

Feynman integrals as A-hypergeometric functions

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ABSTRACT: We show that the Lee-Pomeransky parametric representation of Feynman integrals can be understood as a solution of a certain Gel'fand-Kapranov-Zelevinsky (GKZ) system. In order to define such GKZ system, we consider the polynomial obtained from the Symanzik polynomials $g = \mathcal{U} + \mathcal{F}$ as having indeterminate coefficients. Noncompact integration cycles can be determined from the coamoeba — the argument mapping — of the algebraic variety associated with g . In general, we add a deformation to g in order to define integrals of generic graphs as linear combinations of their canonical series. We evaluate several Feynman integrals with arbitrary non-integer powers in the propagators using the canonical series algorithm.

KEYWORDS: Scattering Amplitudes, Differential and Algebraic Geometry, Perturbative QCD

ARXIV EPRINT: [1907.00507](https://arxiv.org/abs/1907.00507)

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1 Introduction

The analytic evaluation of Feynman integrals in dimensional regularization is still one of the main challenges to compute higher order corrections to observables in collider experiments. Methods for evaluating Feynman integrals involve a good understanding of

their analytic properties. These have been important from the very beginning in order to develop techniques to evaluate them. Long time ago, some of these properties led to the recognition that Feynman integrals satisfy systems of differential equations analogous to those of hypergeometric functions.¹ The modern method to evaluate Feynman integrals is indeed based on differential equations. It is the combination of Integration by Parts (IBP) identities and differential equations [3–8]. The result is particularly simple [9] if the system of differential equations evaluates to combinations of Multiple Polylogarithms [10–13].

In favorable cases, Feynman integrals can be evaluated in terms of classical hypergeometric functions and their generalizations such as Appell, Horn, and Lauricella hypergeometric functions. Typically, these functions appear as infinite sums through the Mellin-Barnes representation of Feynman integrals (see e.g., [14–16]). Once evaluated in terms of hypergeometric functions, they can be utilized to recover the ϵ -expansion [17–20], although it is a nontrivial task when hypergeometric functions of many variables appear (see e.g., [21, 22]). More recently, the Mellin-Barnes representation has been used to find systems of differential equations without resorting to IBP identities [23], through the differential reduction method [17, 23–30], which is based on refs. [31, 32]. Using the Mellin-Barnes representation it has been shown that a large class of Feynman integrals can be expressed in terms of Horn-type functions [33]. The differential reduction approach has led to the application of techniques coming from D -module theory as has been recently shown in ref. [34]. This point of view is close to the one we will adopt in this paper.

A-hypergeometric functions were introduced by Gel'fand-Kapranov-Zelevinsky (GKZ) in 1990 [35] as a generalization of the well-known Appell, Lauricella, and Horn series. One important aspect of the theory is the study of polynomials with indeterminate coefficients associated with an integer matrix A [36]. This matrix and a vector of complex parameters furnish a system of partial differential equations (PDEs) known as a GKZ system. Solutions of these systems of PDEs are called A-hypergeometric functions. They can be represented as Euler-type integrals and hence these integrals define A-hypergeometric functions [35, 37–39]. On the other hand, series solutions can be computed by a generalization of the Frobenius method known as the *canonical series algorithm* due to Saito, Sturmfels, and Takayama [40]. GKZ systems and Feynman integrals have been object of recent interest to mathematicians, who have studied the relation among GKZ systems, Feynman integrals, and their regularization [41, 42]. Maximal cuts in this language have been studied in ref. [43].

In this paper, we will consider the parametric representation of Feynman integrals employed by Lee and Pomeransky to relate critical points of the sum of the Symanzik polynomials and the number of master integrals arising from IBPs [44]. This representation is based on the polynomial $g = \mathcal{U} + \mathcal{F}$, where \mathcal{U} and \mathcal{F} are the first and second Symanzik polynomials, respectively. We will show that the polynomial g defines a GKZ system, which is constructed by considering its coefficients to be indeterminate. We will use this information to obtain series expansions of the Euler integrals solutions using the Saito-Sturmfels-Takayama canonical series algorithm. Our definition of a GKZ system based on g

¹It is due to Regge the conjecture that Feynman integrals belong to a generalization hypergeometric functions [1] and hence that Feynman integrals satisfy analogous systems of differential equations. See ref. [2].

allows the evaluation of integrals with arbitrary noninteger powers in the propagators using computational algebra. The polynomial g may lead to a matrix of codimension 0. In such cases, we introduce a deformation of g to ensure a canonical series representation. Once the canonical series are computed and integration constants are obtained, Feynman integrals can be recovered at the end of the computation by taking the limit of the deformation going to zero and setting its coefficients to their kinematic values.

This paper is organized as follows: in section 2 we review the GKZ framework, present their solutions, and introduce the canonical series algorithm. We end this section with examples. In section 3 we introduce scalar Feynman integrals and furnish a GKZ system based on them suited for canonical series. We give examples at the end of section 3. We give our conclusions in section 4.

2 A-hypergeometric functions and their representations

In this section, we review basic aspects of the GKZ approach to hypergeometric functions. The main references are the book [40] and the lectures [45]. Reviews of the main concepts can also be found in refs. [46, 47].

This review contains four main parts. We first introduce polynomials with indeterminate coefficients (toric polynomials), which is one of the main ideas behind the GKZ approach. Polynomials with indeterminate coefficients lead to generalizations of discriminants, resultants, and determinants through the study of toric varieties² [36].

In the second part, we associate a system of partial differential equations (PDEs) to polynomials with indeterminate coefficients. We introduce Euler integral representations of solutions of GKZ systems. The system of PDEs may be formally defined as a holonomic ideal in a Weyl algebra D and it is the proper language for computational algebra purposes [40].

In the third part, we introduce series representations of solutions of GKZ systems and review the Saito-Sturmfels-Takayama algorithm to compute canonical series [40]. The fourth part contains examples of the methods introduced. A short example using the computer algebra system *Macaulay2* can be found in appendix B.

The connection with Feynman integrals will be made in section 3. The Saito-Sturmfels-Takayama algorithm will be our main tool to evaluate Feynman integrals in section 3.

2.1 Notation

Throughout this paper we will employ multi-index notation, i.e.,

$$z^\alpha := z_1^{\alpha_1} \dots z_N^{\alpha_N}, \quad c^\gamma := c_1^{\gamma_1} \dots c_n^{\gamma_n},$$

where $\alpha \in \mathbb{K}^N$, $\gamma \in \mathbb{K}^n$. For polynomials $b_1(z), \dots, b_M(z)$ in the variables $z = (z_1, \dots, z_N)$, the multi-index notation for products of polynomials reads

$$b(z)^\beta := b_1(z)^{\beta_1} b_2(z)^{\beta_2} \dots b_M(z)^{\beta_M},$$

²An algebraic toric variety is of the form \mathbb{C}_*^n , where $\mathbb{C}_* = \mathbb{C} \setminus \{0\}$.

where $\beta \in \mathbb{K}^M$. Typically we will have $\mathbb{K} = \mathbb{C}$. For $\beta = (1, \dots, 1)$, we simply write $b(z)^{(1, \dots, 1)} = b(z)$. In the case $M = 1$, we write $b_1(z)^{\beta_1} = b(z)^\beta$. We call polynomials with indeterminate coefficients *toric polynomials*³ and emphasize their dependence on their coefficients c by writing $b(c, z)$. The multi-index notation for differential operators reads $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, where $\partial_i = \partial/\partial c_i$. Euler operators are defined by

$$\theta_i := c_i \frac{\partial}{\partial c_i} = c_i \partial_i.$$

Integrals where toric polynomials are involved will be identified by a subscript, e.g., I_b indicates that the integral under consideration has a toric polynomial $b(c, z)$ on its integrand. Finally, the Pochhammer symbol is defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)}, \quad a \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}.$$

Useful identities following from this definition are summarized in appendix A.

2.2 Polynomials, varieties, and their coamoebas

Let us consider q Laurent polynomials in N variables of the form

$$b_i(z) = \sum_{j=1}^{n_i} c_{ij} z^{\alpha_{ij}}, \quad c_{ij} \in \mathbb{C}_*, \quad i = 1, \dots, q, \quad (2.1)$$

where $\mathbb{C}_* = \mathbb{C} \setminus \{0\}$, $\alpha_{ij} \in \mathbb{Z}^N$, and n_i is the length of the set of exponent vectors $A_i = \{\alpha_{i1}, \dots, \alpha_{ik}, \dots, \alpha_{in_i}\}$ associated with the polynomial $b_i(z)$. By abuse of notation, we denote by A_i the $N \times n_i$ configuration matrix of the exponent vectors of the i -th polynomial as

$$A_i = (\alpha_{i1} \cdots \alpha_{ik} \cdots \alpha_{in_i}), \quad \alpha_{ik} \in \mathbb{Z}^N, \quad (2.2)$$

where each (column) vector α_{ik} is associated with a monomial term $c_{ik} z^{\alpha_{ik}}$ in $b_i(z)$. Therefore $|A_i| = n_i$ is the total number of monomials (columns in A_i).⁴

Let us give an example. Suppose we have a single polynomial in N variables with n monomial terms $b^{\text{ex}}(z) = \sum_{j=1}^n c_j z^{\alpha_j}$, where we have labeled the coefficients simply by c_j , $j = 1, \dots, n$. The $N \times n$ configuration matrix of $b^{\text{ex}}(z)$ reads

$$A^{\text{ex}} = \left(\begin{array}{c|c|c|c|c} c_1 & c_2 & \cdots & c_{n-1} & c_n \\ \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,n-1} & \alpha_{1,n} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \alpha_{N,1} & \alpha_{N,2} & \cdots & \alpha_{N,n-1} & \alpha_{N,n} \end{array} \right) \begin{array}{l} z_1 \\ \vdots \\ z_N \end{array}. \quad (2.3)$$

³The reason of this nomenclature is that polynomials with indeterminate coefficients will be associated with an integer matrix, which can be interpreted as representing a toric variety.

⁴Here we are adopting a rather unusual route by defining first Laurent polynomials and then the matrices A_i . The reason behind this is that in Feynman integrals we first consider polynomials with determinate coefficients and then consider the indeterminate case. In the mathematics literature, typically one first considers a configuration matrix and associates a polynomial to it.

The monomial term related with the first column reads

$$c_1 z^{\alpha_1} = c_1 z_1^{\alpha_{1,1}} \cdots z_N^{\alpha_{N,1}}.$$

With the above identifications of the columns and rows, the arrangement of the rows and columns (terms) is irrelevant as they define the same polynomial.

Let us now consider the product

$$b(z) := b_1(z) \cdots b_q(z), \tag{2.4}$$

where each polynomial and its configuration A_i are taken independently, in other words we do not expand $b(z)$. Expanding the polynomials would lead to a single polynomial and hence it is a special case of the above. Let $n := n_1 + \cdots + n_q$ be the total number of monomials and let us define the $(N + q) \times n$ matrix

$$A := \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ A_1 & A_2 & \dots & A_q \end{pmatrix} \tag{2.5}$$

associated with $b(z)$. Here $0 = (0, \dots, 0)$ and $1 = (1, \dots, 1)$ are row vectors of length $|A_i|$. We define the codimension of A as

$$\text{co}(A) := n - N - q. \tag{2.6}$$

The matrix A is the main object of study of the GKZ approach to hypergeometric functions. This definition allows us to consider integrals and PDEs later. This matrix is interpreted as representing a toric variety.⁵

For later purpose we introduce the Newton polytopes of $b_i(z)$ following ref. [38]. For each $b_i(z)$, the Newton polytope is the convex hull $\Delta_{b_i} = \text{conv}(\alpha_{i1}, \dots, \alpha_{in_i})$ in \mathbb{R}^N . Like any other polytope, we can represent Δ_{b_i} as the intersection of a finite number of halfspaces:⁶

$$\Delta_{b_i} = \bigcap_{j=1}^{M_i} \{ \sigma \in \mathbb{R}^N : \mu_j^i \cdot \sigma \geq \nu_j^i \}, \tag{2.7}$$

where $\mu_j^i \in \mathbb{Z}^N$ are primitive integer vectors in the inward normal direction of the facets of Δ_{b_i} , and the $\nu_j^i \in \mathbb{Z}$ are integers (see figure 1). Here the product $X \cdot Y$ stands for the standard scalar product in \mathbb{R}^N . The polytope of $b(z)$ is the Minkowski sum⁷

$$\Delta_b = \Delta_{b_1 b_2 \cdots b_q} = \Delta_{b_1} + \cdots + \Delta_{b_q}. \tag{2.8}$$

⁵See ref. [48] for an introduction to toric ideals.

⁶Any polytope has a vertex representation and a half space representation. Going from one representation to another involves conversion algorithms that are beyond the scope of the present work. The interested reader can consult e.g., ref. [49].

⁷For two polytopes P and Q in \mathbb{R}^N , their Minkowski sum $P + Q$ is the set of all vectors $p + q$, such that $p \in P, q \in Q$ [49].

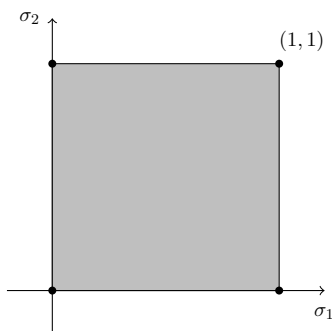


Figure 1. Polytope of $b(z) = 1 + z_1 + z_2 + cz_1z_2$. Written as intersection of half planes, we have $\Delta_b = \{\sigma_1 \geq 0\} \cap \{\sigma_2 \geq 0\} \cap \{-\sigma_1 \geq -1\} \cap \{-\sigma_2 \geq -1\}$.

Let us now review the concept of a coamoeba of a variety. Consider the ideal I generated by some polynomials $f_1, \dots, f_j \in \mathbb{C}[z_1, \dots, z_N]$. The zero set of the ideal $I = \langle f_1, \dots, f_j \rangle$ defines the algebraic variety

$$\mathcal{V}(I) := \{z \in \mathbb{C}^N : f_i(z) = 0, \text{ for all } f_i \in I\}. \tag{2.9}$$

Let us consider the case of a single polynomial f . The amoeba \mathcal{A}_f of the algebraic variety $\mathcal{V}(f)$ is the image of $\mathcal{V}(f)$ under the log mapping

$$\mathcal{A}_f := \text{Log}(\mathcal{V}(f)), \tag{2.10}$$

where $\text{Log}(z) = (\log|z_1|, \dots, \log|z_N|)$. Amoebas were introduced by GKZ in ref. [36]. Similarly, the coamoeba of $\mathcal{V}(f)$ is the image of $\mathcal{V}(f)$ under the argument mapping

$$\mathcal{A}'_f := \text{Arg}(\mathcal{V}(f)), \tag{2.11}$$

where $\text{Arg}(z) = (\arg(z_1), \dots, \arg(z_N))$. Here $\arg(z)$ is defined as usual. For example, the real positive line is given by $\mathbb{R}_+ = \text{Arg}^{-1}(0)$.

Unlike the log mapping, the argument mapping is multivalued and we can think of it as a multiple periodic subset of \mathbb{R}^N or equivalently it can be viewed as the N -dimensional algebraic torus $\mathbb{T}^N = (\mathbb{R}/2\pi\mathbb{Z})^N$ [38]. Coamoebas were introduced by Passare in 2004.⁸ Our motivation to introduce coamoebas is that they define noncompact integration cycles for Euler-Mellin integral representations of A-hypergeometric functions. These cycles are particularly useful in generic cases where polynomials may vanish on the integration region. In many cases, coamoebas are difficult to study analytically and one has to consider a rough version, which is called the lopsided coamoeba [52–55]. The relation between A-hypergeometric functions and coamoebas has been studied in refs. [37, 38, 55, 56]. See refs. [46, 52] for an introduction to coamoebas.

⁸Coamoebas have appeared later in the physics literature under the name of algae in the context of dimer models, see e.g., [50, 51].

2.3 GKZ systems and A-hypergeometric functions

We denote by $H_A(\kappa)$ the GKZ system associated with a matrix A and a vector of parameters κ . It is defined by the following data:

1. A $(N + q) \times n$ matrix A such that the vector $(1, \dots, 1)$ lies in its row span. Hence, there is a vector $\xi \in \mathbb{Z}^{N+q}$, such that $\xi A = (1, \dots, 1)$. By definition this matrix is obtained from

$$b(z) = b_1(z) \cdots b_q(z). \tag{2.12}$$

2. A system of partial differential equations (PDEs) associated with A . Let $u, v \in \mathbb{N}^n$ and consider

$$\left(\partial^u - \partial^v\right)F(c) = 0, \quad \text{where } Au = Av, \tag{2.13}$$

$$\left(\sum_{j=1}^n a_{ij}\theta_j - \kappa_i\right)F(c) = 0, \quad i = 1, \dots, N + q, \tag{2.14}$$

where a_{ij} denotes the components of A . Recall that $\theta_j = c_j \partial / \partial c_j$ and $\partial^u = \partial_1^{u_1} \cdots \partial_n^{u_n}$.

3. A vector of parameters $\kappa = (\kappa_1, \dots, \kappa_{N+q})$, $\kappa_i \in \mathbb{K}$.

A holomorphic function $F(c)$ or formal series is called A -hypergeometric if it satisfies the above system of PDEs.

GKZ systems can be rigorously defined in the language of holonomic ideals in the ring of differential operators with polynomial coefficients — the so called Weyl algebra $D = \mathbb{K}\langle c_1, \dots, c_n, \partial_1, \dots, \partial_n \rangle$ modulo commutation rules. In this sense, the *toric ideal* associated with A is defined by

$$I_A := \langle \partial^u - \partial^v : Au = Av, \quad u, v \in \mathbb{N}^n \rangle \subset \mathbb{K}[\partial_1, \dots, \partial_n], \tag{2.15}$$

where $\mathbb{K}[\partial_1, \dots, \partial_n]$ is a commutative polynomial ring. In addition, we construct the ideal generated by the column vectors κ^T and $\theta = (\theta_1, \dots, \theta_n)^T$. This ideal is given by

$$\langle A\theta - \kappa^T \rangle \subset \mathbb{K}[\theta_1, \dots, \theta_n], \tag{2.16}$$

where each generator of the ideal has the form $\sum_{j=1}^n a_{ij}\theta_j - \kappa_i$. The GKZ system $H_A(\kappa)$ denotes the left ideal on the Weyl algebra D generated by I_A and $\langle A\theta - \kappa^T \rangle$. In this language a holomorphic function $F(c)$ or formal series is called A -hypergeometric of degree κ if $H_A(\kappa) \bullet F(c) = 0$, where \bullet denotes the action of the Weyl algebra on polynomials [40]. The language of holonomic ideals and D -modules is the appropriate one to treat the problem using computational algebra.

Let us denote by $\text{vol}(A)$ the normalized volume — w.r.t. the volume of the standard simplex which is equal to 1 — of the convex hull of A .⁹ Then, for generic parameters¹⁰ κ ,

⁹For the case $q = 1$, this normalization implies that $\text{vol}(A) = N! \text{vol}(\Delta_b)$ with Δ_b defined in section 2.2.

¹⁰Here, generic parameters $\kappa \in \mathbb{K}^{N+q}$ are to be understood as ranging over nonempty open algebraic (Zariski) subsets of \mathbb{K}^{N+q} .

the rank of the system satisfies the inequality (Theorem 3.5.1 in [40])

$$\text{rank}(H_A(\kappa)) \geq \text{vol}(A), \tag{2.17}$$

which corresponds to the dimension of the solution space. This number can also be computed from the degree of the toric ideal I_A .

For generic parameters $\text{rank}(H_A(\kappa)) = \text{vol}(A)$. In cases where the parameters have certain special values (nongeneric) the rank of the system can jump and we have $\text{rank}(H_A(\kappa)) > \text{vol}(A)$ (example 4.2.7 in ref. [40]).

2.3.1 Euler-type integral solutions

Solutions of GKZ systems have representations as Euler-type integrals. Accordingly, we call them A -hypergeometric functions. They can be constructed by taking the vector of parameters

$$\kappa = (-\beta, -\alpha), \quad \beta \in \mathbb{C}^q, \quad \alpha \in \mathbb{C}^N, \tag{2.18}$$

and then write the integral associated with A as follows:

$$I_b(\kappa) = \int_{\Omega} \frac{z^\alpha}{b(c, z)^\beta} d\eta_N, \quad d\eta_N = \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \dots \wedge \frac{dz_N}{z_N}, \tag{2.19}$$

where the integration cycle is such that $\Omega \subset (\mathbb{C}_*)^N \setminus \mathcal{V}(b)$. It is usually assumed that the cycles are compact [35].

In ref. [38], Berkesh, Forsgård, and Passare (BFP) constructed explicit noncompact cycles for these type integrals. In order to reach the above type of integral, BFP proceed in three steps. *First*, we consider that $b(z)$ does not vanish in the positive orthant and consider the *Euler-Mellin* integral

$$I(\kappa) = \int_{\mathbb{R}_+^N} \frac{z^\alpha}{b(z)^\beta} d\eta_N = \int_{\mathbb{R}_+^N} \frac{e^{(\alpha, x)}}{b(e^x)^\beta} dx, \quad dx = dx_1 \wedge \dots \wedge dx_N, \tag{2.20}$$

where in this case polynomial coefficients are fixed. BFP showed that if the polynomials $b(z)$ are nonvanishing,¹¹ then eq. (2.20) converges and defines an analytic function with parameters $\kappa = (-\beta, -\alpha)$ on the tube domain

$$\{(\alpha, \beta) \in \mathbb{C}^{N+q} \mid \tau := \text{Re } \beta \in \mathbb{R}_+^q, \quad \sigma := \text{Re } \alpha \in \text{int}(\tau \Delta_b)\}, \tag{2.21}$$

where $\text{int}(\tau \Delta_b)$ is the interior of the weighted Minkowski sum of the Newton polytopes of b_j weighted by τ , i.e., $\tau \Delta_b = \sum_{j=1}^q \tau_j \Delta_{b_j}$. The *second* step is to consider the less favorable case where the polynomials $b(z)$ vanish on the positive orthant. Here, we can take a connected component Θ of $\mathbb{R}^N \setminus \overline{\mathcal{A}'_b}$, where $\overline{\mathcal{A}'_b}$ denotes the closure of the coamoeba of b and consider the integral

$$I(\kappa) = \int_{\text{Arg}^{-1}\theta} \frac{z^\alpha}{b(z)^\beta} d\eta_N = \int_{\mathbb{R}^N} \frac{e^{\alpha \cdot (x+i\theta)}}{b(e^{x+i\theta})^\beta} dx, \tag{2.22}$$

¹¹Technically, these polynomials should be completely nonvanishing in the sense of BFP i.e., not vanishing on the faces of the polytope of $b(z)$. See definition 2.1 of ref. [38]. See also [46].

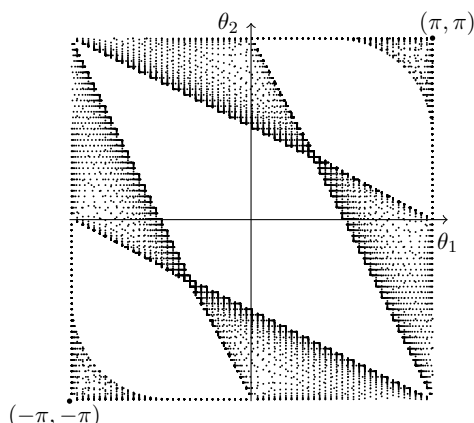


Figure 2. Coamoeba (shaded) of $b = 1 + z_1 z_2 + z_1 z_2^2 + z_1^2 z_2^2 + c_4 z_1^2 z_2$ drawn in the fundamental domain $[-\pi, \pi] \times [-\pi, \pi]$ of \mathbb{T}^2 in \mathbb{R}^2 . This case corresponds to $c = (1, 1, 1, 1, c_4)$, with c_4 near 1. Here, we have three connected components, one of which contains $(0, 0)$ (center of the figure).

where $\theta \in \Theta$ is a representative of the connected component of complement of coamoeba of $b(z)$. This is essentially a change of variables and a slight perturbation of θ does not impact the result of the integral. We write an analogue of Theorem 2.4 in ref. [38], where the proof can be found.

Theorem (Berkesh, Forsgård, and Passare). *For nonvanishing polynomials, b_1, \dots, b_q in $\text{Arg}^{-1}\theta$ the integral (2.22) admits a meromorphic continuation of the form*

$$I(\kappa) = \Phi_b^\Theta(\alpha, \beta) \prod_{k=1}^M \Gamma(\mu_k \cdot \alpha - \nu_k \cdot \beta), \quad (2.23)$$

where $\Phi_b^\Theta(\alpha, \beta)$ is an entire function and Θ is a connected component. μ_k, ν_k can be recovered from the Newton polytope $\Delta_{b_1 \dots b_q}$ (see eq. (2.7)).

The *third* step is the transition to A-hypergeometric functions by promoting the coefficients of $b(z)$ in eq. (2.22) to indeterminate, therefore we consider them as variables. The integral¹²

$$I_b(\kappa) = \int_{\text{Arg}^{-1}\theta} \frac{z^\alpha}{b(c, z)^\beta} d\eta_N \quad (2.24)$$

is a representation of an A-hypergeometric function (Theorem 4.2 in [38]). For generic parameters κ , eq. (2.24) provides a basis of solutions of $H_A(\kappa)$, where each integral is evaluated on a representative of Θ for each connected component of $\mathbb{R}^N \setminus \overline{\mathcal{A}}'_b$ (see figure 2¹³).

2.4 A-hypergeometric canonical series

Saito, Sturmfels, and Takayama (SST) generalized the Frobenius method using Gröbner deformations in order to deal with regular holonomic systems (chapters 2 and 3 of [40]).

¹²Here $c \in \mathbb{C}^n \setminus \Sigma_A$, where Σ_A is the singular locus of all A-hypergeometric functions.

¹³We thank Jens Forsgård for providing his MATHEMATICA package to draw coamoebas and lopsided coamoebas.

Roughly speaking, it consists on taking certain initial ideals of I_A (eq. (2.15)) with respect to a weight $w \in \mathbb{R}^n$ and generate the series solutions from it. The role of the *indicial equation* in the Frobenius method is played by an *indicial ideal* $\text{ind}_w(I_A)$ with respect to w , together with the ideal $\langle A\theta - \kappa^T \rangle$. As in the Frobenius method, the problem consist of finding the roots γ of those ideals and generate the coefficients of the series. Series thus obtained belong to the Nilsson ring, i.e, series of the form

$$f = \sum_{\alpha, \beta} k_{\alpha\beta} c^\alpha \log(c)^\beta. \tag{2.25}$$

In this paper, we will be chiefly interested in the case where the resulting series are *logarithm-free*, i.e., where $\beta = 0$ in the above equation. Let us turn our attention to these cases. Let

$$\mathcal{L} := \{u \in \mathbb{Z}^n : Au = 0\} \tag{2.26}$$

be a lattice of rank m and let $\kappa^T = A\gamma^T$.¹⁴ For $u \in \mathcal{L}$ we can write $u = u_+ - u_-$, where $u_\pm \in \mathbb{N}^n$ have disjoint support. Now, for $\gamma \in \mathbb{C}^n$ we define the following quantities expressed as falling factorials

$$[\gamma]_{u_-} := \prod_{i:u_i < 0} \prod_{j=1}^{-u_i} (\gamma_i - j + 1) = \prod_{i:u_i < 0} (-1)^{-u_i} (\gamma_i)_{-u_i}, \tag{2.27}$$

$$[\gamma + u]_{u_+} := \prod_{i:u_i > 0} \prod_{j=1}^{u_i} (\gamma_i + u_i - j + 1) = \prod_{i:u_i > 0} \prod_{j=1}^{u_i} (\gamma_i + j) = \prod_{i:u_i > 0} (\gamma_i + 1)_{u_i}, \tag{2.28}$$

where $(a)_x$ are Pochhammer symbols. Then for $\gamma \in \mathbb{C}^n$, such that no element in γ is a nonnegative integer, the series

$$\phi_\gamma := \sum_{u \in \mathcal{L}} \frac{[\gamma]_{u_-}}{[\gamma + u]_{u_+}} c^{(\gamma+u)} \tag{2.29}$$

is a solution of $H_A(\kappa)$ (see proposition 3.4.1 and Theorem 3.4.2 in ref. [40]). When the vectors γ contain negative integers, we define the negative support of γ as

$$\overline{\text{supp}}(\gamma) = \{i \in \{1, \dots, n\} : \gamma_i \in \mathbb{Z}_{<0}\}. \tag{2.30}$$

In those cases, we consider the lattice

$$\mathcal{N}_\gamma := \{u \in \mathcal{L} | \overline{\text{supp}}(\gamma + u) = \overline{\text{supp}}(\gamma)\} \tag{2.31}$$

and perform the sum over \mathcal{N}_γ in eq. (2.29). In this paper, we will assume that the vectors κ are generic and thus none of the roots γ are negative integers.¹⁵ This implies that $\overline{\text{supp}}(\gamma) = \emptyset$ and therefore the sum runs over $u \in \mathcal{L}$.

The roots γ can be obtained by finding the roots of an ideal known as *fake indicial ideal*. Accordingly, the roots are called *fake exponents* and we give the algorithm to compute

¹⁴Notice that we will not be interested in general vectors $\gamma \in \mathbb{C}^n$ but only those which are the roots of an ideal of $H_A(\kappa)$ with respect to some weight vector w . See Algorithm below.

¹⁵Corollary 3.4.3 in ref. [40].

them below. Series thus obtained are called *canonical series*. The series obtained from this algorithm have a common domain of convergence $\mathcal{U}_w \in \mathbb{C}^n$, which is characterized by a weight vector w . For generic roots γ , canonical series provide a basis of holomorphic solutions of $H_A(\kappa)$ in \mathcal{U}_w (for a description of this domain see Theorem 2.5.16 in ref. [40] and Theorem 3.19 in ref. [45]).

The relation with Euler integrals goes as follows. Suppose we find r different roots. The general solution of the integral (2.19) is given by a linear combination of its canonical series [40], i.e.,

$$I_b(\kappa) = K_1\phi_1 + \dots + K_r\phi_r, \tag{2.32}$$

where the canonical series ϕ_r can be computed independently of the cycle Ω . The integration cycle plays a role in the computation of the integration constants K_i . We discuss how to obtain integrations constants for the particular case of Feynman integrals in the next section.

We proceed with the canonical series algorithm to compute fake exponents γ . A comprehensive review of this algorithm can be found in ref. [45]. We start with some definitions.

Definitions.

Toric ideal. Let D be a Weyl algebra over \mathbb{K} , which is a free (noncommutative) associative \mathbb{K} -algebra generated by $\mathbb{K}\langle c_1, \dots, c_n, \partial_1, \dots, \partial_n \rangle$ modulo the commutation rules

$$c_i c_j = c_j c_i, \quad \partial_i \partial_j = \partial_j \partial_i, \quad \partial_i c_j = \delta_{ij}. \tag{2.33}$$

Let A be as in eq. (2.5), then

$$I_A := \langle \partial^u - \partial^v : Au = Av, \quad u, v \in \mathbb{N}^n \rangle \tag{2.34}$$

generates the *toric ideal* associated with A . Toric ideals can be computed using reduced Gröbner bases [57].

Initial ideal. Let $w \in \mathbb{R}^n$ be a weight vector and let I_A be a toric ideal. For w generic, the ideal $\text{in}_w(I_A)$ is a monomial ideal generated by the leading terms of I_A with respect to the partial ordering \preceq_w .

Standard pairs. Let $R = \mathbb{K}[\partial_1, \dots, \partial_n]$ and let I be a monomial ideal in R . Furthermore, let ∂^α be a monomial and $F \subseteq \{1, \dots, n\}$, where $\alpha \in \mathbb{N}^n$. A standard pair of a monomial ideal I is a pair (∂^α, F) satisfying three conditions:

1. $\alpha_i = 0$ for all $i \in F$,
2. for all choices of integers $\beta_j \geq 0$, the monomial $\partial^\alpha \prod_{j \in F} \partial_j^{\beta_j} \notin I$,
3. for all $l \notin F$, there exist $\beta_j \geq 0$ such that $\partial^\alpha \partial_l^{\beta_l} \prod_{j \in F} \partial_j^{\beta_j} \in I$.

Let us denote by $\mathcal{S}(I)$ the set of all standard pairs of I . The decomposition of I into irreducible monomial ideals can be obtained from the identity.

$$I = \bigcap_{(\partial^\alpha, F) \in \mathcal{S}(I)} \langle \partial_i^{\alpha_i+1} : i \in F \rangle. \tag{2.35}$$

2.4.1 Algorithm (Saito-Sturmfels-Takayama)

In this algorithm we will set $\mathbb{K} = \mathbb{C}$.

Input: matrix A , weight vector w , and complex parameters κ .

Output: roots of the fake indicial ideal $\text{fin}_w(H_A(\kappa))$.

1. Compute the toric ideal associated with A

$$I_A = \langle \partial^u - \partial^v : Au = Av, \quad u, v \in \mathbb{N}^n \rangle. \quad (2.36)$$

Notice that this is an ideal in the commutative polynomial ring $\mathbb{C}[\partial_1, \dots, \partial_n]$. This ideal is the input to compute the combinatorial object of standard pairs \mathcal{S} and the initial ideal with respect to w .

2. Let $w \in \mathbb{R}^n$ be a generic weight vector. Compute the initial ideal $\text{in}_w(I_A)$ with respect to w and obtain its standard pairs $\mathcal{S}(\text{in}_w(I_A))$. Standard pairs are combinatorial objects which tells us the types of solutions.

3. Use the standard pairs to construct the indicial ideal

$$\text{ind}_w(I_A) = \bigcap_{(\partial^a, F) \in \mathcal{S}(\text{in}_w(I_A))} \langle (\theta_j - a_j), j \notin F \rangle \subset \mathbb{C}[\theta_1, \theta_2, \dots, \theta_n], \quad (2.37)$$

where $\theta_i = c_i \partial_i$.

4. Write the ideal $\langle A\theta - \kappa^T \rangle \subset \mathbb{C}[\theta_1, \theta_2, \dots, \theta_n]$.

5. The fake indicial ideal with respect to w is given by

$$\text{fin}_w(H_A(\kappa)) := \text{ind}_w(I_A) + \langle A\theta - \kappa^T \rangle. \quad (2.38)$$

6. Compute the roots of $\text{fin}_w(H_A(\kappa))$. These are called fake exponents and we denote them by γ .

The canonical series are then given by eq. (2.29). In order to write solutions, we compute the kernel of A to obtain the generating lattice

$$\mathcal{L} := \ker_{\mathbb{Z}} A. \quad (2.39)$$

Finally, we set $[\gamma]_{u_-} = 0$ whenever $w \cdot u < 0$ (see Lemma 3.17 in ref. [45]). Clearly, this property allows us to choose certain weights w that simplify the sums.

We can use *Macaulay2* to perform the above operations. An example is provided in appendix B.

2.5 Examples

In order to illustrate the methods, we will take two representations of the Gauss hypergeometric function.

2.5.1 Double integral Gauss hypergeometric function

The first part of this example follows [37], where the reader can find details. Suppose we are interested in the following integral

$$I(\alpha) = \int_{\mathbb{R}_+^2} \frac{z_1^{\alpha_1} z_2^{\alpha_2}}{(1 + z_1 + z_2 + cz_1 z_2)^\beta} \frac{dz_1 dz_2}{z_1 z_2}. \quad (2.40)$$

From the point of view we are adopting, we should think on the more general situation where the polynomial in the denominator has indeterminate coefficients, i.e.,

$$b(c, z) = c_1 + c_2 z_1 + c_3 z_2 + c_4 z_1 z_2. \quad (2.41)$$

Its associated configuration matrix reads

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \quad (2.42)$$

Then we consider the integral

$$I_b(-\beta, -\alpha) = \int_{\Omega} \frac{z_1^{\alpha_1} z_2^{\alpha_2}}{(c_1 + c_2 z_1 + c_3 z_2 + c_4 z_1 z_2)^\beta} \frac{dz_1 dz_2}{z_1 z_2}, \quad (2.43)$$

where Ω is a suitable cycle of integration. Following [37] the Newton polytope of $b(c, z)$ can be represented by the inequalities (see eq. (2.7))

$$\Delta_b = \{\sigma_1 \geq 0\} \cap \{\sigma_2 \geq 0\} \cap \{-\sigma_1 \geq -1\} \cap \{-\sigma_2 \geq -1\}, \quad (2.44)$$

and hence, we can read off the vectors μ_i and numbers ν_i

$$\begin{aligned} \mu_1 &= (1, 0), & \mu_2 &= (0, 1), & \mu_3 &= (-1, 0), & \mu_4 &= (0, -1), \\ \nu_1 &= 0, & \nu_2 &= 0, & \nu_3 &= -1, & \nu_4 &= -1. \end{aligned} \quad (2.45)$$

Integration cycles can be obtained taking $\theta = (\arg(c_1/c_2), \arg(c_1/c_3))$. The BFP Theorem states that

$$I_b(-\beta, -\alpha) = \Phi_b^\Theta(\alpha, \beta, c) \Gamma(\alpha_1) \Gamma(\alpha_2) \Gamma(\beta - \alpha_1) \Gamma(\beta - \alpha_2), \quad (2.46)$$

where $\Theta \in \mathbb{T}^2 \setminus \overline{\mathcal{A}'_b}$. Taking the representative $\theta \in \Theta$, the entire function reads

$$\Phi_b^\Theta(\alpha, \beta, c) = \frac{c_1^{\alpha_1 + \alpha_2 - \beta} c_2^{-\alpha_2} c_3^{-\alpha_3}}{\Gamma(\beta)^2} {}_2F_1(\alpha_1, \alpha_2, \beta; 1 - \frac{c_1 c_4}{c_2 c_3}). \quad (2.47)$$

Taking the point $c' = (1, 1, 1, c)$, we have the coamoeba of $b(z)$ shown in figure 3, where we see that $(0, 0) \notin \overline{\mathcal{A}'_b}$. There is one connected component of $\mathbb{R}^2 \setminus \overline{\mathcal{A}'_b}$ at this point, therefore we have a single solution of $H_A(-\beta, -\alpha)$. Notice that at c' , we have $\Omega = \text{Arg}^{-1}\theta = \text{Arg}^{-1}(\arg(c_1/c_2), \arg(c_1/c_3)) = \text{Arg}^{-1}(0, 0) = (0, \infty)^2$. Hence, at c' we have

$$I(\alpha) = \frac{\Gamma(\alpha_1) \Gamma(\alpha_2) \Gamma(\beta - \alpha_1) \Gamma(\beta - \alpha_2)}{\Gamma(\beta)^2} {}_2F_1(\alpha_1, \alpha_2, \beta; 1 - c). \quad (2.48)$$

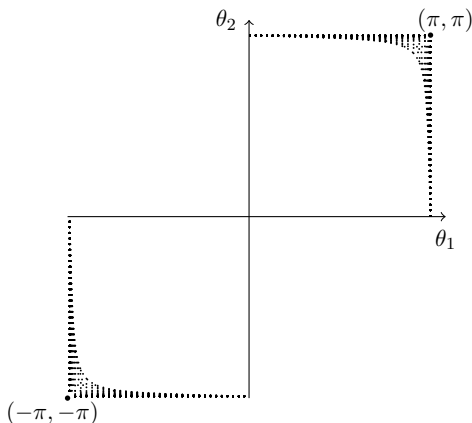


Figure 3. A connected component of the coamoeba (shaded) of $b(z) = 1 + z_1 + z_2 + cz_1z_2$ drawn in the fundamental domain $[-\pi, \pi] \times [-\pi, \pi]$ of \mathbb{T}^2 in \mathbb{R}^2 . The coefficient c is near 1.

It is instructive to recover this result from the canonical series algorithm. Let $w = (0, 1, 1, 1)$,¹⁶ and $\kappa = (-\beta, -\alpha_1, -\alpha_2)$. The toric ideal associated with \mathbf{A} reads

$$I_{\mathbf{A}} = \langle \partial_2 \partial_3 - \partial_1 \partial_4 \rangle. \tag{2.49}$$

We have $\text{in}_w(I_{\mathbf{A}}) = \langle \partial_2 \partial_3 \rangle$, which we use to obtain its standard pairs

$$\mathcal{S}(\text{in}_w(I_{\mathbf{A}})) = \{(1, \{1, 3, 4\}), (1, \{1, 2, 4\})\}, \tag{2.50}$$

therefore

$$\text{ind}_w(I_{\mathbf{A}}) = \langle \theta_2 \rangle \cap \langle \theta_3 \rangle. \tag{2.51}$$

Thus, we obtain

$$\text{fin}_w(H_{\mathbf{A}}(\kappa)) = \langle \theta_2 \theta_3, \theta_1 + \theta_2 + \theta_3 + \theta_4 + \beta, \theta_2 + \theta_4 + \alpha_1, \theta_3 + \theta_4 + \alpha_2 \rangle. \tag{2.52}$$

Computing its roots leads to

$$\{\gamma_i\} = \{(\alpha_1 - \beta, \alpha_2 - \alpha_1, 0, -\alpha_2), (\alpha_2 - \beta, 0, \alpha_1 - \alpha_2, -\alpha_1)\}. \tag{2.53}$$

In addition, computing $\ker(\mathbf{A})$ we have $\mathcal{L} = \mathbb{Z}(1, -1, -1, 1)$. Let $u = n(1, -1, -1, 1)$, then $u \cdot w = -n$, which means that $[\gamma]_{u_-} = 0$ for $n > 1$. In order to start the sum from $n = 0$, we set $u = n(-1, 1, 1, -1)$ such that $[\gamma]_{u_-} = 0$ for $n < 0$. We then write

$$u = u_+ - u_- = (0, n, n, 0) - (n, 0, 0, n). \tag{2.54}$$

Let us work out the case of γ_1 . Eq. (2.29) gives

$$\phi_1 = \sum_{n=0}^{\infty} \frac{[\gamma_1]_{u_-}}{[\gamma_1 + u]_{u_+}} (c^{u+\gamma_1}) = c^{\gamma_1} \sum_{n=0}^{\infty} \frac{[(\alpha_1 - \beta, \alpha_2 - \alpha_1, 0, -\alpha_2)]_{u_-}}{[(\alpha_1 - \beta, \alpha_2 - \alpha_1, 0, -\alpha_2) + n(-1, 1, 1, -1)]_{u_+}} (c^u), \tag{2.55}$$

¹⁶Choices of w are tied with the property that for $w \cdot u < 0$ we have $[\gamma]_{u_-} = 0$. See Algorithm in section 2.4.1.

hence

$$\phi_1 = c^{\gamma_1} \sum_{n=0}^{\infty} \frac{(\beta - \alpha_1)_n (\alpha_2)_n}{(\alpha_2 - \alpha_1 + 1)_n (1)_n} \left(\frac{c_2 c_3}{c_1 c_4} \right)^n = c^{\gamma_1} {}_2F_1(\beta - \alpha_1, \alpha_2; \alpha_2 - \alpha_1 + 1; (c_2 c_3)/(c_1 c_4)), \quad (2.56)$$

where we have used the series representation of the Gauss hypergeometric function (A.3). Similarly, for γ_2 we have

$$\phi_2 = c^{\gamma_2} \sum_{n=0}^{\infty} \frac{(\beta - \alpha_2)_n (\alpha_1)_n}{(\alpha_1 - \alpha_2 + 1)_n (1)_n} \left(\frac{c_2 c_3}{c_1 c_4} \right)^n = c^{\gamma_2} {}_2F_1(\beta - \alpha_2, \alpha_1; \alpha_1 - \alpha_2 + 1; (c_2 c_3)/(c_1 c_4)). \quad (2.57)$$

Therefore, the solution as linear combination of canonical series reads

$$I_b(-\beta, -\alpha) = K_1 c^{\gamma_1} {}_2F_1(\beta - \alpha_1, \alpha_2; \alpha_2 - \alpha_1 + 1; (c_2 c_3)/(c_1 c_4)) + K_2 c^{\gamma_2} {}_2F_1(\beta - \alpha_2, \alpha_1; \alpha_1 - \alpha_2 + 1; (c_2 c_3)/(c_1 c_4)). \quad (2.58)$$

We proceed to compute the integration constants. We take the noncompact cycle $\Omega = \mathbb{R}_+^2$. From the roots, we observe that if $c_3 = 0$, then ϕ_2 vanishes and if $c_2 = 0$, ϕ_1 vanishes. Hence we make

$$\int_{\mathbb{R}_+^2} \frac{z_1^{\alpha_1} z_2^{\alpha_2}}{(c_1 + c_2 z_1 + c_4 z_1 z_2)} \frac{dz_1 dz_2}{z_1 z_2} = (K_1 \phi_1 + K_2 \phi_2) |_{c_3 \rightarrow 0}, \quad (2.59)$$

which leads to

$$K_1 = \frac{\Gamma(\alpha_2) \Gamma(\alpha_1 - \alpha_2) \Gamma(\beta - \alpha_1)}{\Gamma(\beta)}. \quad (2.60)$$

Similarly, we obtain

$$K_2 = \frac{\Gamma(\alpha_1) \Gamma(\alpha_2 - \alpha_1) \Gamma(\beta - \alpha_2)}{\Gamma(\beta)}. \quad (2.61)$$

Let us now recover the original integrals by setting $c_1 = c_2 = c_3 = 1$, $c_4 = c$ in eq. (2.58). We have

$$I(\alpha) = c^{-\alpha_2} \frac{\Gamma(\alpha_2) \Gamma(\alpha_1 - \alpha_2) \Gamma(\beta - \alpha_1)}{\Gamma(\beta)} {}_2F_1(\beta - \alpha_1, \alpha_2; \alpha_2 - \alpha_1 + 1; 1/c) + c^{-\alpha_1} \frac{\Gamma(\alpha_1) \Gamma(\alpha_2 - \alpha_1) \Gamma(\beta - \alpha_2)}{\Gamma(\beta)} {}_2F_1(\beta - \alpha_2, \alpha_1; \alpha_1 - \alpha_2 + 1; 1/c), \quad (2.62)$$

which equals eq. (2.46) after using the identity (A.9) with $z = 1 - c$.

2.5.2 Single integral Gauss hypergeometric function

Let us now study the univariate integral of the Gauss hypergeometric function. We have

$$I(-\beta_1, -\beta_2, -\alpha) = \int_{\mathbb{R}_+} \frac{z^\alpha}{(1+z)^{\beta_1} (1+c z)^{\beta_2}} \frac{dz}{z}. \quad (2.63)$$

In order to evaluate the integral we consider the (toric) polynomial

$$b(z)^\beta = (c_1 + c_2 z)^{\beta_1} (c_3 + c_4 z)^{\beta_2} \iff \mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad (2.64)$$

and its associated GKZ system. The integral under consideration reads

$$I_b(\kappa) = \int_{\Omega} \frac{z^\alpha}{(c_1 + c_2 z)^{\beta_1} (c_3 + c_4 z)^{\beta_2}} \frac{dz}{z}, \quad (2.65)$$

where $\kappa = (-\beta_1, -\beta_2, -\alpha)$. We have $\mathcal{L} = \mathbb{Z}(1, -1, -1, 1)$. Taking the weight vector $w = (0, 1, 1, 1)$, we obtain

$$\text{fin}_w(H_{\mathbf{A}}(\kappa)) = \langle \theta_2 \theta_3, \beta_1 + \theta_1 + \theta_2, \beta_2 + \theta_3 + \theta_4, \alpha + \theta_2 + \theta_4 \rangle, \quad (2.66)$$

$$\{\gamma_i\} = \{(-\beta_1, 0, \alpha - \beta_2, -\alpha), (\alpha - \beta_1 - \beta_2, \beta_2 - \alpha, 0, -\beta_2)\}. \quad (2.67)$$

Hence, we have the two series solutions

$$\phi_1 = c^{\gamma_1} {}_2F_1(\beta_1, \alpha; \alpha - \beta_2 + 1; \frac{c_2 c_3}{c_1 c_4}), \quad (2.68)$$

$$\phi_2 = c^{\gamma_2} {}_2F_1(\beta_1 + \beta_2 - \alpha, \beta_2; \beta_2 - \alpha + 1; \frac{c_2 c_3}{c_1 c_4}). \quad (2.69)$$

Specializing to our case, we have $c_1 = c_2 = c_3 = 1$ and $c_4 = c$, therefore the solution of the integral is given by

$$I(-\beta_1, -\beta_2, -\alpha) = K_1 c^{-\alpha} {}_2F_1(\beta_1, \alpha; \alpha - \beta_2 + 1; 1/c) + K_2 c^{-\beta_2} {}_2F_1(\beta_1 + \beta_2 - \alpha, \beta_2; \beta_2 - \alpha + 1; 1/c), \quad (2.70)$$

where the constants can be obtained by setting c_2 and c_3 to zero in eq. (2.65) choosing $\Omega = \mathbb{R}_+$. The final result reads

$$I(-\beta_1, -\beta_2, -\alpha) = \frac{\Gamma(\alpha)\Gamma(\beta_2 - \alpha)}{\Gamma(\beta_2)} c^{-\alpha} {}_2F_1(\beta_1, \alpha; \alpha - \beta_2 + 1; 1/c) + c^{-\beta_2} \frac{\Gamma(\alpha - \beta_2)\Gamma(\beta_1 + \beta_2 - \alpha)}{\Gamma(\beta_1)} {}_2F_1(\beta_1 + \beta_2 - \alpha, \beta_2; \beta_2 - \alpha + 1; 1/c), \quad (2.71)$$

which evaluates to (2.63) after using eq. (A.9) with $z = 1 - c$.

3 Feynman integrals as \mathbf{A} -hypergeometric functions

In this section we will interpret Feynman integrals as particular points of \mathbf{A} -hypergeometric functions. First, we will introduce the parametric representation based on $g = \mathcal{U} + \mathcal{F}$ used in ref. [44] by Lee and Pomeransky. Later, we will give our proposal for defining GKZ systems based on g . Examples will be presented at the end of this section.

3.1 Lee-Pomeransky representation of Feynman integrals

Let us consider a typical Feynman integral in Euclidean space in dimensional regularization. A L -loop integral with N propagators and E independent external momenta may be written as follows

$$I_F(\alpha) = \int_{\mathbb{R}^L} \left(\prod_{i=1}^L \frac{d^d k_i}{\pi^{d/2}} \right) \frac{1}{D_1^{\alpha_1} \dots D_N^{\alpha_N}}, \quad (3.1)$$

where the inverse propagators are of the form

$$D_i = (M_i)^{rs} k_r \cdot k_s + 2(Q_i)^{rs} k_r \cdot p_s + J_i. \quad (3.2)$$

The matrices, M_i , Q_i , and J_i have dimensions $L \times L$, $L \times E$, and 1×1 , respectively. Integration over loop momenta through Feynman parameters and a Mellin transform leads to the Lee-Pomeransky parametric representation [44]

$$I_F(\alpha) = \xi_{\Gamma_\alpha} \int_{\mathbb{R}_+^N} \left(\prod_{i=1}^N \frac{dz_i}{z_i} z_i^{\alpha_i} \right) \frac{1}{g(z)^{d/2}} = \xi_{\Gamma_\alpha} \int_{\mathbb{R}_+^N} \frac{z^\alpha}{g(z)^{d/2}} d\eta_N, \quad (3.3)$$

where we have used the multi-index notation in the second equality and $\mathbb{R}_+ = (0, \infty)$. The overall factor and the polynomial $g(z)$ are defined as follows:

$$\xi_{\Gamma_\alpha} := \frac{\Gamma(d/2)}{\Gamma((L+1)d/2 - \sum_{i=1}^N \alpha_i) \prod_{i=1}^N \Gamma(\alpha_i)}, \quad (3.4)$$

$$g(z) := \mathcal{U} + \mathcal{F}, \quad (3.5)$$

where \mathcal{U} and \mathcal{F} are the Symanzik polynomials. In order to compute them, we construct the matrices

$$M^{rs} = \sum_{i=1}^N z_i M_i^{rs}, \quad Q^r = \sum_{i=1}^N z_i Q_i^{rs} p_s, \quad J = \sum_{i=1}^N z_i J_i. \quad (3.6)$$

We then have

$$\mathcal{U} = \det(M), \quad \mathcal{F} = \det(M) \left(J - (M^{-1})^{ij} Q^i \cdot Q^j \right), \quad (3.7)$$

where \mathcal{F} is appropriately scaled in order to make it dimensionless. The polynomial \mathcal{U} is a homogeneous polynomial of degree L and the polynomial \mathcal{F} is homogeneous of degree $L+1$. Hence $g(z)$ is an inhomogeneous polynomial of degree $L+1$. In Euclidean kinematics, the Symanzik polynomials \mathcal{U} , \mathcal{F} are positive semi-definite functions of the Feynman parameters. These polynomials can also be obtained from the topology of the graphs and their properties are summarized in refs. [58–60].

3.2 Feynman integrals and canonical series

A suitable definition for computational algebra purposes would be to consider Feynman integrals as a linear combinations of their canonical series, whenever we can compute them. As they stand, Feynman integrals will not satisfy the system of PDEs associated with a GKZ system (eqs. (2.13)–(2.14)) because the polynomial $g(z)$ has fixed coefficients.

Accordingly, the first step in our construction will be to take $g(z)$ and consider its associated toric polynomial $g(c, z)$. We will demand that this polynomial furnish GKZ system in the sense of eq. (2.29). In particular, this means that $g(c, z)$ must lead to matrices of $\text{co}(\mathbf{A}) > 0$ in order to compute a generating lattice and follow the canonical series algorithm. Since we have a single polynomial, the codimension of the matrix \mathbf{A} associated with $g(c, z)$ is given by $\text{co}(\mathbf{A}) = n - N - 1$, where n is the number of monomial terms in $g(c, z)$ with common exponent vectors. Therefore, we will have $\text{co}(\mathbf{A}) = 0$ when the number of terms in $g(c, z)$ equals the number of rows in \mathbf{A} , or equivalently when $n = N + 1$. A typical example of this situation is the polynomial of the L -loop massless cantaloupe graph (figure 6). For instance, let us consider the case $L = 1$, i.e., the massless bubble graph (figure 4). The $g(z)$ polynomial of this graph is given by $g(z) = z_1 + z_2 + sz_1z_2$, therefore

$$g^{\text{bubble}}(c, z) = c_2z_1 + c_3z_2 + c_4z_1z_2 \iff \mathbf{A}^{\text{bubble}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}. \quad (3.8)$$

The codimension of $\mathbf{A}^{\text{bubble}}$ is zero and hence $\ker \mathbf{A} = \emptyset$. Therefore, we cannot use this matrix to represent a solution of a GKZ system as canonical series in the sense of eq. (2.29). However, in this simple example we know that the result of the massless bubble integral can be recovered from the one-mass bubble integral by taking the limit of the mass going to zero. Furthermore, as we will see in the examples, the matrix associated with the one-mass bubble integral is given by

$$\mathbf{A}^{\text{one-mass}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}, \quad (3.9)$$

which has $\text{co}(\mathbf{A}^{\text{one-mass}}) = 1$. Since the one-mass bubble evaluates to a Gauss hypergeometric function, we may consider the massless bubble as a special case of a ${}_2F_1(a, b, c; p^2/m^2)$ function, for some a, b, c depending on the powers of the propagators and the dimension. An equivalent alternative provided by GKZ systems is to deform the polynomial by adding an arbitrary constant $r(z) = c_1$ to $g^{\text{bubble}}(c, z)$ and consider instead

$$g_r(c, z) = c_1 + c_2z_1 + c_4z_2 + c_4z_1z_2 \iff \mathbf{A}^r = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad (3.10)$$

where the constant c_1 can be set to zero at the end of the computation. This matrix corresponds to the GKZ system of a Gauss hypergeometric function. We will work out explicitly this example in section 3.3.1. There, we will take the limit $c_1 \rightarrow 0$ of a Gauss

hypergeometric function as we would do for the mass in the case of a one-mass bubble. The choice of a deformation can be made systematically as we will see for the L -loop cantaloupe graph.

Therefore, in cases where $g(c, z)$ leads to a configuration matrix of $\text{co}(\mathbf{A}) = 0$, we will consider a deformation of $g(c, z)$ demanding that it defines a solution of a GKZ system in the sense of eq. (2.29), thus allowing us to construct canonical series solutions. Recall that $\text{deg}(\mathcal{U}) = L$ and $\text{deg}(\mathcal{F}) = L + 1$. We may choose any polynomial $r(z)$ of $\text{deg}(r) < L$ and define

$$g_r(c, z) := r(c, z) + \mathcal{U}(c) + \mathcal{F}(c), \tag{3.11}$$

with the requirement that the associated matrix has $\text{co}(\mathbf{A}) > 0$. Here $r(c, z)$ denotes the toric polynomial associated with $r(z)$. Similarly, $\mathcal{U}(c)$ and $\mathcal{F}(c)$ denote the Symanzik polynomials where the coefficients now are considered as variables. We have checked for the integral of a massless L -loop cantaloupe graph, up to 5-loop, that this definition leads to well behaved GKZ systems and that we can recover the original integral using canonical series.

The second step in our construction will be to consider Euler-type solutions and series solutions of the GKZ systems furnished with $g_r(c, z)$. Let us first define the class of Euler integrals associated with the Feynman integral (3.3). We will drop the overall factor ξ_{Γ_α} , which is a nonzero constant independent of the kinematics and instead consider

$$I_F(\alpha)/\xi_{\Gamma_\alpha} \mapsto I_{g_r}(-d/2, -\alpha) := \int_{\Omega} \frac{z^\alpha}{g_r(c, z)^{d/2}} d\eta_N, \tag{3.12}$$

which is a toric generalization of eq. (3.3), i.e., where the deformation has been introduced and the coefficients of $g_r(z)$ are considered as variables. Notice that at this point, the noncompact cycle Ω will not be in general \mathbb{R}_+^N and it will be determined by the coamoeba of $g_r(c, z)$ (section 2.3.1).

Let us first show that the class of integrals obtained from $g_r(c, z)$ is \mathbf{A} -hypergeometric.¹⁷ We have the following theorem, which is a specialization of theorem 2.7 in ref. [35] for the case of a single polynomial $g_r(c, z)$. That this theorem still holds for noncompact cycles is discussed in refs. [38, 39].

Theorem. *Let $g_r(c, z)$ be the deformed polynomial in N variables obtained from $g(c, z) = \mathcal{U}(c) + \mathcal{F}(c)$, where $\mathcal{F}(c)$ and $\mathcal{U}(c)$ are obtained by considering the coefficients appearing in the Symanzik polynomials as variables. $g_r(c, z)$ is obtained by introducing a deformation $r(c, z)$ demanding that its matrix satisfies $\text{co}(\mathbf{A}) > 0$. Let $A = (a_1 \ a_2 \ \dots \ a_n)$ be the configuration matrix associated with $g_r(c, z)$ and consider the polynomial with indeterminate generic coefficients*

$$g_r(c, z) = \sum_{i=1}^n c_i z^{a_i}, \quad c_i \in \mathbb{C}_*. \tag{3.13}$$

¹⁷A related construction — which does not introduce deformations — has been considered in ref. [41] for the special case where the powers of the propagators are given by $\alpha = (1, \dots, 1)$. Here we consider generic powers in the propagators as dictated by canonical series solutions.

Let A be its associated $(N + 1) \times n$ matrix

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \end{pmatrix}. \quad (3.14)$$

The Euler-Mellin integral

$$I_{g_r}(\kappa) = \int_{\Omega} \frac{z^{\alpha}}{g_r(c, z)^{d/2}} d\eta_N \quad (3.15)$$

is a solution of the A -hypergeometric system $H_A(\kappa)$ of degree $\kappa = (-d/2, -\alpha)$. Noncompact cycles Ω can be obtained by taking the coamoeba of $g_r(c, z)$ and choosing representatives θ of connected components $\Theta \in \mathbb{R}^N \setminus \overline{\mathcal{A}}_{g_r}$ (see section (2.3.1)).

Proof. The proof goes along the lines of ref. [39] specializing to the nonhomogeneous case and the case of a single polynomial. Let us consider first eq. (2.13). We have

$$\begin{aligned} (\partial_1)^{u_1} (\partial_2)^{u_2} \dots (\partial_n)^{u_n} I_{g_r}(\kappa) &= (-d/2) (\partial_1)^{u_1} (\partial_2)^{u_2} \dots (\partial_n)^{u_n-1} \int_{\Omega} d\eta_N z^{\alpha} g_r(c, z)^{-d/2-1} z^{a_i} \\ &= (-d/2)_{(|u_n|)} (\partial_1)^{u_1} (\partial_2)^{u_2} \dots (\partial_{n-1})^{u_{n-1}} \int_{\Omega} d\eta_N z^{\alpha} g_r(c, z)^{-d/2-|u_n|} z^{|u_n|a_n} \\ &= (-d/2)_{(|u_1|+\dots+|u_n|)} \int_{\Omega} d\eta_N z^{\alpha} g_r(c, z)^{-d/2-|u_1|-\dots-|u_n|} z^{|u_1|a_1+\dots+|u_n|a_n}, \end{aligned}$$

where $(\rho)_{(x)}$ denotes the falling factorial. Similarly

$$\begin{aligned} (\partial_1)^{v_1} (\partial_2)^{v_2} \dots (\partial_n)^{v_n} I_{g_r}(\kappa) \\ = (-d/2)_{(|v_1|+\dots+|v_n|)} \int_{\Omega} d\eta_N z^{\alpha} g_r(c, z)^{-d/2-|v_1|-\dots-|v_n|} z^{|v_1|a_1+\dots+|v_n|a_n}, \end{aligned}$$

and from $Au = Av$, the result $(\partial^u - \partial^v)I_{g_r}(\kappa) = 0$ follows.

Let us now focus on the second set of differential equations (2.14). Consider the first row of A in eq. (3.14). We have

$$\begin{aligned} (c_1 \partial_1 + c_2 \partial_2 + \dots + c_n \partial_n) I_{g_r}(\kappa) &= \int_{\Omega} d\eta_N z^{\alpha} (-d/2) g_r(c, z)^{-d/2-1} (c_1 z^{a_1} + \dots + c_n z^{a_n}) \\ &= (-d/2) I_{g_r}(\kappa). \end{aligned}$$

Similarly, for $i > 1$

$$\begin{aligned} \sum_{j=1}^n a_{ij} c_j \partial_j I_{g_r}(\kappa) &= \sum_{j=1}^n a_{ij} \int_{\Omega} d\eta_N z^{\alpha} (-d/2) g_r(c, z)^{-d/2-1} (c_j z^{a_j}) \\ &= \int_{\Omega} d\eta_N z^{\alpha} (-d/2) g_r(c, z)^{-d/2-1} \left(z_i \frac{\partial}{\partial z_i} g_r(c, z) \right) \\ &= \int_{\Omega} d\eta_N z^{\alpha} \left(z_i \frac{\partial}{\partial z_i} g_r(c, z)^{-d/2} \right) \\ &= -\alpha_i I_{g_r}(\kappa), \end{aligned}$$

where we have used integration by parts in the last equality. This completes the proof.

In this way, we have generated a class of integrals related with the Lee-Pomeransky representation of Feynman integrals. Feynman integrals will correspond to special cases (points) of A-hypergeometric functions whenever the cycle Ω can be taken as \mathbb{R}_+^N and the coefficients c can be taken as functions of the kinematic invariants. The behavior of the integral as the coefficients c vary can be studied through eq. (2.22). This representation is also useful as it provides noncompact cycles at the cost of imposing conditions on κ . However, this can be sorted out by analytic continuation and it can be shown that, as a function of the variables c , eq. (2.22) is A-hypergeometric everywhere. Here c is taken in $\mathbb{C}^n \setminus \Sigma_A$, where Σ_A denotes the singular locus of all A-hypergeometric functions (Theorem 4.2 in [38]). The main difficulty in this approach is to choose a representative of some connected component Θ in $\mathbb{R}^N \setminus \overline{\mathcal{A}'_{g_r}}$. In other words, we have to select a point such that the cycle is nonvanishing on the set $\text{Arg}^{-1}(\theta)$, where $\theta \in \Theta$. The integration region of the Feynman integral, namely \mathbb{R}_+^N , suggest taking any $\theta = (\arg(f_1(c)), \dots, \arg(f_N(c)))$, $f_i(c) > 0$ provided $\theta \notin \overline{\mathcal{A}'_{g_r}}$ (see e.g. figure 3). This gives one connected component of $\text{vol}(\mathbf{A})$ many ones and a possible integration cycle.¹⁸

On the other hand, logarithm-free canonical series allows us to study the behavior of the solution space under variations of the parameters κ at nonsingular points [57, 61]. More important, once we have identified the above integrals as solutions of a GKZ system, we can use the statement in eq. (2.32) relating Euler-type integrals and canonical series. For our case, this statement reads

$$\int_{\Omega} \frac{z^\alpha}{g_r(c, z)^{d/2}} d\eta_N = K_1 \phi_1 + \dots + K_M \phi_M, \tag{3.16}$$

where ϕ_1, \dots, ϕ_M are the canonical series associated with \mathbf{A} . Fake exponents γ for each ϕ_1, \dots, ϕ_M can be obtained from the SST algorithm in section 2.4.1. This equality is fundamental for our purposes as it allows both to take the limit of the deformation to zero and computing the integrations constants. Let us discuss these limits.

The form of the series solution for some fake exponent γ reads

$$\phi_\gamma = c^\gamma \sum_{u \in \mathcal{L}} \frac{[\gamma]_{u_-}}{[\gamma + u]_{u_+}} c^u. \tag{3.17}$$

These are characterized by a weight vector $w \in \mathbb{R}^n$, which selects a common domain of convergence \mathcal{U}_w (see Theorem 2.5.16 in [40]). In addition, we have the restriction $[\gamma]_{u_-} = 0$ for $u \cdot w < 0$. Taking the limit of the deformation to zero amounts to take some of the coefficients in $c = (c_1, \dots, c_n)$ to zero in eq. (3.17) and similarly for the remaining canonical series. On the l.h.s. of eq. (3.16), taking this limit amounts to recover the undeformed integral we started with. Let us clarify this point. The integral in the l.h.s. of (3.16) is a well defined A-hypergeometric function provided c are generic and the cycle is taken in some $\theta \notin \overline{\mathcal{A}'_{g_r}}$. The r.h.s. of (3.16) is an asymptotic expansion of such integral. Therefore, the limits on the l.h.s. correspond to special values of the A-hypergeometric functions in the r.h.s.. A judicious choice of the deformation ensures that this limit can be taken systematically as we will see in the examples.

¹⁸For all real positive coefficients in $g(z)$, i.e., the nontoric polynomial, we can ensure that 0 is in \mathbb{T}^N [38].

In general, this limit will not be smooth as it will require the evaluation of A-hypergeometric functions on their singular points¹⁹ and therefore we may require analytic continuation of the corresponding A-hypergeometric function. This can be seen as follows. A generic canonical series solutions can be understood as a function in $\text{co}(\mathbf{A})$ variables. Setting one of these variables to zero may require a transformation of a variable to, say, its inverse and hence we require analytic continuation. A judicious choice of the weight vector w can simplify taking this limit, since it provides the condition $[\gamma]_{u_-} = 0$ for $u \cdot w < 0$, thus determining the arrangement of the powers of c^u in eq. (3.17). As we will see for the L -loop cantaloupe graph, we can systematically choose a deformation and a weight vector to take this limit. However, we will leave as a conjecture that this this limit can always be taken.

For integration constants, we will use the information available on the fake exponents following ref. [40]. The initial series in eq. (3.17) are given by c^γ . Therefore, the positions of the 0's on each γ will tell us which elements in c we have to set to zero in order to compute the integration constants.

Let us give some final comments. Notice that in order to recover the original Feynman integral — in agreement with our definition of the polynomials associated with the GKZ system and of the integral (3.15) — we must set coefficients c to their kinematic values at the end of the computation. The integral (3.15) can be understood as a holomorphic function on \mathcal{U}_w . This observation gives us a nice way of deducing the long known fact that convergent Feynman integrals are functions of a Nilsson class [2], which is clear from their canonical series.

We end with our prescription to compute integration constants.

Integration constants. Suppose that we obtain M fake exponents from the SST algorithm. Let us isolate an exponent, say, $\gamma = (\gamma^1, \dots, \gamma^n)$ and suppose it contains $k < n$ zeros in positions $\sigma_1, \dots, \sigma_k$. Take the coefficients associated with those positions to zero and consider

$$I_{g_r}(\kappa) \Big|_{\substack{c_{\sigma_1} \rightarrow 0 \\ \vdots \\ c_{\sigma_k} \rightarrow 0}} = (K_1 \phi_1 + \dots + K_M \phi_M) \Big|_{\substack{c_{\sigma_1} \rightarrow 0 \\ \vdots \\ c_{\sigma_k} \rightarrow 0}}, \tag{3.18}$$

where we take $\Omega = \mathbb{R}_+^N$ as the cycle in the l.h.s.. This procedure will compute a single coefficient. We repeat the process until we have computed all of them.

3.3 Examples of $\text{co}(\mathbf{A}) = 0$

The purpose of $\text{co}(\mathbf{A}) = 0$ examples is to show how to deal with the appearance of a deformation.

Structure of the examples. In the following examples we will omit the overall gamma factors ξ_{Γ_α} and hence consider

$$I(\alpha) := I_F(\alpha) / \xi_{\Gamma_\alpha}$$

¹⁹See ref. [37] and the example in section 3.3.1.

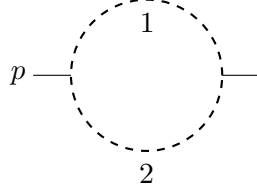


Figure 4. Bubble graph: $s = -p^2$.

along with its toric version $I_{g_r}(\kappa)$. The vector κ has the form

$$\kappa = (-d/2, -\alpha_1, \dots, -\alpha_N),$$

where N is the number of propagators. We assume that the powers of the propagators have generic noninteger complex values. This simplifies the discussion since the sum runs over $\mathcal{L} = \ker_{\mathbb{Z}} \mathbf{A}$. The kernel of \mathbf{A} leads to a rank $\text{co}(\mathbf{A})$ lattice. For a single generator, we write

$$\mathcal{L} := \mathbb{Z}(a_1, \dots, a_n), \quad u = n(a_1, \dots, a_n), \quad a_i \in \mathbb{Z}.$$

We set

$$\beta = d/2.$$

In order to indicate the i -th component of a root vector γ_r we write γ_r^i .

3.3.1 Massless bubble

The simplest example is the massless bubble (figure 4) with inverse propagators

$$D_1 = (k)^2, \quad D_2 = (k - p)^2, \quad (3.19)$$

where $s = -p^2$. We have

$$g(z) = z_1 + z_2 + s z_1 z_2. \quad (3.20)$$

This polynomial leads to a matrix of codimension $\text{co}(\mathbf{A}) = 0$, hence we introduce a deformation $r(z) = c_1$ and consider instead

$$g_r(c, z) = c_1 + c_2 z_1 + c_3 z_2 + c_4 z_1 z_2 \iff \mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \quad (3.21)$$

Thus, we have

$$I_{g_r}(\kappa) = \int_{\text{Arg}^{-1}\theta} \frac{z_1^{\alpha_1} z_2^{\alpha_2}}{(c_1 + c_2 z_1 + c_3 z_2 + c_4 z_1 z_2)^\beta} \frac{dz_1}{z_1} \frac{dz_2}{z_2}, \quad (3.22)$$

for $\theta = (\arg(c_1/c_3), \arg(c_1/c_2))$. We have studied this integral in section 2.5, where we established that

$$I_{g_r}(\kappa) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta - \alpha_1)\Gamma(\beta - \alpha_2)}{\Gamma(\beta)^2} c_1^{\alpha_1 + \alpha_2 - \beta} c_2^{-\alpha_1} c_3^{-\alpha_2} {}_2F_1\left(\alpha_1, \alpha_2; \beta; 1 - \frac{c_1 c_4}{c_2 c_3}\right). \quad (3.23)$$

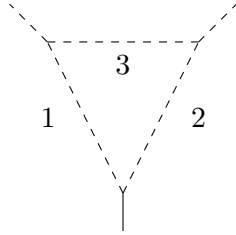


Figure 5. Triangle graph: $s = -(p_1 + p_2)^2$.

In order to recover our Feynman integral we have to take the limit $c_1 \rightarrow 0$. The limit has to be taken carefully as c_1 appears both as a factor and in the argument of the hypergeometric function. Using the identity (A.7) we have

$${}_2F_1(\alpha_1, \alpha_2; \beta; 1 - \frac{c_1 c_4}{c_2 c_3}) = \left(\frac{c_1 c_4}{c_2 c_3}\right)^{(\beta - \alpha_1 - \alpha_2)} {}_2F_1\left(\beta - \alpha_1, \beta - \alpha_2; \beta; 1 - \frac{c_1 c_4}{c_2 c_3}\right). \quad (3.24)$$

Therefore

$$I_{gr}(\kappa) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta - \alpha_1)\Gamma(\beta - \alpha_2)}{\Gamma(\beta)^2} \times c_2^{\alpha_2 - \beta} c_3^{\alpha_1 - \beta} c_4^{\beta - \alpha_1 - \alpha_2} {}_2F_1\left(\beta - \alpha_1, \beta - \alpha_2; \beta; 1 - \frac{c_1 c_4}{c_2 c_3}\right). \quad (3.25)$$

Finally, taking the limit $c_1 = 0$, setting $c_2 = c_3 = 1$, and $c_4 = s$ we recover the desired result

$$I(\alpha) = \frac{\Gamma(\beta - \alpha_1)\Gamma(\beta - \alpha_2)\Gamma(\alpha_1 + \alpha_2 - \beta)}{\Gamma(\beta)} s^{(\beta - \alpha_1 - \alpha_2)}, \quad (3.26)$$

where we have used identity (A.6). Notice that in order to take this limit, we have evaluated eq. (3.23) on one of the singular points of the Gauss hypergeometric function, thus requiring an Euler transformation.

3.3.2 The single-scale massless triangle graph

Let us now turn our attention to those limits through canonical series. Let us consider the triangle graph in figure 5. The inverse propagators read

$$D_1 = (k_1 - p_1)^2, \quad D_2 = (k_1 + p_2), \quad D_3 = k_1^2, \quad (3.27)$$

where $-p_1^2 = -p_2^2 = 0$. Computing the relevant polynomial and taking the deformation $r(z) = c_1$ leads to

$$g_r(c, z) = c_1 + c_2 z_1 + c_3 z_2 + c_4 z_3 + c_5 z_1 z_2 \iff \mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad (3.28)$$

where, at the end of the computation, we will make the identifications $c_2 = c_3 = c_4 = 1$ and $c_5 = s = -(p_1 + p_2)^2$. The integral reads

$$I_{g_r}(\kappa) = \int_{\Omega} d\eta_3 \frac{z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}}{(c_1 + c_2 z_1 + c_3 z_2 + c_4 z_3 + c_5 z_1 z_2)^\beta}. \quad (3.29)$$

Computing $\ker(\mathbf{A})$, we have

$$\mathcal{L} = \mathbb{Z}(1, -1, -1, 0, 1) \Rightarrow u = n(1, -1, -1, 0, 1). \quad (3.30)$$

Choosing $w = (1, 0, 0, 0, 0)$, we have $u \cdot w = n$, hence $[\gamma_i]_{u_-} = 0$ for $n < 0$. Setting $A = \alpha_1 + \alpha_2 + \alpha_3$, the fake indicial ideal and its roots read

$$\text{fin}_w(H_{\mathbf{A}}(\kappa)) = \langle \theta_1 \theta_5, \beta + \theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5, \alpha_1 + \theta_2 + \theta_5, \alpha_2 + \theta_3 + \theta_5, \alpha_3 + \theta_4 \rangle, \quad (3.31)$$

$$\{\gamma_i\} = \{(0, \alpha_2 + \alpha_3 - \beta, \alpha_1 + \alpha_3 - \beta, -\alpha_3, -\alpha + \beta), (A - \beta, -\alpha_1, -\alpha_2, -\alpha_3, 0)\}. \quad (3.32)$$

Inserting the roots in eq. (2.29) gives

$$\phi_i = c^{\gamma_i} \sum_{n \geq 0} \frac{[\gamma_i]_{(0,n,n,0,0)}}{[\gamma_i + (n, -n, -n, 0, n)]_{(n,0,0,0,n)}} \left(\frac{c_1 c_5}{c_2 c_3} \right)^n, \quad (3.33)$$

which leads to the series

$$\phi_1 = c^{\gamma_1} \sum_{n \geq 0} \frac{(\beta - \alpha_2 - \alpha_3)_n (\beta - \alpha_1 - \alpha_3)_n}{(1)_n (\beta - A + 1)_n} \left(\frac{c_1 c_5}{c_2 c_3} \right)^n, \quad (3.34)$$

$$\phi_2 = c^{\gamma_2} \sum_{n \geq 0} \frac{(\alpha_1)_n (\alpha_2)_n}{(-\beta + A + 1)_n (1)_n} \left(\frac{c_1 c_5}{c_2 c_3} \right)^n. \quad (3.35)$$

Integration constants can be easily computed by setting $c_1 = 0$ and $c_5 = 0$ in eq. (3.29) with $\Omega = \mathbb{R}_+^3$. We write them collectively as

$$K_r = \frac{1}{\Gamma(\beta)} \prod_{i \neq 0} \Gamma(-\gamma_r^i). \quad (3.36)$$

Taking the limit $c_1 \rightarrow 0$ and setting $c_2 = c_3 = c_4 = 1$, $c_5 = s$, we obtain

$$I(\alpha) = \frac{\Gamma(\beta - \alpha_2 - \alpha_3) \Gamma(\beta - \alpha_1 - \alpha_3) \Gamma(\alpha_3) \Gamma(\beta - A)}{\Gamma(\beta)} s^{\beta - A}. \quad (3.37)$$

Let us remark that in this example we have set c_1 to zero in order to compute one of the integration constants — as can be seen from the roots — therefore reaching a tautology. We can fix this by choosing an appropriate weight vector such that none of the roots contain zero in position 1. This leads to a more complicated version of the canonical series which will require analytic continuation in order to take the limit $c_1 \rightarrow 0$. Since we want to interpret Feynman integrals with codimension zero matrix as a certain limiting cases of \mathbf{A} -hypergeometric functions, we do not worry about this situation. In fact, as we will see in the next example, we can take advantage of this by choosing a weight vector such that all massless cantaloupe graphs are defined by the coefficients of a linear combination of two Gauss hypergeometric functions.

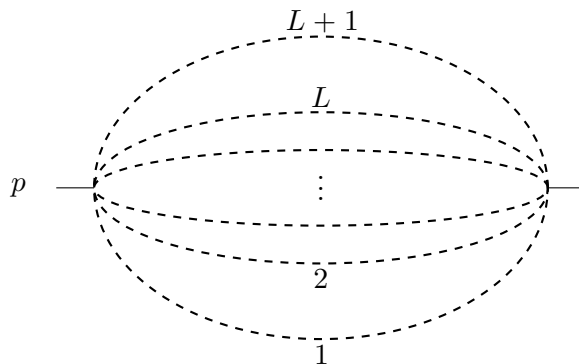


Figure 6. Massless cantaloupe graph.

3.3.3 The massless L -loop cantaloupe graph

Let us consider the case of the L -loop cantaloupe graph (a.k.a. banana graph) shown in figure 6. We parametrize the inverse propagators as follows

$$D_1 = (k_1 - p)^2, \quad D_2 = (k_2 - k_1)^2, \dots, D_L = (k_L - k_{L-1})^2, \quad D_{L+1} = (k_L)^2. \quad (3.38)$$

The polynomial $g(z)$ of this graph can be written in general as

$$g(z_1, \dots, z_{L+1}) = \sum_{i=1}^{L+1} \prod_{j \neq i}^L z_j + s \prod_{i=1}^{L+1} z_i, \quad (3.39)$$

where $s = -p^2$. The integral to be computed reads

$$I(\alpha) = \int_{\mathbb{R}_+^{L+1}} d\eta_{L+1} \frac{z_1^{\alpha_1} \dots z_{L+1}^{\alpha_{L+1}}}{g(z)^\beta}. \quad (3.40)$$

It is easy to check that the $g(z)$ polynomials of this graph lead to codimension zero matrices and hence they fall under the class of problems where we must introduce a deformation to define a GKZ system in the sense of eq. (2.29). In order to perform such deformation systematically, let us introduce some notation. Let 1_i denote a sequence of 1's of length i and similarly for 0_j . We have the relation $i + j = L + 1$. Furthermore, let

$$v := (1_{L-1}, 0_2). \quad (3.41)$$

At each loop, we set a deformation monomial

$$r(z) = c_1 z^v, \quad (3.42)$$

hence we have

$$g_r(c, z) = c_1 z^v + \sum_{i=1}^{L+1} c_{L+3-i} \prod_{j \neq i}^L z_j + c_{L+3} \prod_{i=1}^{L+1} z_i, \quad (3.43)$$

where $c_{L+3} = s$. Let us give an example. For $L = 3$, $v = (1, 1, 0, 0)$ and $r(z) = c_1 z_1 z_2$, then we have the deformed toric polynomial

$$g_r(c, z) = c_1 z_1 z_2 + c_2 z_1 z_2 z_3 + c_3 z_1 z_2 z_4 + c_4 z_1 z_3 z_4 + c_5 z_2 z_3 z_4 + c_6 z_1 z_2 z_3 z_4. \quad (3.44)$$

After introducing the deformation, the $(L + 2) \times (L + 3)$ matrix associated with the L -loop cantaloupe graph can be written in the general form

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ & 1_{L+1} & & 0 & 1_1 \\ & & 1_L & & 0 & 1_2 \\ & & & \vdots & & \\ & & & & 1_3 & 0 & 1_{L-1} \\ 0 & 1 & 0 & & & 1_L \\ 0 & 0 & 1 & & & 1_L \end{pmatrix}. \quad (3.45)$$

Notice that each row contains $L + 3$ elements. There are L rows with have a single zero and 2 rows with two zeros. The integral under consideration is given by

$$I_{g_r}(\kappa) = \int_{\Omega} d\eta_{L+1} \frac{z_1^{\alpha_1} \dots z_{L+1}^{\alpha_{L+1}}}{g_r(c, z)^\beta}, \quad (3.46)$$

where $\kappa = (-\beta, -\alpha_1, \dots, -\alpha_{L+1})$. Computing the kernel of the above matrix leads to

$$\mathcal{L} = \mathbb{Z}(1, -1, -1, 0_{L-1}, 1), \quad (3.47)$$

where by definition $0_0 := \emptyset$. We choose $w = (1, 0_{L+2})$, thus obtaining

$$\text{fin}_w(H_A(\kappa)) = \langle \theta_1 \theta_{L+3} \rangle + \langle A\theta - \kappa^T \rangle. \quad (3.48)$$

The roots can be written as

$$\{\gamma_i\} = \left\{ \left(0, \alpha_{L+1} - \beta, \dots, \alpha_1 - \beta, L\beta - \sum_{i=1}^{L+1} \alpha_i \right), \right. \\ \left. \left(\sum_{i=1}^L \alpha_i - L\beta, (L-1)\beta - \sum_{i=1}^L \alpha_i, (L-1)\beta - \sum_{i \neq L}^{L+1} \alpha_i, -\beta + \alpha_{L-1}, \dots, -\beta + \alpha_1, 0 \right) \right\}, \quad (3.49)$$

which lead to the canonical series

$$\phi_1 = c^{\gamma_1} {}_2F_1 \left(\beta - \alpha_{L+1}, \beta - \alpha_L, L\beta - \sum_{i=1}^{L+1} \alpha_i + 1; x \right), \quad (3.50)$$

$$\phi_2 = c^{\gamma_2} {}_2F_1 \left(-(L-1)\beta + \sum_{i=1}^L \alpha_i, -(L-1)\beta + \sum_{i \neq L}^{L+1} \alpha_i; \sum_{i=1}^L \alpha_i - L\beta + 1; x \right), \quad (3.51)$$

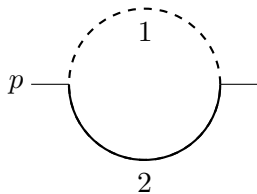


Figure 7. Single mass bubble graph.

where

$$x = \frac{c_1 c_{L+3}}{c_2 c_3} \quad (3.52)$$

The relevant integration constant reads

$$K_1 = \frac{\Gamma(-L\beta + \sum_{i=1}^{L+1} \alpha_i)}{\Gamma(\beta)} \prod_{i=1}^{L+1} \Gamma(\beta - \alpha_i), \quad (3.53)$$

which after a change of variables corresponds to the formula of the Mellin transform of a linear function. Setting $c_1 = 0$, $c_2 = \dots = c_{L+2} = 1$, and $c_{L+3} = s$ we arrive at

$$I(\alpha) = s^{(L\beta - \sum_{i=1}^L \alpha_i)} \frac{\Gamma(-L\beta + \sum_{i=1}^{L+1} \alpha_i)}{\Gamma(\beta)} \prod_{i=1}^{L+1} \Gamma(\beta - \alpha_i) \quad (3.54)$$

The formulas for the fake indicial ideal and its roots have been checked up to 5-loop. In this example, we have seen that deforming $g(z)$ leads to a GKZ system of $\text{co}(\mathbf{A}) = 1$ and taking the limit of the deformation to zero at the end of the computation allows us to interpret Feynman integrals as the limit of a linear combination of their canonical series.

3.4 Examples of $\text{co}(\mathbf{A}) = 1$

3.4.1 One-mass bubble

Let us work now with the bubble integral with one massive internal line (figure 7). The inverse propagators of this integral read

$$D_1 = (k^2), \quad D_2 = (k - p)^2 + m^2. \quad (3.55)$$

Omitting the overall Γ factors we have the Lee-Pomeransky representation

$$I(\alpha) = \int_{\mathbb{R}_+^2} \frac{z_1^{\alpha_1} z_2^{\alpha_2}}{(z_1 + z_2 + (m^2 + s)z_1 z_2 + m^2 z_2^2)^\beta} \frac{dz_1}{z_1} \frac{dz_2}{z_2}, \quad (3.56)$$

where $s = -p^2$. Therefore, the corresponding toric polynomial and the associated matrix \mathbf{A} read

$$g(c, z) = c_1 z_1 + c_2 z_2 + c_3 z_1 z_2 + c_4 z_2^2 \iff \mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}. \quad (3.57)$$

Now, let us consider the related problem from the GKZ point of view. We are interested in the more general situation

$$I_g(\kappa) = \int_{\Omega} \frac{z_1^{\alpha_1} z_2^{\alpha_2}}{(c_1 z_1 + c_2 z_2 + c_3 z_1 z_2 + c_4 z_2^2)^\beta} \frac{dz_1}{z_1} \frac{dz_2}{z_2}. \quad (3.58)$$

Taking $w = (0, 1, 1, 1)$, we obtain the following fake indicial ideal

$$\text{fin}_w(H_A(\kappa)) = \langle \theta_2 \theta_3, \beta + \theta_1 + \theta_2 + \theta_3 + \theta_4, \alpha_1 + \theta_1 + \theta_3, \alpha_2 + \theta_2 + \theta_3 + 2\theta_4 \rangle. \quad (3.59)$$

Then, the roots of this ideal read

$$\{\gamma_i\} = \{(-\alpha_1, 2\alpha_1 + \alpha_2 - 2\beta, 0, -\alpha_1 - \alpha_2 + \beta), (\alpha_1 + \alpha_2 - 2\beta, 0, -2\alpha_1 - \alpha_2 + 2\beta, \alpha_1 - \beta)\}. \quad (3.60)$$

In addition, we have $\mathcal{L} = \mathbb{Z}(-1, 1, 1, -1)$, and hence $u = n(-1, 1, 1, -1)$. We also have $u \cdot w = n$, which implies $[\gamma_i]_{u_-} = 0$ for $n < 0$. The canonical series simplify to the functions

$$\phi_1 = c^{\gamma_1} {}_2F_1 \left(\alpha_1, \alpha_1 + \alpha_2 - \beta; 2\alpha_1 + \alpha_2 - 2\beta + 1; \frac{c_2 c_3}{c_1 c_4} \right), \quad (3.61)$$

$$\phi_2 = c^{\gamma_2} {}_2F_1 \left(2\beta - \alpha_1 - \alpha_2, \beta - \alpha_1; 2\beta - 2\alpha_1 - \alpha_2 + 1; \frac{c_2 c_3}{c_1 c_4} \right). \quad (3.62)$$

The result of the integral is a linear combination of $\phi_{1,2}$. Constants of integration can be obtained by integrating (3.58) taking $c_3 = 0$ and $\Omega = \mathbb{R}_+^2$, and similarly for c_2 . This leads to the multiplying factors

$$K_1 = \frac{\Gamma(\alpha_1) \Gamma(\alpha_2 + \alpha_1 - \beta) \Gamma(2\beta - 2\alpha_1 - \alpha_2)}{\Gamma(\beta)}, \quad (3.63)$$

$$K_2 = \frac{\Gamma(\beta - \alpha_1) \Gamma(2\alpha_1 + \alpha_2 - 2\beta) \Gamma(2\beta - \alpha_1 - \alpha_2)}{\Gamma(\beta)}. \quad (3.64)$$

Setting $c_3 = (s + m^2)$ and $c_4 = m^2$, the resulting integral reads

$$I(\alpha) = (m^2)^{\beta - \alpha_1 - \alpha_2} \left(K_1 {}_2F_1 \left(\alpha_1, \alpha_1 + \alpha_2 - \beta; 2\alpha_1 + \alpha_2 - 2\beta + 1; 1 + s/m^2 \right) \right. \\ \left. + (1 + s/m^2)^{2\beta - 2\alpha_1 - \alpha_2} K_2 {}_2F_1 \left(2\beta - \alpha_1 - \alpha_2, \beta - \alpha_1; 2\beta - 2\alpha_1 - \alpha_2 + 1; 1 + s/m^2 \right) \right). \quad (3.65)$$

The form of the result as a sum of two hypergeometric functions is reminiscent of the negative dimension approach [62]. Using the eq. (A.8), we obtain

$$I(\alpha) = (m^2)^{\beta - \alpha_1 - \alpha_2} \frac{\Gamma(\alpha_1) \Gamma(\beta - \alpha_1) \Gamma(2\beta - \alpha_1 - \alpha_2) \Gamma(-\beta + \alpha_1 + \alpha_2)}{\Gamma(\beta)^2} \\ \times {}_2F_1 \left(\alpha_1, -\beta + \alpha_1 + \alpha_2; \beta; -s/m^2 \right). \quad (3.66)$$

3.4.2 One-mass sunset

We now consider the single mass sunset graph with the constraint $s = -p^2 = m^2$ (figure 8). The inverse propagators read

$$D_1 = (k_1 - p_1)^2, \quad D_2 = (k_2 - k_1)^2 + m^2, \quad D_3 = k_2^2, \quad (3.67)$$

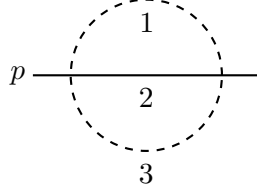


Figure 8. Sunset graph.

which lead to the integral

$$I(\alpha) = \int_{\mathbb{R}_+^3} d\eta_3 \frac{z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}}{(z_1 z_2 + z_1 z_3 + z_2 z_3 + m^2 z_1 z_2^2 + m^2 z_2^2 z_3)^{\beta}}. \quad (3.68)$$

We consider the following toric polynomial

$$g(c, z) = c_1 z_1 z_2 + c_2 z_1 z_3 + c_3 z_2 z_3 + c_4 z_1 z_2^2 + c_5 z_2^2 z_3 \iff \mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}, \quad (3.69)$$

such that the integral under consideration becomes

$$I_g(\kappa) = \int_{\Omega} d\eta_3 \frac{z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}}{(c_1 z_1 z_2 + c_2 z_1 z_3 + c_3 z_2 z_3 + c_4 z_1 z_2^2 + c_5 z_2^2 z_3)^{\beta}}. \quad (3.70)$$

Choosing the weight vector $w = (0, 1, 1, 1, 1)$, we find

$$\text{fin}_w(H_{\mathbf{A}}(\kappa)) = \langle \theta_3 \theta_4 \rangle + \langle \mathbf{A} \theta - \kappa^T \rangle, \quad (3.71)$$

$$\{\gamma_i\} = \{(\alpha_3 - \beta, -\alpha_1 - \alpha_3 + \beta, 2\alpha_1 + \alpha_2 + \alpha_3 - 3\beta, 0, -\alpha_1 - \alpha_2 - \alpha_3 + 2\beta), \quad (3.72)$$

$$(2\alpha_1 + \alpha_2 + 2\alpha_3 - 4\beta, -\alpha_1 - \alpha_3 + \beta, 0, -2\alpha_1 - \alpha_2 - \alpha_3 + 3\beta, \alpha_1 - \beta)\}.$$

From $\mathcal{L} = \mathbb{Z}(1, 0, -1, -1, 1)$, we have $u = n(1, 0, -1, -1, 1)$, hence $u \cdot w = -n$. In order to start the sum from $n = 0$, we set $u \rightarrow -u$ and thus $u = n(0, 0, 1, 1, 0) - n(1, 0, 0, 0, 1)$. We obtain

$$\phi_1 = c^{\gamma_1} \sum_{n \geq 0} \frac{(\beta - \alpha_3)_n (\alpha_1 + \alpha_2 + \alpha_3 - 2\beta)_n}{(2\alpha_1 + \alpha_2 + \alpha_3 - 3\beta + 1)_n (1)_n} \left(\frac{c_3 c_4}{c_1 c_5} \right)^n, \quad (3.73)$$

$$\phi_2 = c^{\gamma_2} \sum_{n \geq 0} \frac{(-2\alpha_1 - \alpha_2 - 2\alpha_3 + 4\beta)_n (\beta - \alpha_1)_n}{(1)_n (-2\alpha_1 - \alpha_2 - \alpha_3 + 3\beta + 1)_n} \left(\frac{c_3 c_4}{c_1 c_5} \right)^n. \quad (3.74)$$

The integration constants can be written collectively as

$$K_r = \frac{1}{\Gamma(\beta)} \prod_{i \neq 0} \Gamma(-\gamma_r^i). \quad (3.75)$$

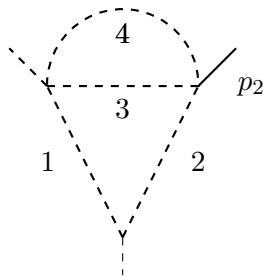


Figure 9. Party hat.

Finally, setting $c_1 = c_2 = c_3 = 1$ and $c_4 = c_5 = m^2$, we arrive at the result

$$I(\alpha) = (m^2)^{-\alpha_1 - \alpha_2 - \alpha_3 + 2\beta} \left(K_1 {}_2F_1(\beta - \alpha_3, \alpha - 2\beta; \alpha + \alpha_1 - 3\beta + 1; 1) + K_2 {}_2F_1(-2\alpha + \alpha_2 + 4\beta, \beta - \alpha_1; -\alpha - \alpha_1 + 3\beta + 1; 1) \right). \quad (3.76)$$

In order to write this result in simpler form let

$$A = \beta - \alpha_3, \quad B = -2\beta + \alpha_1 + \alpha_2 + \alpha_3, \quad C = 2\beta - \alpha_1 - \alpha_3,$$

hence we can write eq. (3.76) as

$$I(\alpha) = (m^2)^{-B} \frac{\Gamma(A)\Gamma(B)\Gamma(\beta - C)\Gamma(C - A)\Gamma(C - B)}{\Gamma(C)\Gamma(\beta)}, \quad (3.77)$$

where we have used identity (A.8).

3.4.3 Single scale party hat

The next example is the party hat graph shown in figure 9. We have the inverse propagators

$$D_1 = k_2^2, \quad D_2 = (k_2 - p_3)^2, \quad D_3 = (k_2 - k_1)^2, \quad D_4 = (k_1 - p_1)^2. \quad (3.78)$$

We will consider the case $-p_1^2 = -p_3^2 = 0$ and $s = p_2^2$. The relevant integral in the Lee-Pomeransky representation reads

$$I(\alpha) = \int_{\mathbb{R}_+^4} d\eta_4 \frac{z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} z_4^{\alpha_4}}{(z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_4 z_3 + s z_2 z_4 z_3)^\beta}. \quad (3.79)$$

Therefore, the toric polynomial has the form

$$g(c, z) = c_1 z_1 z_3 + c_2 z_1 z_4 + c_3 z_2 z_3 + c_4 z_2 z_4 + c_5 z_3 z_4 + c_6 z_2 z_3 z_4, \quad (3.80)$$

where $c_6 = s$. We associate the following matrix to $g(c, z)$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}. \quad (3.81)$$

Taking $w = (0, 1, 1, 1, 1, 1)$ leads to the fake indicial ideal (see example appendix B)

$$\begin{aligned} \text{fin}_w(H_A(\kappa)) = \langle & \theta_2 \theta_3, \beta + \theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 + \theta_6, \alpha_1 + \theta_1 + \theta_2, \alpha_2 + \theta_3 + \theta_4 + \theta_6, \\ & \alpha_3 + \theta_1 + \theta_3 + \theta_5 + \theta_6, \alpha_4 + \theta_2 + \theta_4 + \theta_5 + \theta_6 \rangle, \end{aligned} \quad (3.82)$$

with roots

$$\begin{aligned} \gamma_i = \{ & (-\alpha_1, 0, \alpha_1 + \alpha_4 - \beta, \alpha_3 - \beta, \alpha_1 + \alpha_2 - \beta, -\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 + 2\beta), \\ & (\alpha_4 - \beta, -\alpha_1 - \alpha_4 + \beta, 0, \alpha_1 + \alpha_3 + \alpha_4 - 2\beta, \alpha_1 + \alpha_2 - \beta, -\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 + 2\beta) \}. \end{aligned} \quad (3.83)$$

Computing $\ker(A)$ leads to

$$\mathcal{L} = \mathbb{Z}(1, -1, -1, 1, 0, 0) \Rightarrow u = n(1, 0, 0, 1, 0, 0) - n(0, 1, 1, 0, 0, 0). \quad (3.84)$$

Since $u \cdot w = -n$, $[\gamma]_{u_-} = 0$ for $n > 0$. Then, the canonical series (2.29) leads to

$$\phi_1 = c^{\gamma_1} {}_2F_1(\alpha_1, \beta - \alpha_3; \alpha_1 + \alpha_4 - \beta + 1; (c_1 c_4)/(c_2 c_5)), \quad (3.85)$$

$$\phi_2 = c^{\gamma_2} {}_2F_1(\beta - \alpha_4, -\alpha_1 - \alpha_3 - \alpha_4 + 2\beta; -\alpha_1 - \alpha_4 + \beta + 1; (c_2 c_4)/(c_1 c_5)). \quad (3.86)$$

Integration constants read

$$K_r = \frac{1}{\Gamma(\beta)} \prod_{i \neq 0} \Gamma(-\gamma_r^i). \quad (3.87)$$

Setting $c_1 = c_2 = c_3 = c_4 = c_5 = 1$, $c_6 = s$, and defining

$$A = \beta - \alpha_4, \quad B = 2\beta - \alpha_1 - \alpha_3 - \alpha_4, \quad C = 2\beta - \alpha_3 - \alpha_4, \quad (3.88)$$

we arrive at

$$I(\alpha) = s^{B-\alpha_2} \frac{\Gamma(A)\Gamma(B)\Gamma(C-A)\Gamma(\alpha_2-B)\Gamma(C-B)\Gamma(B-C+\beta-\alpha_2)}{\Gamma(C)\Gamma(\beta)}, \quad (3.89)$$

where we have used identity (A.8).

3.4.4 On-shell massless box

Let us now consider the massless box integral shown in figure 10. The inverse propagators are given by

$$D_1 = (k - p_1)^2, \quad D_2 = (k + p_2 + p_3)^2, \quad D_3 = (k + p_2)^2, \quad D_4 = k_1^2, \quad (3.90)$$

where $-p_i^2 = 0$, $i = 1, \dots, 4$. We obtain

$$g(z) = z_1 + z_2 + z_3 + z_4 + s z_1 z_3 + t z_2 z_4, \quad (3.91)$$

where $s = -(p_1 + p_2)^2$ and $t = -(p_2 + p_3)^2$ are the usual Mandelstam invariants. From this polynomial we obtain the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}. \quad (3.92)$$

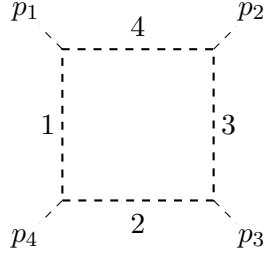


Figure 10. On shell massless box.

Let us consider the more general problem

$$I_g(\kappa) = \int_{\Omega} \frac{z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} z_4^{\alpha_4}}{(c_1 z_1 + c_2 z_2 + c_3 z_3 + c_4 z_4 + c_5 z_3 z_1 + c_6 z_2 z_4)^\beta} d\eta_4. \quad (3.93)$$

Choosing $w = (0, 1, 0, 0, 0, 0)$, the fake indicial ideal reads

$$\text{fin}_w(H_A(\kappa)) = \langle \theta_2 \theta_4 \theta_5 \rangle + \langle A\theta - \kappa^T \rangle, \quad (3.94)$$

which lead to the roots

$$\begin{aligned} \{\gamma_i\} = \{ & (-\alpha_1, \alpha_1 + \alpha_3 + \alpha_4 - \beta, -\alpha_3, \alpha_1 + \alpha_2 + \alpha_3 - \beta, 0, -\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 + \beta), \\ & (\alpha_2 + \alpha_3 - \beta, \alpha_4 - \alpha_2, \alpha_1 + \alpha_2 - \beta, 0, -\alpha_1 - \alpha_2 - \alpha_3 + \beta, -\alpha_4), \\ & (\alpha_3 + \alpha_4 - \beta, 0, \alpha_1 + \alpha_4 - \beta, \alpha_2 - \alpha_4, -\alpha_1 - \alpha_3 - \alpha_4 + \beta, -\alpha_2) \} \end{aligned} \quad (3.95)$$

Computing $\ker A$ leads to $u = n(-1, 1, -1, 1, 1, -1)$, hence $u \cdot w = n$. Therefore $[\gamma]_{u_-} = 0$ for $n < 0$. We also have

$$u_- = (n, 0, n, 0, 0, n), \quad u_+ = (0, n, 0, n, n, 0). \quad (3.96)$$

We can now write the canonical series (2.29) as

$$\phi_1 = c^{\gamma_1} \sum_{n \geq 0} \frac{(\alpha_1)_n (\alpha_3)_n (-\beta + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)_n}{(1)_n (-\beta + \alpha_1 + \alpha_2 + \alpha_3 + 1)_n (-\beta + \alpha_1 + \alpha_3 + \alpha_4 + 1)_n} \left(\frac{c_2 c_4 c_5}{c_1 c_3 c_6} \right)^n, \quad (3.97)$$

$$\phi_2 = c^{\gamma_2} \sum_{n \geq 0} \frac{(\alpha_4)_n (\beta - \alpha_1 - \alpha_2)_n (\beta - \alpha_2 - \alpha_3)_n}{(1)_n (-\alpha_2 + \alpha_4 + 1)_n (\beta - \alpha_1 - \alpha_2 - \alpha_3 + 1)_n} \left(\frac{c_2 c_4 c_5}{c_1 c_3 c_6} \right)^n, \quad (3.98)$$

$$\phi_3 = c^{\gamma_3} \sum_{n \geq 0} \frac{(\alpha_2)_n (\beta - \alpha_1 - \alpha_4)_n (\beta - \alpha_3 - \alpha_4)_n}{(1)_n (\alpha_2 - \alpha_4 + 1)_n (\beta - \alpha_1 - \alpha_3 - \alpha_4 + 1)_n} \left(\frac{c_2 c_4 c_5}{c_1 c_3 c_6} \right)^n. \quad (3.99)$$

The constants of integration read

$$K_r = \frac{1}{\Gamma(\beta)} \prod_{i \neq 0} \Gamma(-\gamma_r^i). \quad (3.100)$$

In this case the canonical series evaluate to hypergeometric functions ${}_3F_2(a, b; c; d, e; x)$ (see definition in eq. (A.4)). Let $A = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ and $x = (c_2 c_4 c_5)/(c_1 c_3 c_6)$. We can

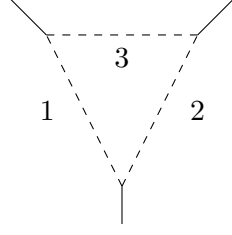


Figure 11. Three-scale triangle: $s_1 = -p_1^2$, $s_2 = -p_2^2$, $s_3 = -(p_1 + p_2)^2$.

write the result in the condensed form

$$\begin{aligned}
 I_b(\kappa, c) = & K_1 c^{\gamma_1} {}_3F_2(\alpha_1, \alpha_3, -\beta + A; -\beta + A - \alpha_4 + 1, -\beta + A - \alpha_2 + 1; x) \\
 & + K_2 c^{\gamma_2} {}_3F_2(\beta - \alpha_1 - \alpha_2, \beta - \alpha_2 - \alpha_3, \alpha_4; \beta - A + \alpha_4 + 1, -\alpha_2 + \alpha_4 + 1; x) \\
 & + K_3 c^{\gamma_3} {}_3F_2(\alpha_2, \beta - \alpha_1 - \alpha_4, \beta - \alpha_3 - \alpha_4; \alpha_2 - \alpha_4 + 1, \beta - A + \alpha_2 + 1; x)
 \end{aligned} \tag{3.101}$$

Setting the constant to the values of the original polynomial, i.e., $c_1 = \dots = c_4 = 1$, $c_5 = s$ and $c_6 = t$, we obtain

$$\begin{aligned}
 I(\alpha) = & K_1 t^{\beta-A} {}_3F_2(\alpha_1, \alpha_3, -\beta + A; -\beta + A - \alpha_4 + 1, -\beta + A - \alpha_2 + 1; s/t) \\
 & + K_2 t^{-\alpha_4} s^{\alpha_4-A+\beta} {}_3F_2(\beta - \alpha_1 - \alpha_2, \beta - \alpha_2 - \alpha_3, \alpha_4; \beta - A + \alpha_4 + 1, -\alpha_2 + \alpha_4 + 1; s/t) \\
 & + K_3 t^{-\alpha_2} s^{\alpha_2-A+\beta} {}_3F_2(\alpha_2, \beta - \alpha_1 - \alpha_4, \beta - \alpha_3 - \alpha_4; \alpha_2 - \alpha_4 + 1, \beta - A + \alpha_2 + 1; s/t).
 \end{aligned} \tag{3.102}$$

Hence, we have recovered the result obtained by the differential reduction method using Gröbner bases given in ref. [63] based on [64, 65]. Interestingly, the computation of integrations constants in ref. [63] follows a similar prescription as ours. However, in our case, because of our choice of w , we only set $c_5 = 0$ to compute K_1 but we do not use $c_6 = 0$ as in [63]. This results also matches nicely the results obtained in ref. [66] using the negative dimension approach.

3.5 A $\text{co}(A) = 2$ example

Finally, let us consider the triangle integral shown in figure 11 where all external momenta are nonvanishing. The inverse propagators read

$$D_1 = (k_1 - p_1)^2, \quad D_2 = (k_1 + p_2), \quad D_3 = k_1^2, \tag{3.103}$$

where we have $-p_1^2 = s_1$, $-p_2^2 = s_2$, and $-(p_1 + p_2)^2 = s_3$. Using momentum conservation, we have

$$g(z) = z_1 + z_2 + z_3 + s_3 z_1 z_2 + s_1 z_1 z_3 + s_2 z_2 z_3. \tag{3.104}$$

The associated matrix reads

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}. \tag{3.105}$$

This matrix has $\text{co}(\mathbf{A}) = 2$ and hence the resulting \mathbf{A} -hypergeometric functions will be functions of two variables. The integral under consideration then reads

$$I_g(\kappa) = \int_{\Omega} d\eta_3 \frac{z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3}}{(c_1 z_1 + c_2 z_2 + c_3 z_3 + c_4 z_1 z_2 + c_5 z_1 z_3 + c_6 z_2 z_3)^{\beta}}. \quad (3.106)$$

Computing $\ker(\mathbf{A})$, we have

$$\mathcal{L} = \{(-1, 0, 1, 1, 0, -1), (-1, 1, 0, 0, 1, -1)\}, \quad (3.107)$$

which means that in this case the lattice is generated by

$$u = m(-1, 0, 1, 1, 0, -1) + n(-1, 1, 0, 0, 1, -1), \quad m, n \in \mathbb{Z}. \quad (3.108)$$

We choose $w = (0, 0, 1, 0, 0, 0)$, which has the advantage of restricting the sum over m . We have $u \cdot w = m$, hence $[\gamma_i]_{u_-} = 0$ for $m < 0$. The fake indicial ideal reads

$$\text{fin}_w(H_{\mathbf{A}}(\kappa)) = \langle \theta_2 \theta_5, \theta_3 \theta_4 \rangle + \langle \mathbf{A}\theta - \kappa^T \rangle, \quad (3.109)$$

with roots

$$\begin{aligned} \{\gamma_i\} = \{ & (-\alpha_1, C - \beta, B - \beta, 0, 0, \beta - A), (\alpha_2 - \beta, C - \beta, 0, \beta - B, 0, -\alpha_3), \\ & (\alpha_3 - \beta, 0, B - \beta, 0, \beta - C, -\alpha_2), (A - 2\beta, 0, 0, \beta - B, \beta - C, \alpha_1 - \beta) \} \end{aligned} \quad (3.110)$$

where $A = \alpha_1 + \alpha_2 + \alpha_3$, $B = \alpha_1 + \alpha_2$, and $C = \alpha_1 + \alpha_3$. Inserting the roots in eq. (2.29) gives four solutions

$$\phi_1 = c^{\gamma_1} \sum_{m \geq 0, n \in \mathbb{Z}} \frac{(\alpha_1)_{m+n} (A - \beta)_{m+n}}{(-\beta + C + 1)_n (-\beta + B + 1)_m (1)_m (1)_n} x^m y^n, \quad (3.111)$$

$$\phi_2 = c^{\gamma_2} \sum_{m \geq 0, n \in \mathbb{Z}} \frac{(\beta - \alpha_2)_{m+n} (\alpha_3)_{m+n}}{(-\beta + C + 1)_n (1)_m (\beta - B + 1)_m (1)_n} x^m y^n, \quad (3.112)$$

$$\phi_3 = c^{\gamma_3} \sum_{m \geq 0, n \in \mathbb{Z}} \frac{(\beta - \alpha_3)_{m+n} (\alpha_2)_{m+n}}{(1)_n (-\beta + B + 1)_m (1)_m (\beta - C + 1)_n} x^m y^n, \quad (3.113)$$

$$\phi_4 = c^{\gamma_4} \sum_{m \geq 0, n \in \mathbb{Z}} \frac{(2\beta - A)_{m+n} (\beta - \alpha_1)_{m+n}}{(1)_n (1)_m (\beta - B + 1)_m (\beta - C + 1)_n} x^m y^n, \quad (3.114)$$

where we have defined $x = (c_3 c_4)/(c_1 c_6)$ and $y = (c_2 c_5)/(c_1 c_6)$. Due to our choice of weight vector, the sum has been only restricted in m . However, it is easy to see that the sum over negative integers n vanish due to the presence of $(1)_n$. Hence the above sums have the form

$$\sum_{m, n \geq 0} \frac{(a)_{m+n} (b)_{m+n}}{(c)_m (d)_n (1)_m (1)_n} x^m y^n \quad (3.115)$$

which are sum representations of the Appell hypergeometric function F_4 (see eq. (A.5)). In order to compute the integration constants we can follow our prescription and set $c_4 = c_5 = 0$ in eq. (3.106) and take $\Omega = \mathbb{R}^3$. This computes the first coefficient in the expansion K_1 . We

follow the same procedure for the remaining integration constants. They can be collectively written as

$$K_r = \frac{1}{\Gamma(\beta)} \sum_{i \neq 0} \Gamma(-\gamma_r^i). \tag{3.116}$$

Setting $c_1 = c_2 = c_3 = 1$ and $c_4 = s_3$, $c_5 = s_1$, and $c_6 = s_2$, we can write the Feynman integral as

$$\begin{aligned} I(\alpha) = & K_1 s_2^{\beta-A} F_4(\alpha_1, A - \beta; -\beta + \alpha_{13} + 1, -\beta + \alpha_{12} + 1; s_3/s_2, s_1/s_2) \\ & + K_2 s_2^{-\alpha_3} s_3^{\beta-B} F_4(\beta - \alpha_2, \alpha_3; C - \beta + 1, -B + \beta + 1; s_3/s_2, s_1/s_2) \\ & + K_3 s_2^{-\alpha_2} s_1^{\beta-C} F_4(\beta - \alpha_3, \alpha_2; B - \beta + 1, -C + \beta + 1; s_3/s_2, s_1/s_2) \\ & + K_4 s_1^{\beta-C} s_2^{\alpha_1-\beta} s_3^{\beta-B} F_4(2\beta - A, \beta - \alpha_1; -B + \beta + 1, -C + \beta + 1; s_3/s_2, s_1/s_2), \end{aligned} \tag{3.117}$$

which agrees with the results obtained via the Mellin-Barnes integral representations [67, 68] and the negative dimension approach [62].

4 Conclusions and outlook

In this paper we have studied the relation between the Lee-Pomeransky representation of Feynman integrals and GKZ systems. We have shown that in generic cases we can associate a matrix A of $\text{co}(A) > 0$ to a deformed polynomial $g_r(c, z) = r(c, z) + \mathcal{U}(c) + \mathcal{F}(c)$, where $r(c, z)$ is introduced to ensure a canonical series representation. $\mathcal{U}(c)$ and $\mathcal{F}(c)$ are toric polynomials associated with the Symanzik polynomials. Under these restrictions, we can interpret a large class of Feynman integrals as furnishing a solution of a GKZ system based on A . The canonical series algorithm then allows us to evaluate integrals with arbitrary powers in the propagators as linear combinations of A -hypergeometric functions. Feynman integrals are recovered at the end of the computation by identifying the coefficients of the toric polynomials with their kinematic values.

Using the canonical series method, we have evaluated several integrals for arbitrary noninteger powers in the propagators. A particularly nontrivial example is the on-shell massless box with arbitrary powers in the propagators. With this method the result was obtained as a particular case of an A -hypergeometric integral and it matches the result based on the recurrence relations method based on Gröbner bases [63] and the negative dimension approach [66]. Another nontrivial example is the $\text{co}(A) = 2$ example of the three-scale massless triangle, which is in agreement with the negative dimension approach as well. It would be interesting to study the relation between those methods and the canonical series algorithm.

Computing canonical series is a straightforward computational algebra problem. However, these series heavily depend on the choice of a weight vector w , which sets the initial ideal and effectively chooses a domain of convergence. This choice is tied with the available information about the integration cycle and ultimately with our ability to compute integration constants. Our recipe of setting $\Omega = \mathbb{R}_+^N$ is the obvious choice and was motivated by the cycles, which one obtains by studying the coamoeba of $g_r(z)$ in Euclidean kinematics.

A full characterization of the coamoeba of $g_r(z)$ might be necessary when non-Euclidean kinematics is considered and going to higher codimensions.

The rank of the system arising from the toric version of $g_r(z)$ is bounded by $\text{vol}(\mathbf{A})$ which, in general, is greater than the number of master integrals arising from IBP identities or from the Euler characteristic [69]. The canonical series algorithm produces $\text{vol}(\mathbf{A})$ hypergeometric series in $\text{co}(\mathbf{A})$ variables. They collapse to simpler expressions once we set the coefficients c to their kinematic values. This typically involves setting one or more of these variables to unity, which amounts to evaluate hypergeometric functions at singular points. At $\text{co}(\mathbf{A}) = 1$ the functions appearing at those limits are Γ -functions. At $\text{co}(\mathbf{A}) = 2$, those limits lead to hypergeometric functions of one variable, which then collapse to simpler expressions. It would be interesting to study the mechanism which relates the number of master integrals and the number of canonical series solutions.

We believe that the application of GKZ systems and canonical series to Feynman integrals is not limited to the Lee-Pomeransky representation. Indeed, it would be interesting to apply these ideas in representations where an algebraic definition of the integration cycles is available. For instance, this is the case of the representation due to Baikov [70], which has recently been studied specially in the context of maximal cuts [71–75]. This approach is closely related to ref. [76], where bases of Pfaffian systems for GKZ systems are constructed using twisted cohomology groups.

We leave these explorations for future work.

Acknowledgments

We thank Einan Gardi for useful discussions and encouragement during the realization of this project. We thank Jens Forsgård for providing his MATHEMATICA notebook for drawing coamoebas. We thank Claude Duhr for useful comments. We thank the anonymous referee for useful comments. We thank the Galileo Galilei Institute in Florence for hospitality and partial financial support for the workshop “Amplitudes in the LHC era” in Autumn 2018. We thank the ETH Institute for Theoretical Studies in the framework of the program “Periods, modular forms and scattering amplitudes” for hospitality and partial financial support. The author’s research is supported by the STFC Consolidated Grant ‘Particle Physics at the Higgs Centre’.

A Useful formulas

Integrals.

$$\int_{\mathbb{R}_+} \frac{z^\alpha}{(a+bz)^\beta} \frac{dz}{z} = \frac{\Gamma(\beta-\alpha)\Gamma(\alpha)}{\Gamma(\beta)} a^{\alpha-\beta} b^{-\alpha}, \tag{A.1}$$

$$\text{Re}(\alpha) > 0, \quad \text{Re}(\beta-\alpha) > 0.$$

$$\int_{\mathbb{R}_+} \frac{z^\alpha}{(1+z)^{\beta_1}(1+cz)^{\beta_2}} \frac{dz}{z} = \frac{\Gamma(\beta_1+\beta_2-\alpha)\Gamma(\alpha)}{\Gamma(\beta_1+\beta_2)} {}_2F_1(\alpha, \beta_2; \beta_1+\beta_2; 1-c), \tag{A.2}$$

$$\text{Re}(\beta_1+\beta_2) > \text{Re}(\alpha) > 0, \quad |\arg c| < \pi$$

Sum representations.

$${}_2F_1(a, b, c; x) = \sum_{n \geq 0} \frac{(a)_n (b)_n}{(c)_n (1)_n} x^n, \quad (\text{A.3})$$

$${}_3F_2(a, b, c; d, e; x) = \sum_{n \geq 0} \frac{(a)_n (b)_n (c)_n}{(d)_n (e)_n (1)_n} x^n, \quad (\text{A.4})$$

$$F_4(a, b, c, d; x, y) = \sum_{m, n \geq 0} \frac{(a)_{m+n} (b)_{m+n}}{(c)_m (d)_n (1)_m (1)_n} x^m y^n. \quad (\text{A.5})$$

Linear transformations.

$${}_2F_1(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad (\text{A.6})$$

$${}_2F_1(a, b, c; z) = (1-z)^{(c-a-b)} {}_2F_1(c-b, c-a; c; z), \quad (\text{A.7})$$

$$\begin{aligned} {}_2F_1(a, b, c; z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b; a+b-c+1; 1-z) \quad (|\arg(1-z)| < \pi) \quad (\text{A.8}) \\ &+ (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} {}_2F_1(c-a, c-b; c-a-b+1; 1-z), \end{aligned}$$

$${}_2F_1(a, b, c; z) = (1-z)^{-a} \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} {}_2F_1\left(a, c-b; a-b+1; \frac{1}{1-z}\right) \quad (|\arg(1-z)| < \pi) \quad (\text{A.9})$$

$$+ (1-z)^{-b} \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} {}_2F_1\left(b, c-a; b-a+1; \frac{1}{1-z}\right).$$

Pochhammer identities.

$$\begin{aligned} (0)_0 &= 1, \\ (0)_m &= 0, \quad m \in \mathbb{N}, & (a)_{-m} &= \frac{(-1)^m}{(1-a)_m}, \\ (0)_{-m} &= \frac{(-1)^m}{(1)_m}, \quad m \in \mathbb{N}, & (a)_{m+n} &= (a)_m (a+m)_n. \end{aligned} \quad (\text{A.10})$$

B Macaulay2 example

In this appendix, we will give a short example using *Macaulay2* [77] of the algorithm to compute fake indicial ideal for the party hat integral. The matrix A associated with this integral is given in eq. (3.81). Our starting point will be to compute the toric ideal I_A . Using the procedure in Part II (Toric Hilbert Schemes) of ref. [57] gives

```
i2 : A={{1, 1, 1, 1, 1, 1}, {1, 1, 0, 0, 0, 0}, {0, 0, 1, 1, 0, 1}, {1, 0, 1,
      0, 1, 1}, {0, 1, 0, 1, 1, 1}};
i3 : toricIdeal A
o3 = ideal(b*c - a*d).
```

We write

$$I_A = \langle \partial_2 \partial_3 - \partial_1 \partial_4 \rangle. \quad (\text{B.1})$$

The next step is to compute the initial ideal of I_A with respect to a weight vector w . Using the built-in *Macaulay2* package *D-modules* to obtain the initial ideal with respect to $w = (0, 1, 1, 1, 1, 1)$ gives

```
i4 : loadPackage "Dmodules";
i5 : D=QQ[a,b,c,d,e,f, Da,Db,Dc, Dd,De,Df,
      WeylAlgebra=>{a=>Da,b=>Db,c=>Dc,d=>Dd,e=>De,f=>Df}];
i6 : IA=ideal(Db*Dc-Da*Dd);
o6 : Ideal of D
i7 : toString inw(IA,{0,-1,-1,-1,-1,-1,0,1,1,1,1,1})
o7 = ideal(Db*Dc)
```

Notice that $\mathbb{K}[\partial_1, \dots, \partial_n]$ is a commutative ring and hence $\text{in}_{(-w,w)}(I_A) = \text{in}_w(I_A)$. We have the monomial ideal

$$\text{in}_w(I_A) = \langle \partial_2 \partial_3 \rangle. \quad (\text{B.2})$$

In *Macaulay2* the `standardPairs` function computes the Standard Pairs of a monomial ideal. We define a monomial ideal in the commutative ring $\mathbb{Q}[a, \dots, f]$ and compute its standard pairs.

```
i8 : R=QQ[vars(0..5)];
i9 : Iinw=monomialIdeal(b*c);
o9 : MonomialIdeal of R
i10 : standardPairs Iinw
o10 = {{1, {d, c, a, e, f}}, {1, {d, b, a, f, e}}}
o10 : List
```

$$\mathcal{S}(\text{in}_{(w)}(I_A)) = \{\{1, \{1, 3, 4, 5, 6\}\}, \{1, \{1, 2, 4, 5, 6\}\}\}, \quad (\text{B.3})$$

$$\text{ind}_w(I_A) = \bigcap_{(\partial^a, K) \in \mathcal{S}(\text{in}_w(I_A))} \langle (\theta_j - a_j), j \notin K \rangle = \langle \theta_2 \rangle \cap \langle \theta_3 \rangle, \quad (\text{B.4})$$

therefore

$$\text{fin}_w(H_A(\kappa)) = \langle \theta_2 \theta_3 \rangle + \langle A\theta - \kappa^T \rangle. \quad (\text{B.5})$$

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