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Feynman's Path Integral

Definition Without Limiting Procedure

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Abstract. Feynman's integral is defined with respect to a pseudomeasure on the space of paths: for instance, let $\mathscr{C}$ be the space of paths $q : T \subset \mathbb{R} \to$ configuration space of the system, let $\mathscr{C}'$ be the topological dual of $\mathscr{C}$; then Feynman's integral for a particle of mass m in a potential V can be written

$$\int_{\mathscr{C}} \exp(iS_{\text{int}}(q)/\hbar)\, dw(\sqrt{m}\, q)$$

where

$$S_{\text{int}}(q) = \int_T V(q(t))\, dt$$

and where dw is a pseudomeasure whose Fourier transform is defined by

$$\mathscr{F} w(\mu) = \exp(-iW(\mu)/2) = \exp\left(-\frac{i}{2}\int_T \int_T \inf(t, t')\, d\mu(t)\, d\mu(t')\right)$$

for $\mu \in \mathscr{C}'$. Pseudomeasures are discussed; several integrals with respect to pseudomeasures are computed.

I. Introduction

The lucid and powerful formalism of quantum mechanics proposed by Feynman [1] has been plagued by the limiting procedure involved in the original definition of Feynman's integral. We propose here a definition which does not rest on a limiting procedure, we show the connection between both definitions of Feynman's integral and we compute several integrals.

Feynman's formalism of quantum mechanics can be summarized in the following table:

1. Quantum experiments $\Rightarrow K(B; A) = \int_X \exp(iF(q)) \ldots$
2. Classical limit of quantum systems $\Rightarrow K(B; A) = \int_X \exp(iS(q)/\hbar) \ldots$

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3. Conservation of total probability $\Rightarrow |K(B;A)| = \left| \int_{\mathscr{C}} \exp(iS_{\text{int}}(q)/\hbar)\, dw(\sqrt{m}\, q) \right|$

4. Compatibility condition for $dw \Rightarrow |K(B;A)| = \left| \sum_{\alpha \in \pi} \chi(\alpha)\, K^{\alpha}(B;A) \right|$

1. An analysis of the quantum interferences of beams of particles going by different paths from A to B leads naturally to the idea that the total probability amplitude $K(B;A)$ for the transition from A to B is the sum of probability amplitudes $\exp(iF(q))$ attached to each possible path q between A and B; hence the idea "sum over paths" or "integral in the space X of paths q".

2. In the classical limit, only the classical path must contribute significantly. It follows that the amplitude attached to each path q is proportional to $\exp(iS(q)/\hbar)$ where S is the action whose first variation vanishes at the classical path; $\hbar$ is Planck's constant; in the classical limit $S(q)/\hbar \to \infty$.

3. The definition of dw and the expression of $K(B;A)$ in terms of dw are established in this paper. The expression written here for $K(B;A)$ with $S_{\text{int}}(q) = \int_T V(q(t))\, dt$ is one of the possible ones for a particle of mass m in a potential V.

4. Feynman's formalism is a global formalism of quantum mechanics. The use of dw brings forth the global aspect of Feynman's integral because it eliminates the division of the time interval into infinitesimal ones necessary in the original definition. Too often the time interval division gives the impression that Feynman's integral is only a method of integrating Schrödinger's equation. Locally, Feynman's and Schrödinger's formalisms are equivalent. Globally they are not equivalent [2]: for instance, the configuration space of a system is often multiply connected. It follows from a compatibility condition on dw that $K(B;A)$ is then a linear combination of partial amplitudes K^{α} where each K^{α} is an integral over paths in the same homotopy class and where the set of coefficients $\{\chi(\alpha)\}$ form a representation of the fundamental group π. The different representations of π correspond to different physical systems. Because there is no unique way to label the homotopy classes by the elements α of the fundamental group, $K(B;A)$ is determined only modulo an overall unobservable phase factor. Thus, Feynman's formalism gives directly an unambiguous answer to global problems. Other formalisms use ad hoc, extraneous conditions to deal with global problems, such as boundary conditions on wave functions, symmetry or antisymmetry property of the wave function, etc. ... and their answers are not necessarily identical with Feynman's.

II. Notation, Basic Definitions and Properties

1. Example: A Particle of Mass m with one Degree of Freedom in a Potential V

In this example, the space of paths X is the space $\mathscr{C}$ of real valued functions q on the time interval $T \equiv (t_a, t_b)$ which take a fixed value $q(t_b) = b$ at t_b. It will be necessary for $\mathscr{C}$ to be a vector space, hence $b = 0$. Indeed: $q, q' \in \mathscr{C} \Rightarrow q + q' \in \mathscr{C}$ only if $(q + q')(t_b) = b$ as well as $q(t_b) = q'(t_b) = b$. The topology on $\mathscr{C}$ is the topology induced by the uniform norm $\|q\| = \sup_{t \in T} |q(t)|$

$$q(t_j) \equiv q_j ,$$

$$t_a = t_1 < \cdots < t_n < t_{n+1} = t_b ,$$

$$\alpha_j \equiv t_{j+1} - t_j .$$

A stands for the pair (q_a, t_a).

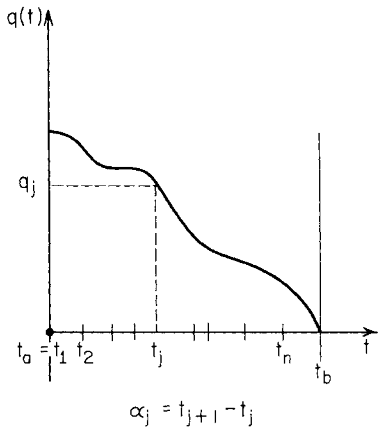


Fig. 1

The topological dual of $\mathscr{C}$ is the space $\mathscr{M}$ of bounded measures on T. For every $\mu \in \mathscr{M}$ and every $q \in \mathscr{C}$

$$\langle \mu, q \rangle = \int_T q(t)\, d\mu(t) \in \mathbb{C} .$$

2. Fourier Transform

Let X be a topological vector space that is Hausdorff and locally convex; let X' be its topological dual; let $x \in X$ and $x' \in X'$; let λ be a measure of X. The Fourier transform $\mathscr{F}\lambda$ of λ is a function on X' defined by

$$\mathscr{F}\lambda(x') = \int_X e^{-i\langle x', x \rangle}\, d\lambda(x) .$$

Note that if λ is an absolutely continuous measure $d\lambda(x)=\lambda'(x)\,dx$; in another terminology, $\mathscr{F}\lambda$ would be called the Fourier transform of λ'. Inverse image of a Fourier transform:

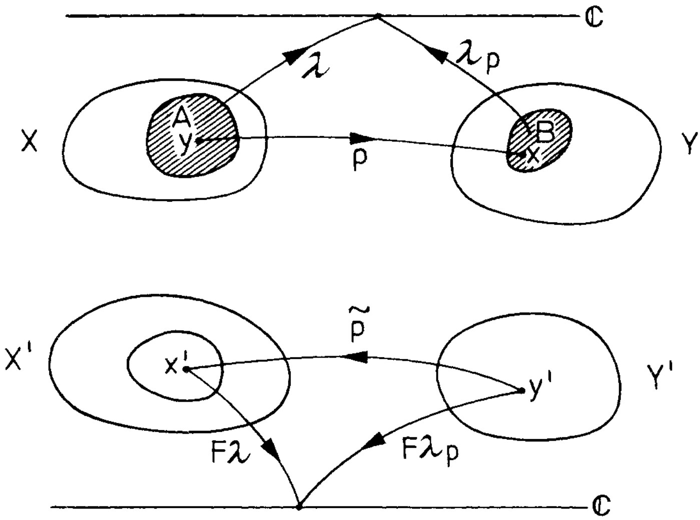


Fig. 2

Let p be a continuous linear mapping $p: X\to Y$, let $\tilde{p}$ be the transposed mapping $\tilde{p}: Y'\to X'$

$$\langle\tilde{p}(y'), x\rangle=\langle y', p(x)\rangle$$
$$\tilde{p}(y')=y'\circ p$$

let λ_p be the image of λ under p, let $B\subset Y$ and let $A=p^{-1}(B)$, let $\chi(A)$ be the characteristic function of $A\subset X$

$$\lambda_p(B)\overset{\text{def.}}{=}\lambda(A)=\int_A d\lambda(x)=\int_X \chi(A)\,d\lambda(x)$$

then:

$$\mathscr{F}\lambda_p=\mathscr{F}\lambda\circ\tilde{p}\,.$$

Applications:

1. Let $p=x'\in X'$

$$x': X\to\mathbb{R}\quad\text{by}\quad x'(x)=\langle x', x\rangle=u$$

then:

$$\mathscr{F}\lambda_{x'}(1)=\mathscr{F}\lambda\circ\tilde{x}'(1)$$
$$=\mathscr{F}\lambda(x')$$
$$\mathscr{F}\lambda(x')=\int_{\mathbb{R}} e^{-iu}\,d\lambda_{x'}(u)\,.$$

2. Let V be a closed subspace of finite codimension of X, let y be the finite dimensional quotient space X/V, let p be the canonical mapping $X \to X/V$; then $\tilde{p}$ is an isomorphism: $(X/V') \to V^0$ where $V^0 \subset X'$ is orthogonal to V:

$$x' \in V^0 \Leftrightarrow \langle x', x \rangle = 0 \qquad \forall x \in V$$

$$\begin{aligned}
\mathscr{F}\lambda_p(y') &= \int_{X/V} \exp(-i\langle y', y\rangle)\, d\lambda_p(y) && \text{by definition} \\
&= \mathscr{F}\lambda \circ \tilde{p}(y') && \text{by a property of inverse images} \\
&= \mathscr{F}\lambda(x') \quad \text{for every} \quad x' \in V^0 && \text{by a property of transposed mappings.}
\end{aligned}$$

Let $F(X)$ be the family of all closed subspaces of X which are of finite codimension, then X' is the union of V^0's orthogonal to V's in $F(X)$ and the above formula gives $\mathscr{F}\lambda$ for all $x' \in X'$ by means of a family of integrals of finite dimensional spaces.

3. *Distributions*

A measure can be considered either as an additive set function or as a distribution of rank zero. The following notation of distributions will be used:

Let λ and μ be two measures on X, their convolution $\lambda * \mu$ is defined by

$$\langle \lambda * \mu, \varphi \rangle = \int_X d\lambda(x) \int_X d\mu(x)\, \varphi(x+y)$$

for φ in the space of $\mathscr{C}'$ functions on X with compact support. The translation operator τ_{x_0} is defined by

$$\langle \tau_{x_0}\lambda, \varphi \rangle = \int_X \varphi(x_0 + x)\, d\lambda(x)\,.$$

If $d\lambda(x) = \lambda'(x)\, dx$ then $\tau_{x_0}\lambda'(x) = \lambda'(x - x_0)$. The dilatation operator ε_α is defined by

$$\langle \varepsilon_\alpha \lambda, \varphi \rangle = \int_X \varphi(\alpha x)\, d\lambda(x)\,.$$

If $d\lambda(x) = \lambda'(x)\, dx$, then $\varepsilon_\alpha \lambda'(x) = \dfrac{1}{\alpha}\lambda'\left(\dfrac{x}{\alpha}\right)$.

Note $\langle \varepsilon_\alpha \lambda * \varepsilon_\beta \lambda, \varphi \rangle = \int_X d\lambda(x) \int_X d\lambda(y)\, \varphi(\alpha x + \beta y)$.

4. *Other Notations*

	General case	Example	One dimensional	Finite dimensional	
Space of integration	$X, x \in X$	$\mathscr{C}, q \in \mathscr{C}$	$\mathbb{R}, Q \in \mathbb{R}$	$\mathbb{R}^n, Q \in \mathbb{R}^n$	X/V
Its dual	$X', x' \in X'$	$\mathscr{M}, \mu \in \mathscr{M}$	$\mathbb{R}, M \in \mathbb{R}$	$\mathbb{R}^n, Q \in \mathbb{R}^n$	$(X/V)'$
Measure	λ		$\bar{\gamma}_\alpha$		μ_v
Promeasure	$\mu = \{\mu_v\}$				
Gaussian pseudomeasure	$\Gamma_{\mathscr{Q}}$	w	$\gamma_\alpha, \gamma_{\mathscr{Q}(x')}$	Γ_n	$\Gamma_{\mathscr{Q}_v}$
Variance	$\mathscr{Q}$	W	$\alpha, \mathscr{Q}(x')$	$\{\alpha_j; j = 1, \dots n\}$	$\mathscr{Q}_v$
Covariance	K	Infimum			

III. Feynman's Integral

We shall show, in this section that Feynman's integral for a particle of mass m with one degree of freedom in a potential V can be written

$$\int_{\mathscr{C}} \exp(iS_{\text{int}}(q)\, dw(\sqrt{m}q) \quad \text{with} \quad \hbar = 1$$

w being defined as follows:

Definition of w. The Fourier transform of w is

$$\mathscr{F} w = \exp\left(-\frac{1}{2} W\right)$$

where W is the quadratic form on the topological dual $\mathscr{M}$ of $\mathscr{C}$ defined by

$$W(\mu) = \int_T \int_T \inf(t, t')\, d\mu(t)\, d\mu(t')$$

$$\inf(t, t') = \begin{cases} t & \text{if} \quad t \leqq t' \\ t' & \text{if} \quad t' \leqq t . \end{cases}$$

We shall discuss the nature of w in Section IV; for the time being we call it a pseudomeasure. In general we call "gaussian pseudomeasure of covariance K" an object whose Fourier transform is $\exp\left(-\frac{i}{2}\mathscr{Q}\right)$ with $\mathscr{Q}(\mu) = \int_T \int_T K(t, t')\, d\mu(t)\, d\mu(t')$. Thus w is a gaussian pseudomeasure of covariance infimum. For clarity, we shall break down the proof in four propositions, each illustrating a different feature of the formalism.

Feynman's integral was defined originally as the limit when $n \to \infty$ of an n-dimensional integral I_n over the space of n-tuple $\{q_1, \dots q_n\}$; namely, the path q was replaced by n of its values $\{q_1, \dots q_n\}$.

Proposition 1. *Let* $p: \mathscr{C} \to \mathbb{R}^n$ *by* $q \mapsto \{Q_1, \dots Q_n\}$ *with* $Q_j = q_j - q_{j+1}$ *and* $q_{n+1} = 0$*; then:*

$$\int_{\mathscr{C}} dw(q) = \int_{\mathbb{R}^n} \exp\left(i \sum_{j=1}^{n} \frac{(q_j - q_{j+1})^2}{2\alpha_j}\right) \prod_{j=1}^{n} \frac{dq_j}{\sqrt{2\pi i \alpha_j}} \equiv I_n$$

with $(i)^{-\frac{1}{2}} = \exp(-i\pi/4)$ *and* $\alpha_j = t_{j+1} - t_j$.

This equality connects both definitions of Feynman's integral.

Proof. Let $d\gamma_\alpha(Q) = \exp\left(i \frac{Q^2}{2\alpha}\right) \frac{dQ}{\sqrt{2\pi i \alpha}}$ with $Q \in \mathbb{R}$, let $d\gamma_{\alpha_1}(Q_1) \times \cdots \times d\gamma_{\alpha_n}(Q_n) \equiv d\Gamma_n(Q_1, \dots Q_n)$; then, by the change of variables $\{q_1, \dots q_n\} \mapsto \{Q_1, \dots Q_n\}$:

$$I_n = \int_{\mathbb{R}^n} d\Gamma_n(Q_1, \dots Q_n).$$

We shall show that $\mathscr{F} w \circ \tilde{p} = \mathscr{F} \Gamma_n$. Indeed: $\tilde{p} : \mathbb{R}^n \to \mathscr{M}$ by $M \mapsto \mu$ such that

$$\langle \tilde{p}(M), q \rangle = \langle M, p(q) \rangle = \sum_{i=1}^{n} M_i Q_i = \sum_{i=1}^{n} (M_i - M_{i+1}) q_i \quad \text{with} \quad M_{n+1} = 0$$

where $\{M_1 \dots M_n\}$ are the coordinates of M in the dual basis. The equation $\langle \mu, q \rangle = \sum_{i=1}^{n} (M_i - M_{i+1}) q_i$ determines $d\mu(t)$

$$d\mu(t) = \sum_{i=1}^{n} (M_i - M_{i+1})\, \delta_{t_i}\, dt$$

where δ is Dirac's measure: $\langle \delta_{t_i}, q \rangle = q(t_i)$. Hence

$$\begin{aligned} W(\mu) &= \int_T \int_T \inf(t, t') \left[\sum_{i=1}^{n} (M_i - M_{i+1})\, \delta_{t_i}\right]^2 dt\, dt' \\ &= \sum_i \sum_j \inf(t_i, t_j)\, (M_i - M_{i+1})\, (M_j - M_{j+1}) \\ &= \sum_{i=1}^{n} (t_j - t_{j-1})\, (M_j)^2 \end{aligned}$$

and

$$\mathscr{F} w \circ \tilde{p}(M) = \exp\left(-\frac{i}{2} W(\mu)\right) = \exp\left[-\frac{i}{2} \sum_{j=1}^{n} (t_j - t_{j-1})\, (M_j)^2\right].$$

On the other hand

$$\begin{aligned} \mathscr{F} \Gamma_n(M) &= \int_{\mathbb{R}^n} \exp\left(-i \sum_j M_j Q_j\right) d\Gamma_n(Q_1, \dots Q_n) \\ &= \exp\left[-\frac{i}{2} \sum_{j=1}^{n} (t_j - t_{j-1})\, (M_j)^2\right] \qquad \text{c.q.f.d.} \end{aligned}$$

By analogy with the theory of distributions, we shall write formally:

$$\int_{\mathscr{C}} dw(q) = \langle w, 1 \rangle$$

$$\int_{\mathscr{C}} \exp(iS_{\text{int}}(q))\, dw(q) = \langle w, \exp(iS_{\text{int}}) \rangle .$$

Because w is defined by its Fourier transform, we shall often use the following pattern to integrate with respect to w:

$$\langle w, \varphi \rangle = \langle \bar{\mathscr{F}} \mathscr{F} w, \varphi \rangle = \langle \mathscr{F} w, \bar{\mathscr{F}} \varphi \rangle = \langle \exp(-iW/2), \bar{\mathscr{F}} \varphi \rangle .$$

For instance

$$\langle w, 1 \rangle = \langle \exp(-iW/2), \bar{\mathscr{F}} 1 \rangle = \langle \exp(-iW/2), \delta \rangle = 1 .$$

Proposition 2. *Consider a free particle of mass $m = 1$ with one degree of freedom; let S_0 be the action for this system, let $K(B; A) \equiv K(b, t_b; a, t_a)$ be the probability amplitude that the particle be at b at time t_b if it is at a at time t_a, let $\bar{q}$ be the classical path from A to B, let $\hbar = 1$; let $\chi(q; q_a = 0)$ be the characteristic function of the subspace of $\mathscr{C}$ which consists of functions q vanishing at t_a:*

$$\chi(q; q_a = 0) = \begin{cases} 1 & \textit{if} \quad q(t_a) \equiv q_a = 0 \\ 0 & \textit{otherwise}; \end{cases}$$

then:

$$K(B; A) = \exp(iS_0(\bar{q})) \int_{\mathscr{C}} \chi(q; q_a = 0)\, dw(q)$$
$$= \exp(iS_0(\bar{q}))/\sqrt{2\pi i(t_b - t_a)} .$$

Proof. In this paper path integration is restricted to spaces of paths which are linear spaces. When we have to consider paths with fixed end points, we make a change of variable $q' \mapsto q$ such that the new variable q vanish at the end points: $q' = \bar{q} + q$ where $\bar{q}$ is a fixed path such that $\bar{q}(t_a) = a$ and $\bar{q}(t_b) = b$; a convenient choice for $\bar{q}$, here, is the classical path from A to B. Because the Taylor expansion of the action S_0 of a free particle terminates at the second variation, $S_0(q')$ is given in terms of the new variable by

$$S_0(\bar{q} + q) = S_0(\bar{q}) + \tfrac{1}{2} S_0''(\bar{q})(q, q) .$$

According to Feynman's original definition

$$K(B; A) = \exp(iS_0(\bar{q})) \int_{\mathbb{R}^n} \exp\left(i \sum_{j=1}^{n} \frac{(q_j - q_{j+1})^2}{2\alpha_j} \right) \prod_{j=1}^{n} \frac{dq_j}{\sqrt{2\pi i \alpha_j}}\, \delta(q_a) .$$

In this case it is not necessary to take the limit for $n \to \infty$ of the n-dimensional integral. By virtue of Proposition 1:

$$K(B; A) = \exp(iS_0(\bar{q})) \int \chi(q; q_a)\, dw(q) = \exp(iS_0(\bar{q}) \langle w, \chi \rangle .$$

The computation of $\langle w, \chi \rangle$ is straightforward:

$\langle w, \chi \rangle = \langle \gamma_{\alpha_1} * \cdots * \gamma_{\alpha_n}, \delta \rangle$	using Proposition 1
$= \langle \gamma_\alpha, \delta \rangle$ with $\alpha = \sum_{j=1}^{n} \alpha_j = t_b - t_a$	using the Fourier transform of γ_α
$= (2\pi i(t_b - t_a))^{-\frac{1}{2}}$	c.q.f.d.

Proposition 3. *Consider a free particle of mass m with one degree of freedom; then:*

$$K(B; A) = \exp(iS_0(\overline{q})) \int_{\mathscr{C}} \chi(q; q_a = 0)\, dw(\sqrt{m}\, q)$$
$$= \exp(iS_0(\overline{q})) \sqrt{m/2\pi i(t_b - t_a)}\,.$$

Proof. Let $q' = \alpha q + q_0$ where α is a constant and q_0 is a fixed path, let $\chi(q'; q'_a = 0)$ represent the same set as $\chi(q; q_a = 0)$; then

$$\chi(q'; q'_a = 0) = \varepsilon_\alpha \tau_{q_0} \chi(q; q_a = 0)\,.$$

Proposition 3 follows readily from:

$$\chi(q; q_a = 0) = \sqrt{m}\, \chi(\sqrt{m}\, q_a = 0)\,.$$

Proposition 4. *This proposition should be considered as a heuristic statement until a theory of integration with respect to pseudomeasures has been developed which determines the class of potentials for which the path integrals are defined. Consider a particle with mass m and one degree of freedom in a potential V; then*

$$K(B; A) = \exp(iS_0(\overline{q})) \sqrt{m} \int_{\mathscr{C}} \exp(iS_{\text{int}}(\overline{q} + q))\, \chi(q; q_a = 0)\, dw(q) \qquad (1)$$

or

$$K(B; A) = \exp(iS(\overline{q})) \int_{\mathscr{C}} \exp(i\Sigma(q))\, \chi(q; q_a\, 0)\, dw(\sqrt{S''(\overline{q})}\, q) \qquad (2)$$

with

$$\Sigma(q) = S(\overline{q} + q) - S(\overline{q}) - \tfrac{1}{2} S''(\overline{q})\,(q, q)\,.$$

Proof. The first equation is proved readily by splitting the action into a free particle term and an interaction term:

$$S = S_0 + S_{\text{int}} \quad \text{with} \quad S_{\text{int}}(q) = \int_{t_a}^{t_b} V(q(t))\, dt\,.$$

The second equation is proved readily by expanding the action around the classical path $\overline{q}$:

$$S(\overline{q} + Q) = S(\overline{q}) + \tfrac{1}{2} S''(\overline{q})\,(q, q) + \Sigma(q)\,.$$

The WKB approximation consists in setting $\Sigma(q)=0$. With this approximation, $K(B;A)$ can immediately be integrated by using Proposition 3:

$$K(B'A)=\exp(iS(\bar{q}))\sqrt{S''(\bar{q})/2\pi i(t_b-t_a)}\,.$$

Because we integrate on the space $\mathscr{C}$ of continuous functions q which are not necessarily differentiable, Eq. (1) is valid only if the potential is velocity independent[1] and Eq. (2) is valid only if $\Sigma(q)$ is velocity independent, i.e. if

$$\frac{\partial^3 V(\bar{q},\dot{\bar{q}})}{\partial\bar{q}^2\,\partial\dot{\bar{q}}}=\frac{\partial^3 V(\bar{q},\dot{\bar{q}})}{\partial q\,\partial\dot{\bar{q}}^2}=\frac{\partial^3 V(\bar{q},\dot{\bar{q}})}{\partial\dot{\bar{q}}^3}=0\,.$$

However, the method proposed here to compute Feynman's path integral is not restricted to integration on the space $\mathscr{C}$ with respect to the pseudomeasure w and hence not limited in general to potentials which satisfy the above conditions. The space of integration is determined by the configuration space of the system. The pseudomeasure is determined in part by the classical limit of the system and in part by the requirement of probability conservation [3]. Whether these requirements are sufficient is an open question. The study of systems whose potential V violates the above conditions, for instance a particle in an arbitrary magnetic field, will bring some light to this question. We shall postpone until section V, paragraph 2, further discussion of the validity of Eqs. (1) and (2).

IV. Promeasures, Pseudomeasures

The study of Feynman's integral is a study of integration on non-locally compact spaces with respect to an object more general than a promeasure [4] called for the time being, pseudomeasure.

1. *Promeasures*

A summary of the theory of promeasures is given here as a point of departure for a discussion of pseudomeasures. Full details on promeasures can be found in Bourbaki [4]. This point of departure is by no means unique, the theory of affine measures is another one [5].

Definition. The theory of promeasures generalizes the theory of integration to spaces which are not locally compact – in particular, to a large class of function spaces. It is restricted to topological vector spaces that are Haussdorff and locally convex, let X, Y... be such spaces; let V,

[1] The word "velocity" refers to the velocity along the classical path; we never use the differential $\dot{q}$ of an arbitrary $q\in\mathscr{C}$, we never take the limit for $n\to\infty$ of such expressions as

$$\sum_{j=1}^{n}(q_{j+1}-q_j)^2/2(t_{j+1}-t_j)\,.$$

$W\ldots$ be closed subspaces of X of finite codimension; let $F(X)$ be the set of all closed subspaces of X of finite codimension, $V \in F(X)$; let p_v be the canonical mapping from X into the quotient space X/V; let μ_v be a finite countably additive measure on X/V such that

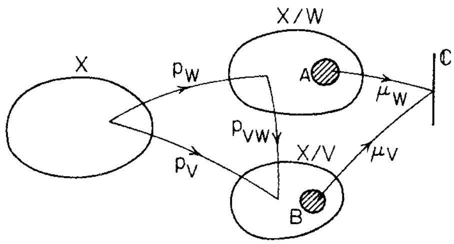
a) $\mu_v(X/V) \equiv \mu(X)$ is independent of V,

b) it satisfies the following reasonable compatibility condition when $W \subset V$:

$$\mu_v = \mu_w \circ p_{vw}^{-1}$$

where p_{vw} is defined by $p_v = p_{vw} \circ p_w$.

Then the family $\mu = \{\mu_v; V \in F(X)\}$ is a promeasure.



$\mu_V(B) = \mu_W(A)$ with $A = p_{VW}^{-1}(B)$

Fig. 3

μ is also called a cylinder measure: the word cylinder comes from the fact that if $X = \mathbb{R}^3$ and V is a line through the origin, X/V is the space of lines parallel to V which generates cylinders.

Example. Let X be the space $\mathscr{C}$ defined in Section I. Let paths q, q' which take the same value for a certain finite set $\{t_1, \ldots t_n\}$ of values of $t \in T$ be considered as equivalent:

$$q \sim q' \Leftrightarrow q(t_j) = q'(t_j) \quad \text{for all } t_j \text{ in the set.}$$

$$q = q' + f \quad \text{with } f(t_j) = 0 \text{ for all } t_j \text{ in the set.}$$

Let V be the space of functions f on T which vanish on the set $\theta_v = \{t_1, \ldots t_v\}$. Then X/V is the space of equivalence classes $\{q'; q' \sim q\} \equiv [q]$; in practice we identify $[q]$ with $\{q(t_1) \ldots q(t_v)\}$ which in turn we identify with $\{Q_1, \ldots Q_v\}$ where $Q_j = q_j - q_{j+1}$.

Let $\mu_v = \bar{\gamma}_{\alpha_1} \times \cdots \times \bar{\gamma}_{\alpha_v}$ where $d\bar{\gamma}_\alpha = \exp\left(-\dfrac{Q^2}{2\alpha}\right)\dfrac{dQ}{\sqrt{2\pi\alpha}}$; then μ is Wiener's promeasure.

Wiener's promeasure is a measure. The phrase "a promeasure is a measure" is a convenient abuse of language: a promeasure is never a measure, it is a family of measures. However, if a measure λ can be defined on X, one can construct in a natural way an associated promeasure $\tilde{\lambda}$ on X; if a promeasure μ on X is equal to the associated promeasure $\tilde{\lambda}$ of a measure λ, one says that the promeasure μ is a measure.

The family of topological vector spaces $\{X/V;\ V \in F(X)\}$ is determined by the topological dual X' of X; indeed: $F(X)$ can be defined by a finite number of elements $x'_1, \dots x'_n \in X'$; moreover the topology which makes X/V Hausdorff is uniquely defined. In the previous example, the space V of functions $f \in \mathscr{C}$ which vanish on the set θ_v is defined by the finite set $\{\delta_{t_\iota} \in \mathscr{M};\ t_\iota \in \theta_v\}$ of Dirac's measures by means of the v equations

$$\langle \delta_{t_\iota}, f \rangle = 0 .$$

Fourier Transform. The structural importance of X' in the construction of a promeasure explains the importance of the Fourier transform of promeasures. The Fourier transform $\mathscr{F}\mu$ of a promeasure μ is the natural generalization of the Fourier transform of a measure:

$$\mathscr{F}\mu(x') = \int_{\mathbb{R}} \exp(-it)\, d\mu_{x'}(t) \qquad \forall x' \in X'$$

where $\mu_{x'}$ is the image by x' of μ – the image of a promeasure being defined in a natural way from the images of the members μ_v which constitute the family μ.

It can be shown that $\mathscr{F}\mu(x') = \int_{X/V} \exp(-i\langle x', x\rangle)\, d\mu_v(x) \qquad \forall x' \in V^0$.

2. *Pseudomeasures*

Superficially, or formally, Feynman's integral can be obtained from Wiener's integral by multiplying the variance α_j by i. Analytical continuation of Wiener's integral has indeed given the first rigorous definitions of Feynman's integral [6]. It is always interesting and puzzling to come across topics where formal multiplication by i gives some correct results although it changes profoundly the mathematics and the physics of the topic considered: for example the formal replacement of t by it changes a positive definite metric into a space time metric, a similar replacement changes an elliptic equation into an hyperbolic equation etc. ... The physical differences between Wiener's integration and Feynman's integration are far-reaching. The former describes systems which obey Laplace's law of probability, the latter describes quantum systems which do not obey Laplace's law of probability but the law of interfering alternatives [1]. How profound are the mathematical differences? The structures of both theories are the same, the building blocks $\bar{\gamma}_\alpha$ and γ_α are different.

In the rest of this paper we shall generalize α to be an arbitrary positive variance and obtain thereby gaussian pseudomeasures of arbitrary variance $\mathscr{Q}$. Wiener's and Feynman's integrals correspond to $\alpha = \Delta t$ and $\mathscr{Q} = W$.

Comparison of $\bar{\gamma}_\alpha$ and γ_α.

1. $\bar{\gamma}_\alpha(0, u) = \int_0^u d\bar{\gamma}_\alpha$ is the error function,

$\gamma_\alpha(0, u) = \int_0^u d\gamma_\alpha$ is a linear combination of Fresnel integrals, $S(u)$ and $C(u)$.

$$\gamma_\alpha(0, u) = \tfrac{1}{2}(S(u) + C(u)) + \frac{i}{2}(S(u) - C(u)).$$

The error function and the Fresnel integrals are related via hypergeometric functions and it can be shown that $\gamma_\alpha(0, u) = \operatorname{erf}(e^{-i\pi/4}u)$ where the analytic function $\operatorname{erf}(z)$ is equal to the error function on the positive real axis. This relation can be used to prove some of the results obtained by analytic continuation of Wiener's integral.

2. $\bar{\gamma}_\alpha$ defines a bounded countably additive positive measure on the σ-ring of Borel sets of $\mathbb{R}$; γ_α does not. It has been shown [7] that the fact that $\int_{\mathbb{R}} |d\gamma_\alpha| = \infty$ makes it impossible to use γ_α in the same manner that $\bar{\gamma}_\alpha$ is used in the construction of Wiener's promeasure. Thus as an additive set function γ_α is not a convenient measure; on the other hand, as a distribution of rank zero, it is an interesting measure:

a) γ_α is a tempered distribution:

Let φ be in the space $\mathscr{S}$ of rapidly decreasing $\mathscr{C}^\infty$ functions on $\mathbb{R}$; then:

$$\langle \gamma_\alpha, \varphi \rangle = \int_{\mathbb{R}} \varphi(Q)\, d\gamma_\alpha(Q) = \int_{\mathbb{R}} \varphi(Q)\, \gamma'_\alpha(Q)\, dQ$$

is always defined. Hence $\gamma_\alpha \in \mathscr{S}'$.

b) γ_α is in the space of operators on $\mathscr{S}'$ [9].

γ_α is in the space $\mathscr{O}_M$ of multiplication operators on $\mathscr{S}'$:
$\langle \gamma'_\alpha T, \varphi \rangle = \langle T, \gamma'_\alpha \varphi \rangle$ is defined for every $T \in \mathscr{S}'$.

γ_α is in the space $\mathscr{O}'_C$ of convolution operators on $\mathscr{S}'$:
$\langle \gamma_\gamma * T, \varphi \rangle$ is defined for every $T \in \mathscr{S}'$.

The spaces $\mathscr{O}_M$ and $\mathscr{O}'_C$ are both subspaces of $\mathscr{S}'$, hence $\langle \gamma_\alpha, \varphi \rangle$ is defined for φ in a space larger than $\mathscr{S}$. Unless stated otherwise, φ is understood in this paper to be either in the dual $\mathscr{O}'_M$ of $\mathscr{O}_M$ or in the space $\mathscr{O}_C$ of which $\mathscr{O}'_C$ is the dual.

Because it is a poor additive set function but a good distribution, we shall not work with γ_α but with its Fourier transform:

$$\mathscr{F}\gamma_\alpha(M) = \int_{\mathbb{R}} \exp(-iMQ)\, d\gamma_\alpha(Q) = \exp(-i\alpha M^2/2).$$

Construction of $\mathscr{F}\Gamma_{\mathscr{Q}}$. Let $\mathscr{Q}$ be a positive quadratic form on the dual X' of a Hausdorff locally convex topological vector space X. For example if $X=\mathscr{C}$,

$$\mathscr{Q}(\mu)=\int\int K(t,t')\,d\mu(t)\,d\mu(t') \qquad \text{where } K \text{ is a positive kernel}.$$

We shall construct the Fourier transform $\mathscr{F}\Gamma_{\mathscr{Q}}$ of a gaussian pseudo-measure $\Gamma_{\mathscr{Q}}$ on X with variance $\mathscr{Q}$ in three steps:

a) X is finite dimensional and $\mathscr{Q}$ is nondegenerate.

There is a unique basis $\{e^j; j=1,\dots n\}$ in X' which diagonalizes $\mathscr{Q}$. Let $\{x'_j\}$ be the coordinates of x' in the basis $\{e^j\}$

$$\mathscr{Q}(x')=\alpha\sum_{j=1}^{n}(x'_j)^2 \qquad \alpha>0\,.$$

Let f be the isomorphism: $X\to\mathbb{R}^n$ by $x\mapsto\{e^j(x)\}$; let $Q\in\mathbb{R}^n$ and let $\{Q^j\}$ be the coordinates of Q in the basis dual to $\{e^j\}$:

$$\langle x', f^{-1}(Q)\rangle=\sum_j x'_j Q^j\,.$$

Let Γ_n be the distribution on $\mathbb{R}^n$ defined by the cartesian product of n γ_α; let $\Gamma_{\mathscr{Q}}$ be the inverse image of Γ_n by f; then:

$$\begin{aligned}\mathscr{F}\Gamma_{\mathscr{Q}}(x')&=\int_X \exp(-i\langle x',x\rangle)\,d\Gamma_{\mathscr{Q}}(x)\\ &=\int_{\mathbb{R}^n}\exp(-i\Sigma x'_j Q^j)\,d\Gamma_n(Q)\\ &=\exp\left(-\frac{i}{2}\alpha\Sigma(x'_j)^2\right)=\exp\left(-\frac{i}{2}\mathscr{Q}(x')\right).\end{aligned}$$

b) X is finite dimensional, $\mathscr{Q}$ may be degenerate.

Let N be the linear subspace of X' which consists of the points x' such that $\mathscr{Q}(x')=0$; let M be the orthogonal of N in X; let j be the canonical injection of M in X and $\tilde{j}: X'\to M'$ be the transposed mapping of j. There exists on M' a non-degenerate positive quadratic form $\overline{\mathscr{Q}}$ such that

$$\mathscr{Q}=\overline{\mathscr{Q}}\circ\tilde{j}\,.$$

Hence we can, by the previous construction, build the distribution $\Gamma_{\overline{\mathscr{Q}}}$ whose Fourier transform is

$$\mathscr{F}\Gamma_{\overline{\mathscr{Q}}}=\exp\left(-\frac{i}{2}\overline{\mathscr{Q}}\right);$$

set $j(\Gamma_{\overline{\mathscr{Q}}})$ equal to $\Gamma_{\mathscr{Q}}$; it follows that

$$\mathscr{F}\Gamma_{\mathscr{Q}}=(\mathscr{F}\Gamma_{\overline{\mathscr{Q}}}\circ\tilde{j})=\exp\left(-\frac{i}{2}\overline{\mathscr{Q}}\circ\tilde{j}\right)=\exp\left(-\frac{i}{2}\mathscr{Q}\right).$$

c) X is infinite dimensional.

Let $V \in F(X)$, let $p_v : X \to X/V$ (see Fig. 2). $\mathcal{Q} \circ \tilde{p}_v$ defines a positive quadratic form $\mathcal{Q}_v$ on $(X/V)'$; hence we can, by the previous construction, build the distribution $\Gamma_{\mathcal{Q}_v}$ whose Fourier transform is

$$\mathcal{F}\Gamma_{\mathcal{Q}_v} = \exp\left(-\frac{i}{2}\mathcal{Q}_v\right).$$

Because $\tilde{p}_v$ is an isomorphism of $(X/V)' \to V^0$, the equation $\mathcal{F}\Gamma_{\mathcal{Q}_v} = \mathcal{F}\Gamma_{\mathcal{Q}} \circ \tilde{p}_v$ defines the restriction of $\mathcal{F}\Gamma_{\mathcal{Q}}$ to $V^0 \subset X'$.

Proposition 5. *The family* $\{\mathcal{F}\Gamma_{\mathcal{Q}_v}; V \in F(X)\}$ *characterizes* $\mathcal{F}\Gamma_{\mathcal{Q}}$ *on* X'.

Proof. 1. $\mathcal{F}\Gamma_{\mathcal{Q}}$ is defined for every $x' \in X'$ because $X' = \bigcup_{V \in F(X)} V^0$.

2. $\mathcal{F}\Gamma_{\mathcal{Q}}$ is coherently defined: if $x' \in V^0 \cap W^0$ then $\mathcal{F}\Gamma_{\mathcal{Q}_v}(x') = \mathcal{F}\Gamma_{\mathcal{Q}_w}(x')$; indeed:

a) The origin $x' = 0$ belongs to each space V^0 associated to a $V \in F(X)$, hence $\mathcal{F}\Gamma_{\mathcal{Q}_v}(0)$ must be independent of V; we have indeed $\mathcal{F}\Gamma_{\mathcal{Q}_v}(0) = 1$.

b) If $W \subset V$,
$$\mathcal{F}\Gamma_{\mathcal{Q}_v} = \mathcal{F}\Gamma_{\mathcal{Q}_w} \circ \tilde{p}_{vw} \qquad \text{c.q.f.d.}$$

In summary we have obtained a well defined Fourier transform $\mathcal{F}\Gamma_{\mathcal{Q}}$ for the system $\{\Gamma_{\mathcal{Q}_v}\}$ of distributions more general than a promeasure.

Remarks. a) When $\Gamma_{\mathcal{Q}_v}$ is a bounded measure on X/V, the coherence conditions satisfied by the family $\{\mathcal{F}\Gamma_{\mathcal{Q}_v}\}$ are equivalent to the coherence conditions satisfied by the promeasure $\Gamma_{\mathcal{Q}} = \{\Gamma_{\mathcal{Q}_v}\}$.

b) We have constructed $\mathcal{F}\Gamma_{\mathcal{Q}}$ using a gaussian distribution γ_α of rank zero; we can repeat this construction using any tempered distribution. This construction suggests the following definition[2] for projective systems of tempered distributions — to be called possibly tempered "prodistributions"[3] (by analogy with *pro*jective systems of bounded measures which are called "*pro*measures"):

Let $\mathcal{F}(X')$ be the linear space of complex valued functions f on X' such that their restrictions to any finite dimensional subspace of X' are continuous functions equivalent to tempered distributions.

Let T_v be a tempered distribution on X/V whose Fourier transform $\mathcal{F}T_v$ is such that

$$\mathcal{F}T_v = f|_{V^0} \circ \tilde{p}_v \quad \text{where} \quad f \in \mathcal{F}(X').$$

The system $\{T_v; V \in F(X)\}$ is by definition, a tempered prodistribution, inverse Fourier transform of f.

[2] This definition has been proposed by Choquet.

[3] Terminology proposed by Dieudonné who stressed the advantages of treating measures throughout this work as distributions rather than additive set functions.

This definition is useful only if the space it defines can be given a good topology: for instance, a topology such that the pseudomeasure w is a measure. As a larger or a smaller space might be more useful than the space proposed here for prodistributions, we shall not use the word prodistribution and shall call w a pseudomeasure until a general theory has been developed.

V. Integration With Respect to Pseudomeasures

A theory of integration with respect to pseudomeasures remains to be done. The following propositions provide only very partial answers, useful mostly for applications.

1. Images of Pseudomeasures under a Linear Continuous Mapping p

Definition. Let $p: X \to Y$ and $\tilde{p}$ be the transposed mapping: $Y' \to X'$. We shall call image of the pseudomeasure $\Gamma_{\mathscr{Q}}$ under p the pseudomeasure $\Gamma_{\mathscr{Q}_p}$ whose Fourier transform is such that

$$\mathscr{F}\Gamma_{\mathscr{Q}_p} \stackrel{\text{def.}}{=} \mathscr{F}\Gamma_{\mathscr{Q}} \circ \tilde{p}\,.$$

Proposition 6. *The image of the gaussian pseudomeasure $\Gamma_{\mathscr{Q}}$ under $x' \in X'$ is equal to the gaussian pseudomeasure $\gamma_{\mathscr{Q}(x')}$ on $\mathbb{R}$ of variance $\mathscr{Q}(x')$.*

Proof. $x' \in X'$ is a linear continuous mapping from $X \to \mathbb{R}$ by $x'(x) = \langle x', x\rangle = u \in R$.

$$\begin{aligned}\mathscr{F}\Gamma_{\mathscr{Q}_{x'}}(1) &= \mathscr{F}\Gamma_{\mathscr{Q}} \circ \tilde{x}'(1)\\ &= \mathscr{F}\Gamma_{\mathscr{Q}}(x') = \exp(-i\,\mathscr{Q}(x')/2)\,.\end{aligned}$$

On the other hand

$$\mathscr{F}\Gamma_{\mathscr{Q}_{x'}}(1) = \int_{\mathbb{R}} \exp(-iu)\, d\Gamma_{\mathscr{Q}_{x'}}(u)\,.$$

It follows that

$$d\Gamma_{\mathscr{Q}_{x'}}(u) = \frac{1}{\sqrt{2\pi i\,\mathscr{Q}(x')}} \exp(iu^2/2\,\mathscr{Q}(x'))\, du = d\gamma_{\mathscr{Q}(x')}(u) \qquad \text{c.q.f.d.}$$

We shall use this proposition to compute several integrals.

2. Some Integrals with Respect to Gaussian Pseudomeasures

$$I(\varphi) = \int_X \varphi(\langle x', x\rangle)\, d\Gamma_{\mathscr{Q}}(x) = \int_{\mathbb{R}} \varphi(u)\, d\gamma_{\mathscr{Q}(x')}(u)\,.$$

The integral over X has been reexpressed as an integral over $\mathbb{R}$ and can be computed by elementary methods. We give three integrals of particular interest:

$$\varphi_1 = \langle x', x\rangle^{2n} \qquad I(\varphi_1) = \frac{1}{2}\,\frac{(2n)!}{n!}\,(i\mathcal{Q}(x'))^n\,,$$

$$\varphi_2 = |\langle x', x\rangle|^n \qquad I(\varphi_2) = \frac{1}{\sqrt{\pi}}\,\Gamma\left(\frac{n+1}{2}\right)(2i\mathcal{Q}(x'))^{n/2}\,,$$

$$\varphi_3 = \langle x', x\rangle^{2n+1} \qquad I(\varphi_3) = 0\,,$$

$$\varphi_4 = \langle x', x\rangle\,\langle y', x\rangle \qquad I(\varphi_4) = \frac{i}{2}\,(\mathcal{Q}(x'+y') - \mathcal{Q}(x') - \mathcal{Q}(y'))\,.$$

$I(\varphi_4)$ is obtained from $I(\varphi_1)$ by writing

$$2\langle x', x\rangle\,\langle y', x\rangle = \langle x'+y', x\rangle^2 - \langle x', x\rangle^2 - \langle y', x\rangle^2\,.$$

In this example, the class of functions φ for which $I(\varphi)$ is defined can immediately be stated: Because $I(\varphi)$ can be reexpressed as an integral over $\mathbb{R}$ equal to $\langle\varphi, \gamma_{\mathcal{Q}(x')}\rangle$, it is defined for $\varphi\in\mathcal{O}'_M$ and $\varphi\in\mathcal{O}_C$ (Section IV, Paragraph 2).

Many integrals encountered in applications of Feynman's path integral to physical problems can, after a change of order of integration, be computed with $I(\varphi)$. We shall justify the change of order of integration which occurs in these cases:

Proposition 7.

$$\int_{\mathscr{C}} d\Gamma_{\mathcal{Q}}\left(\int_T q(t)\,m(t)\,dt\right)^2 = \int_T\int_T dt\,dt'\,m(t)\,m(t')\left(\int_{\mathscr{C}} q(t)\,q(t')\,d\Gamma_{\mathcal{Q}}\right).$$

Proof. Set J the value of $I(\varphi_1)$ for $n=0$, $X=\mathscr{C}$ and $d\mu(t)=m(t)\,dt$;

$$\begin{aligned} J &= \int_{\mathscr{C}} \langle\mu, q\rangle^2\,d\Gamma_{\mathcal{Q}} = \int_{\mathscr{C}} d\Gamma_{\mathcal{Q}}\left(\int_T q(t)\,m(t)\,dt\right)^2 \\ &= i\,\mathcal{Q}(\mu) \\ &= i\int_T\int_T K(t,t')\,m(t)\,m(t')\,dt\,dt'\,. \end{aligned}$$

On the other hand, set

$$L = \int_T\int_T m(t)\,m(t')\,dt\,dt'\left(\int_{\mathscr{C}} q(t)\,q(t')\,d\Gamma_{\mathcal{Q}}(q)\right)$$

$$\begin{aligned} \int_{\mathscr{C}} q(t)\,q(t')\,d\Gamma_{\mathcal{Q}}(q) &= \int_{\mathscr{C}} \langle\delta_t, q\rangle\,\langle\delta_{t'}, q\rangle\,d\Gamma_{\mathcal{Q}}(q) \\ &= \frac{i}{2}\,(\mathcal{Q}(\delta_t+\delta_{t'}) - \mathcal{Q}(\delta_t) - \mathcal{Q}(\delta_{t'})) \\ &= iK(t,t')\,. \end{aligned}$$

Hence $L=J$.

This proposition can readily be extended to integrals of the following type

$$J^n = \int d\Gamma_{\mathscr{D}} \left(\int_T q(t)\, m_1(t)\, dt \right)^{n_1} \left(\int_T q(t)\, m_2(t)\, dt \right)^{n_2} \dots \left(\int_T q(t)\, m_p(t) \right)^{n_p}$$

by an obvious generalization of $I(\varphi_4)$.

Remark. When one replaces q by a finite number of its values $\{q(t_j) = \langle \delta_{t_j}, q \rangle\}$, one is often lead to changes of order of integration which are justified by this proposition[4].

Remark. The probability amplitude $K(B; A)$ (Section III, Proposition 4) can often be computed in terms of $I(\varphi)$:

$$\Sigma(q) = \sum_{n=3}^{\infty} \frac{1}{n!} S^{(n)}(\bar{q})\, q^n .$$

When it is justified to integrate $K(B; A)$ term by term[5], and to expand $\Sigma(q)$ in a power series[6] then

$$K(B; A) = \sum_{p=0}^{\infty} \sum_{n=3}^{\infty} \frac{1}{p!} \left(\frac{i S^{(n)}(\bar{q})}{n!} \right)^p J^{np}$$

where J^{np} is an integral of the J^n type.

One method to determine the range of applicability of Eq. (2) for the calculation of $K(B; A)$ is to compute the radius of convergence of this double series.

3. *Integration with Respect to Pseudomeasures*

$\langle \varphi, \Gamma_{\mathscr{D}} \rangle = \langle \bar{\bar{\mathscr{F}}} \varphi, \mathscr{F} \Gamma_{\mathscr{D}} \rangle$ and the existence of $\langle \varphi, \Gamma_{\mathscr{D}} \rangle$ can be reformulated in terms of the existence of $\langle \bar{\bar{\mathscr{F}}} \varphi, \mathscr{F} \Gamma_{\mathscr{D}} \rangle$. In general, the existence of an integral with respect to an arbitrary pseudomeasure – not necessarily gaussian – can be reformulated in terms of the existence of an integral of the inverse Fourier transform of the integrand with respect to the Fourier transform of the pseudomeasure. It is conjectured that the integral exists if $\bar{\bar{\mathscr{F}}} \varphi$ is an element of $\mathscr{F}(X')$, the linear space of complex valued functions on X' such that their restrictions to any finite dimensional subspace of X' are continuous functions equivalent to tempered

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[4] In the case $X = \mathscr{C}$, $I(\varphi)$ has previously been computed by such manipulations; a wrong answer, still quoted, has been obtained. The error is not due to the change of order of integration but to an error in the computation of $I(\varphi_4)$ which is very laborious when T is first divided in a finite number of intervals.

[5] For instance, if the potential V is a polynome, $\Sigma(q)$ consists of a finite number of terms.

[6] For instance, if V is proportional to a coupling constant smaller than 1.

distributions; X' is the dual of the space of integration X.

As a simple example, we shall compute $I(\varphi)$:

$$I(\varphi) = \langle \bar{\mathscr{F}}\varphi, \mathscr{F}\Gamma_{\mathscr{Q}} \rangle = \int_{\mathbb{R}} du \frac{1}{2\pi} \int dt \exp(iut)\,\varphi(t) \int_{\mathscr{C}} \exp(-iu\langle u, q\rangle)\, d\Gamma_{\mathscr{Q}}$$

$$= \frac{1}{2\pi} \int dt\, \varphi(t) \int_{\mathbb{R}} du \exp\left(iut - \frac{i}{2} u^2\, \mathscr{Q}(\mu)\right)$$

$$= \frac{1}{\sqrt{2\pi i\, \mathscr{Q}(\mu)}} \int_{\mathbb{R}} dt\, \varphi(t) \exp(it^2/2\, \mathscr{Q}(\mu)).$$

VI. Conclusion

This formalism can be extended in many directions:

1. In this paper the space X is a linear space. This implies that the configuration space S of the physical system must be a linear space; indeed:

let $q \in X : \mathbb{R} \to S$,
$q, q' \in X \Rightarrow q + q' \in X$ which in turn implies,
$q(t), q'(t) \in S \Rightarrow q(t) + q'(t) \in S$.

The recent work of Eells and Ellworthy on Wiener's integral opens up a way to study Feynman's integral for systems whose configuration space is a Riemannian manifold.

2. The theory of Fourier transforms on locally compact groups is well developed [9] and provides a natural framework for extending this paper to configuration spaces which are locally compact groups [7].

3. Extension to systems with n degrees of freedom is straightforward and follows the usual pattern: let the configuration space S of the system be an n-dimensional linear space with norm $\| \;\|_S$, let

$q : T \subset \mathbb{R} \to S$,
$q \in \mathscr{C} \times \cdots \times \mathscr{C}$ (n times) $\equiv \mathscr{C}^n$.
The norm on $\mathscr{C}^n$ is defined by $\|q\| = \sup_{t \in T} \|q(t)\|_S$.

One can treat a variety of problems with the formalism presented here; for instance, one can show that a product pseudomeasure on $\mathscr{C}^n$

[7] This extension has been suggested by Bott.

which remains a product pseudomeasure under orthogonal transformations on $\mathscr{C}^n$ is a gaussian pseudomeasure.

4. The formalism developed here for particle physics is well suited for fields, particularly gauge fields because it provides a well defined procedure to restrict the domain of integration [10].

Since its inception in 1942, Feynman's path integral has not received the attention it deserves [8]; it is hoped that this presentation of Feynman's integral will contribute to its acceptance by those who can bring forth its potentialities.

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References

1. Feynman, R. P., Hibbs, A. R.: Quantum mechanics and path integrals. New York: McGraw Hill Book Comp. 1965.
2. Laidlaw, M. G. G., DeWitt, C. Morette: Phys. Rev. D **3**, 1375—1378 (1971). — Laidlaw, M. G. G.: Ph. D. Thesis, University of North Carolina, 1971.
3. DeWitt, C. Morette: Ann. Inst. H. Poincaré **11**, 153—206 (1969). A misprint occurs on p. 162 in the line following «développons S en série de Taylor»; it should read:

$$\bar{S}(q_2, q_1') = \bar{S}(q_2, q_1) - (q_1'^{\mu} - q_1^{\mu}) \frac{\partial \bar{S}}{\partial q_1'^{\mu}}\bigg|_{q_1' = q_1} + O|q_1' - q_1|.$$

4. Bourbaki, N.: Eléments de mathematiques. Chapter IX, Volume VI — also referred to as Fasicule 35 or No. 1343 of the Actualités Scientifiques et Industrielles — Paris, Hermann 1969. See also Friedrichs, K. O., Shapiro, H. N. *et al.*: Integration of functionals. Seminar Notes of the Institute of Mathematical Sciences of New York University 1957.
5. Choquet, G.: Mesures coniques, affines et cylindriques. Conferenza, Istituto di Alta Matematica, 1968.
6. Nelson, E.: J. Math. Phys. **5**, 332—343 (1964).

[8] Dyson [11]; Gel'fand and Yaglom [12] give an interesting review and bibliography of the development of Feynman's integral from 1942–1956. The basic textbooks of quantum mechanics leave aside Feynman's formalism, it is usually presented only in the more specialized texts: for instance, A. Katz "Classical mechanics, quantum mechanics, field theory", Academic Press, 1965; G. Rosen "Formulations of classical and quantum dynamical theory", Academic Press 1969. The powerful Feynman rules, diagrams and integrals (not the path integral but the integrals related to the diagrams) are widely used but their derivation is rarely the original one using Feynman's path integral.

7. Cameron, R. H.: J. Math. Phys. **39**, 126—140 (1960).
8. Schwartz, L.: Théorie des distributions. Paris: Hermann 1966.
9. Rudin, W.: Fourier analysis on groups. New York: Interscience Publ. 1962.
10. Faddeef, L. D., Popov, V. N.: Phys. Letters **25** B, 29 (1967). See also Ref. [3], pp. 196, 200—202.
11. Dyson, F. J.: Missed opportunities. 1972. J. W. Gibbs lecture; Bull. Am. Math. Soc. to appear in 1972.
12. Gel'fand, I. M., Yaglom, A. M.: J. Math. Phys. **1**, 48—69 (1960) (translated from Uspekhi Mat. Nauk **11**, 77, 1956.).

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