

FIBONACCI POLYNOMIALS OF ORDER k , MULTINOMIAL EXPANSIONS AND PROBABILITY

ANDREAS N. PHILIPPOU

Department of Mathematics
University of Patras
Patras, Greece

COSTAS GEORGHIOU

School of Engineering
University of Patras
Patras, Greece

GEORGE N. PHILIPPOU

Department of General Studies
Higher Technical Institute
Nicosia, Cyprus

(Received October 31, 1982)

ABSTRACT. The Fibonacci polynomials of order k are introduced and two expansions of them are obtained, in terms of the multinomial and binomial coefficients, respectively. A relation between them and probability is also established. The present work generalizes results of [2] - [4] and [5].

KEY WORDS AND PHRASES. *Fibonacci polynomials of order k , expansions, multinomial and binomial coefficients, probability.*

1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. 10A40.

1. INTRODUCTION.

In the sequel, k is a fixed integer greater than or equal to 2, x is a positive and finite real number, and n is a nonnegative integer unless otherwise specified. Motivated by [1] we introduced the Fibonacci polynomials of order k , to be denoted by $f_n^{(k)}(x)$, and study some of their properties. First we observe that $f_n^{(k)}(x)$ are generalized polynomials, appropriate extensions for the Fibonacci and Pell numbers of order k [3], [4], and identical to the r -bonacci polynomials $R_n(x)$ ($n \geq -(r-2)$) of [1] for $k=r$ and $n \geq 0$. Then we state and prove a theorem, which provides two expansions of $f_n^{(k)}(x)$ ($n \geq 1$) in terms of the multinomial and binomial coefficients, respectively. Hoggatt and Bicknell [1], among other results, give another expansion of $f_n^{(k)}(x)$, in terms of the

elements of the left - justified k-nomial triangle. The latter, however, are less widely known and used than the multinomial and binomial coefficients, and on this account our expansions may be considered better. As a corollary to our theorem, we derive several results of [2]-[4] and [5]. We also obtain a relation between $f_n^{(k)}(x)$ ($n \geq 1$) and probability.

2. THE FIBONACCI POLYNOMIALS OF ORDER K AND MULTINOMIAL COEFFICIENTS.

In this section, we introduce the Fibonacci polynomials of order k and derive two expansions of them in terms of the multinomial and binomial coefficients, respectively. The proof is along the lines of [2] and [4].

DEFINITION. The sequence of polynomials $\{f_n^{(k)}(x)\}_{n=0}^\infty$ is said to be the sequence of Fibonacci polynomials of order k if $f_0^{(k)}(x)=0$, $f_1^{(k)}(x)=1$, and

$$f_n^{(k)}(x) = \begin{cases} \sum_{i=1}^n x^{k-i} f_{n-i}^{(k)}(x) & \text{if } 2 \leq n \leq k \\ k \sum_{i=1}^k x^{k-i} f_{n-i}^{(k)}(x) & \text{if } n \geq k + 1. \end{cases} \tag{2.1}$$

If $f_n^{(r)}(x)=0$ for $-(r-2) \leq n \leq -1$, Hoggatt and Bicknell [1] call $R_n(x)=f_n^{(r)}(x)$ ($n \geq -(r-2)$) r-fibonacci polynomials.

Denoting by $F_n(x)$, $f_n^{(k)}$ and $P_n^{(k)}$, respectively, the Fibonacci polynomials [5], the Fibonacci numbers of order k [3], and the Pell numbers of order k [4], it follows from (2.1) that

$$f_n^{(2)}(x) = F_n(x), \quad f_n^{(k)}(1) = f_n^{(k)} \quad \text{and} \quad F_n^{(k)}(2) = P_n^{(k)}. \tag{2.2}$$

We now proceed to show the following lemma.

LEMMA. Let $\{f_n^{(k)}(x)\}_{n=0}^\infty$ be the sequence of Fibonacci polynomials of order k, and denote its generating function by $g_k(s;x)$. Then, for $|s| < x/(1+x^k)$,

$$g_k(s;x) = \frac{s(1-\frac{s}{x})}{1-\frac{s}{x}(1+x^k-s^k)} = \frac{s}{1-x^k[\frac{s}{x}+\frac{s^2}{x^2}+\dots+\frac{s^k}{x^k}]}$$

PROOF. We see from the definition that $f_2^{(k)}(x)=x^{k-1}$, $xf_n^{(k)}(x)-f_{n-1}^{(k)}(x)=x^k f_{n-1}^{(k)}(x)$ for $3 \leq n \leq k+1$, and $xf_n^{(k)}(x)-f_{n-1}^{(k)}(x)=x^k f_{n-1}^{(k)}(x)-f_{n-1-k}^{(k)}(x)$ for $n \geq k+2$. Therefore,

$$f_n^{(k)}(x) = \begin{cases} \frac{1}{x}(1+x^k)f_{n-1}^{(k)}(x), & 3 \leq n \leq k+1 \\ \frac{1}{x}(1+x^k)f_{n-1}^{(k)}(x) - \frac{1}{x}f_{n-1-k}^{(k)}(x), & n \geq k+2 \end{cases}$$

$$= \begin{cases} [\frac{1}{x}(1+x^k)]^{n-2}x^{k-1}, & 2 \leq n \leq k+1 \\ \frac{1}{x}(1+x^k)f_{n-1}^{(k)}(x) - \frac{1}{x}f_{n-1-k}^{(k)}(x), & n \geq k+2. \end{cases} \tag{2.3}$$

It may be seen, by means of (2.3) and induction on n, that

$$f_n^{(k)}(x) \leq [\frac{1}{x}(1+x^k)]^{n-2}x^{k-1}, \quad n \geq 2, \tag{2.4}$$

which implies the convergence of $g_k(s;x)$ for $|s| < x/(1+x^k)$. Next, by means of (2.3), we observe that

$$g_k(s;x) = \sum_{n=0}^{\infty} s^n f_n^{(k)}(x)$$

$$= s + \sum_{n=2}^{k-1} s^n [\frac{1}{x}(1+x^k)]^{n-2}x^{k-1} + \sum_{n=k+2}^{\infty} s^n f_n^{(k)}(x), \tag{2.5}$$

and

$$\sum_{n=k+2}^{\infty} s^n f_n^{(k)}(x) = \frac{1}{x}(1+x^k) \sum_{n=k+2}^{\infty} s^n f_{n-1}^{(k)}(x) - \frac{1}{x} \sum_{n=k+2}^{\infty} s^n f_{n-1-k}^{(k)}(x)$$

$$= \frac{s}{x}(1+x^k) \left\{ \sum_{n=0}^{\infty} s^n f_n^{(k)}(x) - s \sum_{n=2}^{\infty} s^n [\frac{1}{x}(1+x^k)]^{n-2}x^{k-1} \right\} - \frac{1}{x} s^{k+1} \sum_{n=1}^{\infty} s^n f_n^{(k)}(x)$$

$$= [\frac{s}{x}(1+x^k) - \frac{s^{k+1}}{x}] g_k(s;x) - \frac{s^2}{x} \sum_{n=2}^{k-1} s^n [\frac{1}{x}(1+x^k)]^{n-2}x^{k-1}. \tag{2.6}$$

The last two relations give

$$g_k(s;x) = s + \frac{s}{x}(1+x^k - s^k) g_k(s;x) - \frac{s^2}{x},$$

so that

$$g_k(s;x) = \frac{s(1-\frac{s}{x})}{1-\frac{s}{x}(1+x^k-s^k)} = \frac{s}{1-x^k [\frac{s}{x} + (\frac{s}{x})^2 + \dots + (\frac{s}{x})^k]}.$$

We will employ the above lemma to establish the following expansions of $f_n^{(k)}(x)$ ($n \geq 1$).

THEOREM. Let $\{f_n^{(k)}(x)\}_{n=0}^{\infty}$ be the Fibonacci polynomials of order k. Then

$$(a) \quad f_{n+1}^{(k)}(x) = \sum_{n_1, \dots, n_k} \binom{n_1 + \dots + n_k}{n_1, \dots, n_k} x^{k(n_1 + \dots + n_k) - n}, \quad n \geq 0,$$

where the summation is over all non-negative integers n_1, \dots, n_k such that $n_1 + 2n_2 + \dots + kn_k = n$;

$$\begin{aligned}
 (b) \quad f_{n+1}^{(k)}(x) &= \left(\frac{1+x}{x}\right)^n \sum_{i=0}^{[n/(k+1)]} (-1)^i \binom{n-ki}{i} x^{ki} (1+x)^{k-(k+1)i} \\
 &\quad - \frac{1}{x} \left(\frac{1+x}{x}\right)^{n-1} \sum_{i=0}^{[(n-1)/(k+1)]} (-1)^i \binom{n-1-ki}{i} x^{ki} (1+x)^{k-(k+1)i}, \quad n \geq 1, \\
 & \hspace{25em} n \geq 1,
 \end{aligned}$$

where, as usual, $[x]$ denotes the greatest integer in x .

PROOF. First we show (a). Let $|s| < x/(1+x^k)$, so that $|x^k[\frac{s}{x} + (\frac{s}{x})^2 + \dots + (\frac{s}{x})^k]| < 1$.

Let $n_i (1 \leq i \leq k)$ be non-negative integers as specified below. Then, using the lemma and the

multinomial theorem, and replacing n by $n - \sum_{i=1}^k (i-1)n_i$, we get,

$$\begin{aligned}
 \sum_{n=0}^{\infty} s^n f_{n+1}^{(k)}(x) &= \{1 - x^k [\frac{s}{x} + (\frac{s}{x})^2 + \dots + (\frac{s}{x})^k]\}^{-1} \\
 &= \sum_{n=0}^{\infty} \{x^k [\frac{s}{x} + (\frac{s}{x})^2 + \dots + (\frac{s}{x})^k]\}^n \\
 &= \sum_{n=0}^{\infty} x^{kn} \sum_{\substack{n_1, \dots, n_k \ni \\ n_1 + \dots + n_k = n}} \binom{n}{n_1, \dots, n_k} \left(\frac{s}{x}\right)^{n_1 + 2n_2 + \dots + kn_k} \\
 &= \sum_{n=0}^{\infty} s^n \sum_{\substack{n_1, \dots, n_k \ni \\ n_1 + 2n_2 + \dots + kn_k = n}} \binom{n_1 + \dots + n_k}{n_1, \dots, n_k} n^{k(n_1 + \dots + n_k) - n}, \quad (2.8)
 \end{aligned}$$

from which (a) follows.

We now proceed to establish (b). Let $0 < s < x/(1+x^k)$, so that $|\frac{s}{x}(1+x^k - s^k)| < 1$.

Then, using the lemma and the binomial theorem, replacing n by $n-ki$, and setting

$$B_n^{(k)}(x) = \left(\frac{1+x}{x}\right)^n \sum_{i=0}^{[n/(k+1)]} (-1)^i \binom{n-ki}{i} x^{ki} (1+x)^{k-(k+1)i}, \quad n \geq 0, \quad (2.9)$$

we get

$$\begin{aligned}
 \sum_{n=0}^{\infty} s^n f_{n+1}^{(k)}(x) &= (1 - \frac{s}{x}) [1 - \frac{s}{x}(1+x^k - s^k)]^{-1} \\
 &= (1 - \frac{s}{x}) \sum_{n=0}^{\infty} \left[\frac{s}{x}(1+x^k - s^k)\right]^n
 \end{aligned}$$

COROLLARY 2.2. Let $\{f_n^{(k)}(x)\}_{n=0}^{\infty}$ be the sequence of Fibonacci polynomials of order k , and denote by N_k the number of trials until the occurrence of the k th consecutive success in independent trials with success probability p ($0 < p < 1$). Then,

$$P(N_k = n+k) = p^{n+k} \left(\frac{1-p}{p}\right)^{n/k} f_{n+1}^{(k)} \left(\left(\frac{1-p}{p}\right)^{1/k}\right), \quad n \geq 0.$$

PROOF. It follows directly from Theorem 3.1 of [3] and part (a) of the present theorem.

In particular, Corollary 2.2. reduces to the following results of [2] and [4], respectively, by means of (2.2).

Let N_k be as above, and set $p = (1+2^k)^{-1}$. Then,

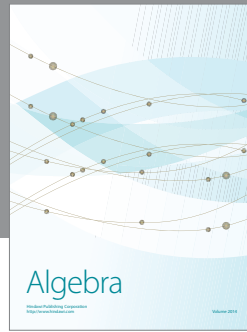
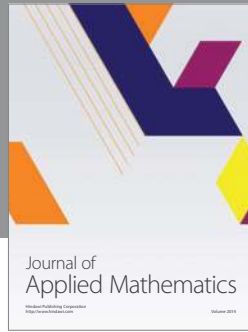
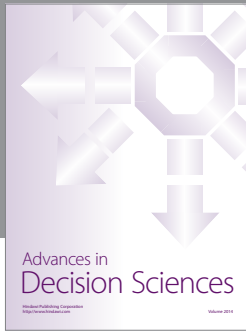
$$P(N_k = n+k) = \frac{2^n}{(1+2^k)^{n+k}} P_{n+1}^{(k)}, \quad n \geq 0. \quad (2.11)$$

Let N_k be as above, and set $p = 1/2$. Then,

$$P(N_k = n+k) = \frac{1}{2^{n+k}} f_{n+1}^{(k)}, \quad n \geq 0. \quad (2.12)$$

REFERENCES

1. HOGGATT, V.E. and BICKNELL, MARGORIE. Generalized Fibonacci Polynomials, The Fibonacci Quarterly 11 (1973), 457-465.
2. PHILIPPOU, A.N. A Note On The Fibonacci Sequence Of Order k and Multinomial Coefficients, The Fibonacci Quarterly 21 (1983), in press.
3. PHILIPPOU, A.N. and MUWAFI, A.A. Waiting for the k th Consecutive Success and the Fibonacci Sequence of Order k , The Fibonacci Quarterly 20 (1982), 28-32.
4. PHILIPPOU, A.N. and PHILIPPOU, G.N. The Pell Sequence of Order k , Multinomial Coefficients, and Probability, Submitted for publication (1982).
5. SWAMY, M.N.S. Problem B - 74, The Fibonacci Quarterly 3 (1965), 236.



Hindawi

Submit your manuscripts at
<http://www.hindawi.com>

