Siberian Mathematical Journal, Vol. 37, No. 3, 1996

FIBONACCI MANIFOLDS AS TWO-FOLD COVERINGS OF THE THREE-DIMENSIONAL SPHERE AND THE MEYERHOFF–NEUMANN CONJECTURE†)

A. Yu. Vesnin and A. D. Mednykh UDC 515.16 + 512.817.7

The present article is devoted to studying geometric properties of three-dimensional compact orientable hyperbolic manifolds with fundamental group a Fibonacci group.

The Fibonacci groups $F(2,m)$ were introduced by Conway [1] and have the following presentation:

$$F(2,m) = \langle x_1, x_2, \dots, x_m : x_i x_{i+1} = x_{i+2}, i \bmod m\rangle.$$

Algebraic properties of the groups $F(2,m)$ and their generalizations were studied in [2–4]. From a geometric point of view, the Fibonacci groups $F(2,2n)$, $n \geq 2$, with an even number of generators are most interesting. It was shown in [5] that for $n \geq 4$ the group $F(2,2n)$ is isomorphic to the fundamental group of a three-dimensional compact orientable hyperbolic manifold. The group $F(2,6)$ is isomorphic to the fundamental group of a three-dimensional compact orientable Euclidean manifold [6]. The group $F(2,4)$ is finite and isomorphic to the fundamental group of the lens space $L(5,2)$ which is a three-dimensional compact orientable spherical manifold. Thus, the group $F(2,2n)$, $n \geq 2$, is realizable as a co-compact discrete group of isometries acting without fixed points on the space X_n, where $X_2 = \mathbb{S}^3$ is the spherical 3-space, $X_3 = \mathbb{E}^3$ is the Euclidean 3-space, and for $n \geq 4$ $X_n = \mathbb{H}^3$ is the Lobachevskiĭ 3-space.

A three-dimensional manifold $M_n = X_n/F(2,2n)$, $n \geq 2$, uniformized by a Fibonacci group is referred to as *a Fibonacci manifold.*

We point out that hyperbolic Fibonacci manifolds M_n, $n \geq 4$, were studied in [7–9].

In the present article we demonstrate that each manifold M_n, $n \geq 2$, can be represented as a two-fold branched covering of the three-dimensional sphere. As a consequence, the Meyerhoff–Neumann conjecture [10] is proven on arithmeticity and volume of the manifold $N = W(3,-2;6,-1)$ obtained in [10] by Dehn surgeries with parameters $(3,-2)$ and $(6,-1)$ on the components of the Whitehead link W.

§ 1. Fibonacci Manifolds as Two-Fold Coverings

To describe geometric properties of the Fibonacci manifolds, we introduce the following family of knots and links. Denote by Th_n, $n \geq 2$, the closure of the 3-string braid $(\sigma_1\sigma_2^{-1})^n$ given in canonical generators [11]. Observe that Th_n is a three-component link if n is divisible by three, while it is a knot otherwise. In particular, Th_2 is the figure-eight knot, Th_3 is the Borromean rings, and Th_4 is the Turk's head knot. It was shown in [12] that the manifolds $S^3 \setminus Th_n$, $n \geq 2$, are hyperbolic. In [9] we established that the hyperbolic volumes of the manifolds $S^3 \setminus Th_n$, $n \geq 2$, coincide with the volumes of the Fibonacci manifolds M_{2n}. The properties of compact manifolds obtained by Dehn surgery on the knots Th_n were studied in [13]. The symmetry groups of the knots and links Th_n are described in [14].

Further we use the terminology of the theory of orbifolds as in [12, Chapter 13].

The main result of this section is the following

†) The research was supported by the Russian Foundation for Basic Research (Grant 96-01-01523) and the INTAS (Grant 94-1474).

Novosibirsk. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 37, No. 3, pp. 534–542, May–June, 1996. Original article submitted September 27, 1994.

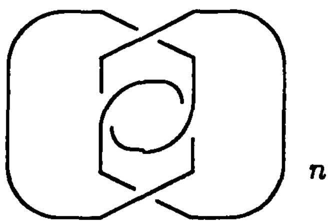

Fig. 1

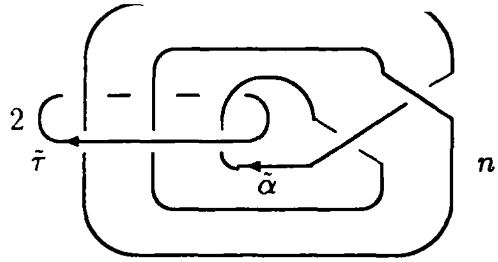

Fig. 2

Theorem 1. *Each Fibonacci manifold M_n, $n \geq 2$, can be represented as a two-fold covering of the three-dimensional sphere branched over the link Th_n.*

PROOF. In virtue of [8], the manifold M_n, $n \geq 2$, can be represented as a cyclic n-fold covering of the orbifold $\mathcal{O}(n)$ whose underlying space is the three-dimensional sphere and whose singular set is the figure-eight knot with index n (Fig. 1).

It is easy to see that the orbifold $\mathcal{O}(n)$ has a rotation symmetry of order two which acts on the singular set of the orbifold $\mathcal{O}(n)$ without fixed points. Factoring the orbifold $\mathcal{O}(n)$ by this symmetry, we obtain an orbifold whose underlying space is the three-dimensional sphere and whose singular set is a two-component link, pictured in Fig. 2, with indices 2 and n on its components.

It was shown in [12, Chapter 13] that the singular set of this orbifold is equivalent to the two-component link 6_2^2, with the notation of [15] (Fig. 3). Therefore, henceforth this orbifold will be symbolized by $6_2^2(2,n)$.

Observe that the above-introduced space X_n, $n \geq 2$, is the universal covering of the manifold M_n and the orbifolds $\mathcal{O}(n)$ and $6_2^2(2,n)$. Denote the respective fundamental groups of the manifold M_n and the orbifolds $\mathcal{O}(n)$ and $6_2^2(2,n)$ by $\Gamma_n = F(2,2n)$, Ω_n, and Π_n. Thus, the groups Γ_n, Ω_n, and Π_n act on X_n as discrete subgroups of the isometry group. Moreover, the following canonical isomorphisms hold: $M_n = X_n/\Gamma_n$, $\mathcal{O}(n) = X_n/\Omega_n$, and $6_2^2(2,n) = X_n/\Pi_n$. Therefore, the sequence of the above-described coverings

$$M_n \xrightarrow{n} \mathcal{O}(n) \xrightarrow{2} 6_2^2(2,n) \tag{1}$$

induces the sequence of group embeddings

$$\Gamma_n \triangleleft \Omega_n \triangleleft \Pi_n, \tag{2}$$

where $|\Pi_n : \Omega_n| = 2$ and $|\Omega_n : \Gamma_n| = n$.

We now use the following presentation of the fundamental group $\pi_1(S^3 \setminus 6_2^2)$ of the link 6_2^2 which was computed in [9]:

$$\langle \tilde{\alpha}, \tilde{\tau} \mid (\tilde{\tau}\tilde{\alpha}^{-1}\tilde{\tau}\tilde{\alpha}\tilde{\tau}^{-1}\tilde{\alpha}\tilde{\tau}^{-1})(\tilde{\tau}^2\tilde{\alpha}^{-1}\tilde{\tau}\tilde{\alpha}\tilde{\tau}^{-1}\tilde{\alpha}\tilde{\tau}^{-2})(\tilde{\tau}\tilde{\alpha}^{-1}\tilde{\tau}\tilde{\alpha}\tilde{\tau}^{-1}\tilde{\alpha}\tilde{\tau}^{-1})^{-1} = \tilde{\alpha}\rangle. \tag{3}$$

Here the generative loops $\tilde{\alpha}$ and $\tilde{\tau}$ are canonically determined from the diagram of the link $6_2^2(2,n)$ in Fig. 2; they correspond to the arcs of the diagram which are denoted by $\tilde{\alpha}$ and $\tilde{\tau}$ [16].

In view of the results of [17] and presentation (3), the fundamental group Π_n of the orbifold $6_2^2(2,n)$ has the following presentation:

$$\langle \alpha, \tau \mid (\tau\alpha^{-1}\tau\alpha\tau^{-1}\alpha\tau^{-1})(\tau^2\alpha^{-1}\tau\alpha\tau^{-1}\alpha\tau^{-2})(\tau\alpha^{-1}\tau\alpha\tau^{-1}\alpha\tau^{-1})^{-1} = \alpha, \alpha^n = \tau^2 = 1\rangle, \tag{4}$$

where the generators α and τ correspond to the loops $\tilde{\alpha}$ and $\tilde{\tau}$ in the group $\pi_1(S^3 \setminus 6_2^2)$.

Consider the group $\mathbb{Z}_n \oplus \mathbb{Z}_2 = \langle a \mid a^n = 1\rangle \oplus \langle t \mid t^2 = 1\rangle$ and, given an n, define an epimorphism $\theta : \Pi_n \to \mathbb{Z}_n \oplus \mathbb{Z}_2$ by the rule

$$\theta(\alpha) = a, \quad \theta(\tau) = t. \tag{5}$$

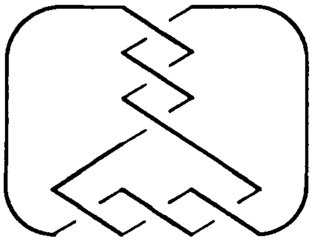

Fig. 3

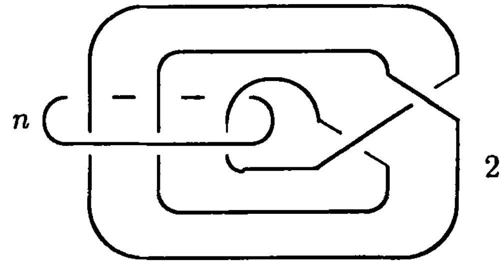

Fig. 4

Observe that under the two-fold covering $\mathcal{O}(n) \to 6_2^2(2,n)$ the loop τ in the fundamental group of the orbifold $6_2^2(2,n)$ lifts to a trivial loop in the fundamental group Ω_n of the orbifold $\mathcal{O}(n)$, while the loop α lifts to some loop that generates a cyclic subgroup of order n in the group Ω_n. Therefore,

$$\Omega_n = \theta^{-1}(\mathbb{Z}_n) = \theta^{-1}(\langle a \mid a^n = 1\rangle). \tag{6}$$

Since under the $2n$-fold covering $M_n \to 6_2^2(2,n)$ the loops α and τ lift to trivial loops in the group Γ_n, we have

$$\Gamma_n = \theta^{-1}(1) = \operatorname{Ker}\theta. \tag{7}$$

For the subgroup $\mathbb{T}_n$ in Π_n defined by

$$\mathbb{T}_n = \theta^{-1}(\mathbb{Z}_2) = \theta^{-1}(\langle t \mid t^2 = 1\rangle), \tag{8}$$

we obtain the sequence of group embeddings

$$\Gamma_n \triangleleft \mathbb{T}_n \triangleleft \Pi_n, \tag{9}$$

where $|\Pi_n : \mathbb{T}_n| = n$ and $|\mathbb{T}_n : \Gamma_n| = 2$.

The group $\mathbb{T}_n$, as a subgroup of the group Π_n, acts by isometries on the universal covering X_n and determines the orbifold $X_n/\mathbb{T}_n$. Relations (9) induce the following sequence of orbifold coverings:

$$M_n = X_n/\Gamma_n \xrightarrow{2} X_n/\mathbb{T}_n \xrightarrow{n} 6_2^2(2,n) = X_n/\Pi_n. \tag{10}$$

We will demonstrate that the underlying space of the orbifold $X_n/\mathbb{T}_n$ is the three-dimensional sphere and that the singular set of the orbifold is the above-defined link Th_n. First, we prove that the covering

$$p : X_n/\mathbb{T}_n \xrightarrow{n} 6_2^2(2,n) = X_n/\Pi_n \tag{11}$$

is cyclic. To this end, we use the following trivial lemma:

Lemma. *Given groups G, K, and L and an epimorphism $\theta : G \to K \oplus L$, suppose that $H = \theta^{-1}(L)$. Then $H \triangleleft G$ and $G/H \cong K$.*

We apply this lemma to the epimorphism $\theta : \Pi_n \to \mathbb{Z}_n \oplus \mathbb{Z}_2$ defined by conditions (5). Since $\mathbb{T}_n = \theta^{-1}(\mathbb{Z}_2)$, we have $\mathbb{T}_n \triangleleft \Pi_n$ and $\Pi_n/\mathbb{T}_n \cong \mathbb{Z}_n$. These conditions amount to the fact that the covering p is a regular n-fold cyclic covering. In view of (8), the covering p is branched over the component of the singular set of the orbifold $6_2^2(2,n)$ which is labelled by the index n.

We note [18] that in the symmetry group of the link 6_2^2 there is an involution (easily seen in Fig. 3) which interchanges the components of the link. Therefore, the singular set of the orbifold $6_2^2(2,n)$ can also be represented in the form pictured in Fig. 4.

As we see from Fig. 4, the component of the link with the index n is unknotted. Thus, p is a standard cyclic covering of the three-dimensional sphere, the underlying space of the orbifold $6_2^2(2,n)$, branched over an unknotted circle. Therefore, the underlying space of the orbifold $X_n/\mathbb{T}_n$ is the three-dimensional sphere. Observe that the component of the singular set of the orbifold $6_2^2(2,n)$ which is labelled in Fig. 4 by the index 2 is the closure of the 3-string braid $\sigma_1\sigma_2^{-1}$. Therefore, its lift to the n-fold cyclic covering $X_n/\mathbb{T}_n$ is the closure of the braid $(\sigma_1\sigma_2^{-1})^n$, i.e. coincident with the link Th_n considered above. So, the underlying space of the orbifold $X_n/\mathbb{T}_n$ is the three-dimensional sphere and the singular set of the orbifold is the link Th_n labelled with index 2. Further we use the notation $Th_n(2) = X_n/\mathbb{T}_n$ for this orbifold.

Comparing sequences (1) and (10), we come to the following diagram:

$$\begin{array}{ccccc} & & M_n & & \\ & {}^{2}\swarrow & & \searrow^{n} & \\ Th_n(2) & & & & \mathcal{O}(n) \\ & {}_{n}\searrow & & \swarrow_{2} & \\ & & 6_2^2(2,n) & & \end{array}$$

As we see from the diagram, the Fibonacci manifold M_n is a two-fold covering of the orbifold $Th_n(2)$, which completes the proof of Theorem 1.

We recall [19] that a *π-orbifold* is an orbifold whose underlying space is the three-dimensional sphere and whose singular set is a knot or a link whose every component has index 2. Theorem 1 enables us to prove the following assertion whose particular instance in the case of $n = 4$ was noted in [20].

Corollary. *For $n \geq 4$ the π-orbifold $Th_n(2)$ is hyperbolic.*

Indeed, in virtue of the results of [5], for $n \geq 4$ the universal covering X_n is the Lobachevskiĭ space $\mathbb{H}^3$. Thus, $\mathbb{T}_n$ is a discrete group of isometries of the space $\mathbb{H}^3$ and hence $Th_n(2) = \mathbb{H}^3/\mathbb{T}_n$, $n \geq 4$, is a hyperbolic orbifold.

§ 2. The Hyperbolic Meyerhoff–Neumann Manifold

In [10] some closed orientable three-dimensional hyperbolic manifold, denoted by $N = W(3,-2;6,-1)$, was constructed by Dehn surgeries with parameters $(3,-2)$ and $(6,-1)$ on the components of the Whitehead link W. The volume of the manifold N has been shown to coincide to within 10^{-50} with the volume of the regular ideal tetrahedron in Lobachevskiĭ space. The authors of [10] raised the question of exact equality between these volumes. In the same article the conjecture was formulated of arithmeticity of the manifold N over the field $\mathbb{Q}(\sqrt{-3})$ [21]. Below the validity of both hypotheses will be established.

First, we reveal some relationship between the Meyerhoff–Neumann manifold N and the hyperbolic Fibonacci manifold M_4.

Theorem 2. *The Fibonacci manifold M_4 is a two-fold unbranched covering of the Meyerhoff–Neumann manifold N.*

Proof. Consider the hyperbolic π-orbifold $Th_4(2) = \mathbb{H}^3/\mathbb{T}_4$ defined at the end of Section 1. Its underlying space is the three-dimensional sphere and its singular set is the Turk's head knot Th_4 pictured in Fig. 5. The orbifold has a rotation symmetry ϱ of order 4 which keeps the singular set invariant. After factorizing the orbifold $Th_4(2)$ by the involution ϱ^2 we obtain a π-orbifold $D(2,2)$ whose singular set is the two-component link pictured in Fig. 6.

Using the Wirtinger presentation and the results of [17], we write down the presentation for the fundamental group Δ of the orbifold $D(2,2)$:

$$\Delta = \langle \alpha, \beta, \tau \mid (\tau\alpha\tau\beta)^2\alpha(\beta\tau\alpha\tau)^2 = \tau\beta\tau, (\beta\alpha)^2\tau\beta\tau(\alpha\beta)^2 = \tau\alpha\tau, \alpha^2 = \beta^2 = \tau^2 = 1\rangle, \qquad (12)$$

where the generators α, β, and τ are canonically determined from the diagram of the link pictured in Fig. 6.

By the Mostow rigidity theorem, the involution ϱ^2 is isotopic to some isometry of the hyperbolic orbifold $Th_4(2)$. Thus the group Δ can be realized as a discrete subgroup of the isometry group of the Lobachevskiĭ space $\mathbb{H}^3$. In this case τ is the lifting of the involution ϱ^2 to the universal covering.

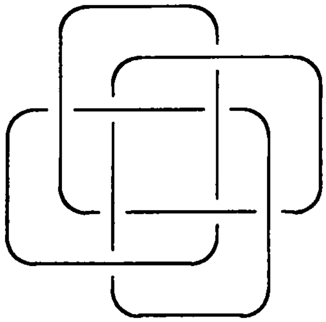

Fig. 5

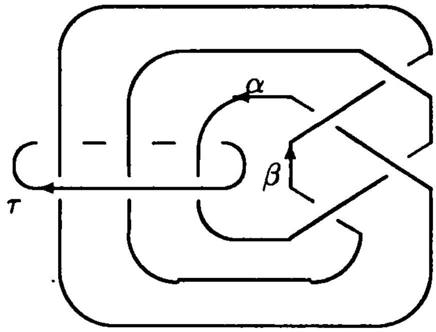

Fig. 6

It follows from Theorem 1 that the Fibonacci manifold $M_4 = \mathbb{H}^3/\Gamma_4$ is a two-fold covering of the orbifold $Th_4(2) = \mathbb{H}^3/\mathbb{T}_4$. By the construction of the orbifold $D(2,2) = \mathbb{H}^3/\Delta$ we thus have the following sequence of orbifold coverings

$$M_4 \xrightarrow{2} Th_4(2) \xrightarrow{2} D(2,2), \tag{13}$$

which induces the sequence of group embeddings

$$\Gamma_4 \triangleleft \mathbb{T}_4 \triangleleft \Delta, \tag{14}$$

where $|\Delta : \mathbb{T}_4| = 2$ and $|\mathbb{T}_4 : \Gamma_4| = 2$.

Consider the epimorphism

$$\theta : \Delta \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 =< a \mid a^2 = 1 > \oplus < t \mid t^2 = 1 >$$

defined by the following rule

$$\theta(\alpha) = \theta(\beta) = a, \quad \theta(\tau) = t. \tag{15}$$

Observe that under the orbifold covering $Th_4(2) \rightarrow D(2,2)$ the loop τ in the fundamental group Δ of the orbifold $D(2,2)$ lifts to a trivial loop in the fundamental group $\mathbb{T}_4$ of the orbifold $Th_4(2)$, while the loops α and β lift to loops generating cyclic subgroups of order two in the group $\mathbb{T}_4$. Thus,

$$\mathbb{T}_4 = \theta^{-1}(\mathbb{Z}_2) = \theta^{-1}(\langle a \mid a^2 = 1 \rangle). \tag{16}$$

Since the loops α, β, and τ lift to trivial loops in the group Γ_4 under the 4-fold covering $M_4 \rightarrow D(2,2)$, we have

$$\Gamma_4 = \theta^{-1}(1) = \operatorname{Ker}\theta. \tag{17}$$

It is obvious that the group

$$\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle a \mid a^2 = 1 \rangle \oplus \langle t \mid t^2 = 1 \rangle$$

includes the cyclic subgroup of order 2 generated by the element $d = a + t$. Define a natural epimorphism $\lambda : \mathbb{Z}_2 \oplus \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$ by setting

$$\lambda(a) = \lambda(t) = d. \tag{18}$$

For the composition $\varphi = \lambda \circ \theta$ of the epimorphisms we infer that the epimorphism

$$\varphi : \Delta \rightarrow \mathbb{Z}_2 = \langle d \mid d^2 = 1 \rangle \tag{19}$$

satisfies the relations

$$\varphi(\alpha) = \varphi(\beta) = \varphi(\tau) = d. \tag{20}$$

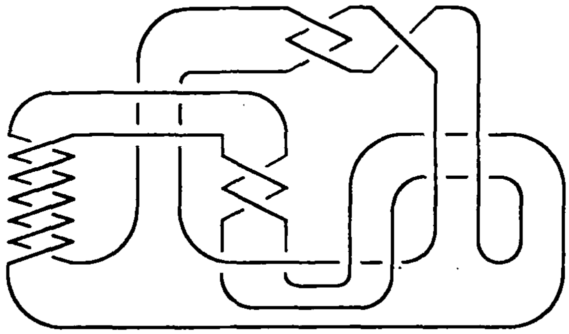

Fig. 7

Put $\Phi = \operatorname{Ker}\varphi$ and consider the orbifold $U = \mathbb{H}^3/\Phi$. By the construction of the epimorphism φ the orbifold covering

$$U = \mathbb{H}^3/\Phi \xrightarrow{2} D(2,2) = \mathbb{H}^3/\Delta \tag{21}$$

is branched over the two components of the singular set of the orbifold $D(2,2)$. Under this covering the loops α, β, and τ lift to trivial loops in the group Φ. Thus, U is a hyperbolic orbifold with the empty singular set; i.e., a hyperbolic manifold.

Demonstrate that $U = N$. To this end, we use the Montesinos algorithm [22], which makes it possible to represent each manifold obtained by a Dehn surgery on a strongly-invertible link as a branched two-fold covering of the three-dimensional sphere. Applying this algorithm to the Whitehead link we in particular infer that the Meyerhoff–Neumann manifold $N = W(3,-2;6,-1)$ is the two-fold covering branched over the two-component link pictured in Fig. 7.

Using Reidemeister moves [16] we directly check that the links pictured in Fig. 6 and 7 are equivalent. Thus, the hyperbolic manifolds U and N are represented as two-fold coverings of the three-dimensional sphere branched over the same link. Therefore, they are homeomorphic and, by the Mostow rigidity theorem, they are isometric. We thus can assume that $N = \mathbb{H}^3/\Phi$. In view of the relation $\varphi = \lambda \circ \theta$ the following group embedding holds

$$\Phi = \operatorname{Ker}\varphi > \Gamma_4 = \operatorname{Ker}\theta \tag{22}$$

which induces the covering of the manifolds

$$M_4 = \mathbb{H}^3/\Gamma_4 \xrightarrow{2} N = \mathbb{H}^3/\Phi. \tag{23}$$

Comparing the sequences of coverings (13), (21), and (23) we come to the following diagram

$$\begin{array}{ccccc} & & M_4 & & \\ & {}^{2}\swarrow & & \searrow^{2} & \\ N & & & & Th_4(2) \\ & {}_{2}\searrow & & \swarrow_{2} & \\ & & D(2,2) & & \end{array}.$$

The groups Γ_4 and Φ are the fundamental groups of the hyperbolic manifolds M_4 and N and so they do not contain elements of finite order. This implies that covering (23) induced by the group embedding (22) is unbranched. The proof of Theorem 2 is complete.

From the above connection between manifolds M_4 and N we derive the following theorem.

Theorem 3. *The hyperbolic Meyerhoff–Neumann manifold $N = W(3,-2;6,-1)$ is arithmetic over the field $\mathbb{Q}(\sqrt{-3})$ and its volume equals the volume of the regular ideal tetrahedron in Lobachevskiĭ space.*

PROOF. Indeed, as shown in [9], the hyperbolic volume of the Fibonacci manifold M_4 equals the doubled volume of the regular ideal tetrahedron in Lobachevskiĭ space. It follows from Theorem 2 that the volume of the manifold N is half of the volume of the manifold M_4 and therefore coincides with the volume of the regular ideal tetrahedron. Moreover, in view of (23), the manifolds M_4 and N are commensurable. According to [5], the manifold M_4 is arithmetic over the field $\mathbb{Q}(\sqrt{-3})$. Hence, the same is true for the manifold N.

In conclusion the authors are glad to express their gratitude to Prof. J. Mennicke, Prof. H. Helling, Prof. J. Montesinos, and Prof. B. Zimmermann for fruitful discussions of the results of the present article.

References

1. J. Conway, "Advanced problem 5327," Amer. Math. Monthly, **72**, 915 (1965).
2. J. Conway, "Solution to Advanced problem 5327," Amer. Math. Monthly, **74**, 91–93 (1967).
3. D. L. Johnson, J. W. Wamsley, and D. Wright, "The Fibonacci groups," Proc. London Math. Soc., **29**, 577–592 (1974).
4. R. M. Thomas, "The Fibonacci groups $F(2, 2m)$," Bull. London Math. Soc., **21**, No. 5, 463–465 (1989).
5. H. Helling, A. C. Kim, and J. Mennicke, A Geometric Study of Fibonacci Groups [Preprint/SFB-343, Diskrete Strukturen in der Mathematik], Bielefeld (1990).
6. B. Zimmermann, "On the Hantzche–Wendt manifold," Monatsh. Math., **110**, No. 3–4, 321–327 (1990).
7. J. Hempel, "The lattice of branched covers over the figure-eight knot," Topology Appl., **34**, No. 2, 183–201 (1990).
8. H. M. Hilden, M. T. Lozano, and J. M. Montesinos, "The arithmeticity of the figure-eight knot orbifolds," in: Topology'90, Walter de Gruyter, Berlin, 1992, pp. 169–183.
9. A. Yu. Vesnin and A. D. Mednykh, "Hyperbolic volumes of Fibonacci manifolds," Sibirsk. Mat. Zh., **36**, No. 2, 266–277 (1995).
10. R. Meyerhoff and W. D. Neumann, "An asymptotic formula for the eta invariants of hyperbolic 3-manifolds," Comment. Math. Helv., **67**, 28–46 (1992).
11. J. S. Birman, "Braids, links, and mapping class groups," Ann. of Math. Stud., No. 82 (1975).
12. W. P. Thurston, The Geometry and Topology of Three-Manifolds, Princeton Univ., Princeton (1980) (Lecture Notes in Math.).
13. S. Kojima and D. D. Long, "Virtual Betti numbers of some hyperbolic 3-manifolds," in: A Fete of Topology, Academic Press Inc., 1988, pp. 417–442.
14. M. Sakuma and J. Weeks, "Examples of canonical decompositions of hyperbolic link complements," Proc. Appl. Math. Workshop., **4**, KAIST, Taejon, 1994, pp. 185–232.
15. D. Rolfsen, Knots and Links, Publish of Perish Inc., Berkeley, Co. (1976).
16. R. H. Crowell and R. H. Fix, Introduction to Knot Theory [Russian translation], Mir, Moscow (1967).
17. A. Haefliger and N. D. Quach, "Une presentation de groupe fundamental d'une orbifold," Astérisque, **116**, 98–107 (1984).
18. C. Adams, M. Hildebrand, and J. Weeks, "Hyperbolic invariants of knots and links," Trans. Amer. Math. Soc., **326**, 1–56 (1991).
19. M. Bolieau and B. Zimmermann, "The π-orbifold group of a link," Math. Z., **200**, 187–208 (1989).
20. M. B. Thistlethwaite, "Knot tabulation and related topics," London Math. Soc. Lecture Note Ser. 93, 1–76 (1985).
21. A. Borel, "Commensurability classes and hyperbolic volumes," Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **8**, 1–33 (1991).
22. J. M. Montesinos and W. Whitten, "Constructions of two-fold branched covering spaces," Pacific J. Math., **125**, 415–446 (1986).

TRANSLATED BY A. YU. VESNIN