_{\omega_1})}{6d(\Gamma)}}\\
&\times\sum_{\Gamma_{n''}\rho''\Gamma_{\omega''}}\sqrt{d(\Gamma_{\omega''})}\langle\Gamma_{\sigma_2}\Gamma_{n_2}\rho_2\Gamma_{\omega_2};S_2|||A^{(20)}|||\Gamma_{\sigma_2}\Gamma_{n''}\rho''\Gamma_{\omega''};S_2\rangle\\
&\times\sum_{\rho_3}\langle\Gamma_{\sigma_2}\Gamma_{n''}\rho''\Gamma_{\omega''};S_2|||\{a^\dagger_\eta\times\tilde a_{\eta'}\}^{\Gamma S}|||\Gamma_{\sigma_1}\Gamma_{n_1'}\rho_1'\Gamma_{\omega_1'};S_1\rangle_{\rho_3}\\
&\times\sum_{\rho_4\rho_5}\Phi_{\rho_3\rho_4}[\Gamma_{\omega''}\tilde\Gamma;\Gamma_{\omega_1'}]\Phi_{\rho_4\rho_5}[\Gamma_{\omega_1'}\tilde\Gamma_{\omega''};\tilde\Gamma]U[\Gamma_{\omega_2}\tilde\Gamma_{\omega''}\Gamma_{\omega_1}\Gamma_{\omega_1'};(20)_{--}\tilde\Gamma\rho_5\rho]\\
&-\left(1-\frac{1}{A}\right)(-1)^{\Gamma}\sqrt{\frac{(\eta+1)(\eta+2)}{2d(\Gamma)}}\sqrt{\frac{d(\Gamma_{\omega_1})}{d(\Gamma_{\omega_1'})}}\\
&\times\sum_{\Gamma''}(-1)^{\Gamma''}\sqrt{d(\Gamma'')}U[(20)(\eta 0)\Gamma''(0\eta');(\eta+2,0)\Gamma]\\
&\times\sum_{\rho_3}\langle\Gamma_{\sigma_2}\Gamma_{n_2}\rho_2\Gamma_{\omega_2};S_2|||\{a^\dagger_{\eta+2}\times\tilde a_{\eta'}\}^{\Gamma''S}|||\Gamma_{\sigma_1}\Gamma_{n_1'}\rho_1'\Gamma_{\omega_1'};S_1\rangle_{\rho_3}\\
&\times U[\Gamma_{\omega_2}\tilde\Gamma''\Gamma_{\omega_1}(20);\Gamma_{\omega_1'}\rho_3{}_-\tilde\Gamma_-\rho]\\
&+\left(1-\frac{1}{A}\right)(-1)^{\eta+\eta'-\Gamma}\sqrt{\frac{(\eta'-1)\eta'(\eta'+1)(\eta'+2)}{24d(\eta 0)}}\sqrt{\frac{d(\Gamma_{\omega_1})}{d(\Gamma_{\omega_1'})}}\\
&\times\sum_{\Gamma''}\sqrt{d(\Gamma'')}U[\Gamma(\eta'0)\Gamma''(0,\eta'-2);(\eta 0)(20)]\\
&\times\sum_{\rho_3}\langle\Gamma_{\sigma_2}\Gamma_{n_2}\rho_2\Gamma_{\omega_2};S_2|||\{a^\dagger_\eta\times\tilde a_{\eta'-2}\}^{\Gamma''S}|||\Gamma_{\sigma_1}\Gamma_{n_1'}\rho_1'\Gamma_{\omega_1'};S_1\rangle_{\rho_3}\\
&\times U[\Gamma_{\omega_2}\tilde\Gamma''\Gamma_{\omega_1}(20);\Gamma_{\omega_1'}\rho_3{}_-\tilde\Gamma_-\rho]\Bigg\}.
\end{aligned}\tag{33}$$

To evaluate this expression, a series of ingredients are necessary: First, numeric values for the SU(3) Racah coefficients $U[\Gamma_1\Gamma_2\Gamma\Gamma_3;\Gamma_{12}\rho_{12}\rho_{12,3}\Gamma_{23}\rho_{23}\rho_{1,23}]$ and $\Phi_{\rho\rho'}[\Gamma_1\Gamma_2;\Gamma_3] = Z[\Gamma_1(00)\Gamma_3\Gamma_2;\Gamma_{1-\rho}\Gamma_{2-\rho'}]$ are required. These may be calculated with a computer code published by Akiyama and Draayer.[34] Second, matrix elements of the symplectic raising operator $A^{(20)}$ and of the $\mathscr{K}$-matrix, which effects the orthonormalization of the symplectic basis states, are needed. Both types can be obtained from the matrix elements of $\mathscr{A}^{(20)}$, a generator of the $u(3)\otimes Weyl$ algebra, as is outlined in Reference 7. Third, matrix elements of the form $\langle\Gamma_{\sigma_2}\Gamma_{n''}\rho''\Gamma_{\omega''};S_2|||\{a^\dagger_\eta\times\tilde{a}_{\eta'}\}^{\Gamma S}|||\Gamma_{\sigma_1}\Gamma_{n_1'}\rho_1'\Gamma_{\omega_1'};S_1\rangle_{\rho_3}$ between states of excitation $N_{n''}$ and $N_{n_1'}$ are requisite ingredients. Here $N_{n''}=N_{n_2}-2$ as a result of the delta function $\delta_{N_{n''},N_{n_2}-2}$, introduced through the symplectic generator $A^{(20)}$ [see also Equation (28)], and $N_{n_1'}=N_{n''}-(\eta-\eta') = N_{\hat{n}_1}-2$. Furthermore, one needs values for the matrix elements of the operators $\{a^\dagger_{\eta+2}\times\tilde{a}_{\eta'}\}^{\Gamma''S}_{\alpha''\Sigma}$ and $\{a^\dagger_\eta\times\tilde{a}_{\eta'-2}\}^{\Gamma''S}_{\alpha''\Sigma}$ between the original bra state of excitation N_{n_2} and ket states with $N_{n_1'}=N_{n_2}-(\eta-\eta'+2)=N_{\hat{n}_1}-2$. Thus the desired matrix element is expressed in terms of known quantities and unit matrix elements involving states of lower excitation. Hence, through repeated application of this recursive process matrix elements of arbitrary one-body operators between symplectic basis states may be related back to valence shell matrix elements, which in turn can be evaluated by means of standard shell model techniques. A user-friendly computer code, which calculates the latter, has been published by Bahri and Draayer.[25]

The above formula has been derived for fermionic unit operators of the form $\{a^\dagger_\eta\times\tilde{a}_{\eta'}\}^{\Gamma S}_{\alpha\Sigma}$ with $\eta\geqslant\eta'$. The recursive process also covers the case $\eta>\eta'$, since the following relation holds:

$$\begin{aligned}\langle\Gamma_{\sigma_2}\Gamma_{n_2}\rho_2\Gamma_{\omega_2};S_2|||\{a^\dagger_\eta\times\tilde{a}_{\eta'}\}^{\Gamma S}|||\Gamma_{\sigma_1}\Gamma_{n_1}\rho_1\Gamma_{\omega_1};S_1\rangle_\rho \\ =(-1)^{\eta+\eta'-\Gamma_{\omega_1}+\Gamma_{\omega_2}+S_1+S_2}\sqrt{\frac{d(\Gamma_{\omega_1})}{d(\Gamma_{\omega_2})}}\sqrt{\frac{2S_1+1}{2S_2+1}} \\ \times\langle\Gamma_{\sigma_1}\Gamma_{n_1}\rho_1\Gamma_{\omega_1};S_1|||\{a^\dagger_{\eta'}\times\tilde{a}_\eta\}^{\tilde{\Gamma}S}|||\Gamma_{\sigma_2}\Gamma_{n_2}\rho_2\Gamma_{\omega_2};S_2\rangle^*_\rho .\end{aligned} \tag{34}$$

A stringent test of Equation (33) is given by the following: One can evaluate the matrix element

$$\langle\Gamma_{\sigma_2}\Gamma_{n_2}\rho_2\Gamma_{\omega_2};S_2|||\{a^\dagger_\eta\times\tilde{a}_{\eta'}\}^{\Gamma S}|||\Gamma_{\sigma_1}\Gamma_{n_1}\rho_1\Gamma_{\omega_1};S_1\rangle_\rho \tag{35}$$

by stepping down on the bra-side, and proceeding analogously to the derivation given above. The result is a recursion formula analogous to Equation (33). Alternatively, using relation (34) in conjunction with Equation (33) yields an expression for the matrix element of Equation (35) which exactly equals the formula that is obtained by stepping down on the ket-side. Another test was carried out by encoding the recursion formula and using the one-body matrix elements so obtained to calculate expectation values of the particle number operator $\hat{N}$ and the symplectic raising and lowering operators $A^{(20)}$ and $B^{(02)}$, as well as the expectation values of $C^{(11)}$ for the valence shell. For $A\to\infty$, perfect agreement with the matrix elements of $A^{(20)}$ and $B^{(02)}$ as evaluated by means of a code that is based on the vector coherent state method outlined in reference 7 was obtained, as well as agreement with the matrix elements of $\hat{N}$ and $C^{(11)}$, which can be evaluated analytically.

VI. CONCLUDING REMARKS

We have introduced a fermion realization of the Sp(6,R) algebra. Specifically, we expressed the symplectic generators in terms of fermion creation and annihilation operators in a Sp(6,R)$\supset$U(3)$\supset$SO(3) basis. The new formalism was employed to derive a recursive process for calculating matrix elements of arbitrary one-body operators. The resulting formula requires as input SU(3) coupling coefficients, matrix elements of symplectic generators, and valence shell matrix elements of one-body unit operators. These ingredients may be calculated using published computer codes; references for those were given. The new formalism allows for the evaluation of physical observables in the nuclear Sp$(6,R)$ model which until now were only available, if at all, as an approximation to the exact result. First applications include symplectic calculations of longitudinal and transverse nuclear form factors,[35] which can be compared to those extracted from experiments.

One can also employ the recursion formula of Equation (33) to derive a recursion relation for $A^{(20)}$ for the special case of a large number of nucleons in the system $(A\to\infty)$. Upon doing so, and comparing the result to the formula that was derived by Rosensteel using a step-down procedure analogous to the one outlined above,[27] we find that exact agreement requires the following relations to hold:

$$(\eta+3)(\eta+4)U[(02)(\eta+2,0)(22)(0\eta);(\eta0)(20)] \\ =(\eta-1)\eta U[(20)(\eta-2,0)(22)(0\eta);(\eta0)(02)], \tag{36}$$

$$(\eta+3)(\eta+4)U[(02)(\eta+2,0)(11)(0\eta);(\eta0)(20)] \\ -(\eta-1)\eta U[(20)(\eta-2,0)(11)(0\eta);(\eta0)(02)]=\sqrt{10\eta(\eta+3)}, \tag{37}$$

$$U[(02)(\eta+2,0)(00)(0\eta);(\eta0)(20)]=U[(20)(\eta-2,0)(00)(0\eta);(\eta0)(02)]=1. \tag{38}$$

While Equation (38) was known previously,[36] the other two relations, Equation (36) and Equation (37), are new and may prove valuable for analytic work that involves SU(3) Racah coefficients.

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APPENDIX A: BOSON CREATION AND ANNIHILATION OPERATORS

Given position $x_i(s)$ and momentum $p_i(s)$ coordinates of the s-th particle ($i=1,2,3$ and $s=1,\ldots,A$), one-body operators $b_i^\dagger(s)$ and $b_i(s)$, which create and annihilate, respectively, one oscillator quantum in the i-th direction of the s-th particle, can be defined as follows:

$$\begin{aligned} b_i^\dagger(s)&\equiv\frac{1}{\sqrt{2}}(x_i(s)-ip_i(s)),\\ b_i(s)&\equiv\frac{1}{\sqrt{2}}(x_i(s)+ip_i(s)). \end{aligned} \tag{A1}$$

These operators are related to each other by Hermitian conjugation $b_i(s)=(b_i^\dagger(s))^\dagger$ and satisfy the standard boson commutation relations:

$$\begin{aligned} [b_i(s),b_j^\dagger(t)]&=\delta_{st}\delta_{ij},\\ [b_i^\dagger(s),b_j^\dagger(t)]&=[b_i(s),b_j(t)]=0. \end{aligned} \tag{A2}$$

They may also be viewed as the components of SU(3) irreducible tensor operators, $b_{1q}^{\dagger(10)}$ and $\tilde{b}_{1q}^{(01)}$ $(q=0,\pm1)$, transforming according to the $(\lambda\mu)=(10)$ and $(\lambda\mu)=(01)$ SU(3) irreps, respectively:

$$b_{1,\pm1}^{\dagger(10)}\equiv\mp\frac{1}{\sqrt{2}}(b_1^\dagger\pm ib_2^\dagger),\quad b_{1,0}^{\dagger(10)}\equiv b_3^\dagger, \tag{A3}$$

and

$$\tilde{b}^{(01)}_{1,\pm 1} \equiv \mp \frac{1}{\sqrt{2}}(b_1 \pm i b_2), \quad \tilde{b}^{(01)}_{1,0} \equiv b_3, \tag{A4}$$

where the subscript $(1,q)$ denotes the angular momentum character $(l=1;\ q=0,\pm 1)$ of the spherical components of the boson creation and annihilation operators. For the sake of simplicity the angular momentum label $l=1$ and the particle index s will be suppressed from here on, except where needed to avoid ambiguities. The operators $b^{\dagger(10)}_q$ and $\tilde{b}^{(01)}_q$ satisfy the conjugation relation $\tilde{b}^{(01)}_q = (-1)^q (b^{\dagger(10)}_{-q})^\dagger$, and their commutator is given by $[\tilde{b}^{(01)}_q, b^{\dagger(10)}_{q'}] = (-1)^q \delta_{q(-q')}$. Matrix elements of the boson creation and annihilation operators take a very simple form in the standard SU(3) basis:

$$\begin{aligned} &\langle(\eta 0)S|||b^{\dagger(10)s=0}|||(\lambda\mu)S'\rangle = \langle(\lambda+1,\mu)S|||b^{\dagger(10)s=0}|||(\eta-1,0)S'\rangle = \sqrt{\eta}\,\delta_{(\lambda\mu)(\eta-1,0)}\delta_{SS'},\\ &\langle(\lambda\mu)S|||\tilde{b}^{(01)s=0}|||(\eta 0)S'\rangle = \langle(\eta-1,0)S|||\tilde{b}^{(01)s=0}_s|||(\lambda+1,\mu)S'\rangle = \sqrt{\eta+2}\,\delta_{(\lambda\mu)(\eta-1,0)}\delta_{SS'}. \end{aligned} \tag{A5}$$

The $b^{\dagger(10)}_q$ and $\tilde{b}^{(01)}_q$ are scalars in spin space, as has been noted explicitly in Equation (A5). [For the definition of the triple-reduced matrix element and the relevant phase conventions refer to Appendix C.]

Since the creation and annihilation operators $b^{\dagger(10)}_q$ and $\tilde{b}^{(01)}_q$ are SU(3) irreducible tensor operators, two or more of them may be coupled to form new SU(3) tensors. Matrix elements of such tensor products can be evaluated with the help of SU(3) reduction rules as given in Appendix C. A relevant example is the matrix element of the one-body operator $\{b^{\dagger(10)s=0} \times \tilde{b}^{(01)s=0}\}^{(11)S=0}$, evaluated between two single-particle states,

$$\begin{aligned} &\langle(\nu 0)\tfrac{1}{2}|||\{b^{\dagger(10)s=0}\times\tilde{b}^{(01)s=0}\}^{(11)S=0}|||(\nu' 0)\tfrac{1}{2}\rangle\\ &= \sum_{(\lambda''\mu'')S''} \Phi_{11}[(10)(01);(11)]\,U[(\nu' 0)(01)(\nu 0)(10);(\lambda''\mu'')(11)]\,U\left(\frac{1}{2}\,0\,\frac{1}{2}\,0;S''0\right)\\ &\quad\times\langle(\nu 0)\tfrac{1}{2}|||b^{\dagger(10)s=0}|||(\lambda''\mu'')S''\rangle\langle(\lambda''\mu'')S''|||\tilde{b}^{(01)s=0}|||(\nu' 0)\tfrac{1}{2}\rangle\\ &= \sqrt{\frac{2}{3}\nu(\nu+3)}\,\delta_{\nu\nu'}, \end{aligned} \tag{A6}$$

where $U[\cdots]$ and $\Phi[\cdots]$ are Racah recoupling coefficients (see Appendix C). For the tensor product of two raising (lowering) operators, acting on a single-particle state, one may proceed analogously to obtain

$$\begin{aligned} &\langle(\nu 0)\tfrac{1}{2}|||\{b^{\dagger(10)s=0}\times b^{\dagger(10)s=0}\}^{(20)S=0}|||(\nu' 0)\tfrac{1}{2}\rangle = \sqrt{\nu(\nu-1)}\,\delta_{\nu,\nu'+2},\\ &\langle(\nu 0)\tfrac{1}{2}|||\{\tilde{b}^{(01)s=0}\times\tilde{b}^{(01)s=0}\}^{(02)S=0}|||(\nu' 0)\tfrac{1}{2}\rangle = \sqrt{(\nu+3)(\nu+4)}\,\delta_{\nu',\nu+2}. \end{aligned} \tag{A7}$$

Matrix elements of a SU(3)-coupled tensor product of two boson operators, acting on a two-particle state, can be evaluated using Millener's reduction rule[37] (see also Appendix C). For the product of a creation and an annihilation operator, coupled to $(\lambda\mu)=(11)$, this method yields

$$\begin{aligned} &\langle\{(\nu_1 0)\times(\nu_2 0)\}(\lambda\mu);\{\tfrac{1}{2}\times\tfrac{1}{2}\}S|||\{b^{\dagger(10)0}(1)\times\tilde{b}^{(01)0}(2)\}^{(11)0}|||\\ &\quad\times\{(\nu_1' 0)\times(\nu_2' 0)\}(\lambda'\mu');\{\tfrac{1}{2}\times\tfrac{1}{2}\}S'\rangle\\ &= \sqrt{\nu_1(\nu_2+3)}\,\delta_{SS'}\,\delta_{\nu_1',\nu_1-1}\,\delta_{\nu_2',\nu_2+1}\left\{\begin{matrix} (\nu_1-1,0) & (10) & (\nu_1 0) & - \\ (\nu_2+1,0) & (01) & (\nu_2 0) & - \\ (\lambda'\mu') & (11) & (\lambda\mu) & - \\ - & - & - & \end{matrix}\right\}, \end{aligned} \tag{A8}$$

whereas for two creation operators, coupled to $(\lambda\mu)=(20)$, we obtain

$$\begin{aligned}&\langle\{(\nu_1 0)\times(\nu_2 0)\}(\lambda\mu);\{\tfrac{1}{2}\times\tfrac{1}{2}\}S|||\{b^{\dagger(10)0}(1)\times b^{\dagger(10)0}(2)\}^{(20)0}|||\\&\times\{(\nu_1' 0)\times(\nu_2' 0)\}(\lambda'\mu');\{\tfrac{1}{2}\times\tfrac{1}{2}\}S'\rangle\\&=\sqrt{\nu_1\nu_2}\,\delta_{SS'}\delta_{\nu_1',\nu_1-1}\delta_{\nu_2',\nu_2-1}\begin{Bmatrix}(\nu_1-1,0) & (10) & (\nu_1 0) & -\\ (\nu_2-1,0) & (10) & (\nu_2 0) & -\\ (\lambda'\mu') & (20) & (\lambda\mu) & -\\ - & - & - & \end{Bmatrix},\end{aligned}\tag{A9}$$

and for two annihilation operators we find

$$\begin{aligned}&\langle\{(\nu_1 0)\times(\nu_2 0)\}(\lambda\mu);\{\tfrac{1}{2}\times\tfrac{1}{2}\}S|||\{\tilde{b}^{(01)0}(1)\times\tilde{b}^{(01)0}(2)\}^{(02)0}|||\{(\nu_1' 0)\times(\nu_2' 0)\}(\lambda'\mu');\{\tfrac{1}{2}\times\tfrac{1}{2}\}S'\rangle\\&=\sqrt{(\nu_1+3)(\nu_2+3)}\,\delta_{SS'}\delta_{\nu_1',\nu_1+1}\delta_{\nu_2',\nu_2+1}\begin{Bmatrix}(\nu_1+1,0) & (01) & (\nu_1 0) & -\\ (\nu_2+1,0) & (01) & (\nu_2 0) & -\\ (\lambda'\mu') & (02) & (\lambda\mu) & -\\ - & - & - & \end{Bmatrix},\end{aligned}\tag{A10}$$

where $b^\dagger(t)$ and $\tilde{b}(t)$ $(t=1,2$ here) act only on that part of the wave function which refers to particle t.

APPENDIX B: FERMION SECOND QUANTIZATION

A general one-body operator that acts symmetrically on a system of A identical particles is given by

$$\mathscr{F}=\sum_s f(\mathbf{r}_s,\boldsymbol{\sigma}_s),\tag{B1}$$

where $\mathbf{r}_s$ and $\boldsymbol{\sigma}_s$ represent the position and spin (or spin–isospin) coordinates, respectively, of the s-th particle. In a fermion second quantization formulation this one-body operator takes the form

$$\mathscr{F}=\sum_{\rho,\rho'}\langle\rho'|f(\mathbf{r},\boldsymbol{\sigma})|\rho\rangle a_{\rho'}^\dagger a_\rho,\tag{B2}$$

where ρ labels the available single-particle states and $a_\rho^\dagger$ and a_ρ are single-particle creation and annihilation operators that satisfy the usual fermion anti-commutation relations:

$$\begin{aligned}\{a_\rho,a_{\rho'}^\dagger\}&=\delta_{\rho,\rho'},\\ \{a_\rho,a_{\rho'}\}&=\{a_\rho^\dagger,a_{\rho'}^\dagger\}=0.\end{aligned}\tag{B3}$$

For fermions in a harmonic oscillator potential, ρ stands for a set of quantum numbers $\rho=\eta l m\frac{1}{2}\sigma$ or $\rho=\eta l\frac{1}{2}jm_j$, depending on whether the states are characterized by an LS- or jj-coupling scheme, respectively. Here η is the principal quantum number (major oscillator shell) of the single-particle level; l, $\frac{1}{2}$, and j label the orbital, spin, and total angular momenta with projections m, σ, and m_j, respectively. (In a spin–isospin formalism, one has $\rho=\eta l m\frac{1}{2}\sigma\frac{1}{2}\tau$ or $\rho=\eta l\frac{1}{2}jm_j\frac{1}{2}\tau$, respectively, where the additional $\frac{1}{2}$ denotes the isospin quantum number with projection τ.) For the present purposes it is most convenient to use the LS-coupling scheme.

Since the single-particle harmonic oscillator wave functions, $|\eta l m\frac{1}{2}\sigma\rangle=a_{\eta l m(1/2)\sigma}^\dagger|-\rangle$, where $|-\rangle$ denotes the particle vacuum, transform irreducibly under a set of physically relevant SU(3) and SU(2) symmetry group operations, the fermion creation operator $a_{\eta l m(1/2)\sigma}^\dagger$ is a double irreducible tensor operator of rank $(\lambda\mu)=(\eta 0)$ in SU(3), which labels its orbital character (with subgroup labels l and m), and of rank $s=\frac{1}{2}$ in SU(2) for the spin part (with subgroup label σ) and should be written as $a^\dagger_{(\eta 0)lm(1/2)\sigma}$. Since $a_\rho=(a^\dagger_\rho)^\dagger$ is not a proper irreducible tensor operator with respect to the above group transformations, it is advantageous to define $\tilde{a}_{(0\eta)lm(1/2)\sigma}\equiv(-1)^{\eta+l+m+(1/2)+\sigma}a_{(\eta 0)l(-m)(1/2)(-\sigma)}$ which is a proper irreducible tensor operator of rank $(\lambda\mu)=(0\eta)$ in SU(3) and rank $s=\frac{1}{2}$ in (spin-)SU(2).

It thus becomes possible to construct tensor products from the fermion creation and annihilation operators, such as

$$\{a^\dagger_{(\eta_1 0)(1/2)}\times\tilde{a}_{(0\eta_2)(1/2)}\}^{(\lambda\mu)S}_{\kappa LM\Sigma}=\sum_{\alpha_1\alpha_2\sigma_1\sigma_2}\langle(\eta_1 0)\alpha_1;(0\eta_2)\alpha_2|(\lambda\mu)\kappa LM\rangle \times\langle\tfrac{1}{2}\sigma_1\,\tfrac{1}{2}\sigma_2|S\Sigma\rangle a^\dagger_{(\eta_1 0)\alpha_1(1/2)\sigma_1}\tilde{a}_{(0\eta_2)\alpha_2(1/2)\sigma_2}, \tag{B4}$$

which moves a particle from the η_2-th major oscillator shell to the η_1-th shell. The possible $(\lambda\mu)$ values are given by the coupling rule $(\eta_1 0)\times(0\eta_2)=\oplus_{k=0}^{\min(\eta_1,\eta_2)}(\eta_1-k,\eta_2-k)$ (see reference 38) and α_i is an abbreviation for the set of possible subgroup labels $\alpha_i=\kappa_i l_i m_i$, where $\kappa_i=1$ must hold here. The total intrinsic spin S can take the values 0 or 1 with projection $\Sigma=0$ or $\Sigma=0$, ±1, respectively. The product $\{a^\dagger_{(\eta_1 0)(1/2)}\times a^\dagger_{(\eta_2 0)(1/2)}\}^{(\lambda\mu)S}_{\kappa LM\Sigma}$, on the other hand, creates a pair of fermions with one particle in the η_1-th shell and one in the η_2-th shell, and the pair is coupled to $(\lambda\mu)\in\{(\eta_1 0)\times(\eta_2 0)\}=\{\oplus_{k=0}^{\min(\eta_1,\eta_2)}(\eta_1+\eta_2-2k,k)\}$ and $S=0$ or 1. Similarly, the product $\{\tilde{a}_{(0\eta_1)(1/2)}\times\tilde{a}_{(0\eta_2)(1/2)}\}^{(\lambda\mu)S}_{\kappa LM\Sigma}$ annihilates a SU(3)-coupled pair of fermions with one particle in the η_1-th shell, one in the η_2-th shell, $(\lambda\mu)\in\{(0\eta_1)\times(0\eta_2)\}=\{\oplus_{k=0}^{\min(\eta_1,\eta_2)}(k,\eta_1+\eta_2-2k)\}$, and $S=0$ or 1. One can furthermore construct a SU(3) irreducible tensor which destroys a pair of fermions in a particular SU(3)-coupled configuration, and creates a new pair configuration:

$$\begin{aligned}&\{\{a^\dagger_{(\eta_1 0)(1/2)}\times a^\dagger_{(\eta_2 0)(1/2)}\}^{(\lambda_1\mu_1)S_1}\times\{\tilde{a}_{(0\nu_1)(1/2)}\times\tilde{a}_{(0\nu_2)(1/2)}\}^{(\lambda_2\mu_2)S_2}\}^{\rho(\lambda\mu)S}_{\kappa LM\Sigma}\\&=\sum_{\alpha_1\alpha_2\Sigma_1\Sigma_2}\langle(\lambda_1\mu_1)\alpha_1;(\lambda_2\mu_2)\alpha_2|(\lambda\mu)\alpha\rangle_\rho\langle S_1\Sigma_1 S_2\Sigma_2|S\Sigma\rangle\\&\quad\times\{a^\dagger_{(\eta_1 0)(1/2)}\times a^\dagger_{(\eta_2 0)(1/2)}\}^{(\lambda_1\mu_1)S_1}_{\alpha_1\Sigma_1}\{\tilde{a}_{(0\nu_1)(1/2)}\times\tilde{a}_{(0\nu_2)(1/2)}\}^{(\lambda_2\mu_2)S_2}_{\alpha_2\Sigma_2},\end{aligned} \tag{B5}$$

where ρ denotes the multiplicity of $(\lambda\mu)$ in the coupling $(\lambda_1\mu_1)\times(\lambda_2\mu_2)\to(\lambda\mu)$.

The Hermitian adjoint of the above products are given by

$$\begin{aligned}(\{a^\dagger_{(\eta_1 0)(1/2)}\times\tilde{a}_{(0\eta_2)(1/2)}\}^{(\lambda\mu)S}_{\kappa LM\Sigma})^\dagger&=(-1)^{\eta_2-\eta_1+L+M+\Sigma}\{a^\dagger_{(\eta_2 0)(1/2)}\times\tilde{a}_{(0\eta_1)(1/2)}\}^{(\mu\lambda)S}_{\kappa L-M-\Sigma},\\(\{a^\dagger_{(\eta_1 0)(1/2)}\times a^\dagger_{(\eta_2 0)(1/2)}\}^{(\lambda\mu)S}_{\kappa LM\Sigma})^\dagger&=(-1)^{-\eta_1-\eta_2-L+M-1+\Sigma}\{\tilde{a}_{(0\eta_2)(1/2)}\times\tilde{a}_{(0\eta_1)(1/2)}\}^{(\mu\lambda)S}_{\kappa L-M-\Sigma},\end{aligned} \tag{B6}$$

and

$$\begin{aligned}&(\{\{a^\dagger_{(\eta_1 0)(1/2)}\times a^\dagger_{(\eta_2 0)(1/2)}\}^{(\lambda_1\mu_1)S_1}\times\{\tilde{a}_{(0\nu_1)(1/2)}\times\tilde{a}_{(0\nu_2)(1/2)}\}^{(\lambda_2\mu_2)S_2}\}^{\rho(\lambda\mu)S}_{\kappa LM\Sigma})^\dagger\\&=(-1)^{\nu_1+\nu_2-\eta_1-\eta_2+(\lambda_1+\mu_1)+(\lambda_2+\mu_2)+(\lambda+\mu)-L+M+\Sigma}\\&\quad\times\sum_{\rho'}(-1)^{\rho'_{\max}-\rho'}\Phi_{\rho'\rho}[(\lambda_2\mu_2)(\lambda_1\mu_1);(\lambda\mu)]\\&\quad\times\{\{a^\dagger_{(\nu_2 0)(1/2)}\times a^\dagger_{(\nu_1 0)(1/2)}\}^{(\mu_2\lambda_2)S_2}\times\{\tilde{a}_{(0\eta_2)(1/2)}\times\tilde{a}_{(0\eta_1)(1/2)}\}^{(\mu_1\lambda_1)S_1}\}^{\rho'(\mu\lambda)S}_{\kappa L-M-\Sigma},\end{aligned} \tag{B7}$$

where the phase matrix Φ is a special case of the recoupling coefficient Z, $\Phi_{\rho\rho'}[(\lambda_1\mu_1)\times(\lambda_2\mu_2);(\lambda\mu)]=Z[(\lambda_1\mu_1)(00)(\lambda\mu)(\lambda_2\mu_2);(\lambda_1\mu_1)_{-}\rho(\lambda_2\mu_2)_{-}\rho']$, and can be evaluated with available computer codes (see also Appendix C). The symbol $\rho'_{\max}$ denotes the maximum multiplicity in the coupling $(\mu_2\lambda_2)\times(\mu_1\lambda_1)\to(\mu\lambda)$, and ρ' takes on values $1,2,\ldots,\rho'_{\max}$.

The following commutation relations, which are required for the derivation of the fermionic expressions for the symplectic generators, can be derived by making use of the commutation rules of the uncoupled components [see Equation (B3)] and SU(3) coupling and recoupling techniques. For clarity we abbreviate $(\lambda_i\mu_i)$ with the symbol Γ_i, $(-1)^{\Gamma_i}\equiv(-1)^{\lambda_i+\mu_i}$, and $\hat{S}\equiv\sqrt{2S+1}$:

$$
\begin{aligned}
&[\{a_\eta^\dagger\times\tilde{a}_{\eta'}\}^{\Gamma_1 S_1}_{\alpha_1\Sigma_1},\{a_\nu^\dagger\times\tilde{a}_{\nu'}\}^{\Gamma_2 S_2}_{\alpha_2\Sigma_2}]\\
&\quad=(-1)^{\eta+\nu}\sum_{s\sigma}\hat{S}_1\hat{S}_2\langle S_1\Sigma_1 S_2\Sigma_2|s\sigma\rangle\begin{Bmatrix}\frac{1}{2}&\frac{1}{2}&S_1\\ s&S_2&\frac{1}{2}\end{Bmatrix}\\
&\quad\times\Bigg\{-\delta_{\eta'\nu}(-1)^{-s-\Gamma_1}\sqrt{\frac{d(\Gamma_1)}{d(\eta 0)}}\sum_{\Gamma\alpha}\{a_\eta^\dagger\times\tilde{a}_{\nu'}\}^{\Gamma s}_{\alpha\sigma}\\
&\quad\times\sum_\rho\langle\Gamma_1\alpha_1;\Gamma_2\alpha_2|\Gamma\alpha\rangle_\rho U[\Gamma_1(\nu 0)\Gamma(0\nu');(\eta 0)_{--}\Gamma_{2-}\rho]\\
&\quad+\delta_{\eta\nu'}(-1)^{S_1+S_2-\Gamma_2}\sqrt{\frac{d(\Gamma_2)}{d(\nu 0)}}\sum_{\Gamma\alpha}\{a_\nu^\dagger\times\tilde{a}_{\eta'}\}^{\Gamma s}_{\alpha\sigma}\\
&\quad\times\sum_\rho\langle\Gamma_2\alpha_2;\Gamma_1\alpha_1|\Gamma\alpha\rangle_\rho U[\Gamma_2(\eta 0)\Gamma(0\eta');(\nu 0)_{--}\Gamma_{1-}\rho]\Bigg\},
\end{aligned}
\tag{B8}
$$

and

$$
\begin{aligned}
&[\{a_\eta^\dagger\times a_{\eta'}^\dagger\}^{\Gamma_1 S_1}_{\alpha_1\Sigma_1},\{a_\nu^\dagger\times\tilde{a}_{\nu'}\}^{\Gamma_2 S_2}_{\alpha_2\Sigma_2}]\\
&\quad=(-1)^{\eta+\nu-\Gamma_2}\sqrt{\frac{d(\Gamma_2)}{d(\nu 0)}}\sum_{s\sigma}\hat{S}_1\hat{S}_2\langle S_2\Sigma_2 S_1\Sigma_1|s\sigma\rangle\begin{Bmatrix}\frac{1}{2}&\frac{1}{2}&S_2\\ s&S_1&\frac{1}{2}\end{Bmatrix}\\
&\quad\times\Bigg\{\delta_{\nu'\eta'}(-1)^{s+S_1-\Gamma_1}\sum_{\Gamma\alpha}\{a_\nu^\dagger\times a_\eta^\dagger\}^{\Gamma s}_{\alpha\sigma}\\
&\quad\times\sum_\rho\langle\Gamma_2\alpha_2;\Gamma_1\alpha_1|\Gamma\alpha\rangle_\rho U[\Gamma_2(\eta' 0)\Gamma(\eta 0);(\nu 0)_{--}\Gamma_{1-}\rho]\\
&\quad+\delta_{\nu'\eta}(-1)^s\sum_{\Gamma\alpha}\{a_\nu^\dagger\times a_{\eta'}^\dagger\}^{\Gamma s}_{\alpha\sigma}\sum_\rho\langle\Gamma_2\alpha_2;\Gamma_1\alpha_1|\Gamma\alpha\rangle_\rho U[\Gamma_2(\eta 0)\Gamma(\eta' 0);(\nu 0)_{--}\Gamma_{1-}\rho]\Bigg\},
\end{aligned}
\tag{B9}
$$

and

$$
\begin{aligned}
&[\{\tilde{a}_\eta\times\tilde{a}_{\eta'}\}^{\Gamma_1 S_1}_{\alpha_1\Sigma_1},\{a_\nu^\dagger\times\tilde{a}_{\nu'}\}^{\Gamma_2 S_2}_{\alpha_2\Sigma_2}]\\
&\quad=-\sum_{s\sigma}\hat{S}_1\hat{S}_2\langle S_1\Sigma_1 S_2\Sigma_2|s\sigma\rangle\begin{Bmatrix}\frac{1}{2}&\frac{1}{2}&S_1\\ s&S_2&\frac{1}{2}\end{Bmatrix}\\
&\quad\times\Bigg\{\delta_{\eta'\nu}(-1)^{\eta'+\eta-\Gamma_1+s}\sqrt{\frac{d(\Gamma_1)}{d(0\eta)}}\sum_{\Gamma\alpha}\{\tilde{a}_\eta\times\tilde{a}_{\nu'}\}^{\Gamma s}_{\alpha\sigma}\\
&\quad\times\sum_\rho\langle\Gamma_1\alpha_1;\Gamma_2\alpha_2|\Gamma\alpha\rangle_\rho U[\Gamma_1(\eta' 0)\Gamma(0\nu');(0\eta)_{--}\Gamma_{2-}\rho]\\
&\quad+\delta_{\eta\nu}(-1)^{S_1-s}\sqrt{\frac{d(\Gamma_1)}{d(0\eta')}}\sum_{\Gamma\alpha}\{\tilde{a}_{\eta'}\times\tilde{a}_{\nu'}\}^{\Gamma s}_{\alpha\sigma}\\
&\quad\times\sum_\rho\langle\Gamma_1\alpha_1;\Gamma_2\alpha_2|\Gamma\alpha\rangle_\rho U[\Gamma_1(\eta 0)\Gamma(0\nu');(0\eta')_{--}\Gamma_{2-}\rho]\Bigg\},
\end{aligned}
\tag{B10}
$$

where $d(\Gamma)=d(\lambda\mu)=\frac{1}{2}(\lambda+1)(\mu+1)(\lambda+\mu+2)$ is the dimension of the SU(3) irrep $\Gamma=(\lambda\mu)$.

In order to derive the recursion relation (33), it is furthermore necessary to evaluate the commutator of the two-body operator $\{\{a^\dagger_\eta\times a^\dagger_{\eta'}\}^{\Gamma_1 S_1}\times\{\tilde{a}_\tau\times\tilde{a}_{\tau'}\}^{\Gamma_2 S_2}\}^{\rho_3\Gamma_3 S_3}_{\alpha_3\Sigma_3}$ with $\{a^\dagger_\nu\times\tilde{a}_{\nu'}\}^{\Gamma_4 S_4}_{\alpha_4\Sigma_4}$, where ρ_3 denotes the multiplicity of Γ_3 in the coupling $\Gamma_1\times\Gamma_2\to\Gamma_3$. We obtain the following expression:

$$
\begin{aligned}
&[\{\{a^\dagger_\eta\times a^\dagger_{\eta'}\}^{\Gamma_1 S_1}\times\{\tilde{a}_\tau\times\tilde{a}_{\tau'}\}^{\Gamma_2 S_2}\}^{\rho_3\Gamma_3 S_3}_{\alpha_3\Sigma_3},\{a^\dagger_\nu\times\tilde{a}_{\nu'}\}^{\Gamma_4 S_4}_{\alpha_4\Sigma_4}]\\
&=-\sum_{\Gamma\rho}\sum_{\rho_5\Gamma_5\alpha_5}(-1)^{\Gamma_1-\Gamma_3+\Gamma_4-\Gamma}\sqrt{\frac{d(\Gamma)d(\Gamma_3)}{d(\Gamma_1)d(\Gamma_4)}}\sum_{\rho'\rho''}\Phi_{\rho\rho'}[\Gamma_2\Gamma_4;\Gamma]\Phi_{\rho'\rho''}[\Gamma\tilde{\Gamma}_2;\Gamma_4]\\
&\quad\times\sum_{\tilde{\rho}}\langle\Gamma_3\alpha_3;\Gamma_4\alpha_4|\Gamma_5\alpha_5\rangle_{\tilde{\rho}}U[\Gamma_3\tilde{\Gamma}_2\Gamma_5\Gamma;\Gamma_1\rho_3\rho_5\Gamma_4\rho''\tilde{\rho}]\\
&\quad\times\sum_{sS_5\Sigma_5}(-1)^{S_1+S_4-S_5+s}\hat{S}_2\hat{S}_3\hat{S}_4\hat{s}\langle S_3\Sigma_3 S_4\Sigma_4|S_5\Sigma_5\rangle\begin{Bmatrix}\frac{1}{2}&\frac{1}{2}&S_2\\ s&S_4&\frac{1}{2}\end{Bmatrix}\begin{Bmatrix}S_2&S_1&S_3\\ S_5&S_4&s\end{Bmatrix}\\
&\quad\times\Bigg\{\delta_{\tau'\nu}(-1)^{\tau'+\tau-\Gamma_2+S_2}\sqrt{\frac{d(\Gamma_2)}{d(0\tau)}}\{\{a^\dagger_\eta\times a^\dagger_{\eta'}\}^{\Gamma_1 S_1}\times\{\tilde{a}_\tau\times\tilde{a}_{\nu'}\}^{\Gamma s}\}^{\rho_5\Gamma_5 S_5}_{\alpha_5\Sigma_5}\\
&\quad\times U[\Gamma_2(\tau'0)\Gamma(0\nu');(0\tau)_{--}\Gamma_{4-}\rho]\\
&\quad+\delta_{\tau\nu}\sqrt{\frac{d(\Gamma_2)}{d(0\tau')}}\{\{a^\dagger_\eta\times a^\dagger_{\eta'}\}^{\Gamma_1 S_1}\times\{\tilde{a}_{\tau'}\times\tilde{a}_{\nu'}\}^{\Gamma s}\}^{\rho_5\Gamma_5 S_5}_{\alpha_5\Sigma_5}U[\Gamma_2(\tau0)\Gamma(0\nu');(0\tau')_{--}\Gamma_{4-}\rho]\Bigg\}\\
&\quad+\sum_{\Gamma\rho}\sum_{\rho_5\Gamma_5\alpha_5}(-1)^{\eta+\nu-\Gamma_4}\sqrt{\frac{d(\Gamma_4)}{d(\nu0)}}\sum_{\tilde{\rho}}\langle\Gamma_4\alpha_4;\Gamma_3\alpha_3|\Gamma_5\alpha_5\rangle_{\tilde{\rho}}U[\Gamma_4\Gamma_1\Gamma_5\Gamma_2;\Gamma\rho\rho_5\Gamma_3\rho_3\tilde{\rho}]\\
&\quad\times\sum_{sS_5\Sigma_5}(-1)^{-S_2-S_4+S_5+s}\hat{S}_1\hat{S}_3\hat{S}_4\hat{s}\langle S_4\Sigma_4 S_3\Sigma_3|S_5\Sigma_5\rangle\begin{Bmatrix}\frac{1}{2}&\frac{1}{2}&S_4\\ s&S_1&\frac{1}{2}\end{Bmatrix}\begin{Bmatrix}S_1&s&S_4\\ S_5&S_3&S_2\end{Bmatrix}\\
&\quad\times\Bigg\{\delta_{\nu'\eta'}(-1)^{\Gamma_1}\{\{a^\dagger_\nu\times a^\dagger_\eta\}^{\Gamma s}\times\{\tilde{a}_\tau\times\tilde{a}_{\tau'}\}^{\Gamma_2 S_2}\}^{\rho_5\Gamma_5 S_5}_{\alpha_5\Sigma_5}U[\Gamma_4(\eta'0)\Gamma(\eta0);(\nu0)_{--}\Gamma_{1-}\rho]\\
&\quad+\delta_{\nu'\eta}(-1)^{S_1}\{\{a^\dagger_\nu\times a^\dagger_{\eta'}\}^{\Gamma s}\times\{\tilde{a}_\tau\times\tilde{a}_{\tau'}\}^{\Gamma_2 S_2}\}^{\rho_5\Gamma_5 S_5}_{\alpha_5\Sigma_5}U[\Gamma_4(\eta0)\Gamma(\eta'0);(\nu0)_{--}\Gamma_{1-}\rho]\Bigg\}.
\end{aligned}
\tag{B11}
$$

APPENDIX C: SU(3) COUPLING AND RECOUPLING RULES

The work presented here makes extensive use of SU(2) and SU(3) coupling techniques and the (generalized) Wigner–Eckart theorem. The relevant definitions of the SU(2) coupling coefficients, along with their phase and normalization conventions, can be found in most textbooks on quantum mechanics. Analytic expressions and numerical values for special cases of these coefficients, as well as useful relations between different coupling coefficients, are compiled in various monographs on angular momentum theory, as for instance in reference 39. In contrast, analogous results for the SU(3) case required here are distributed over about a dozen different articles. For the convenience of the reader and as a practical reference for future analytical and numeric work in this field, we list the basic definitions and relations in this appendix. We also point the reader to sources of additional information and to publicly available computer codes. Specifically, we consider the coupling of two, three, and four irreducible SU(3) representations and give the Wigner–Eckart theorem for SU(3). The results presented here are primarily taken from publications by

Hecht *et al.*,[28,36,40–42] Draayer *et al.*,[34,43–45] O'Reilly,[38] Millener,[37] and Vergados.[46] Several new relations are included as well. The phase conventions used here are those employed in reference 44.

1. SU(3) Wigner coefficients: Coupling of two SU(3) irreps

If α represents a set of labels used to distinguish orthonormal basis states within a given irreducible SU(3) representation $\Gamma=(\lambda\mu)$, the Wigner coefficients $\langle\Gamma_1\alpha_1;\Gamma_2\alpha_2|\Gamma\alpha\rangle_\rho$ are defined as the elements of a unitary transformation between coupled and uncoupled orthonormal irreps of SU(3) in the α-scheme:[44]

$$|\Gamma\alpha\rangle_\rho=\sum_{\alpha_1\alpha_2}\langle\Gamma_1\alpha_1;\Gamma_2\alpha_2|\Gamma\alpha\rangle_\rho|\Gamma_1\alpha_1\rangle|\Gamma_2\alpha_2\rangle, \tag{C1}$$

and the inverse transformation is given by

$$|\Gamma_1\alpha_1\rangle|\Gamma_2\alpha_2\rangle=\sum_{\rho\Gamma\alpha}\langle\Gamma_1\alpha_1;\Gamma_2\alpha_2|\Gamma\alpha\rangle_\rho|\Gamma\alpha\rangle_\rho. \tag{C2}$$

Here $\alpha=\epsilon\Lambda M_\Lambda$ for the SU(3)⊃SU(2)⊗U(1) (canonical) group chain and $\alpha=\kappa lm$ for the SU(3)⊃SO(3) reduction. The subgroup chains impose certain restrictions on the above couplings, for example, $\epsilon=\epsilon_1+\epsilon_2$, $M_\Lambda=M_{\Lambda_1}+M_{\Lambda_2}$, and $\Lambda=\Lambda_1+\Lambda_2,\ldots,|\Lambda_1-\Lambda_2|$ must hold for the canonical group chain, and the usual angular momentum coupling rules, $l=l_1+l_2,\ldots,|l_1-l_2|$, and $m=m_1+m_2$ apply for the chain containing SO(3). The conjugates of the relevant SU(3) basis states are given by[44]

$$|\Gamma\alpha\rangle^*=(-1)^{\chi_\alpha}|\tilde{\Gamma}\bar{\alpha}\rangle, \tag{C3}$$

where $\tilde{\Gamma}=(\mu\lambda)$ is the conjugate irrep to $\Gamma=(\lambda\mu)$, and $\bar{\alpha}$ and χ_i depend on the selected subgroup chain:[47]

$$\begin{gathered}\text{for }\ \alpha=\epsilon\Lambda M_\Lambda,\ \text{ one has }\ \bar{\alpha}=(-\epsilon)\Lambda(-M_\Lambda)\ \text{ and }\ \chi_i=\frac{1}{3}(\lambda_i-\mu_i)-\frac{1}{6}\epsilon_i-M_{\Lambda_i};\\ \text{for }\ \alpha=\kappa lm\ \text{ one has }\ \bar{\alpha}=\kappa l(-m)\ \text{ and }\ \chi_i=(\lambda_i-\mu_i)+l_i-m_i.\end{gathered} \tag{C4}$$

The outer multiplicity label $\rho=1,2,\ldots,\rho_{\max}$ is used to distinguish multiple occurrences of a given Γ in the direct product $\Gamma_1\times\Gamma_2$: $\rho=1,2,\ldots,\rho_{\max}$, where $\rho_{\max}$ denotes the number of possible couplings $\Gamma_1\times\Gamma_2$, and the possible Γ irreps in the product can be obtained by coupling the appropriate Young diagrams.[48] O'Reilly[38] determines a closed formula for the decomposition of the outer product $(p,q)\times(r,s)$ of finite-dimensional irreps of SU(3) for arbitrary positive integers p, q, r, and s:

$$(r,s)\times(p,q)=\bigoplus_{k=0}^{\min(q,r+s)}\ \bigoplus_{j=0}^{\min(s,p,r+s-k)}\ \bigoplus_{i=\max(0,j-s+k)}^{\min(p-j+k,r)}(r+p-j-2i+k,s+q+i-j-2k), \tag{C5}$$

and furthermore derives necessary and sufficient conditions for a SU(3) irrep (m,n) to appear as summand in the product $(r,s)\times(p,q)$.

Draayer and Akiyama[44] give a prescription for the unique determination, including the phases, of all SU(3)⊃SU(2)⊗U(1) Wigner coefficients and derive their relevant conjugation and symmetry properties. Since the SU(3)⊃SO(3) reduction is related to the SU(3)⊃SU(2)⊗U(1) reduction via the coefficients of the transformation between the $\alpha=\kappa lm$ and the $\alpha=\epsilon\Lambda M_\Lambda$ schemes,[44,45] it suffices to determine the conjugation relationship and symmetry properties for the SU(3)⊃SU(2)⊗U(1) chain only. The corresponding SU(3)⊃SO(3) results follow then from the known relationships among the transformation brackets between the two schemes. These relations are given in reference 44 and a computer code which allows for a numerical determination of $SU(3)\supset SU(2)\otimes U(1)$ and $SU(3)\supset SO(3)$ Wigner coefficients, as well as SU(3) Racah coefficients, is published in reference 34. Analytic expressions for Wigner coefficients which are of particular interest in p-shell and ds-shell nuclear shell-model calculations are tabulated in reference 28 for the canonical subgroup chain and in references 46, 41 for the $SU(3)\supset SO(3)$ chain.

The most important of the symmetry relations of the SU(3) Wigner coefficients is the one that involves a $1\leftrightarrow 3$ interchange of the quantum labels:

$$\langle\Gamma_1\alpha_1;\Gamma_2\alpha_2|\Gamma_3\alpha_3\rangle_\rho=(-1)^{\varphi+\chi_2}\sqrt{\frac{\dim(\Gamma_3)}{\dim(\Gamma_1)}}\,\langle\Gamma_3\alpha_3;\tilde{\Gamma}_2\overline{\alpha_2}|\Gamma_1\alpha_1\rangle_\rho, \tag{C6}$$

where $\varphi=(\lambda_1+\mu_1)+(\lambda_2+\mu_2)-(\lambda_3+\mu_3)$. The dimension of the SU(3) irrep Γ is given by $\dim(\Gamma)=\dim(\lambda\mu)=\frac{1}{2}(\lambda+1)(\mu+1)(\lambda+\mu+2)$, and $\overline{\alpha_i}$ and χ_i are as defined in Equation (C4).

The $1\leftrightarrow 2$ interchange is more complicated:

$$\langle\Gamma_1\alpha_1;\Gamma_2\alpha_2|\Gamma_3\alpha_3\rangle_\rho=\sum_{\rho'}\Phi_{\rho\rho'}[\Gamma_1,\Gamma_2;\Gamma_3]\langle\Gamma_2\alpha_2;\Gamma_1\alpha_1|\Gamma_3\alpha_3\rangle_{\rho'}, \tag{C7}$$

since it requires a phase matrix Φ, which is a special case of the recoupling coefficient Z that occurs in the coupling of three SU(3) irreps:

$$\Phi_{\rho\rho'}[\Gamma_1,\Gamma_2;\Gamma_3]=Z[\Gamma_1(00)\Gamma_3\Gamma_2;\Gamma_{1-}\rho\Gamma_{2-}\rho']. \tag{C8}$$

If the SU(3) coupling $\Gamma_1\times\Gamma_2\to\Gamma_3$ is unique, that is, when $\rho_{\max}=1$, the matrix reduces to a simple phase factor: $\Phi_{11}[\Gamma_1,\Gamma_2;\Gamma_3]=(-1)^{\varphi}=(-1)^{\Gamma_1+\Gamma_2-\Gamma_3}$, where $(-1)^{\Gamma_i}\equiv(-1)^{\lambda_i+\mu_i}$ for $\Gamma_i=(\lambda_i\mu_i)$.

Another set of useful relations involves the conjugate irrep $\tilde{\Gamma}=(\mu\lambda)$ of $\Gamma=(\lambda\mu)$; for the canonical group chain,[44]

$$\langle\Gamma_1\alpha_1;\Gamma_2\alpha_2|\Gamma_3\alpha_3\rangle_\rho=(-1)^{\varphi+\rho_{\max}-\rho}\langle\tilde{\Gamma}_1\overline{\alpha_1};\tilde{\Gamma}_2\overline{\alpha_2}|\tilde{\Gamma}_3\overline{\alpha_3}\rangle_\rho \tag{C9}$$

holds and for $\alpha=\kappa lm$ one has[44]

$$\langle\Gamma_1\alpha_1;\Gamma_2\alpha_2|\Gamma_3\alpha_3\rangle_\rho=(-1)^{\varphi+\rho_{\max}-\rho+l_1+l_2-l_3}\langle\tilde{\Gamma}_1\overline{\alpha_1};\tilde{\Gamma}_2\overline{\alpha_2}|\tilde{\Gamma}_3\overline{\alpha_3}\rangle_\rho. \tag{C10}$$

With the phase and normalization conventions of reference 44, one obtains the following orthonormality relations for the Wigner SU(3) coupling coefficients:

$$\sum_{\alpha_1\alpha_2}\langle\Gamma_1\alpha_1;\Gamma_2\alpha_2|\Gamma\alpha\rangle_\rho\langle\Gamma_1\alpha_1;\Gamma_2\alpha_2|\Gamma'\alpha\rangle_{\rho'}=\delta_{\Gamma'\Gamma}\delta_{\rho'\rho} \tag{C11}$$

and

$$\sum_{\rho\Gamma\alpha}\langle\Gamma_1\alpha_1;\Gamma_2\alpha_2|\Gamma\alpha\rangle_\rho\langle\Gamma_1\alpha_1';\Gamma_2\alpha_2'|\Gamma\alpha\rangle_\rho=\delta_{\alpha_1\alpha_1'}\delta_{\alpha_2\alpha_2'}. \tag{C12}$$

It is possible to factor out the dependence of the above $SU(3)\supset SO(3)$ and $SU(3)\supset SU(2)\otimes U(1)$ Wigner coupling coefficients on the M or M_Λ subgroup labels, respectively, by defining so-called double-barred or "reduced" SU(3) coupling coefficients:

$$\langle\Gamma_1\kappa_1l_1m_1;\Gamma_2\kappa_2l_2m_2|\Gamma\kappa lm\rangle_\rho=\underbrace{\langle\Gamma_1\kappa_1l_1;\Gamma_2\kappa_2l_2||\Gamma\kappa l\rangle_\rho}_{\text{reduced Wigner coefficient}}\underbrace{\langle l_1m_1,l_2m_2|lm\rangle}_{\text{geometric part}}, \tag{C13}$$

for $\alpha=\kappa lm$, and

$$\langle \Gamma_1\epsilon_1\Lambda_1 M_{\Lambda_1};\Gamma_2\epsilon_2\Lambda_2 M_{\Lambda_2}|\Gamma\epsilon\Lambda M_\Lambda\rangle_\rho = \underbrace{\langle \Gamma_1\epsilon_1\Lambda_1;\Gamma_2\epsilon_2\Lambda_2||\Gamma_3\epsilon\Lambda\rangle_\rho}_{\text{reduced Wigner coefficient}}\underbrace{\langle \Lambda_1 M_{\Lambda_1},\Lambda_2 M_{\Lambda_2}|\Lambda M_\Lambda\rangle}_{\text{geometric part}}, \tag{C14}$$

for $\alpha=\epsilon\Lambda M_\Lambda$. The ''geometric'' part $\langle$--|-$\rangle$ is simply a SU(2) Clebsch–Gordan coefficient. From the unitarity of the full SU(3) Wigner and the ordinary SU(2) Clebsch–Gordan coefficients it follows that the double-bar coefficients are also unitary. With the phase convention introduced by Draayer and Akiyama[44] they become real, and therefore orthogonal.

2. SU(3) Racah coefficients: Coupling of three SU(3) irreps

Coupling of three SU(3) irreps Γ_1, Γ_2, and Γ_3 to a given resulting SU(3) irrep Γ can be accomplished in three different ways, depending on the order of the coupling: (i) $\{\Gamma_1\otimes\Gamma_2\}\otimes\Gamma_3\to\Gamma$, or (ii) $\Gamma_1\otimes\{\Gamma_2\otimes\Gamma_3\}\to\Gamma$, or (iii) $\{\Gamma_1\otimes\Gamma_3\}\otimes\Gamma_2\to\Gamma$. The transformation from one coupling order to another requires the introduction of a so-called SU(3)-Racah or 6-($\lambda\mu$) coefficient. More specifically, recoupling from scheme (i) to (ii) involves a unitary transformation with coefficients $U[\Gamma_1\Gamma_2\Gamma\Gamma_3;\Gamma_{12}\rho_{12}\rho_{12,3}\Gamma_{23}\rho_{23}\rho_{1,23}]$ (see reference 28):

$$|\{\{\Gamma_1\times\Gamma_2\}^{\rho_{12}\Gamma_{12}}\times\Gamma_3\}_\alpha^{\rho_{12,3}\Gamma}\rangle = \sum_{\Gamma_{23}\rho_{1,23}\rho_{23}} U[\Gamma_1\Gamma_2\Gamma\Gamma_3;\Gamma_{12}\rho_{12}\rho_{12,3}\Gamma_{23}\rho_{23}\rho_{1,23}]|\{\Gamma_1\times\{\Gamma_2\times\Gamma_3\}^{\rho_{23}\Gamma_{23}}\}_\alpha^{\rho_{1,23}\Gamma}\rangle, \tag{C15}$$

and the inverse transformation is given by[36]

$$|\{\Gamma_1\times\{\Gamma_2\times\Gamma_3\}^{\rho_{23}\Gamma_{23}}\}_\alpha^{\rho_{1,23}\Gamma}\rangle = \sum_{\Gamma_{12}\rho_{12}\rho_{12,3}} U[\Gamma_1\Gamma_2\Gamma\Gamma_3;\Gamma_{12}\rho_{12}\rho_{12,3}\Gamma_{23}\rho_{23}\rho_{1,23}]|\{\{\Gamma_1\times\Gamma_2\}^{\rho_{12}\Gamma_{12}}\times\Gamma_3\}_\alpha^{\rho_{12,3}\Gamma}\rangle. \tag{C16}$$

The notation is a straightforward generalization of that introduced by Racah[39] for the $6\text{-}j$ symbols of SU(2). Whenever a coupling is multiplicity-free, the associated ρ-label may be omitted, provided the notation remains unambiguous.

Similarly, the transformation from scheme (i) to scheme (iii) requires a transformation coefficient $Z[\Gamma_2\Gamma_1\Gamma\Gamma_3;\Gamma_{12}\rho_{12}\rho_{12,3}\Gamma_{13}\rho_{13}\rho_{13,2}]$ which is defined through[37]

$$|\{\{\Gamma_1\times\Gamma_2\}^{\rho_{12}\Gamma_{12}}\times\Gamma_3\}_\alpha^{\rho_{12,3}\Gamma}\rangle = \sum_{\Gamma_{13}\rho_{13}\rho_{13,2}} Z[\Gamma_2\Gamma_1\Gamma\Gamma_3;\Gamma_{12}\rho_{12}\rho_{12,3}\Gamma_{13}\rho_{13}\rho_{13,2}]|\{\{\Gamma_1\times\Gamma_3\}^{\rho_{13}\Gamma_{13}}\times\Gamma_2\}_\alpha^{\rho_{13,2}\Gamma}\rangle. \tag{C17}$$

The U- and Z-functions depend only on the SU(3) representations involved in the coupling and not on the specific subgroup chain chosen to specify the states. Most of the SU(3) recoupling coefficients needed for ds-shell nuclear shell model calculations have been tabulated by Hecht,[28] and a computer code that evaluates SU(3) Racah coefficients for arbitrary couplings and multiplicity was developed by Akiyama and Draayer.[34] Special cases and symmetry properties of the SU(3) coefficients can be found in reference 36. Most notably, one has $U[\cdots]=1$ whenever Γ_1 or Γ_2 or Γ_3 or $\Gamma=(00)$, and

$$U[\Gamma_1\Gamma_2\Gamma_1\tilde{\Gamma}_2;\Gamma_{12}\rho\rho'(00)_{--}]=(-1)^{\Gamma_1+\Gamma_2+\Gamma_{12}}\delta_{\rho\rho'}\sqrt{\frac{d(\Gamma_{12})}{d(\Gamma_1)d(\Gamma_2)}}. \tag{C18}$$

Under conjugation the SU(3) Racah coefficients exhibit the following symmetry:

$$U[\Gamma_1\Gamma_2\Gamma\Gamma_3;\Gamma_{12}\rho_{12}\rho_{12,3}\Gamma_{23}\rho_{23}\rho_{1,23}]$$
$$=(-1)^{\rho_{12}^{\max}-\rho_{12}+\rho_{12,3}^{\max}-\rho_{12,3}+\rho_{23}^{\max}-\rho_{23}+\rho_{1,23}^{\max}-\rho_{1,23}}U[\tilde{\Gamma}_1\tilde{\Gamma}_2\tilde{\Gamma}\tilde{\Gamma}_3;\tilde{\Gamma}_{12}\rho_{12}\rho_{12,3}\tilde{\Gamma}_{23}\rho_{23}\rho_{1,23}], \tag{C19}$$

$$Z[\Gamma_2\Gamma_1\Gamma\Gamma_3;\Gamma_{12}\rho_{12}\rho_{12,3}\Gamma_{13}\rho_{13}\rho_{13,2}]$$
$$=(-1)^{\rho_{12}^{\max}-\rho_{12}+\rho_{12,3}^{\max}-\rho_{12,3}+\rho_{13}^{\max}-\rho_{13}+\rho_{13,2}^{\max}-\rho_{13,2}}Z[\tilde{\Gamma}_2\tilde{\Gamma}_1\tilde{\Gamma}\tilde{\Gamma}_3;\tilde{\Gamma}_{12}\rho_{12}\rho_{12,3}\tilde{\Gamma}_{13}\rho_{13}\rho_{13,2}].$$

A straightforward generalization of the relations between SU(2) unitary recoupling coefficients and SU(2) Wigner coefficients leads to the corresponding relationships between SU(3) unitary (Racah) recoupling and SU(3) Wigner coefficients.

$$U[\Gamma_1\Gamma_2\Gamma\Gamma_3;\Gamma_{12}\rho_{12}\rho_{12,3}\Gamma_{23}\rho_{23}\rho_{1,23}]$$
$$=\sum_{\alpha_1\alpha_2\alpha_3\alpha_{12}\alpha_{23}}\langle\Gamma_1\alpha_1;\Gamma_2\alpha_2|\Gamma_{12}\alpha_{12}\rangle_{\rho_{12}}\langle\Gamma_{12}\alpha_{12};\Gamma_3\alpha_3|\Gamma\alpha\rangle_{\rho_{12,3}}$$
$$\times\langle\Gamma_2\alpha_2;\Gamma_3\alpha_3|\Gamma_{23}\alpha_{23}\rangle_{\rho_{23}}\langle\Gamma_1\alpha_1;\Gamma_{23}\alpha_{23}|\Gamma\alpha\rangle_{\rho_{1,23}}, \tag{C20}$$

for $\alpha_i=\epsilon_i\Lambda_iM_i$ or $\alpha_i=\kappa_il_im_i$.

The Z- and U-coefficients are related to each other as follows:

$$Z[\Gamma_2\Gamma_1\Gamma\Gamma_3;\Gamma_{12}\rho_{12}\rho_{12,3}\Gamma_{13}\rho_{13}\rho_{13,2}]$$
$$=\sum_{\Gamma_{23}\rho_{23}\rho_{1,23}}U[\Gamma_1\Gamma_2\Gamma\Gamma_3;\Gamma_{12}\rho_{12}\rho_{12,3}\Gamma_{23}\rho_{23}\rho_{1,23}]\sum_{\rho'_{23}}\Phi_{\rho_{23}\rho'_{23}}[\Gamma_2,\Gamma_3;\Gamma_{23}]$$
$$\times U[\Gamma_1\Gamma_3\Gamma\Gamma_2;\Gamma_{13}\rho_{13}\rho_{13,2}\Gamma_{23}\rho'_{23}\rho_{1,23}], \tag{C21}$$

with the geometrical phase $\Phi_{\rho\rho'}$ as defined in Equation (C8). Further useful relations for the U- and Z-recoupling coefficients are given in reference 28, for the SU(3)⊃SU(2)⊗U(1) chain, and in references 46, 37, for the SU(3)⊃SO(3) chain. The following one, which holds for both cases, is especially important:

$$\sum_{\rho_{1,23}}\langle\Gamma_1\alpha_1;\Gamma_{23}\alpha_{23}|\Gamma\alpha\rangle_{\rho_{1,23}}U[\Gamma_1\Gamma_2\Gamma\Gamma_3;\Gamma_{12}\rho_{12}\rho_{12,3}\Gamma_{23}\rho_{23}\rho_{1,23}]$$
$$=\sum_{\alpha_2\alpha_3\alpha_{12}}\langle\Gamma_2\alpha_2;\Gamma_3\alpha_3|\Gamma_{23}\alpha_{23}\rangle_{\rho_{23}}\langle\Gamma_1\alpha_1;\Gamma_2\alpha_2|\Gamma_{12}\alpha_{12}\rangle_{\rho_{12}}\langle\Gamma_{12}\alpha_{12};\Gamma_3\alpha_3|\Gamma\alpha\rangle_{\rho_{12,3}}. \tag{C22}$$

A similar relation is given by Millener for the Z-coefficient.[37]

3. SU(3) 9-($\lambda\mu$) coefficients: Coupling of four SU(3) irreps

If the coupling of four SU(3) irreps Γ_1, Γ_2, Γ_3 and Γ_4 is required, the resulting irrep Γ may be constructed in three different ways: (i) $\{\Gamma_1\otimes\Gamma_2\}\otimes\{\Gamma_3\otimes\Gamma_4\}\rightarrow\Gamma$, or (ii) $\{\Gamma_1\otimes\Gamma_3\}\otimes\{\Gamma_2\otimes\Gamma_4\}\rightarrow\Gamma$, or (iii) $\{\Gamma_1\otimes\Gamma_4\}\otimes\{\Gamma_2\otimes\Gamma_3\}\rightarrow\Gamma$. In analogy to the SU(2) Jahn–Hope symbol it is possible to define a unitary SU(3) or 9-($\lambda\mu$) symbol, which effects the transformation from one coupling order to another. In particular, for the transition from scheme (i) to (ii) one defines[36]

$$|\{\{\Gamma_1\times\Gamma_2\}^{\rho_{12}\Gamma_{12}}\times\{\Gamma_3\times\Gamma_4\}^{\rho_{34}\Gamma_{34}}\}_{\alpha}^{\rho_{12,34}\Gamma}\rangle=\sum_{\rho_{13}\Gamma_{13}\rho_{24}\Gamma_{24}\rho_{13,24}}\begin{Bmatrix}\Gamma_1 & \Gamma_2 & \Gamma_{12} & \rho_{12}\\ \Gamma_3 & \Gamma_4 & \Gamma_{34} & \rho_{34}\\ \Gamma_{13} & \Gamma_{24} & \Gamma & \rho_{13,24}\\ \rho_{13} & \rho_{24} & \rho_{12,34} & \end{Bmatrix}$$
$$\times|\{\{\Gamma_1\times\Gamma_3\}^{\rho_{13}\Gamma_{13}}\times\{\Gamma_2\times\Gamma_4\}^{\rho_{24}\Gamma_{24}}\}_{\alpha}^{\rho_{13,24}\Gamma}\rangle. \tag{C23}$$

Similarly, one obtains for the transformation from scheme (i) to (iii),

$$|\{\{\Gamma_1\times\Gamma_2\}^{\rho_{12}\Gamma_{12}}\times\{\Gamma_3\times\Gamma_4\}^{\rho_{34}\Gamma_{34}}\}_{\alpha}^{\rho_{12,34}\Gamma}\rangle$$
$$=\sum_{\rho_{14}\Gamma_{14}\rho_{23}\Gamma_{23}\rho_{14,23}}\begin{Bmatrix}\Gamma_1 & \Gamma_2 & \Gamma_{12} & \rho_{12}\\ \Gamma_4 & \Gamma_3 & \Gamma_{34} & \rho_{34}\\ \Gamma_{14} & \Gamma_{23} & \Gamma & \rho_{14,23}\\ \rho_{14} & \rho_{23} & \rho_{12,34} & \end{Bmatrix}$$
$$\times\sum_{\rho'_{34}}\Phi_{\rho_{34}\rho'_{34}}(\Gamma_3,\Gamma_4;\Gamma_{34})|\{\{\Gamma_1\times\Gamma_4\}^{\rho_{14}\Gamma_{14}}\times\{\Gamma_2\times\Gamma_3\}^{\rho_{23}\Gamma_{23}}\}_{\alpha}^{\rho_{14,23}\Gamma}\rangle. \tag{C24}$$

The unitary 9-$(\lambda\mu)$ symbols can be expressed as a sum over SU(3) Wigner coupling coefficients:

$$\begin{Bmatrix}\Gamma_1 & \Gamma_2 & \Gamma_{12} & \rho_{12}\\ \Gamma_3 & \Gamma_4 & \Gamma_{34} & \rho_{34}\\ \Gamma_{13} & \Gamma_{24} & \Gamma & \rho_{13,24}\\ \rho_{13} & \rho_{24} & \rho_{12,34} & \end{Bmatrix}$$
$$=\sum_{\alpha_i\alpha_{ik}}\langle\Gamma_1\alpha_1;\Gamma_2\alpha_2|\Gamma_{12}\alpha_{12}\rangle_{\rho_{12}}\langle\Gamma_3\alpha_3;\Gamma_4\alpha_4|\Gamma_{34}\alpha_{34}\rangle_{\rho_{34}}\langle\Gamma_{13}\alpha_{13};\Gamma_{24}\alpha_{24}|\Gamma\alpha\rangle_{\rho_{13,24}}$$
$$\times\langle\Gamma_1\alpha_1;\Gamma_3\alpha_3|\Gamma_{13}\alpha_{13}\rangle_{\rho_{13}}\langle\Gamma_2\alpha_2;\Gamma_4\alpha_4|\Gamma_{24}\alpha_{24}\rangle_{\rho_{24}}\langle\Gamma_{12}\alpha_{12};\Gamma_{34}\alpha_{34}|\Gamma\alpha\rangle_{\rho_{12,34}}, \tag{C25}$$

or in terms of the Racah Z and U coefficients defined above,[37]

$$\begin{Bmatrix}\Gamma_1 & \Gamma_2 & \Gamma_{12} & \rho_{12}\\ \Gamma_3 & \Gamma_4 & \Gamma_{34} & \rho_{34}\\ \Gamma_{13} & \Gamma_{24} & \Gamma & \rho_{13,24}\\ \rho_{13} & \rho_{24} & \rho_{12,34} & \end{Bmatrix}$$
$$=\sum_{\Gamma_0\rho_{13,2}\rho_{04}\rho_{12,3}}U[\Gamma_{13}\Gamma_2\Gamma\Gamma_4;\Gamma_0\rho_{13,2}\rho_{04}\Gamma_{24}\rho_{24}\rho_{13,24}]Z[\Gamma_2\Gamma_1\Gamma_0\Gamma_3;\Gamma_{12}\rho_{12}\rho_{12,3}\Gamma_{13}\rho_{13}\rho_{13,2}]$$
$$\times U[\Gamma_{12}\Gamma_3\Gamma\Gamma_4;\Gamma_0\rho_{12,3}\rho_{04}\Gamma_{34}\rho_{34}\rho_{12,34}]. \tag{C26}$$

Various symmetry relations and special cases of the 9-$(\lambda\mu)$ coefficients can be found in references 40, 36, and a computer code which provides numerical values for these recoupling coefficients is publicly available.[34]

4. Matrix elements and the Wigner–Eckart theorem for SU(3)

The Wigner–Eckart theorem for the group SU(2) yields SU(2)-reduced (double-bar) matrix elements of an SO(3) irreducible tensor operator. Analogously, the generalized Wigner–Eckart theorem allows one to express matrix elements of SU(3) irreducible tensor operators as a sum over ρ of the product of a ρ-dependent generalized reduced matrix element multiplied by the corresponding Wigner coefficient:[44]

$$\langle\Gamma_3\alpha_3|T^{\Gamma_2\alpha_2}|\Gamma_1\alpha_1\rangle=\sum_{\rho}\langle\Gamma_1\alpha_1;\Gamma_2\alpha_2|\Gamma_3\alpha_3\rangle_{\rho}\langle\Gamma_3|||T^{\Gamma_2}|||\Gamma_1\rangle_{\rho}. \tag{C27}$$

Note that the triple-bar matrix element is independent of the chosen subgroup chain.

In analogy to the well-known reduction rules for SU(2),[39] one can derive expressions for the triple-bar matrix elements of a SU(3)-coupled tensor product acting on a one-component system:

$$\langle\Gamma|||\{T^{\Gamma_1}\times T^{\Gamma_2}\}^{\rho_3\Gamma_3}|||\Gamma'\rangle_{\rho}=\sum_{\rho_1\rho_2\Gamma''}\langle\Gamma|||T^{\Gamma_1}|||\Gamma''\rangle_{\rho_1}\langle\Gamma''|||T^{\Gamma_2}|||\Gamma'\rangle_{\rho_2}$$
$$\times\sum_{\rho_3'}\Phi_{\rho_3\rho_3'}[\Gamma_1\Gamma_2;\Gamma_3]U[\Gamma'\Gamma_2\Gamma\Gamma_1;\Gamma''\rho_2\rho_1\Gamma_3\rho_3'\rho]. \tag{C28}$$

Often one has to consider a quantum-mechanical system which consists of two subsystems, 1 and 2 (for example, protons and neutrons). Reduced matrix elements for the irreducible tensor product of two operators, $R^{\Gamma_r}(1)$ and $S^{\Gamma_s}(2)$, which depend only on variables of the first and second subsystem, respectively, may be evaluated with the help of the following expression:

$$\langle\{\Gamma_1\times\Gamma_2\}\rho\Gamma|||\{R^{\Gamma_r}(1)\times S^{\Gamma_s}(2)\}^{\rho_t\Gamma_t}|||\{\Gamma_1'\times\Gamma_2'\}\rho'\Gamma'\rangle_{\tilde{\rho}}$$
$$=\sum_{\rho_1\rho_2}\begin{Bmatrix}\Gamma_1' & \Gamma_r & \Gamma_1 & \rho_1\\ \Gamma_2' & \Gamma_s & \Gamma_2 & \rho_2\\ \Gamma' & \Gamma_t & \Gamma & \tilde{\rho}\\ \rho' & \rho_t & \rho & \end{Bmatrix}\langle\Gamma_1|||R^{\Gamma_r}(1)|||\Gamma_1'\rangle_{\rho_1}\langle\Gamma_2|||S^{\Gamma_s}(2)|||\Gamma_2'\rangle_{\rho_2}. \tag{C29}$$

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