Ferromagnetic Ising measures on large locally tree-like graphs

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Ising measure

An *Ising model on the finite graph* $G = (V, E)$ is defined by the following distribution over $\underline{x} = \{x_i, i \in V\}$, with $x_i \in \{-1, +1\}$

$$\mu(\underline{x}) = \frac{1}{Z(\beta, B)} \exp\Big\{\beta \sum_{(i,j)\in E} x_i x_j + B \sum_{i \in V} x_i\Big\}.$$

- β is inverse temperature parameter, and B is external magnetic field.
- $Z(\beta, B)$ is called partition function.
- Ferromagnetic if $\beta > 0$, antiferromagnetic otherwise.

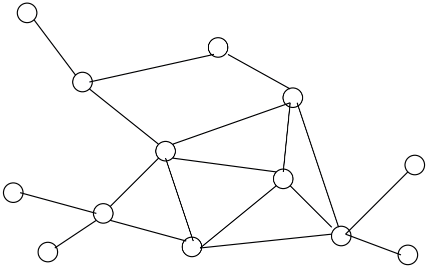
Statistical Physics models on locally tree-like graphs

- There are numerous examples from combinatorics, computer science and statistical inference which correspond to nearest neighbor Gibbs measures for large β:

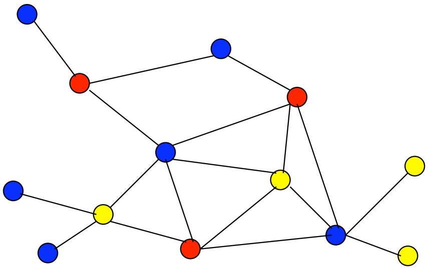
Independent set or hard-core model on $G = (V, E)$ is

$$\mu_G^\lambda(\underline{x}) = \frac{1}{Z_G(\lambda)} \prod_{(i,j)\in E} \mathbb{I}\{x_i x_j \neq 1\} \prod_{i\in V} \lambda^{x_i},$$

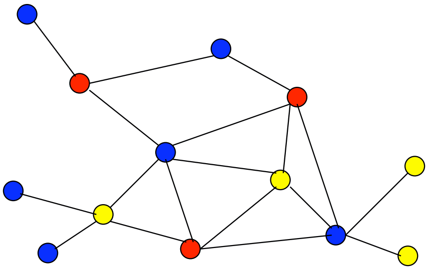
with $x_i \in \{0, 1\}$.



A proper q-coloring of a graph $G = (V, E)$ is an assignment of colors $x_i \in \{1, 2, \ldots, q\}$ for every $i \in V$.



$$\mu_G(\underline{x}) = \frac{1}{Z_G} \prod_{(i,j)\in E} \mathbb{I}\{x_i \neq x_j\}.$$



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Many other examples:

- Communications (LDPC; XORSAT)
- Artificial intelligence (Bayesian networks; Graphical models)
- Statistics (Compressed sensing)
- ......

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- It is also conjectured that in many of these models the limit measure can be expressed as a convex combination of simpler components.

- Recently there has been a lot of interest in also the Ising model on non-lattice complex networks.

``The motivation behind studies of spin models on networks is usually either that they can be regarded as simple models of opinion formation in social networks or that they provide general insight into the effects of network topology on phase transition processes.''

[M. Newman '03]

Phase transition phenomenon: Ising model

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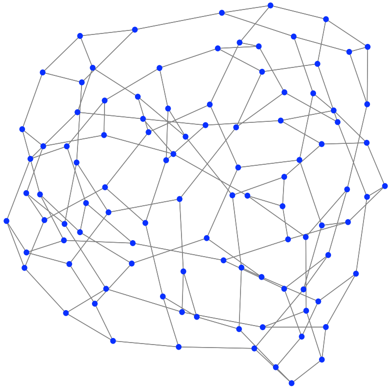
- Regular graph sequences, that converge locally weakly to a tree, have been considered in [Montanari, Mossel, Sly '11].

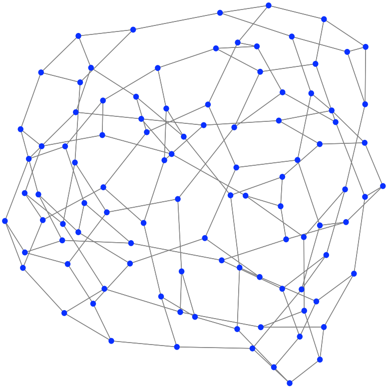
Locally tree-like graphs

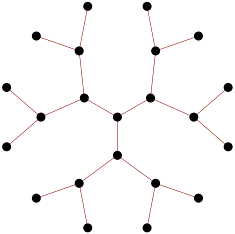
Random 3-regular graph



Locally tree-like graphs

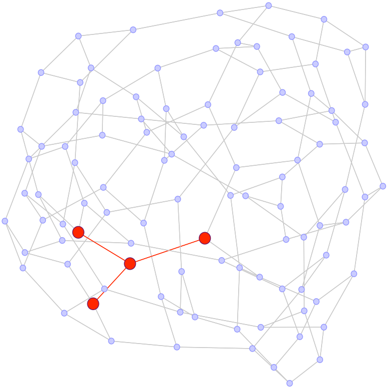
Random 3-regular graph and first few generations of T_3

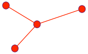




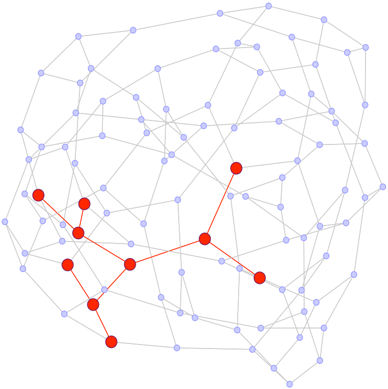
Locally tree-like graphs

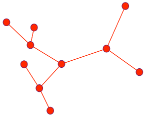
Random 3-regular graph and consider balls of radius 1 and we ask whether they are isomorphic to that of T_3





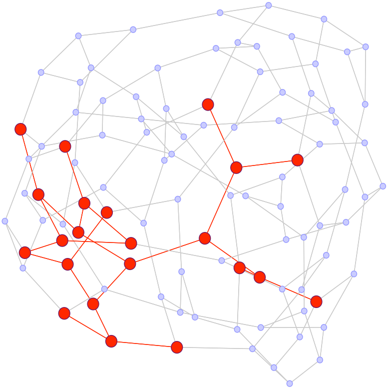
Random 3-regular graph and consider balls of radius 2 and we ask whether they are isomorphic to that of T_3

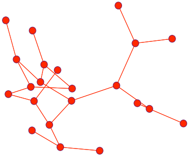




Locally tree-like graphs

Random 3-regular graph and consider balls of radius 3 and we ask whether they are isomorphic to that of T_3





Locally tree-like graphs

Definition

A random graph sequence $G_n = (V_n, E_n)$ converges locally to T_r, if for all $t \geq 0$,

$$\lim_{n\to\infty} \mathbb{P}_n(\mathbf{B}^t_{I_n} \ncong \mathsf{T}^t_r) = 0.$$

$\mathbb{P}_n$ is the joint law of the graph G_n, and $I_n \in V_n$, uniformly at random.

Can make definition with general (random) limiting tree.

Convergence notion due to [Benjamini, Schramm '01]. Many properties are proved in [Aldous, Lyons '07]

Uniform sparsity assumption: the degrees Δ_{I_n} of G_n are U.I.

Results from [MMS '11]

$$\mu_n(\cdot) \to \frac{1}{2}\nu_{+,\mathsf{T}_r}(\cdot) + \frac{1}{2}\nu_{-,\mathsf{T}_r}(\cdot), \text{ for } B = 0 \text{ and any } \beta \geq 0.$$

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[The case $B \neq 0$ is much simpler (less interesting), follows from [D., Montanari '10]]

Edge-expansion condition

Definition

A finite graph $G = (V, E)$ is a (δ, λ) *edge-expander* if, for any set of vertices $S \subseteq V$, with $|S| \leq \delta|V|$, $|E(S, S^c)| := |\partial S| \geq \lambda|S|$.

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- Example. Consider m identical disjoint r-regular graphs on n/m vertices. Condition on $\sum_i x_i > 0$. Probability of sum of the spins being positive in each component is $O(m^{-1/2})$. Thus $\mu_{n,+} \Rightarrow (1-q)\nu_{+,\mathsf{T}_r} + q\nu_{-,\mathsf{T}_r}$ with $q = 1/2 - O(m^{-1/2})$.

[MMS '11]

Similarly one can construct connected version of this example.

Heuristics for [MMS '11]

- The neighborhood $\mathbf{B}_i$ of a vertex $i \in V_n$, w.r.t. graph distance is assumed to converge to a neighborhood of an infinite regular tree T_r.
- It is natural to assume that $\mu_{n,\mathbf{B}_i}(\cdot)$ converges to the marginal of a neighborhood of the root for some Ising Gibbs measure on T_r.

Heuristics for [MMS '11]

- However for large β there are (uncountably) many Gibbs measures, so a-priori not clear which one to choose.

Ising measure:

$$\mu(\underline{x}) = \frac{1}{Z(\beta, B)} \exp\left\{\beta \sum_{(i,j)\in E} x_i x_j + B \sum_{i\in V} x_i\right\}.$$

- $\beta = 0 \Rightarrow$ independence.
- $\beta = \infty$, and $B = 0 \Rightarrow$ with prob. $1/2$ all spins are $+$, and with prob. $1/2$, all of them are $-$.

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- Of particular applied interest are the following examples:
  - Erdős-Rényi graphs $\rightarrow$ GW trees.
  - Graphs with fixed degree distribution $\rightarrow$ GW trees.
  - Random uniform k-partite graphs $\rightarrow$ MGW trees.

Results [BD12]

Theorem (1)

Suppose $G_n \overset{\text{LWC}}{\Longrightarrow} \mathsf{T} \sim \mu$ *and* $\beta \mapsto U(\beta, 0) \in \mathcal{C}$. *Then*

$$\mu_n(\cdot) \Longrightarrow \frac{1}{2}\nu_{+,\mathsf{T}}(\cdot) + \frac{1}{2}\nu_{-,\mathsf{T}}(\cdot), \; B = 0, \; \beta \geq 0, \mathsf{T} \sim \mu$$

$$U(\beta, B) := \frac{1}{2}\mathbb{E}_\mu\Big[\sum_{i \in \partial\phi} \nu_{+,\mathsf{T}}^{\beta,B}\langle x_\phi \cdot x_i\rangle\Big], \; \partial\phi := \{i \in V, \; i \sim \phi\}.$$

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Theorem (2)

Let $G_n \stackrel{\text{LWC}}{\Longrightarrow} \mu$. Assume that for every $0 < \delta < 1/2$, $\{G_n\}_{n\in\mathbb{N}}$ are $(\delta, 1/2, \lambda_\delta)$ edge-expanders for some $\lambda_\delta > 0$, with uniform bounded degrees. Also assume $\beta \mapsto U(\beta\ , 0) \in \mathcal{C}$ and μ ergodic then

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- We confirm that the relevant MGW trees are ergodic, and the corresponding configuration models are edge-expanders. (minimum degree ≥ 3 needed)

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$$\mu_{n,\pm}(\cdot) \Longrightarrow \nu_{\pm,\mathsf{T}}(\cdot), B = 0,\ \ \beta \geq 0, \mathsf{T} \sim \mu.$$

- Obtained a result for the subsequential limits of $\mu_{n,+}$ for any general μ.

Local weak limit and ergodicity

- $\mu_n \Rightarrow \nu$ locally:

for all $t > 0$, the joint law of $(B^t_{I_n}, \underline{x}_{B^t_{I_n}}) \Rightarrow (\mathsf{T}^t, \underline{x}_{\mathsf{T}^t})$, $\mathsf{T} \sim \mu$.

Local weak limit and ergodicity

- Unimodularity and Ergodicity: any LWC limit μ is unimodular.

$$\int \sum_{x \in V} f(G, o, x) d\mu([G, o]) = \int \sum_{x \in V} f(G, x, o) d\mu([G, o]).$$

Choose a rooted graph G according to the measure μ biased by the degree of the root, and perform SRW by moving the root uniformly among the adjacent vertices. This Markov chain is reversible and stationary.

[Aldous, Lyons '07]

We call μ ergodic, if the Markov chain is ergodic too.

Proof strategy of Theorem (1) [following [MMS '11]]

1. Upon showing that $\{\mu_n\}$ is tight, reduces the problem to identification of the limit points.

2. The probability of agreement between neighboring spins in a ball in G_n is asymptotically the same as in the measure $\nu_{+,\mathsf{T}}$ (or $\nu_{-,\mathsf{T}}$) on the infinite tree. This is the quantity $U(\beta, 0)$.

$$\Big[\sum_{i \in \partial\phi} \nu_{+,\mathsf{T}} \langle x_\phi \cdot x_i \rangle\Big] = \lim_{n\to\infty} \frac{1}{n} \sum_{(i,j)\in E} \mu_n \langle x_i \cdot x_j \rangle.$$

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3. Can verify that the probability of agreement between neighboring spins among all extremal Gibbs measures on the tree, is maximized only by $\nu_{+,\mathsf{T}}$ and $\nu_{-,\mathsf{T}}$.

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4. By symmetry w.r.t global sign flips, $\mu_n \Longrightarrow \frac{1}{2}\nu_{+,\mathsf{T}} + \frac{1}{2}\nu_{-,\mathsf{T}}$.

Proof strategy of Theorem (2) [following [MMS '11]]

- As in Theorem (1) progress by first showing Steps 1-3.
- For $\mu_{n,\pm}$, the edge-expansion property of G_n rules out that simultaneously a positive fraction of the vertices have their neighborhood in the "$+$ state" and another positive fraction in the "$-$ state".

Key challenges in our proof

Key estimates in the proofs of Step 2 of Theorem (1), and Step 4 of Theorem (2) in [MMS '11] involve explicit calculations which crucially rely on regularity of both graph sequence, and the limiting tree.

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- For k-regular infinite tree, root-magnetization can be represented as the largest zero of a real analytic function.

- No such representation known for any other tree measure.

Solution and interesting byproducts [BD12]

- For $\beta > \beta_c$, continuity of root magnetization under $\nu_{+,\mathsf{T}}(\cdot)$ is shown for a large class of limiting measures using a more robust argument. This includes MGW trees.

$$U(\beta, B) := \frac{1}{2}\mathbb{E}_\mu\Big[\sum_{i \in \partial\phi} \nu_{+,\mathsf{T}}^{\beta,B}\langle x_\phi \cdot x_i\rangle\Big]$$

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Byproducts:

- Continuity of percolation probability for *random cluster model*, with $q = 2$, and *wired boundary condition*.

- Uniqueness of the *splitting Gibbs measure* on UMGW random trees, for $B = 0$ and any boundary condition strictly larger (stochastically dominating) than the free boundary condition.

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- A contradiction with the expander assumption whenever a positive fraction of the edges have one end in the "$+$ state" and another in the "$-$ state".

Key challenges in Theorem (2)

The proof of [MMS '11] for $\mu_{n,+}(\cdot)$ relies on choosing functionals $F_l(\cdot)$ of the spin configurations on G_n, which approximate the indicator on the vertices that are in "$-$ state", and whose values concentrate as $n \to \infty$ followed by $l \to \infty$.

- A contradiction with the expander assumption whenever a positive fraction of the edges have one end in the "$+$ state" and another in the "$-$ state".

- Regularity of the graphs G_n, and their limit, indicates how to get $F_l(\cdot)$, and allows explicit computations involving them.

Solution for non-regular tree

- $F_l(\cdot)$ is defined via *average occupation measure* of the simple random walk on the tree.

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- $F_l(\cdot)$ is defined via *average occupation measure* of the simple random walk on the tree.

- Tools used are *unimodularity* of the limiting tree, and properties of simple random walk on it.

Some open questions

- Extension to Potts model.

$$\mu(\underline{x}) = \frac{1}{Z_G(\beta, B)} \exp\left\{\beta \sum_{(i,j)\in E} \delta_{x_i,x_j} + B \sum_{i\in V} \delta_{x_i,1}\right\}.$$

Some open questions

- Large Deviation for the root magnetization of μ_n, Ising measure on G_n:

  For regular case exponential concentration,

$$\mathbb{P}\Big(\Big|\frac{1}{n}\sum_i x_i \pm m_*\Big| \leq \delta\Big) \geq \frac{1}{2} - e^{-nC(\delta)}.$$

Some open questions

- Relax expander condition:

  Theorem (2) does not hold for Erdős-Rényi graph sequence.

Thank you!