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Journal

FIBONACCI QUARTERLY, 16(6)

ISSN

0015-0517

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Publication Date

1978

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Peer reviewed

24. O. Struve, *The Universe* (Cambridge, Mass.: MIT Press, 1962).
 25. Brian Marsden, letter to the author dated 1976.
 26. B. A. Read, *The Fibonacci Quarterly*, Vol. 8 (1970), pp. 428-438.
 27. F. X. Byrne, *Bull. Amer. Astron. Soc.*, Vol. 6 (1974), pp. 426-427.
 28. L. H. Wasserman, *et al.*, *Bull. Amer. Astron. Soc.*, Vol. 9 (1977), p. 498.

FIBONACCI NUMBERS IN COIN TOSSING SEQUENCES

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The Fibonacci numbers and their generating function appear in a natural way in the problem of computing the expected number [2] of tosses of a fair coin until two consecutive heads appear. The problem of finding the expected number of tosses of a p -coin until k consecutive heads appear leads to classical generalizations of the Fibonacci numbers.

First consider tossing a fair coin and waiting for two consecutive heads. Let O_n be the set of all sequences of H and T of length n which terminate in HH and have no other occurrence of two consecutive heads. Let S_n be the number of sequences in O_n . Any sequence in O_n either begins with T , followed by a sequence in O_{n-1} , or begins with HT followed by a sequence in O_{n-2} . Thus,

$$(1) \quad S_n = S_{n-1} + S_{n-2}, \quad S_1 = 0, \quad S_2 = 1.$$

Consequently, $S_{n-2} = F_n$, the n th Fibonacci number. The probability of termination in n trials is $S_n/2^n$. Letting

$$g(x) = \sum_2^{\infty} S_n x^n,$$

and using the generating function $(1 - x - x^2)^{-1}$ for the Fibonacci numbers, yields $g(x) = x^2/(1 - x - x^2)$. Hence, the expected number of trials is

$$\sum_{n=1}^{\infty} n S_n / 2^n = (1/2)g'(1/2) = 6.$$

We generalize this result to the following

Theorem: Consider tossing a p -coin, $Pr(H) = p$, repeatedly until k consecutive heads appear. If P_n is the probability of terminating in exactly n trials (tosses), then the generating function

$$(2) \quad G(x) = \sum_k^{\infty} P_n x^n \text{ is given by } G(x) = \frac{(px)^k (1 - px)}{1 - x + \frac{(1-p)}{p}(px)^{k+1}}$$

The expected number of trials, $G'(1)$ is

$$(3) \quad 1/p + 1/p^2 + \cdots + 1/p^k = \frac{1}{1-p} \left[\frac{1}{p^k} - 1 \right].$$

Proof: Let O_n be the set of all sequences of H and T of length n which terminate in k heads and have no other occurrence of k consecutive heads. Let S_n be the number of sequences in O_n and $P_n = Pr(O_n)$ be the probability of the event O_n . One possibility is that a sequence in O_n begins with a T , followed by a sequence in O_{n-1} ; the probability of this is

$$\Pr(T)\Pr(O_{n-1}) = qP_{n-1}, \quad q = 1 - p.$$

The next possibility to consider is that a sequence in O_n begins with HT , followed by a sequence in O_{n-2} ; this has probability

$$\Pr(HT)\Pr(O_{n-2}) = qpP_{n-2}.$$

Continuing in this way, the last possibility to be considered is that a sequence in O_n begins with $k-1$ H 's followed by a T and then by a sequence in O_{n-k} , the probability of which is $qp^{k-1}P_{n-k}$. Hence, the recursion:

$$(4) \quad \begin{aligned} P_n &= qP_{n-1} + qpP_{n-2} + \cdots + qp^{k-1}P_{n-k}, \\ P_1 &= P_2 = \cdots = P_{k-1} = 0, \quad P_k = p^k. \end{aligned}$$

(Note that the probability of achieving k heads with k tosses is p^k , while with less than k tosses it is impossible.) The technique to find the generating function for the Fibonacci numbers applies to finding

$$G(x) = \sum_k P_n x^n.$$

Consider

$$H(x) = \sum_{n=k}^{\infty} P_{n+1} x^n;$$

then

$$xH(x) = \sum_k P_{n+1} x^{n+1} = \sum_k P_n x^n - P_k x^k = G(x) - (px)^k.$$

Hence,

$$H(x) = [G(x) - (px)^k]/x.$$

On the other hand,

$$\begin{aligned} H(x) &= \sum_k P_{n+1} x^n = \sum_k (qP_n + qpP_{n-1} + \cdots + qp^{k-1}P_{n-k+1}) x^n \\ &= q \sum_k P_n x^n + qp x \sum_k P_{n-1} x^{n-1} + \cdots + q(px)^{k-1} \sum_k P_{n-k+1} x^{n-k+1}, \end{aligned}$$

and recalling that $P_j = 0$ for $j < k$,

$$\begin{aligned} &= q \sum_k P_n x^n + qp x \sum_k P_n x^n + \cdots + q(px)^{k-1} \sum_k P_n x^n \\ &= qG[1 + px + \cdots + (px)^{k-1}] = qG \left[\frac{1 - (px)^k}{1 - px} \right]. \end{aligned}$$

Solving for G yields (2).

In the case $p = 1/2$, the combinatorial numbers $S_n = 2^n P_n$ satisfy the recursion $S_n = S_{n-1} + S_{n-2} + \cdots + S_{n-k}$. For these numbers, the generating function $(1 - x - x^2 - \cdots - x^k)^{-1}$ was found by V. Schlegel in 1894. See [1, Chap. XVII] for this and other classical references.

An alternate solution to the problem can be obtained as follows. Consider a sequence of experiments: Toss a p -coin X_1 times, until a sequence of $k-1$ heads occurs. Then toss the p -coin once more and if it comes up heads, set $Y = 1$. If not, toss the p -coin X_2 times until a sequence of $k-1$ heads occurs again, and then toss the p -coin once more and if it comes up heads, set $Y = 2$. If not, continue on in this fashion until finally the value of Y is set. At this time, we have observed a sequence of k heads in a row for the first time, and we have tossed the coin $Y + X_1 + X_2 + \cdots + X_Y$ times. The X_i are independent, identically distributed random variables and Y is independent

of all of the X_i . Let E_k = the expected number of tosses to observe k heads in a row. Let $Z = X_1 + \dots + X_Y$. Then,

$$\begin{aligned} E_k &= E(Y + Z) = E(Y) + E(Z) \\ &= E(Y) + E(Z|Y = 1)Pr(Y = 1) + E(Z|Y = 2)Pr(Y = 2) + \dots \\ &= E(Y) + \sum_{n=1}^{\infty} E(Z|Y = n)Pr(Y = n) = E(Y) + \sum_{n=1}^{\infty} nE(X_1)Pr(Y = n) \\ &= E(Y) + E(X_1)E(Y). \end{aligned}$$

But $E(Y)$ = the expected number of tosses to observe a head = $1/p$, and $E(X_1) = E_{k-1}$. Thus $E_k = 1/p + (1/p)E_{k-1}$, which yields (3).

REFERENCE

1. L. E. Dickson, *History of the Theory of Numbers*, Vol. I (1919; Chelsea reprint 1966).
2. W. Feller, *Introduction to Probability Theory and Its Applications*, Vol. I (New York: John Wiley & Sons, 1968).

STRONG DIVISIBILITY SEQUENCES WITH NONZERO INITIAL TERM

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In 1936, Marshall Hall [1] introduced the notion of a k th order linear divisibility sequence as a sequence of rational integers $u_0, u_1, \dots, u_n, \dots$ satisfying a linear recurrence relation

$$(1) \quad u_{n+k} = a_1 u_{n+k-1} + \dots + a_k u_n,$$

where a_1, a_2, \dots, a_k are rational integers and $u_m | u_n$ whenever $m | n$. Some divisibility sequences satisfy a stronger divisibility property, expressible in terms of greatest common divisors as follows:

$$(u_m, u_n) = u_{(m, n)}$$

for all positive integers m and n . We call such a sequence a *strong divisibility sequence*. An example is the Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, \dots$.

It is well known that for any positive integer m , a linear recurrence sequence $\{u_n\}$ is periodic modulo m . That is, there exists a positive integer M depending on m and a_1, a_2, \dots, a_k such that

$$(2) \quad u_{n+M} \equiv u_n \pmod{m}$$

for all $n \geq n_0[m, a_1, a_2, \dots, a_k]$; in particular, $n_0 = 0$ if $(a_k, m) = 1$.

Hall [1] proved that a linear divisibility sequence $\{u_n\}$ with $u_0 \neq 0$ is *degenerate* in the sense that the totality of primes dividing the terms of $\{u_n\}$ is finite. One should expect a stronger conclusion for a linear strong divisibility sequence having $u_0 \neq 0$. The purpose of this note is to prove that such a sequence must be, in the strictest sense, periodic. That is, there must exist a positive integer M depending on a_1, a_2, \dots, a_k such that

$$u_{n+M} = u_n, \quad n = 0, 1, \dots$$