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FIBONACCI NUMBERS OF GRAPHS: II

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(Submitted April 1982)

1. INTRODUCTION

In [9], the Fibonacci number $f(G)$ of a (simple) graph G is introduced as the total number of all Fibonacci subsets S of the vertex set $V(G)$ of G , where a Fibonacci subset S is a (possibly empty) subset of $V(G)$ such that any two vertices of S are not adjacent. In *Graph Theory* [6, p. 257] a Fibonacci subset is called an independent set of vertices. From [9] we have the elementary inequality

$$F_{n+1} \leq f(G) \leq 2^{n-1} + 1, \tag{1.1}$$

where F_n denotes the usual Fibonacci numbers with

$$F_0 = F_1 = 1, F_n = F_{n-1} + F_{n-2},$$

and G is a tree with n vertices. Furthermore, several problems are formulated concerning the Fibonacci numbers of some special graphs. The present aim is to derive a formula for $f(T_n(t))$, where $T_n(t)$ is the full t -ary tree with height n : [$T_0(t)$ is the empty tree.]

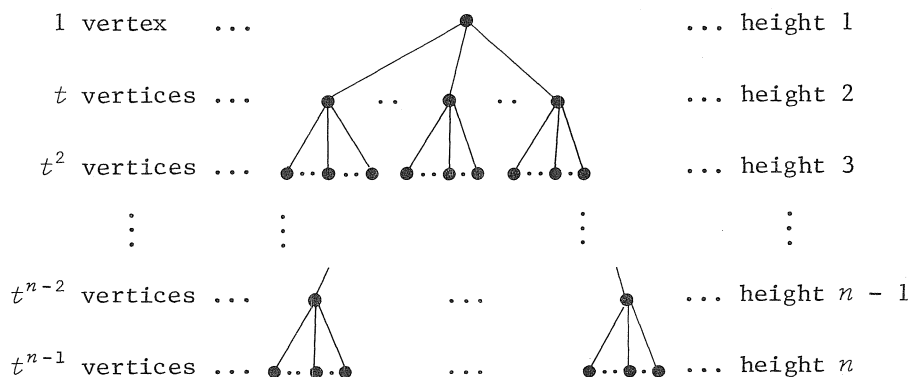


FIGURE 1

For $t = 1$, one can see immediately that $f(T_n(t)) = F_{n+1}$, so the interesting cases are $t \geq 2$. In Section 2, for $t = 2, 3, 4$, the asymptotic formula

$$f(T_n(t)) \sim A(t) \cdot k(t)^{t^n} \quad (n \rightarrow \infty) \tag{1.2}$$

is proved, where $A(t)$ and $k(t)$ denote constants (depending on t) with

FIBONACCI NUMBERS OF GRAPHS: II

$$2^{1/(1-t)} < A(t) < 1 < k(t) < 2^{1/(t-1)}.$$

In Section 3, it is proved that for $t \geq 5$ such an asymptotic formula does not hold; we show that for $t \geq 5$:

$$\begin{aligned} f(T_{2m}(t)) &\sim B(t) \cdot k(t)^{t^{2m}} \\ f(T_{2m+1}(t)) &\sim C(t) \cdot k(t)^{t^{2m+1}} \end{aligned} \tag{1.3}$$

where $B(t) > C(t)$ are constants depending on t with

$$\lim_{t \rightarrow \infty} B(t) = \lim_{t \rightarrow \infty} C(t) = 1.$$

In Section 4, we establish an asymptotic formula for the average value S_n of the Fibonacci number of binary trees with n vertices (where all such trees are regarded equally likely). For the sake of brevity, we restrict our considerations to the important case of binary trees; however, the methods would even be applicable in the very general case of so-called "simply generated families of trees" introduced by Meir and Moon [8].

By a version of Darboux's method (see Bender's survey [1]), we derive

$$S_n \sim G \cdot r^n \quad (n \rightarrow \infty), \tag{1.4}$$

where $G = 1, 12928\dots$ and $r = 1, 63742\dots$ are numerical constants.

2. FIBONACCI NUMBERS OF t -ARY TREES ($t = 2, 3, 4$)

By a simple argument (compare [9]), the following recursion holds for the Fibonacci number $x_n = f(T_n(t))$,

$$x_{n+1} = x_n^t + x_{n-1}^{t^2} \text{ with } x_0 = 1, x_1 = 2. \tag{2.1}$$

We proceed as in [4] and put $y_n = \log x_n$; by (2.1),

$$y_{n+1} = ty_n + \alpha_n \text{ with } \alpha_n = \log \left(1 + \frac{x_{n-1}^{t^2}}{x_n^t} \right). \tag{2.2}$$

Because of

$$x_{n-1}^{t^2} < (x_{n-1}^t + x_{n-2}^{t^2})^t = x_n^t,$$

the estimate

$$0 < \alpha_n < \log 2 \tag{2.3}$$

results. The solution of recursion (2.2) is given by

$$y_n = t^n \left(\frac{\alpha_0}{t} + \frac{\alpha_1}{t^2} + \dots + \frac{\alpha_{n-1}}{t^n} \right).$$

It is now convenient to extend the series in α_i to infinity [because of (2.3)]

the series is convergent]:

$$Y_n: = \sum_{i=0}^{\infty} t^{n-1-i} \alpha_i. \quad (2.4)$$

For the difference

$$r_n: = Y_n - y_n = \sum_{i=n}^{\infty} t^{n-1-i} \alpha_i,$$

we have

$$0 < r_n \leq \frac{\log 2}{t-1}. \quad (2.5)$$

Therefore, we obtain

$$x_n = e^{Y_n - r_n} = e^{-r_n} \cdot k(t)^{t^n}, \quad (2.6)$$

where

$$k(t) = \exp\left(\sum_{i=0}^{\infty} t^{-i-1} \alpha_i\right) \quad (2.7)$$

and $1 < k(t) < 2^{1/(t-1)}$ by (2.3).

In the following, we investigate the factor e^{-r_n} of (2.6); we set

$$q_n = x_n^t / x_{n+1}$$

and obtain the recursion

$$q_{n+1} = \frac{1}{1 + q_n^t}, \quad q_0 = \frac{1}{2} \quad (2.8)$$

from (2.1). It is useful to split up the sequence (q_n) into two complementary subsequences

$$\begin{aligned} (g_m): &= (q_{2m}) = (q_0, q_2, \dots), \\ (u_m): &= (q_{2m+1}) = (q_1, q_3, \dots). \end{aligned} \quad (2.9)$$

Lemma 1

The following inequalities hold for the subsequences (g_m) and (u_m) of (q_n) :

- (i) $g_{m+1} > g_m$ for all $m = 0, 1, 2, 3, \dots$
- (ii) $u_{m+1} < u_m$ for all $m = 0, 1, 2, 3, \dots$
- (iii) $u_m > g_m$ for all $m = 0, 1, 2, 3, \dots$

Proof: Let $q_{n-2} > q_n$; then $1 + q_{n-2}^t > 1 + q_n^t$, $1/(1 + q_{n-2}^t) < 1/(1 + q_n^t)$, and so

$$\frac{1}{1 + \left(\frac{1}{1 + q_{n-2}^t}\right)^t} > \frac{1}{1 + \left(\frac{1}{1 + q_n^t}\right)^t}.$$

Applying (2.8), we have proved:

$$\text{If } q_{n-2} > q_n, \text{ then } q_n > q_{n+2}. \quad (2.10)$$

Because of $g_1 > g_0$, (i) is proved by induction; (ii) and (iii) follow by a similar argument.

By Lemma 1, (g_m) and (u_m) are monotone sequences with the obvious bounds

$$\frac{1}{2} \leq g_m < u_m \leq 1. \quad (2.11)$$

So the sequences (g_m) and (u_m) must be convergent to limits g and u (depending on t). The following proposition shows that $g = u$ in the cases $t = 2, 3, 4$.

Proposition 1

For $t = 2, 3, 4$, the sequence (q_n) is convergent to a limit $w(t)$, where $w(t)$ is the unique root of the equation $w^{t+1} + w - 1 = 0$ with $\frac{1}{2} \leq w(t) \leq 1$.

Proof: By Lemma 1 we only have to show that (g_m) and (u_m) are convergent to the same limit. For (g_m) and (u_m) the following system of recursions holds:

$$\begin{aligned} u_m &= \frac{1}{1 + g_m^t} \\ g_{m+1} &= \frac{1}{1 + u_m^t}. \end{aligned} \quad (2.12)$$

Taking the limit $m \rightarrow \infty$, we obtain

$$u = \frac{1}{1 + g^t}, \quad g = \frac{1}{1 + u^t}, \quad \text{with } \frac{1}{2} \leq g \leq u \leq 1. \quad (2.13)$$

Let us start with the case $t = 2$. By (2.13), we have $ug^2 = 1 - u$, $gu^2 = 1 - g$, and therefore, $u - u^2 = g - g^2$. Because the function $x \mapsto x - x^2$ is strictly decreasing in the interval $[\frac{1}{2}, 1]$, $u = g$ follows immediately.

In the case $t = 3$, we derive in a similar way the relation $u^2 - u^3 = g^2 - g^3$. Since the function $x \mapsto x^2 - x^3$ is strictly decreasing in the interval $[\frac{2}{3}, 1]$ and $\frac{2}{3} < g_4 = 0,684\dots$, we obtain $u = g$ again.

Since the function $x \mapsto x^3 - x^4$ is strictly decreasing in the interval $[\frac{3}{4}, 1]$ and $g_{73} = 0,7500138\dots > \frac{3}{4}$, we obtain $u = g$ in the case $t = 4$, too.

So $u = g$ in all considered cases; therefore, a limit $w(t)$ of (q_n) exists for $t = 2, 3, 4$, and $w(t)$ fulfills the equation

$$w = \frac{1}{1 + w^t}.$$

Since the function $f(w) = w^{t+1} + w - 1$ is strictly monotone in the interval $[\frac{1}{2}, 1]$ and $f(\frac{1}{2}) < 0$, $f(1) > 0$, there exists a unique root of this equation in the interval $[\frac{1}{2}, 1]$, which is the limit $w(t)$ from above.

By (2.2) we derive

$$\lim_{n \rightarrow \infty} \alpha_n = \log(1 + w(t)^t). \tag{2.14}$$

Because of

$$\left| r_n - \frac{1}{t-1} \log(1 + w(t)^t) \right| \leq \sum_{i=n}^{\infty} t^{n-1-i} |\alpha_i - \log(1 + w(t)^t)| < \frac{\varepsilon}{t-1}$$

[for all $\varepsilon > 0$, $n \geq n_0(\varepsilon)$], the sequence (r_n) is convergent; so

$$\lim_{n \rightarrow \infty} e^{-r_n} = (1 + w(t)^t)^{-1/(t-1)} = w(t)^{1/(t-1)} = (1 - w(t))^{1/(t^2-1)} \tag{2.15}$$

results. Altogether we have established:

Theorem 1

Let $T_n(t)$ be the full t -ary tree ($t = 2, 3$, or 4) with height n . Then, the Fibonacci number $f(T_n(t))$ fulfills the following asymptotic formula:

$$f(T_n(t)) \sim A(t) \cdot k(t)^{t^n} \quad (n \rightarrow \infty)$$

where $A(t) = w(t)^{1/(t-1)}$ and $k(t)$, defined by (2.7), are constants (only depending on t) bounded by

$$2^{1/(1-t)} < A(t) < 1 < k(t) < 2^{1/(t-1)};$$

$w(t)$ is the unique root of $w^{t+1} + w - 1 = 0$ with $\frac{1}{2} < w(t) < 1$.

Remark: The numerical values of $w(t)$ are

$$w(2) = 0, 68233\dots, w(3) = 0, 72449\dots, \text{ and } w(4) = 0, 75488\dots$$

3. FIBONACCI NUMBERS OF t -ARY TREES ($t \geq 5$)

In this section we consider t -ary trees with $t \geq 5$. Let (g_m) , (u_m) be the subsequences of (q_n) defined by (2.9). (g_m) and (u_m) are convergent to limits g and u , respectively (depending on t). We shall prove that $g \neq u$; therefore, (q_n) has two accumulation points. For g, u the following system of equations holds,

$$u = \frac{1}{1 + g^t}, \quad g = \frac{1}{1 + u^t}, \tag{3.1}$$

and $g = u$ if and only if u or g is the unique solution of

FIBONACCI NUMBERS OF GRAPHS: II

$$w^{t+1} + w - 1 = 0 \tag{3.2}$$

in the interval $\left[\frac{1}{2}, 1\right]$. If (u', g') and (u'', g'') are two pairs fulfilling (3.1) with $u' < u''$, then $g' > g''$. Let (\bar{u}, \bar{g}) denote the pair of solutions with minimal g and maximal u .

Lemma 2

The subsequence (g_m) of (q_n) is convergent to the limit \bar{g} and the subsequence (u_m) to the limit \bar{u} .

Proof: First we show that $g_m < \bar{g}$ implies $u_m > \bar{u}$ and $g_{m+1} < \bar{g}$.

Because of $g_m < \bar{g}$, we obtain $1 + g_m^t < 1 + \bar{g}^t = 1/\bar{u}$, and so $u_m > \bar{u}$. From $u_m > \bar{u}$, it follows that $1 + u_m^t > 1 + \bar{u}^t = 1/\bar{g}$, hence $g_{m+1} < \bar{g}$.

Using the fact that $g_0 = \frac{1}{2} < \bar{g}$, we obtain, by induction,

$$\lim_{m \rightarrow \infty} g_m \leq \bar{g} \quad \text{and} \quad \lim_{m \rightarrow \infty} u_m \geq \bar{u}.$$

By the definition of (\bar{u}, \bar{g}) , it follows that

$$\lim_{m \rightarrow \infty} g_m = \bar{g} \quad \text{and} \quad \lim_{m \rightarrow \infty} u_m = \bar{u},$$

and the Lemma is proved.

Lemma 3

Let $t \geq 5$ be a positive integer; then there exists a solution (u, g) of the system (3.1) with

$$\frac{1}{2} < g < \frac{1}{2} + \frac{1}{t}.$$

Proof: System (3.1) is equivalent to the equation

$$g = \frac{1}{1 + \frac{1}{(1 + g^t)^t}} \tag{3.3}$$

We consider the function

$$\varphi_t(g) = \frac{(1 + g^t)^t}{1 + (1 + g^t)^t} - g,$$

and obtain $\varphi_t\left(\frac{1}{2}\right) > 0$; in the sequel, we show $\varphi_t\left(\frac{1}{2} + \frac{1}{t}\right) < 0$. For $t = 5$ or 6 , this inequality can be shown by direct computation:

$$\varphi_5\left(\frac{7}{10}\right) = -0, 01502 \quad \text{and} \quad \varphi_6\left(\frac{2}{3}\right) = -0, 04306.$$

Let us assume $t \geq 7$ in the sequel. By elementary manipulations, the inequality

$$\frac{1}{1 + \frac{1}{\left(1 + \left(\frac{1}{2} + \frac{1}{t}\right)^t\right)^t}} - \frac{1}{2} - \frac{1}{t} < 0$$

is equivalent to $1 + \left(1 + \left(\frac{1}{2} + \frac{1}{t}\right)^t\right)^{-t} > \frac{2t}{t+2}$ or

$$\left(1 + \left(\frac{1}{2} + \frac{1}{t}\right)^t\right)^t < \frac{t+2}{t-2}. \tag{3.4}$$

Because of $\left(1 + \frac{2}{t}\right)^{t/2} < e$, it is sufficient to prove

$$\left(1 + \frac{e^2}{2^t}\right)^t < \frac{t+2}{t-2}. \tag{3.5}$$

We have

$$\left(1 + \frac{e^2}{2^t}\right)^t \leq \exp\left(\frac{e^{2t}}{2^t}\right)$$

and

$$\exp\left(\frac{e^{2t}}{2^t}\right) \leq \frac{t+2}{t-2} \text{ for } t \geq 7.$$

So $\varphi_t\left(\frac{1}{2} + \frac{1}{t}\right) < 0$, and the Lemma is proved, because the continuous function φ has a root between $\frac{1}{2}$ and $\frac{1}{2} + \frac{1}{t}$.

Equation (3.3) is equivalent to

$$g(g^t + 1)^t - (g^t + 1)^t + g = 0. \tag{3.6}$$

The polynomial on the left-hand side of (3.6) is divisible by $g^{t+1} + g - 1$. Because of $\left(\frac{3}{4}\right)^{t+1} + \frac{3}{4} - 1 < 0$ (for $t \geq 5$), the unique solution $w(t)$ of $g^{t+1} + g - 1 = 0$ is contained in the interval $\left[\frac{3}{4}, 1\right]$. By Lemma 3, we have found a pair of solutions (u, g) with $u \neq g$ such that $\frac{1}{2} < g < \frac{3}{4} < w(t) < u < 1$. We denote by $(u(t), g(t))$, $t \geq 5$, the pair of solutions of (3.1) such that $g(t)$ is minimal and $u(t)$ is maximal. Because of $g(t) < \frac{1}{2} + \frac{1}{t}$ and $u(t) = 1 + g(t)^{-t}$ for $t \geq 5$, we obtain

$$\lim_{t \rightarrow \infty} g(t) = \frac{1}{2}, \quad \lim_{t \rightarrow \infty} u(t) = 1. \tag{3.7}$$

Altogether, we have proved:

Theorem 2

Let $T_n(t)$ be the full t -ary tree (for $t \geq 5$) with height n . Then the Fibonacci numbers fulfill the following asymptotic formulas, respectively:

$$f(T_{2m}(t)) \sim B(t) \cdot k(t)^{t^{2m}},$$

$$f(T_{2m+1}(t)) \sim C(t) \cdot k(t)^{t^{2m+1}},$$

where

$$C(t) = (g(t)^t u(t))^{1/(t^2-1)} = (1 - u(t))^{1/(t^2-1)},$$

$$B(t) = (g(t)u(t)^t)^{1/(t^2-1)} = (1 - g(t))^{1/(t^2-1)},$$

and $k(t)$, defined by (2.7), are constants (only depending on t) bounded by

$$2^{1/(1-t)} < C(t) < B(t) < 1 < k(t) < 2^{1/(t-1)};$$

$g(t)$ is the minimal root and $u(t)$ the maximal root of

$$x(x^t + 1)^t - (x^t + 1)^t + x = 0$$

in the interval $[\frac{1}{2}, 1]$; furthermore,

$$\lim_{t \rightarrow \infty} B(t) = \lim_{t \rightarrow \infty} C(t) = 1.$$

Remark: In [2], similar recurrences are treated by a slightly different method. The recursion for (q_n) can be considered as a fixed-point problem and our results can be derived in principal by studying this fixed-point problem.

4. THE AVERAGE FIBONACCI NUMBER OF BINARY TREES

The family β of all binary trees is defined by the following formal equation (\square is the symbol for a leaf and \circ for an internal node):

$$\beta = \square + \begin{array}{c} \circ \\ / \quad \backslash \\ \beta \quad \beta \end{array}; \tag{4.1}$$

(this notation is due to Ph. Flajolet [3]). The generating function

$$B(z) = \sum_{n \geq 0} b_n z^n$$

of the numbers of binary trees with n internal nodes is given by

$$B(z) = \frac{1 - \sqrt{1 - 4z}}{2z} \tag{4.2}$$

and, therefore,

$$b_n = \frac{1}{n+1} \binom{2n}{n}. \tag{4.3}$$

FIBONACCI NUMBERS OF GRAPHS: II

For technical reasons, we consider the family β^* of all binary trees with leaves removed; β^* fulfills

$$\beta^* = \circ + \begin{array}{c} \circ \\ / \backslash \\ \beta^* \end{array} + \begin{array}{c} \circ \\ | \\ \beta^* \end{array} + \begin{array}{c} \circ \\ / \backslash \\ \beta^* \beta^* \end{array} \quad (4.4)$$

Let β_n be the family of binary trees t with n internal nodes, and let

$$f(z) = \sum_{n \geq 1} f_n z^n \quad \text{and} \quad g(z) = \sum_{n \geq 1} g_n z^n$$

be generating functions of

$$\begin{aligned} f_n &= \sum_{T \in \beta_n} \text{card}\{S : S \subseteq V(T); S \text{ a Fibonacci subset} \\ &\quad \text{not containing the root}\}, \\ g_n &= \sum_{T \in \beta_n} \text{card}\{S : S \subseteq V(T); S \text{ a Fibonacci subset} \\ &\quad \text{containing the root}\}. \end{aligned} \quad (4.5)$$

Obviously, the average value of the Fibonacci number of a binary tree with n internal nodes is given by

$$S_n = \frac{h_n}{b_n} \quad \text{with} \quad h_n = f_n + g_n. \quad (4.6)$$

The remainder of this paper is devoted to the asymptotic evaluation of S_n . By Stirling's approximation of the factorials, the well-known formula

$$b_n \sim \frac{1}{\sqrt{\pi}} 2^{2n} n^{-3/2} \quad (n \rightarrow \infty) \quad (4.7)$$

holds and we can restrict our attention to h_n .

For the generating functions, we obtain

$$\begin{aligned} f &= z + z(f + g) + z(f + g) + z(f + g)^2 \\ g &= z + zf + zf + zf^2. \end{aligned} \quad (4.8)$$

[The contributions of (4.8) correspond to the terms in (4.4).] Setting

$$y(z) = 1 + f(z) + g(z),$$

we derive, by some elementary manipulations,

$$z^3 y^4 + (2z^2 + z)y^2 - y + (z + 1) = 0. \quad (4.9)$$

Now we want to apply Theorem 5 of [1]; for this purpose, we have to determine the singularity ρ of $y(z)$ nearest to the origin. (4.9) is an implicit representation of $y(z)$. Abbreviating the left-hand side of (4.9) by $F(z, y)$, the singularity ρ (nearest to the origin) and $\sigma = y(\rho)$ are given as solutions of the following system of algebraic equations:

FIBONACCI NUMBERS OF GRAPHS: II

$$\begin{aligned} F(z, y) &= 0, \\ \frac{\partial F}{\partial y}(z, y) &= 0. \end{aligned} \tag{4.10}$$

Now ρ and σ are simple roots of the above equations. By a theorem of Pringsheim [7, p. 389], ρ and σ are positive (real) numbers. Using the two-dimensional version of Newton's algorithm (starting with $z_0 = 0, 2$ and $y_0 = 1$), we obtain the following numerical values:

$$\rho = 0, 15268\dots \quad \text{and} \quad \sigma = 2, 15254\dots \tag{4.11}$$

Now Theorem 5 of [1] allows us to formulate the following:

Proposition 2

$$\begin{aligned} h_n &\sim \left(\frac{\rho \cdot F_z(\rho, \sigma)}{2\pi \cdot F_{yy}(\rho, \sigma)} \right)^{1/2} \cdot \rho^{-n} \cdot n^{-3/2}, \\ &\sim (0, 63713\dots) \quad (0, 15268\dots)^{-n} \cdot n^{-3/2}. \end{aligned} \quad (n \rightarrow \infty) \tag{4.12}$$

Altogether, we have proved:

Theorem 3

The average value S_n of the Fibonacci number of a binary tree with n internal nodes fulfills asymptotically

$$S_n \sim G \cdot r^n \quad (n \rightarrow \infty),$$

where $G = 1, 12928\dots$ and $r = 1, 63742\dots$ are numerical constants.

ACKNOWLEDGMENT

We would like to express our thanks for helpful discussions to Professor G. Baron and Dr. Ch. Buchta.

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FIBONACCI NUMBERS OF GRAPHS: II

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CORRIGENDA TO "SOME SEQUENCES LIKE FIBONACCI'S"

B. H. NEUMANN and L. G. WILSON

The Fibonacci Quarterly, Vol. 17, No. 1, 1979, pp. 80-83

The following changes should be made in the above article. These errors are the responsibility of the editorial staff and were recently brought to the editor's attention by the authors.

- p. 80, at the end of formula (1), add superscript "n".
- p. 81, in formula (7), replace the second "y" by "t".
- p. 82, in the line following (8), add subscript "d" to the last "a".
- p. 82, in the line following (10), add subscript "d" to the last "α".
- p. 83, line 3, insert "growth" between "slower" and "rate".
- p. 83, end of text and reference, delete "t" from the name "Johnson".

Gerald E. Bergum

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