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Fibonacci-Like Sequence and its Properties

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Abstract

The Fibonacci sequence is a source of many nice and interesting identities. A similar interpretation exists for Lucas sequence. In this paper, we study Fibonacci-Like sequence that is defined by the recurrence $S_n = S_{n-1} + S_{n-2}$, for all $n \geq 2$, $S_0 = 2, S_1 = 2$. The associated initial conditions are the sum of initial conditions of Fibonacci and Lucas sequences respectively. We shall define Binet's formula and generating function of Fibonacci-Like sequence. Mainly, Inducion method and Binet's formula will be used to establish properties of Fibonacci-Like sequence.

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1. INTRODUCTION

As illustrated in the tome by Koshy [11], the Fibonacci and Lucas numbers are arguably two of the most interesting sequences in all of mathematics. Many identities have been documented in an extensive list that appears in the work of Vajda [10], where they are proved by algebraic means, even though combinatorial proofs of many of these interesting identities are also given [3]. The sequence of Fibonacci numbers F_n is defined by

$$F_n = F_{n-1} + F_{n-2},\ n \ge 2,\ F_0 = 0, F_1 = 1. \tag{1.1}$$

The sequence of Lucas numbers L_n is defined by

$$L_n = L_{n-1} + L_{n-2},\ n \ge 2,\ L_0 = 2, L_1 = 1. \tag{1.2}$$

The Binet's formula for Fibonacci sequence is given by

$$F_n = \frac{\phi^n - \bar{\phi}^n}{\phi - \bar{\phi}} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}, \tag{1.3}$$

where $\phi = \frac{1+\sqrt{5}}{2} =$ Golden ratio ≈ 1.618 and $\bar{\phi} = \frac{1-\sqrt{5}}{2} \approx -0.618.$

In this paper, we present properties of Fibonacci-like sequence that is defined by

$$S_n = S_{n-1} + S_{n-2}, \text{ for all } n \ge 2,\ S_0 = 2 \text{ and } S_1 = 2. \tag{1.4}$$

Here initial conditions S_0 and S_1 are the sum of initial conditions of Fibonacci and Lucas sequences respectively, i.e. $S_0 = F_0 + L_0,\ S_1 = F_1 + L_1.$

The few terms of the sequence S_n are 2, 2, 4, 6, 10, 16, 26 and so on.

2. PRELIMINARY RESULTS OF FIBONACCI-LIKE SEQUENCE

We will need to introduce some basic results of Fibonacci-Like sequence and Fibonacci sequence.

The Fibonacci-Like sequence may also be written as

$$S_n = \sum_{k=0}^{[\frac{n}{2}]} 2\binom{n-k}{k} = 2\binom{n}{0} + 2\binom{n-1}{1} + 2\binom{n-2}{2} + \cdots + 2\binom{n/2}{n/2}, \tag{2.1}$$

where [x] is defined as the greatest integer less than or equal to x.

The relation between Fibonacci sequence and Fibonacci-Like sequence be written as $S_n = 2F_n$. (2.2)

The recurrence relation (1.1) has the characteristic equation $x^2 - x - 1 = 0$, which produces two roots as

$$\alpha = \frac{1+\sqrt{5}}{2} = \text{ Golden ratio } \approx 1.618 = \phi,\ \beta = \frac{1-\sqrt{5}}{2} \approx -0.618. \tag{2.3}$$

Now notice a few things about α and β:

$$\alpha + \beta = 1,\ \alpha - \beta = \sqrt{5} \text{ and } \alpha\beta = -1. \tag{2.4}$$

Using these two roots, we obtain Binet's formula of recurrence relation (1.4)

$$S_n = 2\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} = \frac{2}{\sqrt{5}}\left(\alpha^{n+1} - \beta^{n+1}\right),$$
$$= \frac{2}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right\}. \tag{2.5}$$

The generating function of S_n is defined by

$$S=\sum_{n=0}^{\infty} S_n x^n = \frac{2}{\left(1-x-x^2\right)}. \tag{2.6}$$

This can be written as

$$\begin{aligned} S &= \frac{2}{(x+\alpha)(x+\beta)} = \frac{2}{(1-\alpha x)(1-\beta x)} \\ &= \frac{2}{\sqrt{5}}\left(\frac{1}{(1-\alpha x)} - \frac{1}{(1-\beta x)}\right) \\ &= \frac{2}{\sqrt{5}}\left(\sum_{k=0}^{\infty}\alpha^{k+1}x^k - \sum_{k=0}^{\infty}\beta^{k+1}x^k\right) = \frac{2}{\sqrt{5}}\sum_{k=0}^{\infty}\left(\alpha^{k+1}-\beta^{k+1}\right)x^k. \end{aligned} \tag{2.7}$$

3. PROPERTIES OF FIBONACCI-LIKE SEQUENCE

Despite its simple appearance the Fibonacci-Like sequence contains a wealth of subtle and fascinating properties [8], [10].

Sums of Fibonacci –Like numbers:

Theorem 3.1: Sum of first n terms of the Fibonacci –Like sequence is defined by

$$S_1+S_2+s_3+\cdots+S_n=\sum_{k=1}^{n} S_k = S_{n+2}-4. \tag{3.1}$$

This identity becomes

$$S_1+S_2+s_3+\cdots+S_{2n}=\sum_{k=1}^{2n} S_k = S_{2n+2}-4. \tag{3.2}$$

Theorem 3.2: Sum of first n terms with odd indices is defined by

$$S_1+S_3+s_5+\cdots+S_{2n-1}=\sum_{k=1}^{n} S_{2k-1} = S_{2n}-2. \tag{3.3}$$

Theorem 3.3: Sum of first n terms with even indices is defined by

$$S_2+S_4+s_6+\cdots+S_{2n-1}=\sum_{k=0}^{n} S_{2k} = S_{2n+1}-2. \tag{3.4}$$

The identities from 3.1 to 3.3 can be derived by induction method.
If we subtract 3.4 termwise from 3.3, we get alternating sum of the first n numbers

$$S_1-S_2+S_3-S_4+\cdots+S_{2n-1}-S_{2n}=S_{2n}-2-S_{2n+1}+2=-S_{2n-1} \tag{3.5}$$

Adding S_{2n+1} to both sides of 2.5, we get

$$S_1-S_2+S_3-S_4+\cdots+S_{2n-1}-S_{2n}+S_{2n+1}=-S_{2n-1}+S_{2n+1}=S_{2n} \tag{3.6}$$

Combining (3.5) and (3.6), we get

$$S_1 - S_2 + S_3 - S_4 + \cdots + (-1)^{n+1} S_n = (-1)^{n+1} S_{n-1} \tag{3.7}$$

Theorem 3.4: Sum of square of first n terms of the Fibonacci –Like sequence is

$$S_1^2 + S_2^2 + S_3^2 + \cdots + S_n^2 = \sum_{k=1}^{n} S_k^2 = S_n S_{n+1} - 4. \tag{3.8}$$

Now we state and prove some nice identities similar to those obtained for Fibonacci and Lucas sequences [1], [2], [3] and [11].

Theorem 3.5: For positive integer n, prove that

$$S_{-n} = (-1)^n S_{n-2}, n \neq 1. \tag{3.9}$$

Theorem 3.6: For positive integer n, prove that

$$S_n^2 + S_{n+1}^2 = 2S_{2n+2} \tag{3.10}$$

Proof: By Binet's formula, first we calculate

$$S_n^2 + S_{n+1}^2 = \frac{4}{5}\left\{(\alpha^{n+1} - \beta^{n+1})^2 + (\alpha^{n+2} - \beta^{n+2})^2\right\} = \frac{4}{5}\left\{\alpha^{2n+2}(1+\alpha^2) + \beta^{2n+2}(1+\beta^2)\right\}.$$

Because $1+\alpha^2 = \sqrt{5}\alpha$ and $1+\beta^2 = -\sqrt{5}\beta,$ we get

$$S_n^2 + S_{n+1}^2 = \frac{4}{\sqrt{5}}\left\{\alpha^{2n+3} - \beta^{2n+3}\right\} = 2S_{2n+2}.$$

Theorem 3.7: For positive integer n, prove that

$$S_{n+1}^2 - S_{n-1}^2 = 2S_{2n+1}. \tag{3.11}$$

Proof: By Binet's formula, first we calculate

$$S_{n+1}^2 - S_{n-1}^2 = \frac{4}{5}\left\{(\alpha^{n+2} - \beta^{n+2})^2 - (\alpha^n - \beta^n)^2\right\} = \frac{4}{5}\left\{\alpha^{2n}(\alpha^4 - 1) + \beta^{2n}(\beta^4 - 1)\right\}.$$

Because $(\alpha^4 - 1) = \sqrt{5}\alpha^2$ *and* $(\beta^4 - 1) = -\sqrt{5}\beta^2,$ we get

$$S_{n+1}^2 - S_{n-1}^2 = \frac{4}{\sqrt{5}}\left\{\alpha^{2n+2} - \beta^{2n+2})\right\} = 2S_{2n+1}.$$

Theorem 3.8: For positive integer n, prove that

$$S_5 + S_8 + S_{11} + \cdots + S_{3n+2} = \frac{S_{3n+4}}{2} - 5. \tag{3.12}$$

Proof: By Binet's formula, we have

$$S_5 + S_8 + S_{11} + \cdots + S_{3n+2}$$
$$= \frac{2}{\sqrt{5}}\left\{(\alpha^6 - \beta^6) + (\alpha^9 - \beta^9) + (\alpha^{12} - \beta^{12}) + \cdots + (\alpha^{3n+3} - \beta^{3n+3})\right\}$$
$$= \frac{2}{\sqrt{5}}\left\{(\alpha^6 + \alpha^9 + \alpha^{12} + \cdots + \alpha^{3n+3}) - (\beta^6 + \beta^9 + \beta^{12} + \cdots + \beta^{3n+3})\right\}$$
$$= \frac{2}{\sqrt{5}}\left\{\frac{(\alpha^{3n+6} - \alpha^6)}{\alpha^3 - 1} + \frac{(\beta^6 - \beta^{3n+6})}{\beta^3 - 1}\right\}$$
$$= \frac{S_{3n+4}}{2} - 5. \quad (\text{since } \alpha^3 - 1 = \alpha, \beta^3 - 1 = \beta)$$

Theorem 3.9: For positive integer n, prove that

$$S_n^3 = \frac{4}{5}\left[S_{3n+2} - 3(-1)^{n+1} S_n\right]. \tag{3.13}$$

This can be derived same as theorem 2.3.

Theorem 3.10: For positive integer n, prove that

$$S_1^3 + S_2^3 + S_3^3 + \cdots + S_n^3$$
$$= \frac{4}{5}\left[\left\{\left(S_5 + S_8 + S_{11} + \cdots + S_{3n+2}\right) - 3\left(S_1 - S_2 + S_3 - S_4 + \cdots + (-1)^{n+1} S_n\right)\right\}\right] \tag{3.14}$$
$$= \frac{1}{5}\left\{2S_{3n+4} - 20 - \left(-1\right)^{n+1} 12 S_{n-1}\right\}. \quad \text{(By 3.8 and 3.12)}$$

This can be derived same as theorem 2.3.

Theorem 3.11: For positive integer n, prove that

$$2S_{m+n+1} = \mathrm{S_m} S_{n+1} + S_{m-1} S_n, m \geq 1, n \geq 0. \tag{3.15}$$

Proof: Let m be fixed and we proceed by inducting on n.
When $n = 0$, then

$$2S_{m+1} = \mathrm{S_m} S_1 + S_{m-1}$$
$$= 2\mathrm{S_m} + 2S_{m-1}S_0 = 2\left(\mathrm{S_m} + S_{m-1}\right) = 2S_{m+1}, \text{ which is true.}$$

When $n = 1$, then

$$2S_{m+2} = \mathrm{S_m} S_2 + S_{m-1} S_1 = 4\mathrm{S_m} + 2S_{m-1} = 2\mathrm{S_m} + 2\mathrm{S_m} + 2S_{m-1}$$
$$= 2\mathrm{S_m} + 2\mathrm{S_{m+1}} = 2\left(\mathrm{S_m} + S_{m+1}\right) = 2S_{m+2}, \text{ which is also true.}$$

Now assume that identity is true for $n = k$ and by assumption,
$2S_{m+k} = \mathrm{S_m} S_k + S_{m-1} S_{k-1}$ and $2S_{m+k+1} = \mathrm{S_m} S_{k+1} + S_{m-1} S_k$.
Adding these two relations, we get

$$2S_{m+k+1}+2S_{m+k} = \mathrm{S_m}\left(S_{k+1}+S_k\right)+S_{m-1}\left(S_k+S_{k-1}\right)$$
$$=2\left(S_{m+k+1}+S_{m+k}\right)= \mathrm{S_m}S_{k+2}+S_{m-1}S_{k+1}$$
$$= \mathrm{S_m}S_{k+2}+S_{m-1}S_{k+1}=S_{m+k+2}, \text{ which is true for } n=k+1.$$

Hence, $2S_{m+n+1} = \mathrm{S_m}S_{n+1}+S_{m-1}S_n.$

Corollary 3.11.1: If m=n, then we get

$$2S_{2n+1} = \mathrm{S_n}S_{n+1}+S_{n-1}S_n = \mathrm{S_n}\left(S_{n+1}+S_{n-1}\right)= S_{n+1}^2-S_{n-1}^2.$$

Corollary 3.11.2: If m=n-1, then we get

$$2S_{2n} = \mathrm{S_{n-1}}S_{n+1}+S_{n-2}S_n = \mathrm{S_{n-1}}\left(S_n+S_{n-1}\right)+S_{n-2}S_n$$
$$=\mathrm{S_n}\left(S_{n-1}+S_{n-2}\right)+S_{n-1}^2=\mathrm{S}_n^2+S_{n-1}^2.$$

Corollary 3.11.3: If m=2n-1, then we get

$$2S_{3n} = \mathrm{S_{2n-1}}S_{n+1}+S_{2n-2}S_n=\mathrm{S_{2n-1}}\left(S_n+S_{n-1}\right)+S_{2n-2}S_n$$
$$=S_n\mathrm{S_{2n}}+S_{2n-1}S_{n-1}.$$

Theorem 3.12: Prove that

$$S_n^2\text{-}\mathrm{S}_{n+r}S_{n-r}=(-1)^{n-r+1}S_{r-1}^2,\ r\geq 1,\ n\geq 1. \tag{3.16}$$

This can be derived same as theorem 2.3.

Theorem 3.13: Prove that

$$\mathrm{S}_{m+1}S_n-\mathrm{S}_mS_{n+1}=2(-1)^{m+1}S_{n-m-1},\ n\geq 1,\ m\geq 1. \tag{3.17}$$

This can be derived same as theorem 2.3.

Theorem 3.14: Prove that

$$\mathrm{S}_{n+1}S_{n-1}-\mathrm{S}_n^2=4(-1)^{n+1},\ n\geq 0. \tag{3.18}$$

This can be derived same as theorem 2.3.

We derive the identity related to binomial coefficients.

Theorem 3.15: Prove that

$$S_{2n}=\sum_{k=0}^{n}\binom{n}{k}S_{n-k}. \tag{3.19}$$

Proof: By recurrence relation, we have

$$S_{2n} = S_{2n-1} + S_{2n-2} = \left(S_{2n-2} + S_{2n-3}\right) + \left(S_{2n-3} + S_{2n-4}\right)$$
$$= S_{2n-2} + 2S_{2n-3} + S_{2n-4}.$$

Again

$$S_{2n} = S_{2n-2} + 2S_{2n-3} + S_{2n-4} = \left(S_{2n-3} + S_{2n-4}\right) + 2\left(S_{2n-4} + S_{2n-5}\right) + \left(S_{2n-5} + S_{2n-6}\right)$$
$$= S_{2n-3} + 3S_{2n-4} + 3S_{2n-5} + S_{2n-6}.$$

Going on n times, we get

$$S_{2n} = S_n + nS_{n-1} + \frac{n(n-1)}{2}S_{n-2} + \ldots + \frac{n(n-1)}{2}S_2 + nS_1 + S_0 = \sum_{k=0}^{n}\binom{n}{k}S_{n-k}.$$

4. CONNECTION FORMULAE

Theorem 4.1: Prove that

$$2L_{n+1} = S_{n+1} + S_{n-1},\ n \geq 1. \tag{4.1}$$

Proof: We shall prove this identity by induction.
It is easy to see that for $n = 1$

$2L_2 = 2\times 3 = 6 = 4 + 2 = S_2 + S_0.$

Now suppose that the identity holds for $n = k-2$ and $n = k-1$. Then,

$$2L_{k-1} = S_{k-1} + S_{k-3}. \tag{4.2}$$

$$2L_k = S_k + S_{k-2}. \tag{4.3}$$

Adding equation (4.2) and equation (4.3), we get

$2L_{k-1} + 2L_k = \left(S_{k-1} + S_k\right) + \left(S_{k-2} + S_{k-3}\right)$

i.e. $2L_{k+1} = S_{k+1} + S_{k-1}.$

Which is precisely our identity when $n = k$.
Hence $2L_{n+1} = S_{n+1} + S_{n-1},\ n \geq 1.$

Theorem 4.2: Prove that

$$2F_{n+1} = S_{n+1} - S_{n-1},\ n \geq 1. \tag{4.4}$$

Proof: We shall prove this identity by induction over n.
For $n = 0$, we see that

$2F_1 = 2\times 1 = 2 = 4 - 2 = S_2 - S_0$, which is true.

Now suppose that the identity holds for $n = k-1$ and $n = k-2$.
Then,

$$2F_k = S_k - S_{k-2}. \tag{4.5}$$

$$2F_{k-1} = S_{k-1} - S_{k-3}. \tag{4.6}$$

Adding equation (4.4) and equation (4.5), we get

$2F_k + 2F_{k-1} = \left(S_k + S_{k-1}\right) - \left(S_{k-2} + S_{k-3}\right)$
i.e. $2F_{k+1} = S_{k+1} - S_{k-1}$,
which is true for $n = k$.
Hence $2F_{n+1} = S_{n+1} - S_{n-1}, n \geq 1$.

Theorem 4.3: Prove that

$$S_{n+1} = L_{n+1} + F_{n+1}. \tag{4.7}$$

Proof: By (4.1) and (4.4), we have
$2F_{n+1} = S_{n+1} - S_{n-1}, n \geq 1$.
$2L_{n+1} = S_{n+1} + S_{n-1}, n \geq 1$.
Adding both above results, we get
$2F_{n+1} + 2L_{n+1} = 2S_{n+1}$, *i.e.* $S_{n+1} = L_{n+1} + F_{n+1}$.

Theorem 4.4: Prove that

$$S_{n-1} = L_{n+1} - F_{n+1}, n \geq 1. \tag{4.8}$$

Proof: Subtracting (4.1) from (4.4), we get
$2L_{n+1} - 2F_{n+1} = 2S_{n-1}$
i.e. $S_{n-1} = L_{n+1} - F_{n+1}$.

Theorem 4.5: Prove that

$$L_{2n} + F_{2n} = S_{2n}. \tag{4.9}$$

Proof: we shall use mathematical induction method over n. For $n = 0$,
$L_0 + F_0 = 2 + 0 = 2 = S_0 = S_{2\times 0}$.**,** which is true for n=0.
For $n = 1$, $L_2 + F_2 = 3 + 1 = 4 = S_2 = S_{2\times 1}$.**,** which is also true for n=1
Assume that the result is true for $n = k$, then $L_{2k} + F_{2k} = S_{2k}$.
Now

$$\begin{aligned} L_{2(k+1)} + F_{2(k+1)} &= \left(L_{2(k+1)-1} + L_{2(k+1)-2}\right) + \left(F_{2(k+1)-1} + F_{2(k+1)-2}\right) \\ &= \left(L_{2(k+1)-1} + F_{2(k+1)-1}\right) + \left(L_{2(k+1)-2} + F_{2(k+1)-2}\right) \\ &= S_{2(k+1)-1} + S_{2(k+1)-2} \qquad \text{(By induction hypothesis)} \\ &= S_{2(k+1)}. \end{aligned}$$

Therefore, $L_{2(k+1)} + F_{2(k+1)} = S_{2(k+1)}$., which is also true for n=k+1.
Hence, the result is true for all n.

5. CONCLUSION

There are many known identities established for Fibonacci and Lucas sequences. This paper describes comparable identities of Fibonacci-Like sequence. We have also developed connection formulas for Fibonacci-Like sequence, Fibonacci sequence and Lucas sequence respectively. It is easy to discover new identities simply by varying the pattern of known identities and using inductive reasoning to guess new results. Of course, the ideas can be extended to more general recurrent sequences in obvious way.

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