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Fibonacci Search with Arbitrary First Evaluation

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FIBONACCI SEARCH WITH ARBITRARY FIRST EVALUATION

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ABSTRACT

The Fibonacci search technique for maximizing a unimodal function of one real variable is generalized to the case of a given first evaluation. This technique is then employed to determine the optimal sequential search technique for the maximization of a concave function.

I. Introduction

A real function $f\colon [a,b] \to R$, where $a < b$, is called

(1.1) *unimodal,*

if there are $\underline{x}$, $\bar{x} \in [a,b]$ such that f is increasing for $x \leq \underline{x}$ and nonincreasing for $x \geq \underline{x}$, decreasing for $x \geq \bar{x}$ and nondecreasing for $x \leq \bar{x}$ (Fig. 1).

(1.2) *If* f *is unimodal, then the interval* $[\underline{x},\bar{x}]$ *consists of all maxima of* f.

Proof: f is constant in $[\underline{x},\bar{x}]$, since it is by definition nonincreasing for $x \geq \underline{x}$ as well as nondecreasing for $x \leq \bar{x}$. If $x < \underline{x}$, then $f(x) < f(\bar{x})$ as f increases in $[a,x]$. If $x > \bar{x}$, then $f(x) < f(\bar{x})$ as f decreases in $[\bar{x},b]$.-

The definition of unimodality is chosen so as to guarantee that

(1.3) *whenever a unimodal function* f *has been evaluated for two arguments* x_1 *and* x_2 *with* $a \leq x_1 < x_2 \leq b$, *then some maximum of* f *must lie in* $[x_1,b]$ *if* $f(x_1) \leq f(x_2)$ *and in* $[a,x_2]$ *if* $f(x_1) \geq f(x_2)$.

Proof: If $f(x_1) \geq f(x_2)$, then x_1 and x_2 cannot be both in that portion of the interval $[a,b]$ in which the function decreases. In other words, $\bar{x}$ cannot lie to the left of x_1. Thus $\bar{x} \in [x_1,b]$,

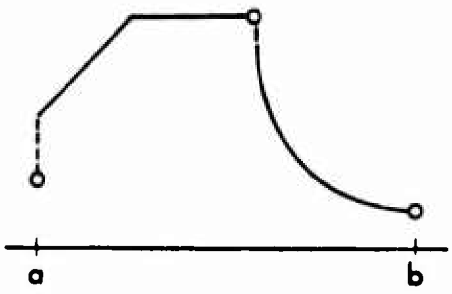

Figure 1. Example of a unimodal function.

and $\bar{x}$ is a maximum of f by (1.2). Similarly, if $f(x_1) \leq f(x_2)$, then $\underline{x} \in [a,x_2]$.

A sequential search based on (1.3) will successively narrow down the interval in which a maximum of f is known to lie. Such an interval is called the

(1.4) *interval of uncertainty.*

Kiefer [3] has asked the question of optimally conducting this search, and answered it by developing his well known Fibonacci search.

The Fibonacci search gives a choice of two arguments for which to make the first evaluation. But what happens if by mistake or for some other reason the first evaluation took place at some argument other than the two optimal ones? How does one optimally proceed from there?

In this paper, we shall therefore ask and answer the question for an optimal sequential search plan with given arbitrary first evaluation. The resulting technique is applied to improving on Fibonacci search for functions known to be concave. The technique may also be of interest in the context of stability of Fibonacci search in the presence of round-off errors as studied by Overholt [6] and Boothroydt [1] (see also Kovalik and Osborne [4]).

2. Length of Uncertainty

In what follows we assume that $a = 0$ and $b = 1$. Furthermore, we shall permit zero distances between two arguments of evaluation, interpreting each such occurrence as evaluating the (not necessarily unique or finite) derivative of the function f. A more careful analysis would take into account the smallest justifiable distance ε between arguments (Kiefer [3], Oliver and Wilde [5]).

By

$$L_k(x), \quad 0 \leq x \leq 1 ,$$

we denote the length to which the interval of uncertainty (1.4) can surely be reduced by k evaluations in addition to a first one at x. Extending a recursive argument due to Johnson [2], we obtain

$$(2.1) \qquad L_k(x) = \min \{ M_k(x), \; M_k(1-x) \} ,$$

where

$$M_k(x) := \min_{x \leq y \leq 1} \max \left\{ (1-x) L_{k-1}\left(\frac{1-y}{1-x}\right), \; y L_{k-1}\left(\frac{x}{y}\right) \right\} .$$

Proof: Let y denote the first function argument over which we have control. If $x \leq y \leq 1$, then the two possible intervals of uncertainty are $[0,y]$ and $[x,1]$. The former contains the point of evaluation x. The best upper bound for the length of the interval of uncertainty after the remaining $k-1$ evaluations is given by

$$(2.2) \qquad y L_{k-1}\left(\frac{x}{y}\right) .$$

Similarly, y is the evaluation point in $[x,1]$, leading to the best upper bound

(2.3) $(1-x)L_{k-1}(\frac{1-y}{1-x})$.

Whether $[0,y]$ or $[x,1]$ is the first interval of uncertainty depends on the result of the evaluation at y: if $f(y) > f(x)$, then $[0,y]$, if $f(y) \le f(x)$, then $[x,1]$. Hence the maximum $M_k(x)$ of the two expressions (2.2) and (2.3) is the best result achievable if y is selected between x and 1. The expression

$$N_k(x) := \min_{0 \le y \le x} \max\left\{xL_{k-1}(\frac{y}{x}),\ (1-y)L_{k-1}(\frac{1-x}{1-y})\right\}$$

analogously describes the best result achievable if y is between 0 and x. Since we control the choice of y, we can choose the smaller one of these two expressions; and this gives

$$L_k(x) = \min\{M_k(x),\ N_k(x)\} .$$

Introducing for $0 \le x \le y \le 1$,

$$S_k(x,y) := \max\left\{(1-x)L_{k-1}(\frac{1-y}{1-x}),\ yL_{k-1}(\frac{x}{y})\right\} ,$$

we have

$$M_k(x) = \min_{x \le y \le 1} S_k(x,y), \quad N_k(x) = \min_{0 \le y \le x} S_k(y,x) .$$

Now for $0 \le x \le y \le 1$,

(2.4) $S_k(x,y) = S_k(1-y,1-x)$.

Therefore, $N_k(x) = M_k(1-x)$, and (2.1) is proved.-

At the beginning, the interval of uncertainty is the entire interval in which the function is to be examined. A single function evaluation at any point x does not change this situation. Hence

$$L_0(x) = 1 .$$

We then have

$$M_1(x) = \min_{x \le y \le 1} \max\{1-x,y\} = \max\{1-x,x\} = M_1(1-x).$$

Hence

$$(2.5) \qquad L_1(x) = \max\{1-x,x\} = \begin{cases} 1-x & \textit{for } 0 \le x \le \tfrac{1}{2} \\ x & \textit{for } \tfrac{1}{2} \le x \le 1 . \end{cases}$$

For $k \ge 2$, we claim (Fig. 2):

$$(2.6) \qquad L_k(x) = \begin{cases} \dfrac{1-x}{F_x} & \textit{for } 0 \le x \le \dfrac{F_{k-1}}{F_{k+1}} \\[2ex] \dfrac{x}{F_{k-1}} & \textit{for } \dfrac{F_{k-1}}{F_{k+1}} \le x \le \tfrac{1}{2} \\[2ex] \dfrac{1-x}{F_{k-1}} & \textit{for } \tfrac{1}{2} \le x \le \dfrac{F_k}{F_{k+1}} \\[2ex] \dfrac{x}{F_k} & \textit{for } \dfrac{F_k}{F_{k+1}} \le x \le 1 , \end{cases}$$

where $F_0 = 1$, $F_1 = 1$, $F_2 = 2$, $F_3 = 3$, ..., $F_k = F_{k-2} + F_{k-1}$ *are the Fibonacci numbers.*

Proof: The case $k = 2$ requires special treatment. From (2.5),

$$yL_1(\tfrac{x}{y}) = \begin{cases} y-x & \text{for } (x,y) \in A_1 := \left\{0 \le \frac{x}{y} \le \tfrac{1}{2}\right\} \\ x & \text{for } (x,y) \in A_2 := \left\{\tfrac{1}{2} \le \frac{x}{y} \le 1\right\} , \end{cases}$$

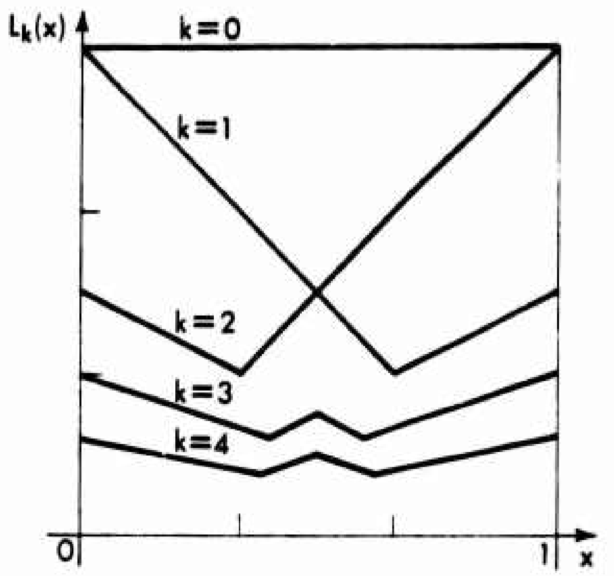

Figure 2. $L_k(x)$ for $k = 0, \ldots, 4$.

$$(1-x)L_1\left(\frac{1-y}{1-x}\right) = \begin{cases} y-x & \text{for} \quad (x,y) \in B_1 := 0 \le \frac{1-y}{1-x} < \frac{1}{2} \\ 1-y & \text{for} \quad (x,y) \in B_2 := \frac{1}{2} \le \frac{1-y}{1-x} < 1 \end{cases} .$$

We are now able to determine $S_2(x,y)$ in each of the four regions $A_i \cap B_j$ separately:

$A_1 \cap B_1$: $S_2(x,y) = \max\{y-x,\ y-x\} = y-x.$

$A_1 \cap B_2$: $S_2(x,y) = \max\{y-x,\ 1-y\} = 1-y.$

$A_2 \cap B_1$: $S_2(x,y) = x$ by (2.4) and $(1-y,1-x) \in A_1 \cap B_2$.

$A_2 \cap B_2$: $S_2(x,y) = \max\{x,1-y\} = \begin{cases} x & \text{if} \quad y \ge 1-x \\ 1-y & \text{if} \quad y \le 1-x. \end{cases}$

The sets A_i and B_j are represented in Fig. 3. They are triangles formed by the line segments marked A_i and B_j, respectively, and the corresponding opposite corner of the triangles. The feathered lines are the minimum lines with respect to constant values of x, i.e. if proceeding vertically the intersection with the feathered lines marks a minimum. The function $M_k(x)$ is defined to be the value of this minimum. Hence

$$M_2(x) = \begin{cases} \frac{1-x}{2} & \text{if} \quad 0 \le x \le \frac{1}{3} \\ x & \text{if} \quad \frac{1}{3} \le x \le 1 . \end{cases}$$

By (2.1) we then have finally

$$L_2(x) = \begin{cases} \frac{1-x}{2} & \text{if} \quad 0 \le x \le \frac{1}{3} \\ x & \text{if} \quad \frac{1}{3} \le x \le \frac{1}{2} \\ 1-x & \text{if} \quad \frac{1}{2} \le x \le \frac{2}{3} \\ x & \text{if} \quad \frac{2}{3} \le x \le 1, \end{cases}$$

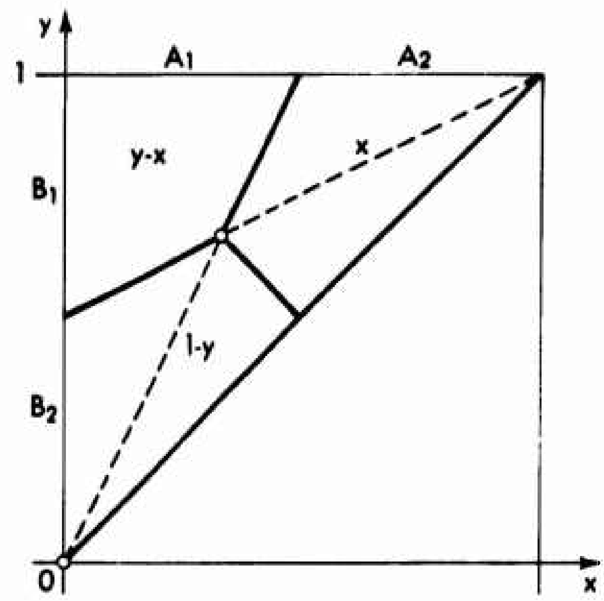

Figure 3. $S_2(x,y)$.

in accordance with (2.6).

The case $k \geq 3$ is now proved by induction over k. We have

$$yL_{k-1}\left(\frac{x}{y}\right) = \begin{cases} \dfrac{y-x}{F_{k-1}} & \text{for } (x,y) \in A_1 := 0 \leq \dfrac{x}{y} \leq \dfrac{F_{k-2}}{F_k} \\ \dfrac{x}{F_{k-2}} & \text{for } (x,y) \in A_2 := \dfrac{F_{k-2}}{F_k} \leq \dfrac{x}{y} \leq \dfrac{1}{2} \\ \dfrac{y-x}{F_{k-2}} & \text{for } (x,y) \in A_3 := \dfrac{1}{2} \leq \dfrac{x}{y} \leq \dfrac{F_{k-1}}{F_k} \\ \dfrac{x}{F_{k-1}} & \text{for } (x,y) \in A_4 := \dfrac{F_{k-1}}{F_k} \leq \dfrac{x}{y} \leq 1 \end{cases}$$

$$(1-x)L_{k-1}\left(\frac{1-y}{1-x}\right) = \begin{cases} \dfrac{y-x}{F_{k-1}} & \text{for } (x,y) \in B_1 := 0 \leq \dfrac{1-y}{1-x} \leq \dfrac{F_{k-2}}{F_k} \\ \dfrac{1-y}{F_{k-2}} & \text{for } (x,y) \in B_2 := \dfrac{F_{k-2}}{F_k} \leq \dfrac{1-y}{1-x} \leq \dfrac{1}{2} \\ \dfrac{y-x}{F_{k-2}} & \text{for } (x,y) \in B_3 := \dfrac{1}{2} \leq \dfrac{1-y}{1-x} \leq \dfrac{F_{k-1}}{F_k} \\ \dfrac{1-y}{F_{k-1}} & \text{for } (x,y) \in B_4 := \dfrac{F_{k-1}}{F_k} \leq \dfrac{1-y}{1-x} \leq 1 \end{cases}$$

we determine $S_k(x,y)$ in all regions $A_i \cap B_j$ with $i \leq j$. For the remaining regions, we use (2.4).

$A_1 \cap B_1$: $S_k(x,y) = \max\left\{\dfrac{y-x}{F_{k-1}}, \dfrac{y-x}{F_{k-1}}\right\} = \dfrac{y-x}{F_{k-1}}$

$A_1 \cap B_2$: $S_k(x,y) = \max\left\{\dfrac{y-x}{F_{k-1}}, \dfrac{1-y}{F_{k-2}}\right\} = \dfrac{1-y}{F_{k-2}}$ since $(x,y) \in B_2$

gives $(1-x)F_{k-2} \leq (1-y)F_k$, and therefore $(y-x)F_{k-2} =$
$(1-x)F_{k-2} - (1-y)F_{k-2} \leq (1-y)F_k - (1-y)F_{k-2} = (1-y)F_{k-1}$.

$A_1 \cap B_3$: $S_k(x,y) = \max\left\{\frac{y-x}{F_{k-1}}, \frac{y-x}{F_{k-2}}\right\} = \frac{y-x}{F_{k-2}}$.

$A_1 \cap B_4$: $S_k(x,y) = \max\left\{\frac{y-x}{F_{k-1}}, \frac{1-y}{F_{k-1}}\right\} = \frac{1-y}{F_{k-1}}$ since $(x,y) \in B_4$

gives $1-x \leq 2(1-y)$ or $y-x \leq 1-y$.

$A_2 \cap B_2$: $S_k(x,y) = \max\left\{\frac{x}{F_{k-2}}, \frac{1-y}{F_{k-2}}\right\} = \frac{1}{F_{k-2}} \max\{x, 1-y\}$.

$A_2 \cap B_3$: $S_k(x,y) = \max\left\{\frac{x}{F_{k-2}}, \frac{y-x}{F_{k-2}}\right\} = \frac{y-x}{F_{k-2}}$ since $(x,y) \in A_2$

gives $2x \leq y$ or $x \leq y - x$.

$A_2 \cap B_4$: $S_k(x,y) = \max\left\{\frac{x}{F_{k-2}}, \frac{1-y}{F_{k-1}}\right\} = \frac{1-y}{F_{k-1}}$ since $(x,y) \in A_2$

gives $2x-y \leq 0$, and since $(x,y) \in B_4$ gives $-xF_{k-1} + yF_k \leq F_{k-2}$.
Indeed, multiplying the former inequality by F_{k-1} and adding it to the latter gives $xF_{k-1} + yF_{k-2} \leq F_{k-2}$.

$A_3 \cap B_3$: $S_k(x,y) = \max\left\{\frac{y-x}{F_{k-2}}, \frac{y-x}{F_{k-2}}\right\} = \frac{y-x}{F_{k-2}}$.

$A_3 \cap B_4$: $S_k(x,y) = \max\left\{\frac{y-x}{F_{k-2}}, \frac{1-y}{F_{k-1}}\right\} = \frac{1-y}{F_{k-1}}$ since $(x,y) \in B_4$

gives $(1-x)F_{k-1} \leq (1-y)F_k$, and therefore $(y-x)F_{k-1} =$
$(1-x)F_{k-1} - (1-y)F_{k-1} \leq (1-y)F_k - (1-y)F_{k-1} = (1-y)F_{k-2}$.

$A_4 \cap B_4$: $S_k(x,y) = \max\left\{\frac{x}{F_{k-1}}, \frac{1-y}{F_{k-1}}\right\} = \frac{1}{F_{k-1}} \max\{x, 1-y\}$.

The schematic representation of $S_k(x,y)$ then is given by Fig. 4. There are breaks along the line $x = 1 - y$ in areas $A_2 \cap B_2$ and $A_4 \cap B_4$. The feathered lines are again those boundaries of linearity regions at which S_k decreases for fixed x. The abscissae of intersection points of feathered lines are therefore critical. The first one of these critical arguments we denote by v. It is the abscissa of the intersection

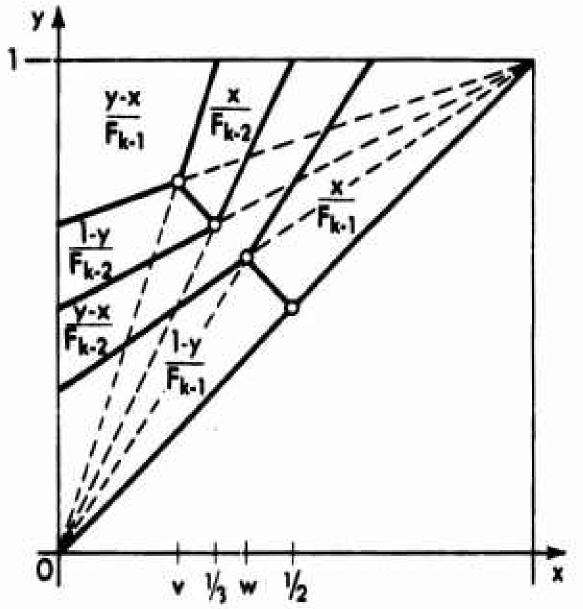

Figure 4. $S_k(x,y)$ and critical arguments.

point of the line

$$(2.7) \qquad \frac{1-y}{1-x} = \frac{F_{k-2}}{F_k} ,$$

which separates B_1 from B_2, and the line

$$(2.8) \qquad \frac{x}{y} = \frac{F_{k-2}}{F_k} ,$$

which separates A_1 from A_2. Elimination of y yields

$$v = \frac{F_{k-2}}{F_k + F_{k-2}} .$$

The next critical argument clearly has the value $\frac{1}{3}$. The third one, which we call w, is the intersection of the line

$$(2.9) \qquad \frac{1-y}{1-x} = \frac{F_{k-1}}{F_k} ,$$

which separates B_3 from B_4, and the line

$$(2.10) \qquad \frac{x}{y} = \frac{F_{k-1}}{F_k} ,$$

which separates A_3 and A_4. Elimination of y yields

$$w = \frac{F_{k-1}}{F_{k+1}} .$$

The last critical argument finally has the value $\frac{1}{2}$.

For $0 \leq x \leq v$ the values of $S_k(x,y)$ at the intersection of the vertical through x with the two feathered lines (2.8) and (2.9) are potential minima. The equations of these lines can be rewritten as

$$\frac{1-y}{F_{k-2}} = \frac{1-x}{F_k} \quad \text{and} \quad \frac{1-y}{F_{k-1}} = \frac{1-x}{F_k} .$$

As these terms also represent the value of $S_k(x,y)$, we have

$M_k(x) = \frac{1-x}{F_k}$ for $0 \le x \le v$.

For $v < x < \frac{1}{3}$ locally minimal points are to be found on line (2.9) and in the area where $S_k(x,y)$ assumes the value $\frac{x}{F_{k-2}}$. Now $x \ge v$ gives $xF_k \ge (1-x)F_{k-2}$ or $\frac{x}{F_{k-2}} \ge \frac{1-x}{F_k}$. Thus $M_k(x) = \frac{1-x}{F_k}$ for $v \le x \le \frac{1}{3}$.

For $\frac{1}{3} \le x \le w$ only the line (2.9) is interesting, and $M_k(x)$ still takes the value $\frac{1-x}{F_k}$.

For $w \le x \le \frac{1}{2}$ and beyond the minimum is assumed within the entire line segment which happens to meet the area in which $S_k(x,y) = \frac{x}{F_{k-1}}$. Thus finally

$$(2.11) \qquad M_k(x) = \begin{cases} \frac{1-x}{F_k} & \textit{for} \quad 0 \le x \le \frac{F_{k-1}}{F_{k+1}} \\ \frac{x}{F_{k-1}} & \textit{for} \quad \frac{F_{k-1}}{F_{k+1}} \le x \le 1 , \end{cases}$$

and (2.6) follows immediately from (2.1).-

Note also that (2.11) implies

$$(2.12) \qquad L_k(x) = \begin{cases} M_k(x) & \text{for} \quad 0 \le x \le \frac{1}{2} \\ M_k(1-x) & \text{for} \quad \frac{1}{2} \le x \le 1 . \end{cases}$$

3. Search Strategy.

In the previous section, we have determined the optimal length of uncertainty $L_k(x)$, which can be achieved in k evaluations in addition to one evaluation at $x \in [0,1]$. We have yet to describe a search strategy which realizes $L_k(x)$. This amounts to specifying the argument y of the first evaluation in addition to x. In view of (2.12), this reduces to determining y such that $M_k(x) = S_k(x,y)$ for given x between 0 and $\frac{1}{2}$, a task which has been performed already while calculating $M_k(x)$.

If $0 \le x \le v$, then there are two optimal solutions y, since $S_k(x,y) = \frac{1-x}{F_k}$ along both feathered lines in Fig. 4. This non-uniqueness is not surprising. Indeed, if $x = 0$, then the evaluation at this argument does not contribute at all towards narrowing the interval of uncertainty, and the optimal continuation is just plain Fibonacci with one evaluation wasted. And in this case there are two optimal arguments, namely the first and second (k-1)-st order Fibonacci points

$$\frac{F_{k-2}}{F_k}, \quad \frac{F_{k-1}}{F_k} \quad .$$

(3.1) *If* $0 < x < \frac{F_{k-2}}{F_k+F_{k-2}}$, *then any of the two* (k-1)-st *order Fibonacci points in the interval* $[x,1]$ *is an optimal evaluation point*

$$y_1 = x + \frac{F_{k-2}}{F_k}(1-x) = \frac{xF_{k-1} + F_{k-2}}{F_k}$$

$$y_2 = x + \frac{F_{k-1}}{F_k}(1-x) = \frac{xF_{k-2} + F_{k-1}}{F_k} \quad .$$

In both intervals $v \le x \le \frac{1}{3}$ and $\frac{1}{3} \le x \le w$, the optimal solution y is unique.

(3.2) *If* $\frac{F_{k-2}}{F_k+F_{k-2}} \leq x \leq \frac{F_{k-1}}{F_{k-2}}$ *then the optimal evaluation point* y *is the first* $(k-1)$-st *order Fibonacci point of the interval* $[x,1]$.

Finally, if $w \leq x \leq \frac{1}{2}$, then the optimal solutions fill an entire interval.

(3.3) Let $\frac{F_{k-1}}{F_{k-2}} \leq x \leq \frac{1}{2}$. *If* y_0 *is such that* x *is the second* $(k-1)$-st *order Fibonacci point in* $[0,y_0]$, *then all points in* $[1-x,y_0]$ *are optimal evaluation points.*

The following rule will always yield an optimal solution:

(3.4) <u>*Theorem:*</u> *An optimal search strategy after an arbitrary first evaluation at* $x_0 \in [a,b]$ *is as follows. If* $c \leq x \leq d$ *are such that* $[c,d]$ *constitutes the interval of uncertainty after* ℓ *additional evaluations, and if* x *is the argument for which the function has been evaluated already, then:*

(i) *If* x *lies between* c *and the first* $(k-\ell)$-th *order Fibonacci points in* $[c,d]$, *then choose* y *as the first* $(k-\ell)$-th *order Fibonacci point in* $[x,d]$.

(ii) *If* x *lies between the two* $(k-\ell)$-th *order Fibonacci points of* $[c,d]$, *then choose* y *as the symmetric image of* x *in* $[c,d]$, *i.e.* $y = c + d - x$.

(iii) *If* x *lies between* d *and the second of the two* $(k-\ell)$-th *order Fibonacci points in* $[c,d]$, *then choose* y *as the second* $(k-\ell)$-th *order Fibonacci point in* $[c,x]$.

We shall refer to any sequential search strategy in keeping with

(3.1, 2, 3), in particular the rule described in Theorem (3.4), as

(3.5) *modified Fibonacci search* .

If the interior of the interval of uncertainty does not contain an argument at which the function has been evaluated already, then the selection of the next evaluation by modified Fibonacci search will be the same as in standard Fibonacci search.

4. Spies

Intervals of uncertainty with nonoptimal evaluation points may be the result of the following situation. Suppose in maximizing a function we avail ourselves of the services of a "spy". This spy operates as follows: every time an interval of uncertainty has been based on the results of prior evaluations, he is consulted, and as a result of this consultation, the interval of uncertainty may sometimes be further reduced (remaining an interval) without additional evaluations. One cannot expect, however, that the remaining evaluation point (if there is any) is in optimal position within the new interval of uncertainty.

In this case, there is a question whether the additional information should be accepted. It is indeed conceivable that reducing the interval of uncertainty and subsequently continuing from a nonoptimal evaluation point would in the final analysis lead to a larger interval of uncertainty than ignoring the additional information and doing a straightforward Fibonacci search. That this is not so, is essentially the content of the following

(4.1) *Theorem: The optimal policy in the presence of an unpredictable spy is to heed his advice and to proceed from the interval of uncertainty so achieved by modified Fibonacci search with respect to the remaining evaluation point if there is any.*

Proof: Let $[c,d]$ be the interval of uncertainty as determined by the previous step of the search, and let $[\bar{c},\bar{d}]$, $c \leq \bar{c} \leq \bar{d} \leq d$, be the interval of uncertainty after consulting the spy. As the spy is unpredictable, there may be no further information forthcoming. This is the worst case, since even if the spy is providing information, it need not be heeded. Thus all we have to show is that we do not worse by proceeding form $[\bar{c},\bar{d}]$ than from any other interval $[c^*,d^*]$ with $[c,d] \supseteq [c^*,d^*] \supseteq [\bar{c},\bar{d}]$.

Now let x be the evaluation point in $[c,d]$. Then we distinguish two cases, depending on whether $x \in [\bar{c},\bar{d}]$ or not. Suppose $x \in [\bar{c},\bar{d}]$, then $x \in [c^*,d^*]$. Working on the latter interval, the best we can do in ℓ remaining steps is reducing the uncertainty to

$$(d^* - c^*)L_\ell\left(\frac{x - c^*}{d^* - c^*}\right) = \begin{cases} \dfrac{d^* - x}{F_\ell} & \text{for } 0 < \dfrac{x - c^*}{d^* - c^*} < \dfrac{F_{\ell-1}}{F_{\ell+1}} \quad (=: I_1) \\[2ex] \dfrac{x - c^*}{F_{\ell-1}} & \text{for } \dfrac{F_{\ell-1}}{F_{\ell+1}} < \dfrac{x - c^*}{d^* - c^*} < \dfrac{1}{2} \quad (=: I_2) \\[2ex] \dfrac{d^* - x}{F_{\ell-1}} & \text{for } \dfrac{1}{2} < \dfrac{x - c^*}{d^* - c^*} < \dfrac{F_\ell}{F_{\ell+1}} \quad (=: I_3) \\[2ex] \dfrac{x - c^*}{F_\ell} & \text{for } \dfrac{F_\ell}{F_{\ell+1}} < \dfrac{x - c^*}{d^* - c^*} < 1 \quad (=: I_4). \end{cases}$$

For all x such that $\frac{x - c^*}{d^* - c^*}$ and $\frac{x - \bar{c}}{\bar{d} - \bar{c}}$ are both in one of the four intervals I_i above,

$$(4.2)\qquad (d^* - c^*)L_\ell\left(\frac{x - c^*}{d^* - c^*}\right) \geq (\bar{d} - \bar{c})L_\ell\left(\frac{x - \bar{c}}{\bar{d} - \bar{c}}\right)$$

is immediate. Of the remaining twelve cases, we need consider only six, as the others follow by symmetry. Let

$$u^* := d^* - c^* \quad\text{and}\quad \bar{u} := \bar{d} - \bar{c}\ .$$

$\frac{x - c^*}{u^*} \in I_1$ and $\frac{x - \bar{c}}{\bar{u}} \in I_2$: $\frac{x - c^*}{u^*} \leq \frac{F_{\ell-1}}{F_{\ell+1}}$ implies $\frac{d^* - x}{u^*} \geq \frac{F_{\ell-1}}{F_{\ell+1}}$

Thus $\frac{d^* - x}{F_\ell} \geq \frac{x - c^*}{F_{\ell-1}} \geq \frac{x - \bar{c}}{F_{\ell-1}}$.

$\frac{x - c^*}{u^*} \in I_1$ and $\frac{x - \bar{c}}{\bar{u}} \in I_4$: $F_\ell \geq F_{\ell-1}$.

Thus $\frac{d^* - x}{F_\ell} \geq \frac{x - c^*}{F_{\ell-1}} > \frac{x - \bar{c}}{F_{\ell-1}} \geq \frac{x - \bar{c}}{F_\ell}$.

$\frac{x - c^*}{u^*} \in I_2$ and $\frac{x - \bar{c}}{\bar{u}} \in I_3$: $x - \bar{c} \geq \frac{\bar{u}}{2}$ gives $x - \bar{c} \geq \bar{d} - x$.

Thus $\frac{x - c^*}{F_{\ell-1}} > \frac{x - \bar{c}}{F_{\ell-1}} > \frac{\bar{d} - x}{F_{\ell-1}}$.

$\frac{x - c^*}{u^*} \in I_2$ and $\frac{x - \bar{c}}{\bar{u}} \in I_4$: $F_\ell \geq F_{\ell-1}$.

Thus $\frac{x - c^*}{F_{\ell-1}} \geq \frac{x - \bar{c}}{F_{\ell-1}} \geq \frac{x - \bar{c}}{F_\ell}$.

$\frac{x - c^*}{u^*} \in I_3$ and $\frac{x - \bar{c}}{\bar{u}} \in I_4$: $\frac{x - c^*}{u^*} \leq \frac{F_\ell}{F_{\ell+1}}$ implies $\frac{d^* - x}{u^*} \leq \frac{F_{\ell-1}}{F_{\ell+1}}$.

Thus $\frac{d^* - x}{F_{\ell-1}} \geq \frac{x - c^*}{F_\ell} \geq \frac{x - \bar{c}}{F_\ell}$.

The case in which $x \in [\bar{c},\bar{d}]$ remains to be considered. Suppose $x < \bar{c} < \bar{d}$. Since we proceed by standard Fibonacci in any interval of uncertainty not containing x in its interior, starting with $[\bar{c},\bar{d}]$ is certainly better than starting with $[x,\bar{d}] \subseteq [c.d]$, and we have already seen that $[x,\bar{d}]$ is better than any interval between $[c.d]$ and $[x,\bar{d}]$.

A spy is called

(4.3) *almost unpredictable* ,

if for each subinterval $[c^*,d^*]$ of the interval of uncertainty $[c.d]$, which results from the evaluation pattern, the spy has the option of reducing it only to an interval $[\bar{c},\bar{d}]$ which contains $[c^*,d^*]$. Plainly, we still have

(4.4) <u>*Theorem:*</u> *The optimal policy in the presence of an almost unpredictable spy is to heed his advice and to proceed from the interval of uncertainty so achieved by modified Fibonacci search with respect to the remaining evaluation point if there is any.*

5. <u>Concave Functions</u>

We shall see that a "spy" is available if the unimodal function to be maximized is known to be concave.

A function $f\colon [a,b] \to R$ is

(5.1) *concave*

in $[a,b]$ if

$$f(\lambda x + \mu y) \geq \lambda f(x) + \mu f(y)$$

holds for all $x,y \in [a,b]$, $\lambda, \mu \geq 0$ and $\lambda + \mu = 1$. The function is

(5.2) *strictly concave*

if

$$f(\lambda x + \mu y) > \lambda f(x) + \mu f(y)$$

holds for all x,y,λ,μ which are as above and satisfy in addition $x \neq y$ and $\lambda,\mu > 0$. We state without proof that

(5.3) *every upper semicontinuous concave function on* $[a,b]$ *is unimodal.*

Without the additional hypothesis of upper semicontinuity, (5.3) does not hold as there are concave functions without maximum on $[a,b]$.

Now consider two points $P_i := (x_i, f(x_i))$ $P_j := (x_j, f(x_j))$, $x_i < x_j$, of the graph

$$G(f) := \{(x,f(x)) : x \in [a,b]\},$$

and let L_{ij} be the straight line through P_i, P_j. Concavity implies that the graph of f lies not below L_{ij} in $[x_i,x_j]$ and not above L_{ij} in the remainder of the interval $[a,b]$. Hence if five points of the graph $G(f)$,

$$P_0 := (x_0, f(x_0)), \ldots, P_4 := (x_4, f(x_4))$$

with

$$x_0 < x_1 < x_2 < x_3 < x_4$$

and

$$f(x_2) > f(x_1), \quad i = 1,2,$$

are known, then that part of the graph $G(f)$ that lies above $[x_1,x_3]$ is contained in the union of the two triangles Δ_1 and Δ_2 formed by L_{01}, L_{12}, L_{23} and L_{12}, L_{23}, L_{34}, respectively. $f(x_2)$ is a lower bound for the maximum value of f. Therefore

(5.4) *a maximum of* f *must lie in the intersection of* $\Delta_1 \cup \Delta_2$ *with the horizontal through* P_2. (Fig. 5)

The information that the function f is concave can thus be used in order to reduce the interval of uncertainty.

In order to complete the description of the proposed search method for concave functions, a few more conventions are necessary. At the ends of the interval $[a,b]$, we pretend that the function has value $-\infty$, and if it has been evaluated there, we pretend that there are two values for the same abscissa, one of the values being infinite. Three evaluations will therefore reduce the interval of uncertainty as indicated in Fig. 6.

We proceed to show that

(5.5) *concavity is an almost unpredictable spy* (4.3)

Proof: Suppose we have five points

$$a \le x_0 \le x_1 < x_2 < x_3 \le x_4 \le b,$$

where x_0 and x_1 may both coincide with the left end-point a, and similarly x_3 and x_4 may coincide with the right end-point b. For

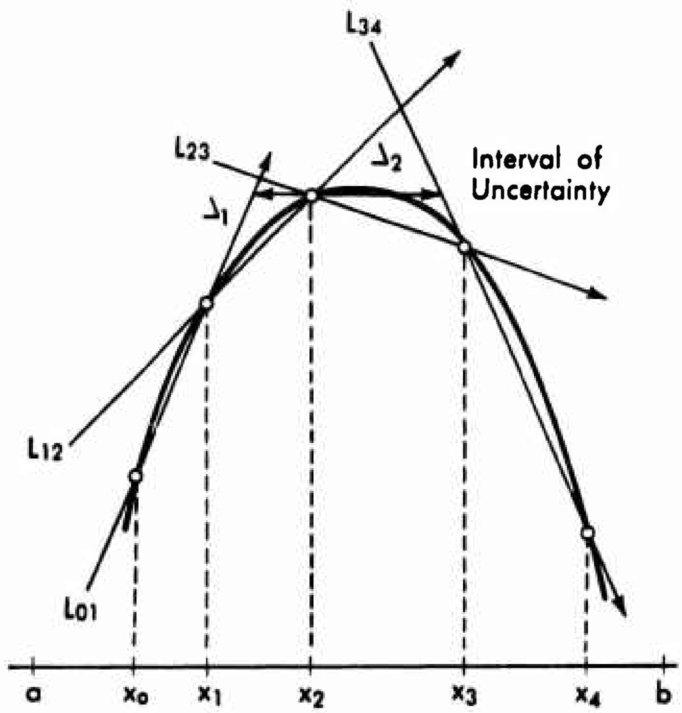

Figure 5. Bounding a concave function by chords.

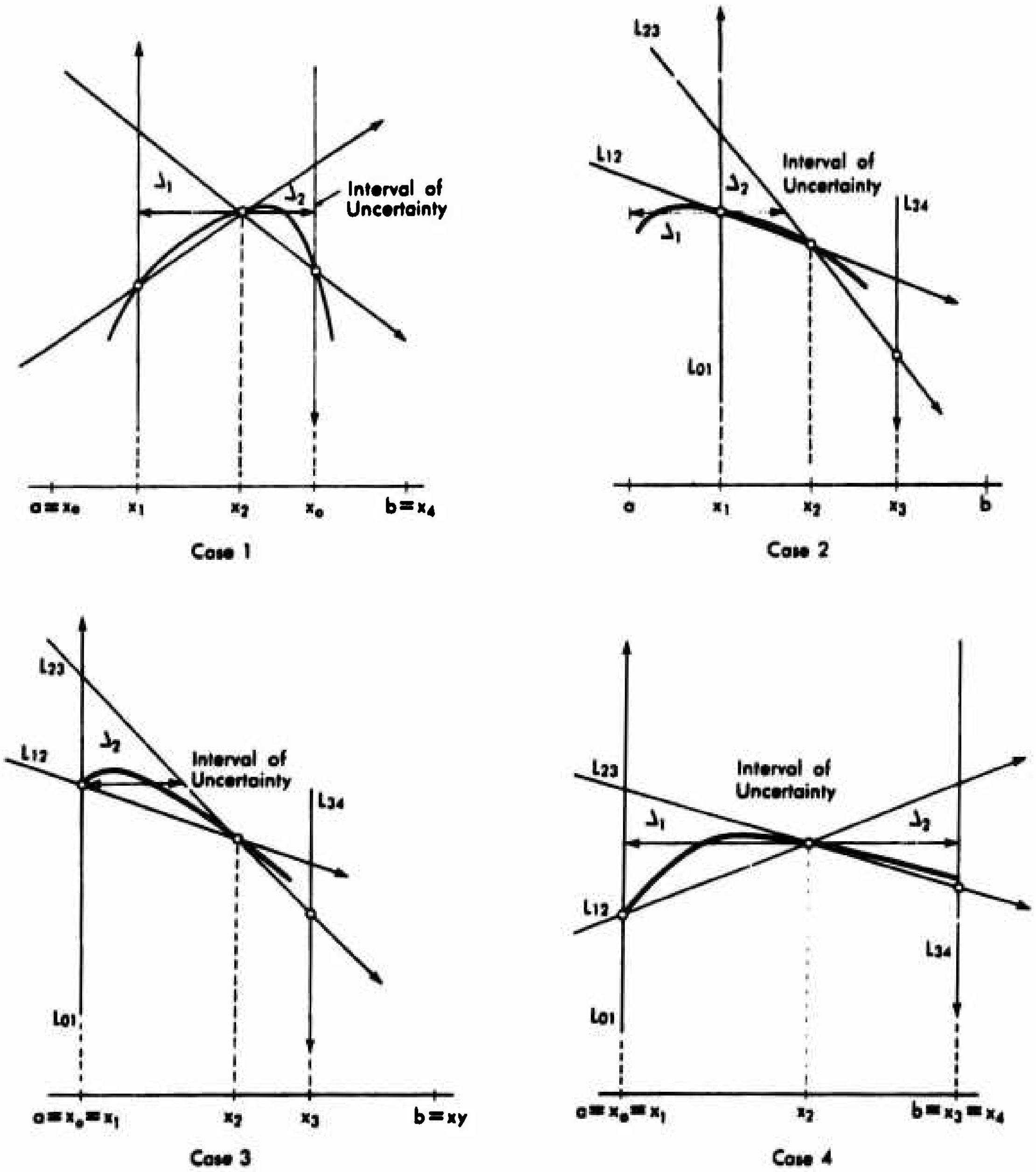

Figure 6. Three evaluations.

x_i with $i \neq 0, 4$, we have finite function values $f(x_i)$, whereas $f(x_0)$ and $f(x_4)$ are possibly infinite, provided $x_0 = a$ or $x_4 = b$, respectively. We suppose furthermore that

$$f(x_0) < f(x_1) < f(x_2) < f(x_3) < f(x_4) .$$

Let $[c,d]$ be the interval of uncertainty that results in view of concavity. Observe that

$$x_2 \varepsilon [c,d] .$$

Now select any x with $c \leq x \leq x_2$, $x_1 < x$, and assume that $f(x)$ satisfies

$$f(x) = f(x_2) + \delta(x - x_2)$$

for some δ with

$$0 \leq \delta \leq \frac{f(x_1) - f(x_1)}{x_2 - x_1} .$$

Then the new interval of uncertainty taking concavity into account will be of the form $[\bar{c},d]$, where

$$\bar{c} = x + \frac{\delta(x-x_1)(x_2-x)}{f(x_2) - f(x_1) - \delta(x_2-x)} > x .$$

The difference $\bar{c} - x$ measures the reduction of uncertainty due to concavity. Now by definition of δ,

$$\bar{c} - x \leq \frac{\delta(x-x_1)(x_2-x)}{f(x_2) - f(x_1) - \delta(x_2-x_1)} \leq \frac{\delta(x_2-x_1)^2}{f(x_2) - f(x_1) - \delta(x_2-x_1)}$$

and the last term, independent of x, goes to zero as δ goes to zero. In other words, the contribution of concavity beyond unimodality becomes arbitrarily small as $f(x)$ approaches $f(x_2)$ from below, but not assuming it.

What happens if $f(x)$ approaches $f(x_2)$ from above? Assume $\delta > 0$ and

$$f(x) = f(x_2) - \delta(x-x_2) .$$

If $m > 0$ denotes the, - possibly infinite -, slope of line L_{01}, then

$$\bar{c} := \begin{cases} c + \dfrac{f(x) - f(x_2)}{m} & \text{for } m = +\infty \\ c & \text{for } m = \infty \end{cases}$$

$$\bar{d} := x_2 + \frac{f(x) - f(x_2)}{f(x_3) - f(x_2)} ,$$

determine the new interval $[\bar{c},\bar{d}]$ of uncertainty, taking into account concavity. The reduction beyond unimodality is the sum of $\bar{c} - c$ and $x_2 - \bar{d}$. Now

$$\bar{c} - c = \frac{-\delta(x-x_2)}{m} \leq \frac{\delta(x_2-x_1)}{m} ,$$

$$x_2 - \bar{d} = \frac{\delta(x-x_2)}{f(x_3) - f(x_2)} \leq \frac{\delta(x_2-x_1)}{f(x_2) - f(x_3)} ,$$

and again the gain beyond unimodality becomes arbitrarily small as $f(x)$ approaches $f(x_2)$ from above without assuming it.

The symmetric argument can be carried out for $x_2 < x \leq d$ and $x < x_3$. This then will establish concavity as an almost independent spy.-

Combining (5.5) with Theorem (4.4) yields

(5.6) *Theorem: Using concavity as a spy in a modified Fibonacci search is the optimal strategy for reducing the interval of uncertainty of concave functions.*

6. Final Remarks

From the proof of Theorem (5.6) it is apparent that the proposed search strategy for concave function is "min sup" rather than "min max". In other words, the problem is not well set. Indeed, it makes probably more sense for concave functions to decrease the uncertainty in the value of the minimum than in its location.

A similar argument as was used for proving (5.5) can be employed to show that fore each $\varepsilon > 0$ and each positive integer k there is a concave function for which the reduction of uncertainty by optimal search is improved by less than ε over unimodal search. In general, however, the improvement will be drastic, in particular if the function is well rounded, so to speak, and has a maximum in the interior.

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