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FIBONACCI NUMBERS, LUCAS NUMBERS AND INTEGRALS OF CERTAIN GAUSSIAN PROCESSES

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ABSTRACT. We study the distributions of integrals of Gaussian processes arising as limiting distributions of test statistics proposed for treating a goodness of fit or symmetry problem. We show that the cumulants of the distributions can be expressed in terms of Fibonacci numbers and Lucas numbers.

1. INTRODUCTION

Let $X_1, \dots, X_n, \dots$ be a sequence of independent and identically distributed d-dimensional random (column) vectors with Fourier transform ϕ. Let $\phi_n(t) = \frac{1}{n}\sum_{k=1}^{n} \exp(it'X_k)$, $t \in \mathbb{R}^d$, be the empirical Fourier transform of the first n observation vectors. It follows from the Glivenko-Cantelli theorem that as $n \to \infty$, the ϕ_n converge almost surely to ϕ uniformly on compact subsets of $\mathbb{R}^d$. This gives the motivation for various statistical procedures based on empirical Fourier transforms. Let us concentrate on two applications. The first one is a goodness of fit problem. Assuming that the distribution of the X_k is unknown, a suitable test statistic for testing the hypothesis that the X_k have the d-variate unit normal distribution N_d with density $f_d(x) = (\frac{1}{2\pi})^{d/2} \exp(-|x|^2/2)$, $x \in \mathbb{R}^d$, and Fourier transform $\phi(t) = \exp(-|t|^2/2)$, $t \in \mathbb{R}^d$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$, is

$$T_{1n} = n \int |\phi_n(t) - \exp(-|t|^2/2)|^2 \, dN_d(t); \tag{1}$$

see Keller [10], considering the case $d = 1$, or Baringhaus and Henze [3], treating the composite hypothesis of multivariate normality. As a second application we mention the problem of testing the hypothesis of symmetry, meaning that the random vectors X_k and $-X_k$ have the same distribution. Then one may suggest the test statistic

$$T_{2n} = n \int [\operatorname{Im} \phi_n(t)]^2 \, dQ(t), \tag{2}$$

where Q is taken to be a distribution symmetric about the origin; see Feuerverger and Mureika [9], considering the case $d = 1$. Choosing $Q = N_d$ we have the

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representations

$$T_{in} = \int G_{in}(t)^2 \, dN_d(t), \qquad i = 1, 2,$$

with

$$G_{1n}(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (\cos(t' X_k) + \sin(t' X_k) - \exp(-|t|^2/2)), \qquad t \in \mathbb{R}^d,$$

and

$$G_{2n}(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \sin(t' X_k), \qquad t \in \mathbb{R}^d.$$

For unit normal X_k the covariance functions $\rho_i(s,t)$ of the empirical processes $\{G_{in}(t), t \in \mathbb{R}^d\}$, $i = 1, 2$, are

$$\rho_1(s,t) = \exp(-|s-t|^2/2) - \exp(-|s|^2/2) \exp(-|t|^2/2), \qquad s, t \in \mathbb{R}^d, \tag{3}$$

and

$$\rho_2(s,t) = \frac{1}{2} [\exp(-|s-t|^2/2) - \exp(-|s+t|^2/2)], \qquad s, t \in \mathbb{R}^d. \tag{4}$$

It will be demonstrated in the next section that in the case of d-variate unit normal X_k the process $\{G_{in}(t), t \in \mathbb{R}^d\}$ has limiting sample path continuous Gaussian processes $\{G_i(t), t \in \mathbb{R}^d\}$ with zero mean and covariance function ρ_i, implying weak convergence of the laws of $T_{in} = \int G_{in}(t)^2 \, dN_d(t)$ to the laws of $T_i = \int G_i(t)^2 \, dN_d(t)$. The present paper aims to study these limit laws. It will be seen in Section 3 and Section 4 that the cumulants can be expressed in terms of Fibonacci and Lucas numbers.

2. Preliminaries

In what follows $\{G_n(t)\}$ and ρ stand for the sequence of processes $\{G_{1n}(t)\}$ with covariance function ρ_1 or for the sequence of processes $\{G_{2n}(t)\}$ with covariance function ρ_2. Let K be any compact subset of $\mathbb{R}^d$. Denote by $\mathcal{C}(K)$ the separable Banach space of real valued continuous functions on K with the supremum norm. G_n restricted to K can be regarded as a random element in $\mathcal{C}(K)$. Applying Theorem 3.1 of Csörgő [4] or a central limit theorem for Banach space valued random variables (see, e.g. Araujo and Giné [2], Corollary 7.17) we obtain the weak convergence of $\{G_n(t),\, t \in K\}$ to some zero mean Gaussian process $\{G(t),\, t \in K\}$ with covariance function $\rho|_{K \times K}$. Adapting the results of Whitt [12] we get the weak convergence of the process $\{G_n(t),\, t \in \mathbb{R}^d\}$ to some zero mean Gaussian process $\{G(t),\, t \in \mathbb{R}^d\}$ with covariance function ρ. Here these processes are regarded as random elements in the separable Fréchet space of all real valued continuous functions on $\mathbb{R}^d$ endowed with the σ-algebra of Borel sets generated by the topology of uniform convergence on compacta. Applying the Continuous Mapping Theorem we see that given any compact subset K in $\mathbb{R}^d$, the integrals $\int_K G_n(t)^2 \, dN_d(t)$ converge in distribution to $\int_K G(t)^2 \, dN_d(t)$. Given any $\eta > 0, \epsilon > 0$ we can choose some compact set $K \subset \mathbb{R}^d$ such that

$$P\left(\int_{K^c} G_n(t)^2 \, dN_d(t) > \epsilon \right) \le \frac{1}{\epsilon} \int_{K^c} \rho(t,t) \, dN_d(t) \le \eta$$

for all n. From this it follows that the integrals $\int G_n^2\,dN_d(t)$ converge in distribution to $T=\int G(t)^2\,dN_d(t)$. To study the distribution of T we introduce the Hilbert-Schmidt operator $B_d : L_2(\mathbb{R}^d, N_d) \to L_2(\mathbb{R}^d, N_d)$, defined by $(B_d f)(t) = \int \rho(s,t) f(s)\,dN_d(s)$, $t\in\mathbb{R}^d$, $f\in L_2(\mathbb{R}^d,N_d)$. Let $\{\lambda_k, e_k, k\geq 1\}$ be a system of eigenvalues and orthonormal eigenfunctions of B_d, that is, $\int e_k^2\,dN_d = 1$, $\int e_k e_l\,dN_d = 0$, $k\neq l$, $\int \rho(s,t)e_k(s)\,dN_d(s) = \lambda_k e_k(t)$. The operator B_d is of trace class with $\sum_{k\geq 1}\lambda_k = \int \rho(s,s)\,dN_d(s) < \infty$, and its Fredholm determinant δ_d is known to be $\delta_d(z) = \prod_{k\geq 1}(1-z\lambda_k)$, $z\in\mathbb{C}$; see Dunford and Schwartz [6]. There is a version of the process $\{G(t), t\in\mathbb{R}^d\}$ which can be regarded as a Gaussian random element in the Hilbert space $L_2(\mathbb{R}^d, N_d)$. Its law is the same as that of $\sum_{k\geq 1}\sqrt{\lambda_k}e_k W_k$, where the W_k are independent standard normal variables; see Araujo and Giné [2], page 157. It follows that $T=\int G^2\,dN_d$ has the same distribution as $\sum_{k\geq 1}\lambda_k W_k^2$. Its Laplace transform is $\delta_d(-2t)^{-1/2}$, $t\geq 0$.

3. The integral $\int G_1(t)^2\,dN_d(t)$

Let $\rho=\rho_1$. To derive the complete system of eigenvalues and orthonormal eigenfunctions of the integral operator B_d, we introduce the integral operator $B_{0d} : L_2(\mathbb{R}^d, N_d) \to L_2(\mathbb{R}^d, N_d)$ defined by

$$B_{0d} f(y) = \int \exp\left(-\frac{1}{2}|x-y|^2\right) f(x)\,dN_d(x), \qquad f\in L_2(\mathbb{R}^d,N_d),\ y\in\mathbb{R}^d.$$

Let us treat the case $d=1$ first. For abbreviation, we put $N=N_1=N(0,1)$ and $b=\frac{1}{2}(\sqrt{5}-1)$. Note, that $b+1$ is the Golden Section number; see Vajda [11]. For verifying some of the subsequent assertions, it is useful to remember the identities $b^2=1-b, b=1/(1+b), b^2=1/(b+2)$. Denoting the Hermite polynomial of degree n by $\mathcal{H}e_n(x) = (-1)^n\exp(x^2/2)\frac{d^n}{dx^n}\exp(-x^2/2)$, $x\in\mathbb{R}$, we have

$$\text{(5)}\quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\left(-\frac{1}{2}(x-y)^2\right)\mathcal{H}e_n(ax)\,dx = (1-a^2)^{n/2}\mathcal{H}e_n\left(\frac{ay}{(1-a^2)^{1/2}}\right),$$

for any real $a, |a|<1$, and $y\in\mathbb{R}$ (see Erdélyi, Magnus, Oberhettinger and Tricomi [8], page 290, formula (17)). Starting from (5) and the orthogonality relations for the Hermite polynomials (see Erdélyi, Magnus, Oberhettinger and Tricomi [8], page 289, formulas (9) and (11)) it can be verified that

(6)

$$\gamma_m = b^{2m+1},\quad g_m(x) = \frac{5^{1/8}}{\sqrt{m!}}\mathcal{H}e_m(5^{1/4}x)\exp\left(-\frac{b}{2}x^2\right), \qquad x\in\mathbb{R},\ m=0,1,\ldots,$$

is a complete system of eigenvalues and orthonormal eigenfunctions of B_{01}. In the case $d>1$ then obviously

(7)

$$g_{d;m_1,\ldots,m_d}(x_1,\ldots,x_d) = \prod_{j=1}^d g_{m_j}(x_j), \qquad x_j\in\mathbb{R},\ m_j\in\{0,1,\ldots\},\ j=1,\ldots,d,$$

is a complete system of orthonormal eigenfunctions of B_{0d}. The eigenvalue associated with the eigenfunction $g_{d;m_1,\ldots,m_d}$ is $\prod_{j=1}^d\gamma_{m_j} = b^{2(m_1+\cdots+m_d)+d}$. This means that the eigenvalues of B_{0d} are b^{2m+d} with multiplicity $\binom{m+d-1}{m}$, $m=0,1,\ldots$. Now we have that the operator B_d is of the form $B_d = B_{0d} - B_{1d}$, where the operator $B_{1d} : L_2(\mathbb{R}^d, N_d) \to L_2(\mathbb{R}^d, N_d)$ is defined by

$$B_{1d}f(y) = \left[\int \exp\left(-\frac{1}{2}|x|^2\right) f(x)\, dN_d(x)\right] \exp\left(-\frac{1}{2}|y|^2\right), \qquad f \in L_2(\mathbb{R}^d, N_d),\ y \in \mathbb{R}^d. \tag{8}$$

Again, let us consider the case $d = 1$ first. For an odd function $f \in L_2(\mathbb{R}, N)$, the integral in (8) vanishes. Since the Hermite polynomials of degree $m = 2l + 1$, $l = 0, 1, \ldots$, are odd functions, it follows that the functions g_m in (6) with $m = 2l + 1$, $l = 0, 1, \ldots$, are also eigenfunctions of B_1. Therefore, b^{4l+3}, $l = 0, 1, \ldots$, are also eigenvalues of B_1. To find the remaining eigenvalues of B_1 we argue as follows. A solution g of

$$B_1 g = \gamma g \tag{9}$$

for some positive γ, with g being orthogonal to these g_{2l+1} for each $l = 0, 1, \ldots$, can be represented in the form $g = \sum_{l=0}^{\infty} a_{2l} g_{2l}$ with real a_{2l}. The series converges in $L_2(\mathbb{R}, N)$. Putting

$$b_{2l} = \int \exp\left(-\frac{1}{2}x^2\right) g_{2l}(x)\, dN(x) = 5^{1/8} \frac{\sqrt{2l!}}{2^l l!} (-1)^l b^{4l+1} \tag{10}$$

(see Abramowitz and Stegun [1], formula 22.13.17) and

$$c = \int \exp\left(-\frac{1}{2}x^2\right) g(x)\, dN(x),$$

we conclude, having such a solution of (9), that

$$\sum_{l=0}^{\infty} a_{2l} g_{2l} (b^{4l+1} - \gamma) = \sum_{l=0}^{\infty} c b_{2l} g_{2l}. \tag{11}$$

It is immediately seen that $c \neq 0$ and that $\gamma \neq b^{4l+1}$, $l = 0, 1, \ldots$. So, $a_{2l} = cb_{2l}/(b^{4l+1} - \gamma)$. Multiplying the left and the right hand sides of

$$g(x) = \sum_{l=0}^{\infty} \frac{cb_{2l}}{b^{4l+1} - \gamma} g_{2l}(x)$$

by $\exp(-x^2/2)$ and then integrating with respect to the standard normal distribution we get

$$1 = \sum_{l=0}^{\infty} \frac{b_{2l}^2}{b^{4l+1} - \gamma}.$$

Thus introducing the meromorphic function

$$\omega_1(z) = 1 + 5^{1/4} z \sum_{l=0}^{\infty} \frac{\frac{(2l)!}{(l!)^2 2^{2l}} b^{8l+2}}{1 - zb^{4l+1}}, \qquad z \in \mathbb{C}, \tag{12}$$

we find that $\xi = 1/\gamma$ is a zero of $\omega_1(z)$. On the other hand, having a zero ξ of $\omega_1(z)$, it is easily seen that its inverse $1/\xi$ is an eigenvalue of B_1 associated with the eigenfunction

$$\sum_{l=0}^{\infty} \frac{b_{2l}}{b^{4l+1} - \gamma} g_{2l}.$$

Although somewhat more intricate, the case $d > 1$ can be treated in a similar manner. In fact, we get that for every d-tuple $(m_1, \ldots, m_d)$ with entries $m_j \in \{0, 1, \ldots\}$ at least one of which is odd, the function $g_{d;m_1,\ldots,m_d}(x_1, \ldots, x_d)$ in (7) is an eigenfunction of B_d with associated eigenvalue $b^{2(m_1+\cdots+m_d)+d}$. On writing a further solution $g \in L_2(\mathbb{R}^d, N_d)$ of $B_{0d} = \gamma g$ with associated eigenvalue γ and being orthonormal to these $g_{d;m_1,\ldots,m_d}(x_1, \ldots, x_d)$ in the form

$$g = \sum_{m_1=0}^{\infty} \cdots \sum_{m_d=0}^{\infty} a_{2m_1,\ldots,2m_d} g_{d;2m_1,\ldots,2m_d},$$

by analogy to (11) we get the identity

$$\begin{aligned}\sum_{m_1=0}^{\infty} \cdots \sum_{m_d=0}^{\infty} & a_{2m_1,\ldots,2m_d}(b^{4(m_1+\cdots+m_d)+d} - \gamma) g_{d;2m_1,\ldots,2m_d} \\ &= \sum_{m_1=0}^{\infty} \cdots \sum_{m_d=0}^{\infty} c b_{2m_1} \ldots b_{2m_d} g_{d;2m_1,\ldots,2m_d},\end{aligned}$$

where $c = \int \exp(-|x|^2/2) g(x)\, dN_d(x)$, and the b_{2m_j} are defined in (10). Then we have to consider two cases.

Case 1: $\gamma = b^{4m+d}$ for some nonnegative integer m. Then $c = 0$, and g is of the form

$$g = \sum_{m_1+\cdots+m_d=m} a_{2m_1,\ldots,2m_d} g_{d;2m_1,\ldots,2m_d} \tag{13}$$

with coefficients $a_{2m_1,\ldots,2m_d}$ satisfying

$$\sum_{m_1+\cdots+m_d=m} a_{2m_1,\ldots,2m_d} b_{2m_1} \ldots b_{2m_d} = 0.$$

The linear space of functions of this form is of dimension $\binom{m+d-1}{m} - 1$. So, $\gamma = b^{4m+d}$ has multiplicity $\binom{m+d-1}{m} - 1$.

Case 2: $\gamma \neq b^{4m+d}$ for every nonnegative integer m. Then $c \neq 0$, and γ is obtained as a solution of the equation

$$1 = \sum_{m_1=0}^{\infty} \cdots \sum_{m_d=0}^{\infty} \frac{b_{2m_1}^2 \ldots b_{2m_d}^2}{b^{4(m_1+\cdots+m_d)+d} - \gamma} = \sum_{m=0}^{\infty} 5^{d/4} (-1)^m \binom{-d/2}{m} \frac{b^{8m+2d}}{b^{4m+d} - \gamma}.$$

Alternatively, introducing the meromorphic function

$$\omega_d(z) = 1 + 5^{d/4} z \sum_{m=0}^{\infty} (-1)^m \binom{-d/2}{m} \frac{b^{8m+2d}}{1 - z b^{4m+d}}, \tag{14}$$

we find that $1/\gamma$ is a simple zero of $\omega_d(z)$. The associated function g is

$$\sum_{m_1=0}^{\infty} \cdots \sum_{m_1=0}^{\infty} \frac{b_{2m_1} \ldots b_{2m_d}}{b^{4(m_1+\cdots+m_d)+d} - \gamma} g_{d;2m_1,\ldots,2m_d}.$$

It is immediately seen that the γ, g found are in fact solutions of $B_d g = \gamma g$.

Let us summarize the results obtained. To this end, note that for a given nonnegative integer m, the number of d-tuples $(m_1, \ldots, m_d)$ of nonnegative integers, at least one of which is odd and satisfies $m_1 + \cdots + m_d = m$, is $\binom{m+d-1}{m} - \binom{m/2+d-1}{m/2}$ if m is even and $\binom{m+d-1}{m}$ if m is odd.

Theorem 1. *Let ξ_m, $m = 1, 2, \ldots$, be the zeros of the meromorphic function $\omega_d(z)$. Then the distribution of $T_1 = \int G_1(t)^2\, dN_d(t)$ is the same as that of*

$$\sum_{k=0}^{\infty} b^{4k+d+2}\chi^2_{\binom{2k+d}{2k+1}}(k,1) + \sum_{l=1}^{\infty} b^{4l+d}\chi^2_{\binom{2l+d-1}{2l}-1}(l,2) + \sum_{m=1}^{\infty} \frac{1}{\xi_m}\chi^2_1(m,3),$$

where the $\chi^2_{\binom{2k+d}{2k+1}}(k,1), \chi^2_{\binom{2l+d-1}{2l}-1}(l,2), \chi^2_1(m,3)$ are independent χ^2-variables, with the respective degrees of freedom indicated in the subscripts. The variables with subscript 0 occurring if $d = 1$ are suppressed.

We deduce from Theorem 1 that the rth cumulant of T_1 is

$$\kappa_r = 2^{r-1}(r-1)!\left\{\sum_{m=1}^{\infty}\left(\frac{1}{\xi_m}\right)^r + \left(\frac{b^r}{1-b^{2r}}\right)^d - \frac{b^{rd}}{1-b^{4r}}\right\}, \qquad r \geq 1. \tag{15}$$

Arguing as in Darling [5] where a related problem is dealt with, it can be seen that the Fredholm determinant of B_d is

$$\delta_d(z) = \omega_d(z)\prod_{m=0}^{\infty}(1 - zb^{2m+d})^{\binom{m+d-1}{m}}, \qquad z \in \mathbb{C}.$$

This gives the representation $[1/\delta_d(-2t)]^{1/2}$, $t \geq 0$, for the Laplace transform of T_1. Expanding the rational expressions $1/(1 - zb^{4m+d})$ in (14) in geometric power series we get

$$\omega_d(z) = \sum_{m=0}^{\infty} z^m/(F(2(m+1)))^{d/2}, \qquad |z| < b^{-d},$$

with $F(m)$ the Fibonacci numbers determined by $F(m+1) = F(m) + F(m-1)$, $m = 1, 2, \ldots,$ $F(0) = 0$, $F(1) = 1$. Using this and expanding $\log[1/\delta_d(-2t)]^{1/2}$ in powers of t, we see that the coefficient of t^r is

$$(-1)^r\frac{2^{r-1}}{r}\left\{\left(\frac{b^r}{1-b^{2r}}\right)^d + \delta_r\right\}, \tag{16}$$

where

$$\delta_r = r\sum_{k=1}^{r}(-1)^k\frac{1}{k}\sum_{\substack{m_1+\cdots+m_k=r\\ m_i=1,2,\ldots,i=1,\ldots,k}}\frac{1}{(F(2(m_1+1))\ldots F(2(m_k+1)))^{d/2}}.$$

Up to the multiplicative constant $(-1)^r r!$ this coefficient is equal to the rth cumulant of T_1. Since

$$\frac{b^r}{1-b^{2r}} = \begin{cases} 1/(\sqrt{5}F(r)) & \text{if } r \text{ is even},\\ 1/L(r) & \text{if } r \text{ is odd},\end{cases}$$

where the $L(r)$, $r = 0, 1, \ldots,$ are the Lucas numbers defined by $L(0) = 2, L(1) = 1, L(r+1) = L(r) + L(r-1)$, $r = 1, 2, \ldots,$ the cumulants of T_1 can be written alternatively in the form

$$\kappa_r = \begin{cases} 2^{r-1}(r-1)!\,[(\sqrt{5}F(r))^{-d} + \delta_r] & \text{if } r \text{ is even},\\ 2^{r-1}(r-1)!\,[L(r)^{-d} + \delta_r] & \text{if } r \text{ is odd}.\end{cases}$$

Comparing (15) and (16) we are able to compute the infinite sums $\sum_{m=1}^{\infty} \xi_m^{-r}$, $r = 1, 2, \ldots$. In fact,

$$\sum_{m=1}^{\infty} \left(\frac{1}{\xi_m}\right)^r = \frac{b^{rd}}{1 - b^{4r}} + \delta_r.$$

The distribution of T_1 is most easily approximated by the distribution of the sum of finitely many of the weighted χ^2-variables occurring in Theorem 1. It is plain that such an approximation should be based on the sum consisting of the variables with the largest weights. So we have calculated by numerical methods the values of the 12 smallest zeros of $\omega_1(z)$ and $\omega_2(z)$. These values are shown in Table 1.

TABLE 1. The 12 smallest zeros ξ_i, $i = 1, \ldots, 12$, of $\omega_d(z)$, $d = 1, 2$

ξ_i	$d = 1$	$d = 2$
1	7.849336555578469	9.980597866810863
2	61.50441057386329	84.24682401892317
3	448.1831773838836	638.7849329947641
4	3179.147341949039	4645.090437279074
5	22268.85055919773	33091.54159992883
6	154929.5194062535	233046.4067592094
7	1073517.332252734	1629672.896578112
8	7419105.877262517	11343464.55464175
9	51182811.59769955	78704778.23373327
10	352654830.8574741	544825391.8014784
11	2427578689.601553	3765058015.432077
12	16699075379.06429	25984849966.06623

The sum of all eigenvalues of B_d is $\int (1 - \exp(-|x|^2))\, dN_d(x) = 1 - 3^{-d/2}$. For $d = 1$, the sum of the eigenvalues b^{4m+3}, $m = 0, 1, \ldots$, is $b/\sqrt{5}$. Adding to it the sum $\sum_{i=1}^{12} 1/\xi_i$, where we use the calculated value stated in the ith row of the first column of Table 1 for ξ_i, we get the value 0.422649730800 agreeing with the exact value $1 - 1/\sqrt{3}$ up to 10 decimal places.

For $d = 2$, the eigenvalues b^{4m+2} and b^{4m} both occur with multiplicity $2m$, $m = 1, 2, \ldots$. The sum of these eigenvalues counted according to their multiplicities is $2/5 + 2b^2/5$. Adding to it the sum $\sum_{i=1}^{12} 1/\xi_i$, where we use the calculated value stated in the ith row of the second column of Table 1 for ξ_i, we get the value 0.666666666660. The exact value is $2/3$.

4. THE INTEGRAL $\int G_2(t)^2\, dN_d(t)$

Let $\rho = \rho_2$. Let us treat the case $d = 1$ first. From Mehler's expansion (see Erdélyi, Magnus, Oberhettinger and Tricomi [7], formula 10.13(22))

$$(17) \qquad \sum_{m=0}^{\infty} \frac{z^m}{m!} \mathcal{H}e_m(\sqrt{2}x) \mathcal{H}e_m(\sqrt{2}y) = (1 - z^2)^{-1/2} \exp\left(\frac{2xyz - (x^2 + y^2)z^2}{1 - z^2}\right)$$

converging uniformly for z in compact subsets of $(-1,+1)$, and for x,y in compact subsets of $\mathbb{R}$, we get by putting $z=b^2, x=5^{1/4}s/\sqrt{2}$ and $y=5^{1/4}y/\sqrt{2}$ that

$$\sum_{m=0}^{\infty} b^{2m+1} g_m(s) g_m(t) = \exp(-(s-t)^2/2). \tag{18}$$

Substituting $z=-b^2, x=5^{1/4}x/\sqrt{2}$ and $y=5^{1/4}t/\sqrt{2}$ in (17) we obtain

$$\sum_{m=0}^{\infty} (-1)^m b^{2m+1} g_m(s) g_m(t) = \exp(-(s+t)^2/2). \tag{19}$$

Subtracting the expressions on the left (right) hand side of (19) from the expressions on the left (right) hand side of (18) we get the representation

$$\sum_{m=0}^{\infty} b^{4m+3} g_m(s) g_m(t) = \frac{1}{2}[\exp(-(s-t)^2/2) - \exp(-(s+t)^2/2)]. \tag{20}$$

From this we deduce that

$$\sum_{m=0}^{\infty} b^{2m+3/2} g_m(t) Z_m, \qquad t \in \mathbb{R},$$

is a version of G_2 with sample path in $L_2(\mathbb{R}, N)$. The square of its norm equals $\sum_{m=0}^{\infty} b^{4m+3} Z_m^2$. Its distribution is the same as that of $T_2 = \int G_2(t)^2\, dN(t)$.

Treating the case $d>1$ we put for abbreviation $e(u) = \exp(-u^2/2)$, $u \in \mathbb{R}$. Then the covariance function ρ_2 can be written as

$$\rho_2(s,t) = \sum_{k=1}^{d} \frac{1}{2}[e(s_k - t_k) - e(s_k + t_k)] \prod_{j=1}^{k-1} e(s_j + t_j) \prod_{l=k+1}^{d} e(s_l - t_l),$$

for $s = (s_1, \ldots, s_d)' \in \mathbb{R}^d$, $t = (t_1, \ldots, t_d)' \in \mathbb{R}^d$. In what follows we use the identities

$$\int e(u-v) g_m(u)\, dN_1(u) = b^{2m+1}, \qquad m = 0,1,2,\ldots,$$

$$\int e(u+v) g_m(u)\, dN_1(u) = \begin{cases} b^{2m+1} & \text{if } m \text{ is even,} \\ -b^{2m+1} & \text{if } m \text{ is odd,} \end{cases}$$

and

$$\int \frac{1}{2}[e(u-v) - e(u+v)] g_m(u)\, dN_1(u) = \begin{cases} 0 & \text{if } m \text{ is even,} \\ b^{2m+1} & \text{if } m \text{ is odd.} \end{cases}$$

Let $\mathbf{m} = (m_1, \ldots, m_d)$ be a d-tuple of nonnegative integers, and denote by $k(\mathbf{m})$ the number of its odd components m_i. It is easily verified that

$$\int \rho_2(s,t) g_{d,m_1,\ldots,m_d}(t)\, dN_d(t) = \begin{cases} 0 & \text{if } k(\mathbf{m}) \text{ is even,} \\ b^{2(m_1+\cdots+m_d)+d} g_{d;m_1,\ldots,m_d}(s) & \text{if } k(\mathbf{m}) \text{ is odd.} \end{cases}$$

If $\mathbb{M}^d$ is the set of all d-tuples of nonnegative integers with odd number $k(\mathbf{m})$, we see that $g_{d;m_1,\ldots,m_d}$, $(m_1, \ldots, m_d) \in \mathbb{M}^d$, is a collection of orthonormal eigenfunctions with associated positive eigenvalues

$$b^{2(m_1+\cdots+m_d)+d}, \qquad (m_1, \ldots, m_d) \in \mathbb{M}^d. \tag{21}$$

The sum of these eigenvalues is

$$\begin{aligned}\sum_{(m_1,\dots,m_d)\in\mathbb{M}^d} b^{2(m_1+\cdots+m_d)+d} &= \sum_{k=1}^{[(d+1)/2]}\sum_{m=0}^{\infty} b^{4m+d+4k-2}\binom{d}{2k-1}\binom{m+d-1}{m}\\ &= \sum_{k=1}^{[(d+1)/2]}\binom{d}{2k-1}\frac{b^{2d}}{(1-b^4)^d}b^{4k-2-d} \qquad (22)\\ &= 5^{-d/2}b^{-d}\frac{1}{2}[(1+b^2)^d-(1-b^2)^d]\\ &= \frac{1}{2}(1-5^{-d/2}).\end{aligned}$$

The sum of all positive eigenvalues of the integral operator B_d is equal to $\int \rho_2(t,t)\,dN_d(t)$. An easy calculation yields $\int \rho_2(t,t)\,dN_d(t) = \frac{1}{2}(1-5^{-d/2})$. Therefore (21) gives the complete system of positive eigenvalues. Summarizing we can state the following result.

Theorem 2. *The distribution of $T_2 = \int G_2(t)^2\,dN_d(t)$ is the same as that of*

$$\sum_{k=1}^{[(d+1)/2]}\sum_{m=0}^{\infty} b^{4m+d+4k-2}\chi^2_{\binom{d}{2k-1}\binom{m+d-1}{m}},$$

where the $\chi^2_{\binom{d}{2k-1}\binom{m+d-1}{m}}$ are independent χ^2-variables, with the degrees of freedom indicated in the subscripts.

By similar manipulations as those in (22) the rth cumulant of T_2 is seen to be

$$\begin{aligned}&2^{r-1}(r-1)!\sum_{k=1}^{[(d+1)/2]}\sum_{m=0}^{\infty} b^{(4m+d+4k-2)r}\binom{d}{2k-1}\binom{m+d-1}{m}\\ &= \begin{cases}\frac{2^{r-2}(r-1)!}{F(2r)^d}[(L(r)/\sqrt{5})^d - F(r)^d] & \text{if } r \text{ is even},\\ \frac{2^{r-2}(r-1)!}{F(2r)^d}[F(r)^d-(L(r)/\sqrt{5})^d] & \text{if } r \text{ is odd}.\end{cases}\end{aligned}$$

5. Concluding remark

The problem of testing the composite hypothesis of multivariate normality can be treated in a similar manner. But now in the case where the hypothesis is true the mean vector and covariance matrix of the true normal distribution are unknown and must be estimated from the data. It can be shown that a test statistic similar to (1) converges in distribution to the integral $\int G(t)^2\,dN_d(t)$ of a certain Gaussian process $\{G(t),\ t\in\mathbb{R}^d\}$ so that the methods of Section 2 apply. However, the covariance function of this process is more complicated. At present we are unable to give a complete solution analogous to those in Section 3 and Section 4.

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