## Fibre-base duality of 5d KK theories

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AbStract: We study circle compactifications of 6d superconformal field theories giving rise to 5 d rank 1 and rank 2 Kaluza-Klein theories. We realise the resulting theories as M-theory compactifications on local Calabi-Yau 3-folds and match the prepotentials from geometry and field theory. One novelty in our approach is that we include explicit dependence on bare gauge couplings and mass parameters in the description which in turn leads to an accurate parametrisation of the prepotential including all parameters of the field theory. We find that the resulting geometries admit "fibre-base" duality which relates their six-dimensional origin with the purely five-dimensional quantum field theory interpretation. The fibre-base duality is realised simply by swapping base and fibre curves of compact surfaces in the local Calabi-Yau which can be viewed as the total space of the anti-canonical bundle over such surfaces. Our results show that such swappings precisely occur for surfaces with a zero self-intersection of the base curve and result in an exchange of the 6 d and 5 d pictures.

Keywords: Conformal Field Models in String Theory, Field Theories in Higher Dimensions, M-Theory, Supersymmetric Gauge Theory

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## 1 Introduction

Five-dimensional $\mathcal{N}=1$ supersymmetric gauge theories play an important role in our understanding of supersymmetric gauge theories in general. Initially, these theories have been studied from various aspects: field theory [1-3], brane constructions [4-6], and geometry via M-theory backgrounds with Calabi-Yau singularities [7]. Recently renewed interest has culminated in significant progress: notably from field theory [8-17], using brane systems [18-41], and from geometry [42-73]. On the one hand, compactifications of such theories give rise to lower dimensional field theories and moreover, what has recently emerged as a major research line, constructing 4 d domain walls within them is a stepping stone for finding UV Lagrangians for a wide class of $4 \mathrm{~d} \mathcal{N}=1$ superconformal theories $[22,74-82]$. On the other hand, it is believed that those 5d field theories that are conformal (SCFTs) arise from circle compactification of 6d SCFTs [16, 47]. Thus a thorough study of these theories will shed light on the landscape of their six-dimensional parents and possibly unveil new dualities [50, 51, 57-59].

One instance where the above $6 \mathrm{~d} / 5 \mathrm{~d}$ relation becomes very transparent and, thus, more tractable is the case of 5 d Kaluza-Klein (KK) theories arising from twisted circle compactifications $[20,21,24,25,28,35,47,50,51,61]$. To be more explicit, in these cases the six-dimensional parent theory emerges as a UV fixed point of a 5 d supersymmetric gauge theory by taking the gauge couplings to infinity or alternatively sending $1 / g_{\mathrm{YM}}^{2} \rightarrow 0$ which translates to decompactifying the 6 d circle. It has recently been conjectured [47] that all 5d SCFTs are connected to 5d KK theories via RG flows and that these flows arise by integrating out BPS particles from the 5 d KK theory. From a geometric engineering perspective arising from an M-theory compactification, such BPS particles correspond to M2-branes wrapping holomorphic curves inside the Calabi-Yau with their masses being identified with the volume of the curves as measured by the Kähler form. Integrating out such states means performing a flop transition in the geometry. In fact, the resulting 5d SCFTs at the end of such RG flows have long been known to arise from M-theory compactifications on non-compact Calabi-Yau three-fold singularities obtained from collapsing surfaces $[2,3,7,47,54]$. The starting point of such RG flows is then an M-theory compactification which lifts to an F-theory compactification as the Calabi-Yau becomes elliptically fibred [61]. This story is well-known in the case of the E-string theory where the starting point of the RG flow corresponds to the total space of the anti-canonical bundle over del Pezzo 9, which is sometimes also called the half-K3 surface. This is a rational elliptic surface obtained from blowing up $\mathbb{P}^{2}$ nine times at different points. Mass-deformations of this theory then result in the chain of 5 d SCFTs corresponding to $\mathcal{N}=1 \mathrm{SU}(2)$ gauge theories with $N_{f}<8$ as first described by Seiberg [1].

Starting from a 5d SCFT, there is yet another mass-deformation one can turn on, corresponding to a non-zero $1 / g_{\mathrm{YM}}^{2}$, which results in a weakly coupled gauge theory description in the IR. Geometrically, this corresponds to the phase where one moves away from the boundary of the Kähler cone such that the volume of the collapsing surface is restored to a non-zero value. This phase is important for studying magnetic monopole strings [83-85] and their BPS spectra as well as for connecting to four-dimensional gauge
theories [86]. Typically, in such phases the Calabi-Yau is the anti-canonical bundle over a bouquet of surfaces which are Hirzebruch surfaces or their blowups. Such surfaces admit a description as a $\mathbb{P}^{1}$ fibration over $\mathbb{P}^{1}$. For fibred surfaces the self-intersection number of the fibre vanishes, so that we can distinguish the base $\mathbb{P}^{1}$ (identified with sections of the fibration) by its non-trivial self-intersection. We henceforth denote the class of the fibre by $f$, and the class of the section with negative self-intersection number ${ }^{1}$ by $e$. Whenever the self-intersection of $e$ is zero, which happens in the case of a Hirzebruch 0 surface $\left(\mathbb{F}_{0}\right)$, the identification of base and fibre is ambiguous and their roles can be swapped. This leads to fibre-base duality among the resulting gauge theory descriptions as first noticed in [87] and from a brane-web perspective in [5]. This duality becomes particularly interesting when one is dealing with the class of 5 d KK theories. Whereas in one frame the theory can be naturally viewed as resulting from a circle-compactification of a six-dimensional theory, as is the viewpoint adopted in [61], there is another frame where the theory admits a description in terms of a purely five-dimensional supersymmetric QFT. These two frames are precisely related through fibre-base duality. The more familiar frame is the six-dimensional frame where the "fibre" is identified with the elliptic fibre of the Calabi-Yau which decomposes into intersecting $\mathbb{P}^{1}$ s over the base $-n$-curves. Here, the volume of the fibral curves can be naturally identified with (combinations of) gauge fugacities of the six-dimensional theory. But there is a dual frame where the same fibral curves can be viewed as gauge nodes in a five-dimensional QFT and it is more natural to identify their volume with the inverse of the square of the gauge coupling. In this frame the volume of the base curves gets identified with gauge fugacities of the five-dimensional theory and it is more natural to swap the roles of base and fibre.

Speaking from a more practical/technical point of view, in the prepotentials of 5 d KK theories presented in [61] the dependence on flavour fugacities corresponding to gauge couplings and masses has been omitted. This is partly due to the fact that, as discussed above, the superconformal fixed points of KK theories and their deformations arise in the limits where these parameters are turned off. However, in order to arrive at a complete picture of fibre-base duality and various other applications, it is beneficial to include the dependence on flavour parameters explicitly in the prepotential. From the geometric viewpoint, this means to associate non-compact divisors of the Calabi-Yau to them and parametrise the Kähler form accordingly. Among the main applications of such a parametrisation would be the computation of 5 d BPS spectra using the 6 d blowup equations of [88-91] which has also been advocated in [92]. In this paper, we employ a parametrisation of the Kähler form which makes both the 6 d and 5 d frames manifest and restores the dependence on gauge couplings and mass parameters in the 5d prepotential. To this end, we introduce two sets of divisors where for ease of exposition we restrict to the case of a single 5 d gauge node. The first set, denoted by $f_{i}$, restricts to the fibres of compact surfaces $S_{i}$ which are at the same time the irreducible components of the elliptic fibre of the Calabi-Yau. The second set, denoted by $F$, restricts to the base curve $e_{i}$ whenever the surface $S_{i}$ is a blowup of $\mathbb{F}_{0}$ and otherwise to $f_{i}$. In the 6 d frame the $f_{i}$ play the role of the fibres of the geometry,

[^0]while in the 5 d frame the $F$ correspond to the fibres. The $F$ is precisely introduced for measuring the gauge coupling strengths $1 / g^{2}$ in the 5 d frame and is dual to the $f_{i}$ in the sense that they measure the volume of the curves $f_{i}$ subject to the constraint that the total volume of the elliptic fibre is expressed as $1 / g^{2}$. This ensures that the infinite coupling limit corresponds to the F-theory limit where the theory becomes six-dimensional.

In this paper we identify the 6 d and 5 d duality frames for the class of rank 1 and rank 2 KK theories, see the table 1 . These frames are connected by exchanging $e$ and $f$ whenever possible as can be explicitly seen in the examples we study. For instance, in section 3.3, equation (3.73) shows the 6 d frame where the $f_{i}$ correspond to the fibres of the geometry and give rise to the roots of the 6 d gauge group. On the other hand, equation (3.75) shows the 5 d frame where $f_{1}$ and $e_{1}$ have been exchanged and the resulting intersection matrix is the Cartan matrix of the 5d gauge group. Another example can be found in section 3.4 where the 6 d frame can be seen in equation (3.87) corresponding to the F-theory geometry over a -4-curve where the $\mathrm{SO}(8)$ gauge group has been modded out by $\mathbb{Z}_{3}$ in a twisted compactification on $S^{1}$. The 5 d dual frame is obtained by swapping $f_{2}$ and $e_{2}$ and gives rise to the Cartan matrix (3.89) of the 5 d gauge group, namely $\mathrm{SU}(4)$. Similar examples can be found in section 4.1: 6d fibre (4.5) vs 5 d fibre (4.6); section 4.2: 6d fibre (4.19) vs 5 d fibre (4.21); section 4.3: 6d fibre (4.34) vs 5d fibre (4.36); section 4.4: 6d fibre (4.47) vs 5 d fibre (4.48). In all cases we study in this paper, the geometric prepotential we obtain by taking the cube of the Kähler form agrees precisely with the expectation from the corresponding $5 \mathrm{~d} \mathcal{N}=1$ QFTs. When discussing fibre-base dual theories, we only present results for one specific 5 d duality frame. Other 5 d frames can be obtained from that one by various geometric operations, but are not just related by $6 \mathrm{~d}-5 \mathrm{~d}$ fibre-base duality.

The organisation of the present paper is as follows. In section 2 we start by reviewing known results about five-dimensional gauge theories including how to obtain their prepotentials from the 5 d and 6 d frames, respectively, as well as geometrically in terms of triple intersections of compact surfaces. We then proceed to describe the geometry in more detail and explain how to define non-compact divisors in order to include the contributions of mass parameters and gauge couplings in the prepotential. This is done in section 2.3. In section 3 we present some instructive examples consisting of the E-string geometry, the geometry of $\mathrm{SU}(n)$ gauge groups over a - 2 -curve, as well as twisted compactifications of the non-Higgsable $\mathrm{SU}(3)$ and $\mathrm{SO}(8)$ theories over -3 and -4 curves, respectively. In section 4 , we proceed to more exotic cases in the class of rank 1 and 2 KK theories, which admit a fibre-base duality between the 6 d frame and one 5 d frame. Thereafter, we exemplify some fibre-base like dualities between dual 5 d theories realised on geometries that are related via an exchange of $e \leftrightarrow f$ in an $\mathbb{F}_{0}$. Finally, we present our conclusions in section 6 . The appendices A and B contain further details about our conventions on Lie algebras, mathematical details on the geometry of Hirzebruch surfaces, and a self-contained exposition about the del Pezzo 9 surface giving rise to the E-string theory.

## 2 5d gauge theories from geometry

An $\mathcal{N}=15 d$ effective gauge theory is characterised by a gauge group $G$ and hypermultiplets in a representation $\mathbf{R}=\oplus \mathbf{R}_{j}$ of $G$. Furthermore, there can be topological data $k$

| 6d theory |  | 5d KK theory | 5d geometry |  |
| :---: | :---: | :---: | :---: | :---: |
| $\begin{gathered} \mathfrak{s p ( 0 ) ^ { ( 1 ) }} \\ 1 \end{gathered}$ | 6d E-string | $\mathrm{SU}(2)+8 \mathrm{~F}$ | $d P_{9}$ |  |
| $\begin{gathered} \mathfrak{s u}(1)^{(1)} \\ \mathbf{2} \end{gathered}$ | $6 \mathrm{~d} \mathcal{N}=(2,0) A_{1}$ theory, rank-1 M-string | $\mathrm{SU}(2)_{0}+1 \mathrm{Adj}$ | $\mathbb{F}_{0}^{1+1} \bigcup_{e-y}^{e-x}$ |  |
| ${\stackrel{y}{s u(1)^{(1)}}}^{( }$ | $6 \mathrm{~d} \mathcal{N}=(2,0) A_{2}$ theory, <br> w/ permutation twist | $\mathrm{SU}(2)_{\pi}+1 \mathrm{Adj}$ | $\mathbb{F}_{1}^{1+1} \bigcirc_{y}^{x}$ | * |
| $\begin{gathered} {\mathfrak{s u}(3)^{(2)}}^{3} \end{gathered}$ | 6d min $\mathrm{SU}(3) \mathrm{SCFT}$, <br> w/ automorphism twist | $\mathrm{SU}(3){ }_{9}$ | $\mathbb{F}_{10} \frac{e}{}{ }^{\text {e }}$ +f $\mathbb{F}_{0}$ |  |
| $\begin{gathered} \mathfrak{s u ( 3 ) ^ { ( 2 ) }} \\ \hline \end{gathered}$ | $6 \mathrm{~d} \mathrm{SU}(3)+6 \mathrm{~F}$ theory, w/ automorphism twist | $\mathrm{Sp}(2)+3 \Lambda^{2}$ | $\mathbb{F}_{6} \frac{e{ }^{\text {e }}+2 \mathrm{f}-2 \sum_{i} x_{i}}{} \mathbb{F}_{0}^{3}$ |  |
| $\begin{gathered} \mathfrak{s u ( 3 ) ^ { ( 2 ) }} \\ 1 \end{gathered}$ | $6 \mathrm{~d} \operatorname{SU}(3)+12 \mathrm{~F}$ theory, w/ automorphism twist | $\left\{\begin{array}{l} \mathrm{SU}(3)_{4}+6 \mathrm{~F} \\ G_{2}+6 \mathrm{~F} \\ \mathrm{Sp}(2)+2 \Lambda^{2}+4 \mathrm{~F} \end{array}\right.$ | $\mathbb{F}_{2} \xrightarrow{e \quad 3 e+4 f-2 \sum_{i} x_{i} \mathbb{F}_{0}^{6}}$ |  |
| $\begin{gathered} \mathfrak{s p}(1)^{(1)} \\ 1 \end{gathered}$ | $6 \mathrm{~d} \operatorname{Sp}(1)+10 \mathrm{~F}$ theory | $\left\{\begin{array}{l} \mathrm{SU}(3)_{0}+10 \mathrm{~F} \\ \mathrm{Sp}(2)+10 \mathrm{~F} \end{array}\right.$ | $\mathbb{F}_{0} \stackrel{2 e+f \quad 2 h-\sum_{i} x_{i}}{\mathbb{F}_{1}^{10}}$ |  |
| $\begin{gathered} \mathfrak{s p}(0)^{(1)} \\ \mathbf{1}-\mathbf{s u}(1)^{(1)} \\ \hline \end{gathered}$ | 6d rank 2 E-string | $\left\{\begin{array}{l}\mathrm{SU}(3)_{\frac{3}{2}}+9 \mathrm{~F} \\ \mathrm{Sp}(2)+1 \Lambda^{2}+8 \mathrm{~F}\end{array}\right.$ | ${ }_{e-z}^{e-w}\left(\mathbb{F}_{0}^{1+1}{ }^{f} \quad{ }^{2 e+3 f-\sum_{i} x_{i}} \mathbb{F}_{1}^{8}\right.$ |  |
| $\stackrel{\mathfrak{s u}^{(2)^{(1)}}}{2}$ | $6 \mathrm{~d} \mathrm{SU}(2)+4 \mathrm{~F}$ theory | $\mathrm{SU}(2) \times \mathrm{SU}(2)+2$ bi-F | $\mathbb{F}_{0}^{4 e^{-}-\sum_{i}^{e} x_{i}} 2^{\stackrel{e}{h}} \mathbb{F}_{2}$ |  |
| $\mathbf{2 s u}^{\mathfrak{s u}(1)^{(1)}}{ }^{\mathfrak{s u}(1)^{(1)}} \mathbf{2}$ | $6 \mathrm{~d} \mathcal{N}=(2,0) A_{2}$ theory | $\mathrm{SU}(3)_{0}+1 \mathrm{Adj}$ | ${ }_{e-y}^{e-x}\left[\mathbb{F}_{0}^{1+1} \stackrel{f-x}{x} 2 \stackrel{f-z}{z} \mathbb{F}_{0}^{1+1}\right]_{e-w}^{e-z}$ |  |
| $\stackrel{\mathfrak{s u}(1)^{(1)}}{\mathbf{2}-2 \xrightarrow{\mathfrak{s u}(1)^{(1)}} \mathbf{2}}$ | $6 \mathrm{~d} \mathcal{N}=(2,0) A_{3}$ theory, w/ permutation twist | $\mathrm{Sp}(2)_{0}+1 \mathrm{Adj}$ | ${ }_{e-y}^{e-x}\left(\mathbb{F}_{0}^{1+1} \stackrel{f-x}{x} 2 \stackrel{2 f-z}{z} \mathbb{F}_{0}^{1+1}\right]_{e-w}^{e-z}$ | * |
|  | $6 \mathrm{~d} \mathcal{N}=(2,0) A_{4}$ theory, w/ permutation twist | $\left\{\begin{array}{l}\mathrm{SU}(3)_{\frac{3}{2}}+1 \mathrm{Sym} \\ \mathrm{Sp}(2)_{\pi}+1 \mathrm{Adj}\end{array}\right.$ | ${ }_{y}^{x}\left(\mathbb{F}_{6}^{1+1} \frac{e}{} \quad 2 e+f\right) \mathbb{F}_{0}$ | * |
| $\mathbf{S u}^{\mathfrak{s u}(2)^{(1)}} \mathbf{2}$ | 6d rank-2 ( $A_{1}, A_{1}$ ) theory, w/ permutation twist | $\mathrm{SU}(3)_{0}+1 \mathrm{Sym}+1 \mathrm{~F}$ | $\mathbb{F}_{1}^{2^{h}} \underline{\sum_{i}^{h} x_{i}}{ }_{2} \xlongequal{e+f-x-2 y} \mathbb{F}_{0}^{1+1} \bigcup_{y}^{x}$ | * |
| $\stackrel{\mathfrak{s u}(1)^{(1)}}{\mathbf{2}-3 \xrightarrow{\mathfrak{s u}(1)^{(1)}} \mathbf{2}}$ | $6 \mathrm{~d} \mathcal{N}=(2,0) D_{4}$ theory, w/ permutation twist | $\left\{\begin{array}{l}G_{2}+1 \mathrm{Adj} \\ \mathrm{SU}(3)_{\frac{15}{2}}+1 \mathrm{~F}\end{array}\right.$ | ${ }_{e-y}^{e-x}\left[\mathbb{F}_{0}^{1+1^{f-x}} \frac{x^{3}}{x} \frac{3 f-x}{x}_{0}^{1+1}\right]_{e-w}^{e-z}$ |  |

Table 1. Rank 1 and rank 2 KK theories [16, 47, 61]. Theories labelled with $\star$ are referred to as non-geometric, in the sense that these are not conventional geometric descriptions. These theories are understood as algebraic proposals that mimic many features of local threefolds, but may not satisfy the consistency conditions reviewed in section 2 .
corresponding to classical Chern-Simons level, as for example in the case of $G=\mathrm{SU}(N \geq 3)$, or discrete $\theta$-angle as in the cases $G=\operatorname{Sp}(N)$. At a generic point in the Coulomb branch, the gauge symmetry $G$ is broken to the maximal torus $\mathrm{U}(1)^{r}$. The Coulomb branch is parametrised by the expectation values of scalar fields $\phi$ in the vector multiplets. Here, the scalar field $\phi$ takes values in the Cartan subalgebra of the gauge group $G$. The low energy abelian action is determined by a prepotential $\mathcal{F}_{5 d}$. The prepotential is 1-loop exact and the full quantum result is a cubic polynomial of the vector multiplet scalar $\phi$ and mass
parameters $m_{f}$, given by:

$$
\begin{equation*}
\mathcal{F}_{5 d}=\frac{1}{2 g^{2}} h_{i j} \phi_{i} \phi_{j}+\frac{\kappa}{6} d_{i j k} \phi_{i} \phi_{j} \phi_{k}+\frac{1}{12}\left(\sum_{\alpha \in \text { root }}|\boldsymbol{\alpha} \cdot \phi|^{3}-\sum_{f} \sum_{\mathbf{w} \in \mathbf{R}_{f}}\left|\mathbf{w} \cdot \phi+m_{f}\right|^{3}\right), \tag{2.1}
\end{equation*}
$$

where $h_{i j}=\operatorname{Tr}\left(t_{i} t_{j}\right)$, and $d_{i j k}=\frac{1}{2} \operatorname{Tr}_{\mathbf{F}}\left(t_{i}\left\{t_{j}, t_{k}\right\}\right)$ with $\mathbf{F}$ in the fundamental representation, and we refer to appendix A. 1 for our Lie algebra conventions.

The 1-loop correction to the prepotential renormalises the gauge coupling. The effective coupling in the Coulomb branch is simply given by a second derivative of the quantum prepotential which fixes the exact metric on the Coulomb branch:

$$
\begin{equation*}
\left(\tau_{\mathrm{eff}}\right)_{i j}=\left(g_{\mathrm{eff}}^{-2}\right)_{i j}=\partial_{i} \partial_{j} \mathcal{F}_{5 d}, \quad d s^{2}=\left(\tau_{\mathrm{eff}}\right)_{i j} d \phi_{i} d \phi_{j} . \tag{2.2}
\end{equation*}
$$

In addition, the exact spectrum of magnetic monopoles on the Coulomb branch can be easily obtained from the quantum prepotential and their tension is given by

$$
\begin{equation*}
\phi_{D_{i}}=\partial_{i} \mathcal{F}_{5 d}, \quad i=1, \ldots, r . \tag{2.3}
\end{equation*}
$$

Moreover, one can compute Chern-Simons couplings:

$$
\begin{equation*}
\kappa_{i j k}=\partial_{i} \partial_{j} \partial_{k} \mathcal{F}_{5 d} \tag{2.4}
\end{equation*}
$$

In the following paragraphs we see how to obtain the prepotential (2.1) from various viewpoints ranging from a parent 6d SCFT to a geometric realisation in terms of intersecting complex rational surfaces.

### 2.1 Prepotentials from 6d SCFTs

In this section we review how 5d prepotentials are obtained from a given 6d SCFT upon twisted circle compactification following [61].

6 d SCFTs can be described on their tensor branch in terms of tensor multiplets $B_{i}$ together with gauge multiplets associated to a gauge algebra $\mathfrak{g}_{i}$. It is known that the $\mathfrak{g}_{i}$ are either simple or trivial algebras. There are, moreover, fundamental BPS string excitations $s^{j}$ charged under the $B_{j}$ such that their charge is given by the Kronecker delta $\delta_{j}^{i}$. Let $\Omega^{i j}$ denote the Dirac pairing between $s^{i}$ and $s^{j}$. Then there is a Green-Schwarz term in the Lagrangian on the tensor branch of the following form

$$
\begin{equation*}
\Omega^{i j} B_{i} \wedge \operatorname{tr}\left(F_{j}^{2}\right) \tag{2.5}
\end{equation*}
$$

where $F_{j}$ is the field strength for the $j$-th gauge group if $\mathfrak{g}_{j}$ is simple and $F_{j}=0$ if $\mathfrak{g}_{j}$ is trivial. In the classification of [93], the $\Omega^{i j}$ matrix is the intersection matrix of $-n$ curves in the base of the F-theory construction. As such it is symmetric, positive definite and all of its entries are integers.

Upon compactification on a circle, there is the possibility to twist by an element of a discrete global symmetry group $\Gamma$. In order to generate the most general discrete symmetry of a 6 d SCFT, one has to combine two kinds of basic discrete symmetries. The first type
acts as an outer automorphisms of the gauge algebras $\mathfrak{g}_{i}$. The outer automorphism acting on the roots of $\mathfrak{g}$ induces an action on the irreducible representations; more precisely, the automorphism twist acts on the Dynkin coefficients of the corresponding weights. Therefore, we see that such an action is an action on representations of the gauge algebra. The second type of automorphism arises from permuting tensor multiplets $i \mapsto \sigma(i)$ such that

$$
\begin{align*}
\mathfrak{g}_{\sigma(i)} & =\mathfrak{g}_{i} \\
\Omega^{\sigma(i) \sigma(j)} & =\Omega^{i j} \tag{2.6}
\end{align*}
$$

for all $i, j$. Modding out by a given permutation $\sigma$ of a 6 d SCFT, generates a different pairing matrix $\Omega_{\sigma}^{\alpha \beta}$; here the indices $\alpha, \beta$ label orbits of nodes $i$, which arise under the repeated action of the permutation $\sigma$. In more detail, the element $\Omega_{\sigma}^{\alpha \beta}$ can be computed by selecting a node $i$, which lies inside the orbit $\alpha$, and summing over all nodes $j$ that constitute the other orbit $\beta$, i.e.

$$
\begin{equation*}
\Omega_{\sigma}^{\alpha \beta}=\sum_{j \in \beta} \Omega^{i j} \tag{2.7}
\end{equation*}
$$

This matrix may not be symmetric, but must be positive definite. One can then combine the action of the two types of twists discussed and denote them by $\sigma,\left\{q_{\alpha}\right\}$ where $q_{\alpha}$ is the degree of the action of the outer automorphism on the gauge algebra corresponding to node $\alpha$. Naturally, one is led to consider the $\mathcal{O}\left(q_{\alpha}\right)$ invariant subalgebra $\mathfrak{h}_{\alpha}=\mathfrak{g}_{\alpha} / \mathcal{O}^{\left(q_{\alpha}\right)}$ of $\mathfrak{g}_{\alpha}$. From the viewpoint of the compactified theory, every node $\alpha$ gives rise to the such defined low energy gauge algebra $\mathfrak{h}_{\alpha}$. For a given 6 d theory, the authors of [61] then define a graph associated to the twisted compactification, such that the value of each node $\alpha$ is $\mathfrak{g}_{\boldsymbol{\Omega}^{\alpha \alpha}}^{\left(q_{\alpha}\right)}$ and the number of edges between two nodes $\alpha$ and $\beta$ is given by $\left|\Omega^{\alpha \beta}\right|$. In the first column of table 1 we summarise all rank 1 and 25 d theories so obtained using the notation of [61].

With these data, the authors of [61] derive the following prepotential for the twisted circle compactification of the 6 d theory

$$
\begin{equation*}
6 \mathcal{F}_{6 d}=\sum_{\alpha, \beta} 3 \Omega_{\sigma}^{\alpha \beta} \phi_{0, \alpha}\left(K_{\beta}^{a b} \phi_{a, \beta} \phi_{b, \beta}\right)+\frac{1}{2}\left(\sum_{\alpha}|\boldsymbol{\alpha} \cdot \phi|^{3}-\sum_{f} \sum_{\mathbf{w}\left(\mathcal{R}_{f}\right)}\left|\mathbf{w}\left(\mathcal{R}_{f}\right) \cdot \phi+m_{f}\right|^{3}\right) \tag{2.8}
\end{equation*}
$$

where $\phi_{0, \alpha}$ is the scalar living in the vector multiplet corresponding to $\mathrm{U}(1)_{\alpha}$, i.e. the affine node. The $\phi_{a, \beta}$ are scalar components of the $\mathrm{U}(1)_{a, \beta}$ vector multiplets that parametrise the Cartan of $\mathfrak{h}_{\beta}$ coming from the degenerate fibre above the curve $\beta$. Here $K_{\beta}^{a b}$ is the Killing form on $\mathfrak{h}_{\beta}$.

Moreover, the hypermultiplets, which are charged under the low energy gauge algebra $\mathfrak{h}$, furnish a representation $\mathcal{R}$ which can be decomposed as $\mathcal{R}=\oplus_{f} \mathcal{R}_{f}$ into irreducible representations of $\mathfrak{h}$. Note that contrary to the prepotential given in (2.1), the so obtained prepotential (2.8) from 6d does not depend on gauge coupling parameters $\frac{1}{g^{2}}$. However, such terms should be there and have been neglected in the initial derivation given in [61]. One central goal of the current paper is to remedy this gap.

### 2.2 Prepotential from geometry

As advocated in [61], the 5d gauge theory resulting from twisted circle compactification can be equivalently described in terms of an M-theory compactification on a Calabi-Yau $X_{S}$.

This Calabi-Yau can be described as a local neighbourhood of an arrangement of surfaces - which each need to be irreducible, compact, and holomorphic - such that these surfaces intersect each other at most pairwise transversely. Because these surfaces can be arranged into families, it is convenient to assign a corresponding label $\alpha$. The irreducible surfaces in each family $\alpha$ are denoted as $S_{a, \alpha}$ where $0 \leq a \leq r_{\alpha}$ and $r_{\alpha}$ denotes the rank of $h_{\alpha}$. The Kähler parameter associated to $S_{a, \alpha}$ are identified with the parameter $\phi_{a, \alpha}$ of the Coulomb branch in the compactified KK theory discussed above. If the invariant algebra $h_{\alpha}$ for the node $\alpha$ is trivial then one associates just one surface $S_{0, \alpha}$.

A key role is played by the shifted prepotential $\mathcal{F}_{\text {trun }}$ which corresponds to the truncated prepotential of a purely five-dimensional QFT where all mass and gauge coupling deformations have been turned off. $\mathcal{F}_{\text {trun }}$ is obtained from $\mathcal{F}_{6 d}$ by performing the following replacement:

$$
\begin{equation*}
\phi_{b, \alpha} \rightarrow \phi_{b, \alpha}-d_{b}^{\vee} \phi_{0, \alpha}, \tag{2.9}
\end{equation*}
$$

for all $1 \leq b \leq r_{\alpha}$ and for all $\alpha$. The $d_{a}^{\vee}$ are the dual Coxeter labels of the affine twisted algebra associated to the 6 d gauge algebra. The prepotential $6 \mathcal{F}_{\text {trun }}$ is composed of cubic terms of the form $\kappa_{a \alpha, b \beta, c \gamma} \phi_{a, \alpha} \phi_{b, \beta} \phi_{c, \gamma}$ and the appearing coefficients are geometrically realised by the following intersection numbers of compact surfaces inside the Calabi-Yau $X_{S}$ :

$$
\begin{align*}
\kappa_{a \alpha, a \alpha, a \alpha} & =S_{a, \alpha} \cap S_{a, \alpha} \cap S_{a, \alpha},  \tag{2.10a}\\
\kappa_{a \alpha, a \alpha, b \beta} & =3 S_{a, \alpha} \cap S_{a, \alpha} \cap S_{b, \beta},  \tag{2.10b}\\
\kappa_{a \alpha, b \beta, c \gamma} & =6 S_{a, \alpha} \cap S_{b, \beta} \cap S_{c, \gamma}, \tag{2.10c}
\end{align*}
$$

and the $(a, \alpha),(b, \beta),(c, \gamma)$ indices are assumed to be distinct and non-equal. For evaluating the triple intersection product of any three such surfaces one has the freedom to evaluate the intersection number inside any one of the three. In more detail, the concept of "gluing curves" needs to be introduced. Denote the intersection locus between two distinct surfaces $S_{a, \alpha}$ and $S_{b, \beta}$ in $X_{S}$ by $\mathcal{L}_{a \alpha, b \beta}$. In general, $\mathcal{L}_{a \alpha, b \beta}$ can be reducible and splits into geometrically irreducible components given by the sum $\sum_{i} \mathcal{L}_{a \alpha, b \beta}^{i}$. Each $\mathcal{L}_{a \alpha, b \beta}^{i}$ can then be associated to an irreducible curve $C_{a, \alpha ; b, \beta}^{i}$ in $S_{a, \alpha}$ as well as an irreducible curve $C_{b, \beta ; a, \alpha}^{i}$ in $S_{b, \beta}$. Put differently, the intersection of $S_{a, \alpha}$ and $S_{b, \beta}$ can be realised by identifying the curves

$$
\begin{equation*}
C_{a, \alpha ; b, \beta}^{i} \sim C_{b, \beta ; a, \alpha}^{i} \tag{2.11}
\end{equation*}
$$

with each other for all $i$. One may think of the process that identifies curves as in (2.11) as an operation in which two surfaces are "gluing together" along such curves. The above description of the local Calabi-Yau $X_{S}$ can be compactly summarised in terms of connected graphs as shown for the case of rank 1 and rank 25 d theories in the last column of table 1. Each node of the graph corresponds to one of the surfaces $S_{a, \alpha}$ which in general are Hirzebruch surfaces and blowups thereof. An edge between two nodes indicates that the corresponding surfaces share one or more curves. If the number of gluing curves is greater than 1 , it is indicated on the edge. The gluing curves are written next to each surface.

Neglecting gauge coupling and mass terms for now, the truncated prepotential $\mathcal{F}_{\text {trun }}$ can be expressed purely geometrically using the above data by defining a Kähler form

$$
\begin{equation*}
J_{\phi}=\sum_{a, \alpha} \phi_{a, \alpha} S_{a, \alpha}, \tag{2.12}
\end{equation*}
$$

by the following identity

$$
\begin{equation*}
\mathcal{F}_{\text {trun }}=\operatorname{vol}\left(X_{S}\right)=\frac{1}{3!} \int_{X_{S}} J_{\phi} \wedge J_{\phi} \wedge J_{\phi} . \tag{2.13}
\end{equation*}
$$

The Kähler form can also be used to measure the volume of a curve $C$ via

$$
\begin{equation*}
\operatorname{vol}(C)=-J_{\phi} \cdot C \tag{2.14}
\end{equation*}
$$

where we have adopted the conventions of [61]. The identification (2.11) implies that

$$
\begin{equation*}
J_{\phi} \cdot C_{a, \alpha ; b, \beta}^{i}=J_{\phi} \cdot C_{b, \beta ; a, \alpha}^{i}, \tag{2.15}
\end{equation*}
$$

and the gluing curves also have to satisfy the 'Calabi-Yau condition' spelled out in (2.19).

### 2.3 The geometry of $X_{S}$

So far, we have only described the compact divisors $S_{a, \alpha}$ which are related to the Coulomb branch parameters $\phi_{a, \alpha}$. In the Calabi-Yau threefold $X_{S}$ there can be further non-compact divisors related to the mass parameters and the gauge couplings. In order to describe such divisors, we discuss the geometry of $X_{S}$ in some more detail.

As every single one of the compact surfaces $S_{a, \alpha}$ sits inside a non-compact Calabi-Yau threefold $X_{S}$, their normal bundle must be equal to their canonical bundle, $K_{S_{a, \alpha}}$. For a single surface $S_{a, \alpha}$, this means that $X_{S}$ is simply the total space of the bundle $K_{S_{a, \alpha}}$. Gluing several surface $S_{a, \alpha}$ along $\mathcal{L}_{a \alpha, b \beta}=S_{a, \alpha} \cap S_{b, \beta}$ implies that we should think of $X_{S}$ as being glued from the total spaces of the bundles $K_{S_{a, \alpha}}$. The surfaces $S_{a, \alpha}$ are the zero loci of sections of the line bundles $K_{S_{a, \alpha}}$ on $X_{S}$.

We can check that this gluing can be done consistently by examining the normal bundle of $\mathcal{L}_{a \alpha, b \beta}$ inside $X_{S}$. This normal bundle is a sum of line bundles which can be described as

$$
\begin{equation*}
N_{\mathcal{L}_{a \alpha, b \beta} \backslash X_{S}}=\left.K_{S_{a, \alpha}}\right|_{\mathcal{L}_{a \alpha, b \beta}} \oplus N_{\mathcal{L}_{a \alpha, b \beta} \backslash S_{a, \alpha}} \tag{2.16}
\end{equation*}
$$

by taking $\mathcal{L}_{a \alpha, b \beta}$ as an algebraic subvariety of $S_{a, \alpha}$, and as

$$
\begin{equation*}
N_{\mathcal{L}_{a \alpha, b \beta} \backslash X_{S}}=\left.K_{S_{b, \beta}}\right|_{\mathcal{L}_{a \alpha, b \beta}} \oplus N_{\mathcal{L}_{a \alpha, b \beta} \backslash S_{b, \beta}} \tag{2.17}
\end{equation*}
$$

by taking $\mathcal{L}_{a \alpha, b \beta}$ as an algebraic subvariety of $S_{b, \beta}$. Under a consistent gluing, these two expression need to be identical. As $\mathcal{L}_{a \alpha, b \beta}$ is given by the intersection of $S_{a, \alpha}$ and $S_{b, \beta}$, adjunction tells us that $N_{\mathcal{L}_{a \alpha, b \beta} \backslash S_{a, \alpha}}$ is the restriction of $K_{S_{\alpha, b}}$ to $\mathcal{L}_{a \alpha, b \beta}$. Hence, we find

$$
\begin{equation*}
\left.K_{S_{a, \alpha}}\right|_{\mathcal{L}_{a \alpha, b \beta}}=N_{\mathcal{L}_{a \alpha, b \beta} \backslash S_{b, \beta}}, \tag{2.18}
\end{equation*}
$$

which shows that the gluing identifies the direction transverse to $S_{a \alpha}$ in $X$ with the normal direction of $\mathcal{L}_{a \alpha, b \beta}$ in $S_{b, \beta}$, so that (2.16) agrees with (2.17).

Using the realisation of $\mathcal{L}_{a \alpha, b \beta} \backslash S_{b, \beta}$ as a complete intersection, it is straightforward to work out using adjunction that

$$
\begin{align*}
\chi\left(\mathcal{L}_{a \alpha, b \beta}\right) & =\int_{\mathcal{L}_{a \alpha, b \beta}} c_{1}\left(\mathcal{L}_{a \alpha, b \beta}\right)=\int_{\mathcal{L}_{a \alpha, b \beta}}-\left[K_{S_{a, \alpha}}\right]-\left[K_{S_{b, \beta}}\right] \\
& =-\int_{X_{S}}\left[K_{S_{a, \alpha}}\right] \cdot\left[K_{S_{b, \beta}}\right] \cdot\left(\left[K_{S_{a, \alpha}}\right]+\left[K_{S_{b, \beta}}\right]\right)  \tag{2.19}\\
& =-\int_{S_{a, \alpha}}\left(\mathcal{L}_{a \alpha, b \beta}\right)^{2}-\int_{S_{b, \beta}}\left(\mathcal{L}_{b \beta, a \alpha}\right)^{2} .
\end{align*}
$$

When $\mathcal{L}_{a \alpha, b \beta}$ is an irreducible curve of genus $g$, the above expression is equal to $2-2 g$, which gives the 'Calabi-Yau condition' of the gluing.

The description of $X_{S}$ we have given makes it clear that given a collection of curves $\left\{C_{a, \alpha}\right\}$ on the surfaces $S_{a, \alpha}$, we can construct a divisor in $X_{S}$ that restricts to these curves by simply taking the inverse image $\pi_{a, \alpha}^{-1}\left(C_{a, \alpha}\right)$ under the projection $\pi_{a, \alpha}: K_{S_{a, \alpha}} \rightarrow S_{a, \alpha}$ for each $C_{a, \alpha}$, and then gluing these to a (reducible) divisor. As the total spaces $K_{S_{a, \alpha}}$ glue together consistently, the only extra consistency condition that arises is when one (or some) of the $\left\{C_{a, \alpha}\right\}$ is a gluing curve, i.e. a component of $\mathcal{L}_{a \alpha, b \beta}$. In this case, the divisor $\pi^{-1}\left(C_{a, \alpha}\right)$ will restrict to non-trivial curves on both $S_{a, \alpha}$ and $S_{b, \beta}$.

Having discussed the inclusion of non-compact divisors into our geometry, we are now ready to employ a parametrisation of the Kähler form which includes dependence on bare gauge couplings and mass parameters. To this end, we introduce divisors $F_{i}$ with the index $i$ running over $i=0, \ldots, n_{0}$. Here $n_{0}$ is the total number of surfaces $S_{a, \alpha}$ which can be written as blowups of $\mathbb{F}_{0}$, which is equal to the total number of bare gauge couplings. The $F_{i}$ are then given by the inverse images $\pi_{a, \alpha}^{-1}\left(e_{a, \alpha}\right) \cup \pi_{b, \beta}^{-1}\left(l f_{b, \beta}\right)$ where $e_{a, \alpha}$ is a base of $\mathbb{F}_{0}$ (or blowups of it) and $l f_{b, \beta}$ denotes the fibre (taken with an integer multiplicity $l$ ) of a surface with non-zero self-intersection number of base curve (i.e. which does not arise from $\mathbb{F}_{0}$ ). In cases where all $S_{a, \alpha}$ admit a description in terms of blowups of $\mathbb{F}_{0}$, the second factor in the inverse image is missing and the $F_{i}$ can be purely written in terms of inverse images of base curves $e_{i}$. The $F_{i}$ are associated with gauge coupling parameters.

In case that there is only one 5 d gauge algebra $\mathfrak{g}$, this can be summarised as follows: suppose there are $n=\operatorname{rk}(\mathfrak{g})$ compact surfaces $S_{i}$ and one is able to identify inside each $S_{i}$ a complex curve $\tilde{f}_{i}$ of zero self-intersection such that

$$
\begin{equation*}
-\tilde{f}_{i} \cdot S_{j}=C_{i j}^{\mathfrak{q}} \tag{2.20}
\end{equation*}
$$

which implies that the $\tilde{f}_{i}$ act as simple roots $\alpha_{i}$ of $\mathfrak{g}$. The non-compact divisor $F$, associated to the gauge coupling $g$, has to intersect to compact surfaces as

$$
\begin{equation*}
-F \cdot S_{i} \cdot S_{j}=h_{i j}^{\mathfrak{g}} . \tag{2.21}
\end{equation*}
$$

Recalling the relation (A.5) between $C_{i j}^{\mathfrak{q}}$ and $h_{i j}^{\mathfrak{q}}$, one finds

$$
\begin{equation*}
\left.F\right|_{S_{i}}=D_{i}^{-1} \tilde{f}_{i}=\frac{2}{\left\langle\alpha_{i}, \alpha_{i}\right\rangle} \tilde{f}_{i}, \tag{2.22}
\end{equation*}
$$

which is verified in all the examples considered in this paper.

We can find similar sets of non-compact divisors, denoted by $N_{f}$, which are associated to bare mass parameters $m_{f}$. The divisors $N_{f}$ can restrict to blowup curves $x_{i}$ or fibres $f_{i}$, but never to base curves $e_{i}$. Now we can parametrise the Kähler form as follows

$$
\begin{equation*}
J=-\sum_{i} \frac{1}{g_{i}^{2}} F_{i}+\sum_{a, \alpha} \phi_{a, \alpha} S_{a, \alpha}+\sum_{f} m_{f} N_{f} \tag{2.23}
\end{equation*}
$$

which results in the geometric prepotential

$$
\begin{equation*}
\mathcal{F}_{\text {geom }}=\frac{1}{3!} \int_{X_{s}} J \wedge J \wedge J \tag{2.24}
\end{equation*}
$$

Henceforth, we will omit the indices $\{a, \alpha\}$ and parametrise all non-compact as well as compact divisors in the geometry purely in terms of the Latin indices $i, j$ and $f$. Using the above parametrisation of the Kähler form, we can then express effective gauge couplings as volumes of various 2 -cycles as follows:

$$
\begin{equation*}
\tau_{i j}=\partial_{i} \partial_{j} \mathcal{F}_{\text {geom }}=\operatorname{vol}\left(S_{i} \bigcap S_{j}\right)=\int_{X_{S}} J \wedge S_{i} \wedge S_{j} \tag{2.25}
\end{equation*}
$$

The parametrisation of the Kähler form is fixed in such a way that $\tau_{i j}$ matches the effective gauge coupling $\partial_{i} \partial_{j} \mathcal{F}_{5 d}$ of a five-dimensional supersymmetric QFT.

## 3 Instructive examples

In this section, inclusion of gauge coupling and mass parameters into the geometric description is explored in a set of selected examples. Among these are untwisted compactification, like the well-studied E-string theory or the 5d affine $A$-type quiver theory. Twisted compactifications are explored for the 6d minimal $\mathrm{SU}(3)$ and $\mathrm{SO}(8)$ SCFTs with outer automorphism twist.

## 3.1 - 1 curve: E-string theory on a circle

Let us start with $6 \mathrm{~d} \mathcal{N}=(1,0)$ E-string theory corresponding to one M5 brane probing an M9 wall. Compactifying this theory on a circle (with a possible twist) gives a $5 \mathrm{~d} \operatorname{Sp}(1)$ gauge theory with $N_{f}=n \leq 8$. Using the fact that $d_{i j k}=0$ for $G=\operatorname{Sp}(1)=\operatorname{SU}(2)$, we obtain from (2.1)

$$
\begin{equation*}
\mathcal{F}_{5 d}=\frac{1}{g^{2}} \phi^{2}+\frac{1}{12}\left(2|2 \phi|^{3}-\sum_{j=1}^{8}\left|\phi+m_{j}\right|^{3}-\sum_{j=1}^{8}\left|\phi-m_{j}\right|^{3}\right) \tag{3.1}
\end{equation*}
$$

For $n=8$, setting all masses to zero, i.e. $m_{j}=0$ for $j=1, \ldots, 8$, we get

$$
\begin{equation*}
\mathcal{F}_{5 d}=\frac{1}{g^{2}} \phi^{2} \tag{3.2}
\end{equation*}
$$

In particular, we see that at infinite coupling, $g=\infty$, the monopole tension $\phi_{D}=\partial \mathcal{F} / \partial \phi$ vanishes regardless of the value of $\phi$. This is a remnant of the fact, that the UV completion of this 5 d theory is in fact a 6 d theory, namely the E-string theory.

Let us now assume that $\phi>m_{j}$ for all $j$. We can then write

$$
\begin{equation*}
\mathcal{F}_{5 d}=\frac{1}{g^{2}} \phi^{2}+\frac{1}{6}\left((8-n) \phi^{3}-3 \phi \sum_{j} m_{j}^{2}\right) . \tag{3.3}
\end{equation*}
$$

The effective gauge coupling is given by

$$
\begin{equation*}
\tau_{\mathrm{eff}}=\left(\partial_{\phi}\right)^{2} \mathcal{F}_{5 d}=\frac{2}{g^{2}}+(8-n) \phi \tag{3.4}
\end{equation*}
$$

which matches the result of [1].
Geometric realisation. The E-string theory in 6 d is engineered geometrically in Ftheory by collapsing a rational elliptic surface (also called $d P_{9}$ surface) $S$ inside a noncompact elliptically fibred Calabi-Yau threefold. If we consider M-Theory on the same geometry, we find the 5 d theory that was discussed as a compactification of the $6 \mathrm{~d} \mathcal{N}=$ $(1,0)$ SCFT on a circle. Following the general discussion of geometric engineering of 5 d gauge theories, we need to exploit a ruling on $S$. Such a ruling is discussed in [7], see also appendix B, where it is shown that it originates from $S$ being a blowup of $\mathbb{F}_{0}$ at 8 points. We can generalise this to surfaces $S=d P_{n+1}$ which are blowups of $\mathbb{F}_{0}$ at $n$ points, and which in turn give rise to a $\operatorname{Sp}(1)$ gauge theory with $n$ flavours. We can model the relevant Calabi-Yau geometry $X_{S}$ as

$$
\begin{equation*}
X_{S}=\mathcal{O}_{d P_{n+1}}\left(K_{d P_{n+1}}\right) \tag{3.5}
\end{equation*}
$$

For every curve in $d P_{n+1}$, there is hence an associated non-compact divisor in $X_{S}$ by simply taking the preimage under the projection $\pi: X_{S} \rightarrow d P_{n+1}$.

Let us denote the fibre and base classes of the ruling by $f$ and $e$, and the $n$ exceptional divisors of the blowups by $x_{i}$. In terms of the $\mathbb{P}^{1}$ fibration on $S$, we can think of the $x_{i}$ as irreducible fibre components, i.e. over $n$ points of the base, the fibre splits into the two components $x_{i}$ and $f-x_{i}$.

The intersections between the curves on $S$ can be summarised as (see appendix B for details)

$$
\begin{equation*}
e \cdot f=1, \quad x_{i} \cdot x_{j}=-\delta_{i j} \tag{3.6}
\end{equation*}
$$

with all others vanishing. Note that this implies that

$$
\begin{equation*}
\left(f-x_{i}\right) \cdot\left(f-x_{j}\right)=-\delta_{i j}, \quad e \cdot\left(f-x_{i}\right)=1 . \tag{3.7}
\end{equation*}
$$

Let us parametrise the Kähler form as

$$
\begin{equation*}
J=a f+b[K]+\sum_{i} c_{i} x_{i} \tag{3.8}
\end{equation*}
$$

where [ $K$ ] is the class of the $d P_{n+1}$ inside $X_{S}$,

$$
\begin{equation*}
\left.[K]\right|_{S}=-c_{1}(S)=-\left.\left(2 e+2 f-\sum_{i} x_{i}\right)\right|_{S} . \tag{3.9}
\end{equation*}
$$

The flavour masses are given by integrating the Kähler form over the curves $x_{i}$ and $f-x_{i}$, they are given by $\phi \pm m_{i}$ [1], where $\phi$ is the Coulomb branch parameter and $m_{i}$ are the mass parameters. We find

$$
\begin{equation*}
-J \cdot x_{i}=b+c_{i} \quad-J \cdot\left(f-x_{i}\right)=b-c_{i} \tag{3.10}
\end{equation*}
$$

so that we are led to associate $b=\phi$ and $c_{i}=m_{i}$. Note that we are in the phase where $\phi>m_{i}$.

We can work out triple intersection numbers involving at least one compact divisor by using that $[K]$ is the Poincare dual to $S$ inside $X$. The geometric prepotential is then

$$
\begin{align*}
\mathcal{F}_{\text {geom }}=\frac{1}{6} J^{3} & =\frac{1}{6}\left([K]^{3} \phi^{3}+3\left([K]^{2} \cdot f\right) \phi^{2} a+\sum_{i=1}^{n} 3\left([K]^{2} \cdot x_{i}\right) \phi^{2} m_{i}+3\left([K] \cdot x_{i}^{2}\right) \phi m_{i}^{2}\right) \\
& =\frac{1}{6}\left((8-n) \phi^{3}-6 \phi^{2} a-\sum_{i=1}^{n} 3 \phi^{2} m_{i}-3 \phi m_{i}^{2}\right) \tag{3.11}
\end{align*}
$$

Comparing to (3.3) we can fix $a=-\frac{1}{g^{2}}-\frac{1}{2} \sum_{i} m_{i}$.
We can now work out the volume of the elliptic curve defining F-theory on $X_{S}$. The class of the elliptic curve inside $S$ is simply $c_{1}(S)$. Its volume is hence given by

$$
\begin{equation*}
\operatorname{Vol}\left(f_{\mathrm{ell}}\right)=-J \cdot f_{\mathrm{ell}}=-J \cdot c_{1}(S)=\frac{2}{g^{2}}+(8-n) \tag{3.12}
\end{equation*}
$$

The SCFT fixed point is reached by setting all $m_{i}$ to zero and letting $g \rightarrow \infty$, which implies for $n=8$ that $\operatorname{Vol}\left(f_{\text {ell }}\right) \rightarrow 0$. This is the F-theory limit and we recover that the marginal $(n=8)$ theory has a SCFT fixed point in 6d.

## 3.2 - 2 curve: $\operatorname{SU}(n)$ without twist

Consider a -2 curve which supports an $\mathfrak{s u}(n)$ gauge algebra. The 5 d KK theory is known to be the affine $\widehat{A}_{n-1}$ quiver gauge theory

with $n \mathrm{SU}(2)$ gauge groups and $\mathrm{SU}(2)_{i} \times \mathrm{SU}(2)_{i+1}$ bifundamental hypermultiplets. There are $n$ gauge couplings $g_{i}$, one for each gauge group, and $n$ mass parameters $m_{i, i+1}$, one for each bifundamental hypermultiplet.

5d gauge theory description. The prepotential of (3.13) is given by

$$
\begin{equation*}
6 \mathcal{F}_{5 d}=\sum_{i=0}^{n-1} \frac{6}{g_{i}^{2}} \phi_{i}^{2}+\sum_{i=0}^{n-1} 8 \phi_{i}^{3}-\frac{1}{2} \sum_{i=0}^{n-1}\left(\left|\phi_{i}+\phi_{i+1} \pm m_{i, i+1}\right|^{3}+\left|-\phi_{i}+\phi_{i+1} \pm m_{i, i+1}\right|^{3}\right), \tag{3.14}
\end{equation*}
$$

with the index identification $i+1=n \sim 0$. One restricts to the dominant Weyl chamber $\phi_{i} \geq 0$ for each $i$ and chooses a suitable Coulomb branch phase

$$
\left\{\begin{array}{ll}
\phi_{i}+\phi_{i+1} \pm m_{i, i+1} & \geq 0,  \tag{3.15}\\
-\phi_{i}+\phi_{i+1}+m_{i, i+1} & \geq 0, \\
\phi_{i}-\phi_{i+1}+m_{i, i+1} & \geq 0,
\end{array} \quad \text { for } i=0,1, \ldots, n-1\right.
$$

The prepotential (3.14) in this phase becomes:

$$
\begin{align*}
6 \mathcal{F}_{5 d}=\sum_{i=0}^{n-1}\left(\frac{6}{g_{i}^{2}} \phi_{i}^{2}+\right. & 6 \phi_{i}^{3}-3 \phi_{i} \phi_{i+1}^{2}-3 \phi_{i+1} \phi_{i}^{2} \\
& \left.\quad-3 m_{i, i+1}\left(\phi_{i}^{2}-2 \phi_{i} \phi_{i+1}+\phi_{i+1}^{2}\right)-3 m_{i, i+1}^{2}\left(\phi_{i}+\phi_{i+1}\right)\right) \tag{3.16}
\end{align*}
$$

up to constant terms $\mathcal{O}\left(m_{i}^{3}\right)$.
5-brane web and toric geometry. The theory (3.13) can be realised as world-volume theory of Type IIB 5 -brane web. In the simplest case of an $\widehat{A}_{1}$ quiver, the periodic 5-brane web is

which describes the phase (3.15). Focusing on the part of the 5 -brane web that describes $\mathrm{SU}(2)$ with 2 fundamental flavours, one can equivalently consider the dual graph. This yields the fan of $\mathbb{F}_{0}$ blown up at two points.


We have labelled the rays of the fan by the associated divisor (or curve) classes as follows: $e$ and $h$ are sections, and $f$ the fibre of the ruling. The classes of the exceptional divisors are denoted by $x_{i}$. As we see below, we can geometrically realise the $\mathrm{SU}(n)$ theories on a -2 curve without twist by combining these surfaces $S_{i}$ as building blocks along a circle. The gluing curves connecting every such surface to the one on the left/right are always given by the toric divisors that point to the left/right, $e-x_{1}$ and $e-x_{2}$ in the above notation. This can already be anticipated from the 5 -brane web.

### 3.2.1 $\mathrm{SU}(2)$

To begin with, consider the untwisted circle compactification of a -2 curve with an $\mathfrak{s u}(2)$ algebra. In the following, and throughout the paper, we employ the following notation for compact surfaces $S_{i}$ in the Calabi-Yau geometry. Namely, a surface $S_{i}$ obtained from blowing up $\mathbb{F}_{n} k$ times is denoted by $\mathbf{i}_{n}^{k}$.

Geometry. Following [61], the geometry for the circle compactified theory reads

$$
\mathbf{0}_{0}^{4} \stackrel{e_{0}-\sum_{i=1}^{e_{0}} x_{i}}{ } 2 \xrightarrow{\begin{array}{l}
e_{1}  \tag{3.19}\\
h_{1}
\end{array}} \mathbf{1}_{2}
$$

First, since the $e_{0}-x_{i}$ curves have self-intersection $\left(e_{0}-x_{i}\right)^{2}=-1$, due to $\mathbb{F}_{0}$, one can flip any $e_{0}-x_{i}$. So, one might choose

$$
\begin{equation*}
\mathbf{0}_{0}^{2} \xlongequal{e_{0}-\sum_{i=3}^{e_{0}} x_{i}} 2 \xlongequal[h_{1}-\sum_{i=1}^{e_{1}} x_{i}^{2}]{\mathbf{1}_{2}^{2}} \tag{3.20}
\end{equation*}
$$

where $x_{1,2}$ have been flopped. Now, by virtue of inverse of the isomorphism (A.22) for $x_{1}$ one finds

$$
\mathbf{0}_{0}^{2} \stackrel{e_{0}-\sum_{i=3}^{e_{0}} x_{i}}{4}-\begin{align*}
& e_{1}-x_{1}  \tag{3.21}\\
& h_{1}-x_{2}
\end{align*} \mathbf{1}_{1}^{2}
$$

where $h_{1}$ is inside $\mathbb{F}_{1}^{2}$. Repeating the isomorphism (A.22) for $x_{2}$ yields

$$
\begin{equation*}
\mathbf{0}_{0}^{2} \stackrel{e_{0}-\sum_{i=3}^{e_{0}} x_{i}}{4} \xlongequal{e_{1}-\sum_{e_{1}}^{2} x_{i=1}} \mathbf{1}_{0}^{2} \tag{3.22}
\end{equation*}
$$

Finally, one splits the blowups symmetrically

$$
\mathbf{0}_{0}^{2} \xlongequal{\substack{e_{0}-x_{1}  \tag{3.23}\\
e_{0}-x_{4}}} 2 \xlongequal{\begin{array}{c}
e_{1}-x_{2} \\
e_{1}-x_{3}
\end{array}} \mathbf{1}_{0}^{2}
$$

which is possible via further flop transitions.
Consistency of geometry. Before exploring the physics associated to the geometry (3.23), one verifies overall consistency.

- Firstly, one verifies the consistency condition (2.15) on gluing curves of (3.23). With $J_{\phi}=\sum_{i} \phi_{i} S_{i}$, the truncated Kähler form, one obtains:

$$
\begin{align*}
& \text { 1st component: }  \tag{3.24a}\\
& \text { 2nd component: }  \tag{3.24b}\\
&\left.S_{0}\right|_{S_{1}} \cdot J_{\phi}=\left.S_{1}\right|_{S_{0}} \cdot J_{\phi}=S_{0}+\phi_{1}, \\
& S_{0} \cdot J_{\phi}=\phi_{0}+\phi_{1} .
\end{align*}
$$

- Secondly, one verifies the Calabi-Yau condition (2.19)

$$
\begin{align*}
\text { 1st component: } & \left(\left.S_{0}\right|_{S_{1}}\right)^{2}+\left(S_{1} \mid S_{0}\right)^{2}=-2  \tag{3.25a}\\
\text { 2nd component: } & \left(\left.S_{0}\right|_{S_{1}}\right)^{2}+\left(\left.S_{1}\right|_{S_{0}}\right)^{2}=-2 \tag{3.25b}
\end{align*}
$$

which is consistent for genus $g=0$.

- Lastly, the fibre intersections are computed to be

$$
-\left(\begin{array}{cc}
f_{0} \cdot K_{S_{0}} & \left.f_{0} \cdot S_{1}\right|_{S_{0}}  \tag{3.26}\\
\left.f_{1} \cdot S_{0}\right|_{S_{1}} & f_{1} \cdot K_{S_{1}}
\end{array}\right)=\left(\begin{array}{cc}
2 & -2 \\
-2 & 2
\end{array}\right)=C_{\widehat{A}_{1}^{(1)}},
$$

which is consistent with 6 d origin of an $\mathfrak{s u}(2)$ gauge algebra.
Prepotential via intersection numbers. The triple intersection numbers for the compact surfaces in (3.23) give rise to the following cubic part of the prepotential:

$$
\begin{equation*}
6 \mathcal{F}_{\text {trun }} \equiv J_{\phi}^{3}=6 \phi_{0}^{3}-6 \phi_{1} \phi_{0}^{2}-6 \phi_{1}^{2} \phi_{0}+6 \phi_{1}^{3} . \tag{3.27}
\end{equation*}
$$

Besides the compact surfaces, one may also add non-compact ones that introduce deformation parameters. For this, one may parametrise the Kähler form as follows

$$
\begin{align*}
& \left.J\right|_{S_{0}}=\phi_{0} K_{S_{0}}+\left.\phi_{1} S_{1}\right|_{S_{0}}+a_{0} f_{0}+b_{0} e_{0}+\sum_{i=0}^{1} M_{i} x_{i},  \tag{3.28a}\\
& \left.J\right|_{S_{1}}=\left.\phi_{0} S_{0}\right|_{S_{1}}+\phi_{1} K_{S_{1}}+a_{1} f_{1}+b_{1} e_{1}+\sum_{i=2}^{3} M_{i} x_{i} . \tag{3.28b}
\end{align*}
$$

recalling that $x_{0} \equiv x_{4}$ and $i=4 \sim 0$. The introduced parameters can be determined via the following two requirements:

- The volume of the blowup curves match the physical mass terms given in (3.14). Based on (3.28), one computes

$$
\begin{array}{rlrl}
\operatorname{Vol}\left(x_{1}\right) & =\phi_{0}-\phi_{1}+M_{1} & \stackrel{!}{=} \phi_{0}-\phi_{1}+m_{0,1} \\
\operatorname{Vol}\left(e_{0}-x_{1}\right) & =\phi_{0}+\phi_{1}-M_{1}-a_{0} & \stackrel{!}{=} \phi_{0}+\phi_{1}-m_{0,1} \\
\operatorname{Vol}\left(x_{2}\right) & =-\phi_{0}+\phi_{1}+M_{2} & & \stackrel{!}{=}-\phi_{0}+\phi_{1}+m_{0,1} \\
\operatorname{Vol}\left(e_{1}-x_{2}\right) & =\phi_{0}+\phi_{1}-M_{2}-a_{1} & \stackrel{!}{=} \phi_{0}+\phi_{1}-m_{0,1} \tag{3.29d}
\end{array}
$$

which is identified as the contributions of the first bifundamental with mass parameter $m_{0,1}$. The second bifundamental, with parameter $m_{1,0}$, contributes as

$$
\begin{array}{rlrl}
\operatorname{Vol}\left(x_{4}\right) & =\phi_{0}-\phi_{1}+M_{4} & & \stackrel{!}{=} \phi_{0}-\phi_{1}+m_{1,0} \\
\operatorname{Vol}\left(e_{0}-x_{4}\right) & =\phi_{0}+\phi_{1}-M_{4}-a_{0} & \stackrel{!}{=} \phi_{0}+\phi_{1}-m_{1,0} \\
\operatorname{Vol}\left(x_{3}\right) & =-\phi_{0}+\phi_{1}+M_{3} & & \stackrel{!}{=}-\phi_{0}+\phi_{1}+m_{1,0} \\
\operatorname{Vol}\left(e_{1}-x_{3}\right) & =\phi_{0}+\phi_{1}-M_{3}-a_{1} & \stackrel{!}{=} \phi_{0}+\phi_{1}-m_{1,0} \tag{3.30d}
\end{array}
$$

The set of linear equations (3.29), (3.30) are straightforwardly solved by

$$
\begin{equation*}
M_{1}=M_{2}=m_{0,1}, \quad M_{3}=M_{4}=m_{1,0}, \quad a_{0}=a_{1}=0 . \tag{3.31}
\end{equation*}
$$

- The effective gauge coupling (2.25) is geometrically given by

$$
\begin{align*}
\left(\tau_{\text {eff }}\right) & =\left(\begin{array}{cc}
\left.J\right|_{S_{0}} \cdot K_{S_{0}} & \left.\left.J\right|_{S_{0}} \cdot S_{1}\right|_{S_{0}} \\
\left.\left.J\right|_{S_{1}} \cdot S_{0}\right|_{S_{1}} & \left.J\right|_{S_{1}} \cdot K_{S_{1}}
\end{array}\right) \\
& =\left(\begin{array}{cc}
6 \phi_{0}-2 \phi_{1}-\left(m_{0,1}+m_{1,0}\right)-2 b_{0} & -2\left(\phi_{0}+\phi_{1}\right)+m_{0,1}+m_{1,0} \\
-2\left(\phi_{0}+\phi_{1}\right)+m_{0,1}+m_{1,0} & 6 \phi_{1}-2 \phi_{0}-\left(m_{0,1}+m_{1,0}\right)-2 b_{1}
\end{array}\right)  \tag{3.32}\\
& \stackrel{!}{=}\left(\frac{\partial^{2} \mathcal{F}_{5 d}}{\partial \phi_{i} \partial \phi_{j}}\right)
\end{align*}
$$

such that comparing to the field theory expectation (2.2) imposes

$$
\begin{equation*}
b_{0}=-\frac{1}{g_{0}^{2}}, \quad b_{1}=-\frac{1}{g_{1}^{2}} . \tag{3.33}
\end{equation*}
$$

Therefore, the Kähler form may be written as

$$
\begin{equation*}
J=-\sum_{i=0}^{1} \frac{1}{g_{i}^{2}} F_{i}+\sum_{i=0}^{1} \phi_{i} S_{i}+\sum_{i=0}^{1} m_{i, i+1} N_{i, i+1}, \tag{3.34a}
\end{equation*}
$$

where the non-compact surfaces $F_{i}, N_{i, i+1}$ restrict to the compact surfaces as follows:

$$
F_{i}{\mid S_{j}}=\delta_{i j} e_{i}, \quad\left\{\begin{array}{l}
N_{0,1} \mid S_{0}=x_{1}  \tag{3.34b}\\
N_{0,1} \mid S_{1}=x_{2}
\end{array}, \quad\left\{\begin{array}{l}
N_{1,0} \mid S_{0}=x_{4} \\
N_{1,0} \mid S_{1}=x_{3}
\end{array} .\right.\right.
$$

Based on (3.34) one computes the volume of the elliptic fibre

$$
\begin{equation*}
\operatorname{Vol}\left(f_{\mathrm{ell}}\right)=\operatorname{Vol}\left(f_{0}\right)+\operatorname{Vol}\left(f_{1}\right)=\frac{1}{g_{0}^{2}}+\frac{1}{g_{1}^{2}}=\tau, \tag{3.35}
\end{equation*}
$$

which equals the modular parameter $\tau$.
Finally, the triple intersection numbers of the compact surfaces with the non-compact ones are

$$
\begin{align*}
F_{i} \cdot S_{j} \cdot S_{k} & = \begin{cases}F_{i} \mid S_{j} \cdot K_{S_{j}}=-2 \delta_{i, j}, & \text { for } j=k, \\
F_{i}\left|S_{j} \cdot S_{k}\right| S_{j}=0, & \text { for } j \neq k,\end{cases}  \tag{3.36a}\\
F_{i} \cdot F_{j} \cdot S_{k} & =0, \quad \forall i, j, k,  \tag{3.36b}\\
N_{i, i+1} \cdot S_{j} \cdot S_{k} & = \begin{cases}N_{i, i+1} \mid S_{j} \cdot K_{S_{j}}=-1 \cdot\left(\delta_{i, j}+\delta_{i+1, j}\right), & \text { for } j=k, \\
N_{i, i+1}\left|S_{j} \cdot S_{j+1}\right| S_{j}=\delta_{i, j}+\delta_{i+1, j}, & \text { for } k=j+1 \sim j-1,\end{cases} \tag{3.36c}
\end{align*}
$$

$$
\begin{align*}
N_{i, i+1} \cdot N_{j, j+1} \cdot S_{k} & =N_{i, i+1}\left|S_{k} \cdot N_{j, j+1}\right|_{S_{k}}=\delta_{i, k} \delta_{j, k}+\delta_{i+1, k} \delta_{j+1, k},  \tag{3.36d}\\
N_{i, i+1} \cdot F_{j} \cdot S_{k} & =0, \quad \forall i, j, k . \tag{3.36e}
\end{align*}
$$

These intersection numbers give rise to the geometric prepotential $6 \mathcal{F}_{\text {geom }}=J^{3}$ which matches (3.16). Therefore, the geometry (3.23) describes the $\widehat{A}_{1}$ KK-theory (3.13) in the phase (3.15).

### 3.2.2 SU(3)

Consider a -2 curve with an $\mathfrak{s u}(3)$ gauge algebra and its untwisted circle compactification.
Geometry. Following [61], the relevant geometry is given by


In $S_{0}=\mathbb{F}_{0}^{6}$, any of the $e_{0}-x_{i}$ curves has self-intersection $\left(e_{0}-x_{i}\right)^{2}=-1$; hence, can be flopped. So, one may choose

i.e. the blowups $x_{1,2,3,4}$ have been flopped. Applying the isomorphism (A.22) separately for $x_{1,2,3,4}$ yields


In $\mathbb{F}_{0}^{4}$, any of the $e_{2}-x_{i}$ curves has self-intersection $\left(e_{2}-x_{i}\right)^{2}=-1$; hence, can be flopped. One may choose

i.e. $x_{1,2}$ have been flopped. Applying the isomorphism (A.22) separately for $x_{1}$ and $x_{2}$ yields


Lastly, the blowups can be distributed symmetrically to yield

which can be achieved via further flop transitions.
Consistency of geometry. Next, the consistency of the geometry (3.42) is verified.

- Firstly, the consistency condition (2.15) for the gluing curves in (3.42) reads

$$
\begin{align*}
& \left.S_{0}\right|_{S_{1}} \cdot J_{\phi}=\left.S_{1}\right|_{S_{0}} \cdot J_{\phi}=\phi_{0}+\phi_{1},  \tag{3.43a}\\
& \left.S_{0}\right|_{S_{2}} \cdot J_{\phi}=\left.S_{2}\right|_{S_{0}} \cdot J_{\phi}=\phi_{0}+\phi_{2},  \tag{3.43b}\\
& \left.S_{1}\right|_{S_{2}} \cdot J_{\phi}=\left.S_{2}\right|_{S_{1}} \cdot J_{\phi}=\phi_{1}+\phi_{2}, \tag{3.43c}
\end{align*}
$$

where $J_{\phi}=\sum_{i} \phi_{i} S_{i}$ denotes the truncated Kähler form for (3.42).

- Secondly, one verifies the Calabi-Yau condition (2.19) and finds

$$
\begin{align*}
& \left(\left.S_{0}\right|_{S_{1}}\right)^{2}+\left(\left.S_{1}\right|_{S_{0}}\right)^{2}=-2,  \tag{3.44a}\\
& \left(\left.S_{0}\right|_{S_{2}}\right)^{2}+\left(S_{2} \mid S_{0}\right)^{2}=-2,  \tag{3.44b}\\
& \left(\left.S_{1}\right|_{S_{2}}\right)^{2}+\left(\left.S_{2}\right|_{S_{1}}\right)^{2}=-2, \tag{3.44c}
\end{align*}
$$

which is consistent for genus $g=0$.

- Lastly, the fibre intersections read

$$
-\left(\begin{array}{ccc}
f_{0} \cdot K_{S_{0}} & \left.f_{0} \cdot S_{1}\right|_{S_{0}} & \left.f_{0} \cdot S_{2}\right|_{S_{0}}  \tag{3.45}\\
\left.f_{1} \cdot S_{0}\right|_{S_{1}} & f_{1} \cdot K_{S_{1}} & \left.f_{1} \cdot S_{2}\right|_{S_{1}} \\
\left.f_{2} \cdot S_{0}\right|_{S_{2}} & f_{2} \cdot S_{1} \mid S_{2} & f_{2} \cdot K_{S_{2}}
\end{array}\right)=\left(\begin{array}{ccc}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-1 & -1 & 2
\end{array}\right)=C_{\widehat{A}_{2}^{(1)}},
$$

which is consistent with 6 d origin of an $\mathfrak{s u}(3)$ gauge algebra.
Prepotential via intersection numbers. For the geometry (3.42), one finds the following cubic part of the prepotential from the intersection numbers of the compact surface:

$$
\begin{equation*}
6 \mathcal{F}_{\text {trun }} \equiv J_{\phi}^{3}=6 \phi_{0}^{3}-3 \phi_{1} \phi_{0}^{2}-3 \phi_{2} \phi_{0}^{2}-3 \phi_{1}^{2} \phi_{0}-3 \phi_{2}^{2} \phi_{0}+6 \phi_{1}^{3}+6 \phi_{2}^{3}-3 \phi_{1} \phi_{2}^{2}-3 \phi_{1}^{2} \phi_{2} . \tag{3.46}
\end{equation*}
$$

Besides the compact surfaces, one may also add non-compact ones that introduce deformation parameters. For this, one may parametrise the Kähler form as follows

$$
\begin{align*}
& \left.J\right|_{S_{0}}=\phi_{0} K_{S_{0}}+\left.\phi_{1} S_{1}\right|_{S_{0}}+\left.\phi_{2} S_{2}\right|_{S_{0}}+a_{0} f_{0}+b_{0} e_{0}+\sum_{i=0}^{1} M_{i} x_{i},  \tag{3.47a}\\
& \left.J\right|_{S_{1}}=\left.\phi_{0} S_{0}\right|_{S_{1}}+\phi_{1} K_{S_{1}}+\left.\phi_{2} S_{2}\right|_{S_{1}}+a_{1} f_{1}+b_{1} e_{1}+\sum_{i=2}^{3} M_{i} x_{i},  \tag{3.47b}\\
& \left.J\right|_{S_{2}}=\left.\phi_{0} S_{0}\right|_{S_{2}}+\left.\phi_{1} S_{1}\right|_{S_{2}}+\phi_{2} K_{S_{2}}+a_{2} f_{2}+b_{2} e_{2}+\sum_{i=4}^{5} M_{i} x_{i}, \tag{3.47c}
\end{align*}
$$

recalling that $x_{0} \equiv x_{6}$ and $i=6 \sim 0$. The additional parameters are determined by two requirements:

- The volume of the blowup curves match the physical mass terms given in (3.14). Based on (3.47), one computes

$$
\begin{array}{rlrl}
\operatorname{Vol}\left(x_{6}\right) & =\phi_{0}-\phi_{2}+M_{6} & & \stackrel{!}{=} \phi_{0}-\phi_{2}+m_{2,0} \\
\operatorname{Vol}\left(e_{0}-x_{6}\right) & =\phi_{0}+\phi_{2}-M_{6}-a_{0} & \stackrel{!}{=} \phi_{0}+\phi_{2}-m_{2,0} \\
\operatorname{Vol}\left(x_{5}\right) & =-\phi_{0}+\phi_{2}+M_{5} & & \stackrel{!}{=}-\phi_{0}+\phi_{2}+m_{2,0} \\
\operatorname{Vol}\left(e_{2}-x_{5}\right) & =\phi_{0}+\phi_{2}-M_{5}-a_{2} & & \stackrel{!}{=} \phi_{0}+\phi_{2}+m_{1,0} \tag{3.48d}
\end{array}
$$

which is identified as the contributions of the $\mathrm{SU}(2)_{2} \times \mathrm{SU}(2)_{0}$ bifundamental with mass parameter $m_{2,0}$. The $\mathrm{SU}(2)_{0} \times \mathrm{SU}(2)_{1}$ bifundamental hypermultiplets, with mass parameters $m_{0,1}$, contributes as

$$
\begin{array}{rlrl}
\operatorname{Vol}\left(x_{1}\right) & =\phi_{0}-\phi_{1}+M_{1} & \stackrel{!}{=} \phi_{0}-\phi_{1}+m_{0,1} \\
\operatorname{Vol}\left(e_{0}-x_{1}\right) & =\phi_{0}+\phi_{1}-M_{1}-a_{0} & \stackrel{!}{=} \phi_{0}+\phi_{1}-m_{0,1} \\
\operatorname{Vol}\left(x_{2}\right) & =-\phi_{0}+\phi_{1}+M_{2} & & \stackrel{!}{=}-\phi_{0}+\phi_{1}+m_{0,1} \\
\operatorname{Vol}\left(e_{1}-x_{2}\right) & =\phi_{0}+\phi_{1}-M_{2}-a_{1} & \stackrel{!}{=} \phi_{0}+\phi_{1}-m_{0,1} \tag{3.49~d}
\end{array}
$$

The remaining blowups have volumes given by

$$
\begin{array}{rlrl}
\operatorname{Vol}\left(x_{3}\right) & =\phi_{1}-\phi_{2}+M_{3} & \stackrel{!}{=} \phi_{1}-\phi_{2}+m_{1,2} \\
\operatorname{Vol}\left(e_{1}-x_{3}\right) & =\phi_{1}+\phi_{2}-M_{3}-a_{1} & \stackrel{!}{=} \phi_{1}+\phi_{2}-m_{1,2} \\
\operatorname{Vol}\left(x_{4}\right) & =-\phi_{1}+\phi_{2}+M_{4} & & \stackrel{!}{=}-\phi_{1}+\phi_{2}+m_{1,2} \\
\operatorname{Vol}\left(e_{2}-x_{4}\right) & =\phi_{1}+\phi_{2}-M_{4}-a_{2} & \stackrel{!}{=} \phi_{1}+\phi_{2}-m_{1,2} \tag{3.50~d}
\end{array}
$$

which identifies them as $\mathrm{SU}(2)_{1} \times \mathrm{SU}(2)_{2}$ bifundamental hypermultiplet with mass parameter $m_{1,2}$. The set of linear equations (3.48)-(3.50) are straightforwardly solved by

$$
\begin{equation*}
M_{5}=M_{6}=m_{2,0}, \quad M_{1}=M_{2}=m_{0,1}, \quad M_{3}=M_{4}=m_{1,2}, \quad a_{0}=a_{1}=a_{2}=0 \tag{3.51}
\end{equation*}
$$

- The effective gauge coupling (2.25) is geometrically given by

$$
\begin{align*}
\left(\tau_{\text {eff }}\right) & =\left(\begin{array}{ccc}
\left.J\right|_{S_{0}} \cdot K_{S_{0}} & \left.\left.J\right|_{S_{0}} \cdot S_{1}\right|_{S_{0}} & \left.\left.J\right|_{S_{0}} \cdot S_{2}\right|_{S_{0}} \\
\left.\left.J\right|_{S_{1}} \cdot S_{0}\right|_{S_{1}} & \left.J\right|_{S_{1}} \cdot K_{S_{1}} & \left.\left.J\right|_{S_{1}} \cdot S_{2}\right|_{S_{1}} \\
\left.\left.J\right|_{S_{2}} \cdot S_{0}\right|_{S_{2}} & \left.\left.J\right|_{S_{2}} \cdot S_{1}\right|_{S_{2}} & \left.J\right|_{S_{2}} \cdot K_{S_{2}}
\end{array}\right) \stackrel{!}{=}\left(\frac{\partial^{2} \mathcal{F}_{5 d}}{\partial \phi_{i} \partial \phi_{j}}\right) \\
\text { with }\left.\quad J\right|_{S_{0}} \cdot K_{S_{0}} & =6 \phi_{0}-\phi_{1}-\phi_{2}-\left(m_{0,1}+m_{2,0}\right)-2 b_{0}, \\
\left.J\right|_{S_{1}} \cdot K_{S_{1}} & =6 \phi_{1}-\phi_{0}-\phi_{2}-\left(m_{0,1}+m_{1,2}\right)-2 b_{1}, \\
\left.J\right|_{S_{2}} \cdot K_{S_{2}} & =6 \phi_{2}-\phi_{0}-\phi_{1}-\left(m_{1,2}+m_{2,0}\right)-2 b_{2}, \\
\left.\left.J\right|_{S_{0}} \cdot S_{1}\right|_{S_{0}} & =\left.\left.J\right|_{S_{1}} \cdot S_{0}\right|_{S_{1}}=-\left(\phi_{0}+\phi_{1}\right)+m_{0,1}, \\
\left.\left.J\right|_{S_{0}} \cdot S_{2}\right|_{S_{0}} & =\left.\left.J\right|_{S_{2}} \cdot S_{0}\right|_{S_{2}}=-\left(\phi_{0}+\phi_{2}\right)+m_{2,0}, \\
\left.\left.J\right|_{S_{1}} \cdot S_{2}\right|_{S_{1}} & =\left.\left.J\right|_{S_{2}} \cdot S_{1}\right|_{S_{2}}=-\left(\phi_{1}+\phi_{2}\right)+m_{1,2}, \tag{3.52}
\end{align*}
$$

such that comparing to the field theory expectation (2.2) imposes

$$
\begin{equation*}
b_{0}=-\frac{1}{g_{0}^{2}}, \quad b_{1}=-\frac{1}{g_{1}^{2}}, \quad b_{2}=-\frac{1}{g_{2}^{2}} . \tag{3.53}
\end{equation*}
$$

Therefore, the Kähler form may be written as

$$
\begin{equation*}
J=-\sum_{i=0}^{2} \frac{1}{g_{i}^{2}} F_{i}+\sum_{i=0}^{2} \phi_{i} S_{i}+\sum_{i=0}^{2} m_{i, i+1} N_{i, i+1}, \tag{3.54a}
\end{equation*}
$$

where the parameter $i+1=3 \sim 0$ is periodically identified. The non-compact surfaces $F_{i}$, $N_{i, i+1}$ restrict to the compact surfaces as follows:

$$
\begin{equation*}
\left.F_{i}\right|_{S_{j}}=\delta_{i j} e_{i},\left.\quad N_{i, i+1}\right|_{S_{j}}=\delta_{i, j} x_{2 i+1}+\delta_{i+1, j} x_{2 i+2} \tag{3.54b}
\end{equation*}
$$

Based on (3.54), the volume of the elliptic fibre is given by

$$
\begin{equation*}
\operatorname{Vol}\left(f_{\mathrm{ell}}\right)=\sum_{i=0}^{2} \operatorname{Vol}\left(f_{i}\right)=\frac{1}{g_{0}^{2}}+\frac{1}{g_{1}^{2}}+\frac{1}{g_{2}^{2}}=\tau, \tag{3.55}
\end{equation*}
$$

which equals the modular parameter $\tau$.
Lastly, the triple intersection numbers of the compact surfaces with the non-compact ones are

$$
\begin{align*}
F_{i} \cdot S_{j} \cdot S_{k} & = \begin{cases}\left.F_{i}\right|_{S_{j}} \cdot K_{S_{j}}=-2 \delta_{i, j}, & \text { for } j=k, \\
F_{i}\left|S_{j} \cdot S_{k}\right| S_{j}=0, & \text { for } j \neq k,\end{cases}  \tag{3.56a}\\
F_{i} \cdot F_{j} \cdot S_{k} & =0, \quad \forall i, j, k,
\end{align*} \begin{array}{ll}
N_{i, i+1} \cdot S_{j} \cdot S_{k} & = \begin{cases}N_{i, i+1} \mid S_{j} \cdot K_{S_{j}}=-1 \cdot\left(\delta_{i, j}+\delta_{i+1, j}\right), & \text { for } j=k, \\
N_{i, i+1}\left|S_{j} \cdot S_{j+1}\right| S_{j}=\delta_{i, j}, & \text { for } k=j+1, \\
N_{i, i+1}\left|S_{j} \cdot S_{j-1}\right| S_{j}=\delta_{i+1, j}, & \text { for } k=j-1, \\
N_{i, i+1}\left|S_{j} \cdot S_{k}\right| S_{j}=0, & \text { for } k \notin\{j, j \pm 1\},\end{cases}  \tag{3.56b}\\
N_{i, i+1} \cdot N_{j, j+1} \cdot S_{k} & =N_{i, i+1}\left|S_{k} \cdot N_{j, j+1}\right| S_{k}=\delta_{i, k} \delta_{j, k}+\delta_{i+1, k} \delta_{j+1, k},  \tag{3.56c}\\
N_{i, i+1} \cdot F_{j} \cdot S_{k} & =0 \quad \forall i, j, k . \tag{3.56d}
\end{array}
$$

These intersection numbers give rise to the geometric prepotential $6 \mathcal{F}_{\text {geom }}=J^{3}$ which matches (3.16). Therefore, the geometry (3.42) describes the $\widehat{A}_{2}$ KK-theory (3.13) in the phase (3.15).

### 3.2.3 SU(even)

For the general case, one proceeds in an analogous manner to sections 3.2.1-3.2.2.

Geometry. The geometric description is based on [61]

and all $4 n$ blowups are inside the 0-th compact surface $S_{0}=\mathbb{F}_{0}^{4 n}$. Since all the blowups are at the gluing of $S_{0}$ with $S_{2 n-1}=\mathbb{F}_{4 n-2}$, one can flop $4 n-2$ out of them and obtains


Next, applying the isomorphism (A.22) $4 n-2$ times reduces $\mathbb{F}_{4 n-2}^{4 n-2} \rightarrow \mathbb{F}_{0}^{4 n-2}$. As a consequence, all these $4 n-2$ blowups are in the gluing curve to $S_{2 n-2}=\mathbb{F}_{4 n-4}$. In detail,

and, again, one can flop $4 n-4$ blowups into $S_{2 n-2}=\mathbb{F}_{4 n-4}$. This leads to

and applying the isomorphism (A.22) $4 n-4$ times reduces $\mathbb{F}_{4 n-4}^{4 n-4} \rightarrow \mathbb{F}_{0}^{4 n-4}$. Thus, the geometry becomes

and one realises that this processes of flop transition and isomorphism (A.22) can be repeated systematically for all compact surfaces $S_{i}$. Therefore, one can reduce all surfaces to $\mathbb{F}_{0}^{2}$ and all blowups can be distributed symmetrically as follows:

which is a fully symmetric arrangement of $\mathbb{F}_{0}^{2}$ surfaces glued along $e-x$ curves to the neighbouring compact surfaces.

Prepotential via intersection numbers. The claim is as follows:

- The compact surfaces $S_{i}$ in (3.62) give rise to the truncated Kähler form

$$
\begin{equation*}
J_{\phi}=\sum_{i=0}^{2 n-1} \phi_{i} S_{i} \tag{3.63}
\end{equation*}
$$

whose intersection numbers induce the (truncated) geometric prepotential

$$
\begin{equation*}
6 \mathcal{F}_{\text {trun }} \equiv J_{\phi}^{3}=\sum_{i=0}^{2 n-1} 6 \phi_{i}^{3}-\sum_{i=0}^{2 n-1} 3 \phi_{i} \phi_{i+1}^{2}-\sum_{i=0}^{2 n-1} 3 \phi_{i} \phi_{i-1}^{2} \tag{3.64}
\end{equation*}
$$

with the identification $\phi_{2 n} \equiv \phi_{0}$.

- One can add $2 n$ non-compact surfaces $F_{i}$ for $i=0,1, \ldots, 2 n-1$ which restrict to the compact surfaces in (3.62) as follows:

$$
\begin{equation*}
\left.F_{i}\right|_{S_{j}}=\delta_{i j} e_{i} \tag{3.65}
\end{equation*}
$$

and further $2 n$ non-compact surfaces $N_{i, i+1}$ for $i=0,1, \ldots, 2 n-1$ which are characterised by

$$
\begin{equation*}
\left.N_{i, i+1}\right|_{S_{j}}=\delta_{i, j} x_{2 i+1}+\delta_{i+1, j} x_{2 i+2} \tag{3.66}
\end{equation*}
$$

The Kähler form is supplemented via

$$
\begin{equation*}
J=-\sum_{i=0}^{2 n-1} \frac{1}{g_{i}^{2}} F_{i}+\sum_{i=0}^{2 n-1} \phi_{i} S_{i}+\sum_{i=0}^{2 n-1} m_{i, i+1} N_{i, i+1} \tag{3.67}
\end{equation*}
$$

and the associated prepotential becomes

$$
\begin{align*}
6 \mathcal{F}_{\text {geom }} \equiv J^{3}=6 \mathcal{F}_{\text {trun }}+\sum_{i=0}^{2 n-1}\left(\frac{6}{g_{i}^{2}} \phi_{i}^{2}-3 m_{i, i+1}\left(\phi_{i}^{2}-\right.\right. & \left.2 \phi_{i} \phi_{i+1}+\phi_{i+1}^{2}\right) \\
& \left.-3 m_{i, i+1}^{2}\left(\phi_{i}+\phi_{i+1}\right)\right) \tag{3.68}
\end{align*}
$$

with the identification $\phi_{2 n} \equiv \phi_{0}$. Hence, the geometric result agrees with prepotential (3.16) of the affine quiver gauge theory (3.13) in the phase (3.15).

### 3.2.4 SU(odd)

The geometry is given by [61]

and one observes that the same reasoning as for $\operatorname{SU}(2 n)$ applies. In brief, the 0 -th surface $S_{0}=\mathbb{F}_{0}^{4 n+2}$ carries all $4 n+2$ blowups. Due to the degree 0 , all curves of the form $e_{0}-x_{i}$ can be flopped into the surface $S_{2 n}=\mathbb{F}_{4 n}$. The isomorphism (A.22) suggests that one flops as many blowups into a given Hirzebruch surface as its degree. Thus, $4 n$ blowups are flopped into $S_{2 n}$ such that the repeated application of the isomorphism (A.22) yields $\mathbb{F}_{4 n}^{4 n} \rightarrow \mathbb{F}_{0}^{4 n}$, but all the blowups are then in the gluing curves to $S_{2 n-1}=\mathbb{F}_{2 n-2}$. Repeating this process for all other compact surfaces yields the claim.

## $3.3-3$ curve: $\mathrm{SU}(3)$ with $\mathbb{Z}_{2}$ twist

Consider a -3 curve with an $\mathfrak{s u}(3)$ gauge algebra such that the 6 d theory has no hypermultiplets, also known as the 6 d minimal $\operatorname{SU}(3)$ SCFT. The circle compactification with a $\mathbb{Z}_{2}$ outer automorphism twist on $\mathfrak{s u}(3)$

$$
\begin{equation*}
{\stackrel{5 u}{ }{ }^{\mathfrak{s u}(3)}{ }^{(2)}}^{(2)} \tag{3.70}
\end{equation*}
$$

is known to be dual to a $5 \mathrm{~d} \operatorname{SU}(3)$ gauge theory with Chern-Simons level $\kappa=9$ [33, 47].
$\mathbf{5 d}$ description. The 5 d gauge theory description is based on the prepotential (2.1), which can be evaluated using appendix A.1.2. Restricting to the Weyl chamber, i.e. $\left\langle\phi, \alpha_{i}\right\rangle \geq 0$ for $i=1,2$, and choosing $\kappa=9$, one finds:

$$
\begin{equation*}
6 \mathcal{F}_{5 d}=\frac{6}{g^{2}}\left(\phi_{0}^{2}-\phi_{0} \phi_{1}+\phi_{1}^{2}\right)+8 \phi_{0}^{3}+24 \phi_{0}^{2} \phi_{1}-30 \phi_{0} \phi_{1}^{2}+8 \phi_{1}^{3}, \tag{3.71}
\end{equation*}
$$

which has been computed before in [33].
Geometry. According to [61], the geometry of the twisted circle compactification is described by

$$
\begin{equation*}
\mathbf{0}_{10} \stackrel{e_{0}}{4 e_{1}+f_{1}} \mathbf{1}_{0} \tag{3.72}
\end{equation*}
$$

and one computes the fibre intersections

$$
-\left(\begin{array}{cc}
f_{0} \cdot K_{S_{0}} & f_{0} \cdot S_{1} \mid S_{0}  \tag{3.73}\\
f_{1} \cdot S_{0} \mid S_{1} & f_{1} \cdot K_{S_{1}}
\end{array}\right)=\left(\begin{array}{cc}
2 & -1 \\
-4 & 2
\end{array}\right)=C_{\widehat{A}_{2}^{(2)}}
$$

which is consistent with the 6 d origin of a -3 curve with an $\mathfrak{s u}(3)$ algebra and $\mathbb{Z}_{2}$ twist. The associated truncated Kähler form $J_{\phi}=\sum_{i=0}^{1} \phi_{i} S_{i}$ to (3.72) gives rise to the following cubic part of the prepotential

$$
\begin{equation*}
6 \mathcal{F}_{\text {trun }} \equiv J_{\phi}^{3}=8 \phi_{0}^{3}+8 \phi_{1}^{3}+24 \phi_{0}^{2} \phi_{1}-30 \phi_{0} \phi_{1}^{2} . \tag{3.74}
\end{equation*}
$$

In order to include the gauge coupling terms, it is instructive to note that

$$
-\left(\begin{array}{cc}
f_{0} \cdot K_{S_{0}} & f_{0} \cdot S_{1} \mid S_{0}  \tag{3.75}\\
\left.e_{1} \cdot S_{0}\right|_{S_{1}} & e_{1} \cdot K_{S_{1}}
\end{array}\right)=\left(\begin{array}{cc}
2 & -1 \\
-1 & 2
\end{array}\right)=C_{A_{2}} \quad \Rightarrow \quad \begin{cases}-J_{\phi} \cdot f_{0} & =\left\langle\alpha_{1}, \phi\right\rangle \\
-J_{\phi} \cdot e_{1} & =\left\langle\alpha_{2}, \phi\right\rangle\end{cases}
$$

which identifies $f_{0}$ and $e_{1}$ with the simple roots $\alpha_{i}$ of the 5 d gauge algebra $\mathfrak{s u}(3)$, see appendix A.1.2. One proceeds by adding a non-compact surface $F$ to the Kähler form

$$
\begin{equation*}
J=-\frac{1}{g^{2}} F+\phi_{0} S_{0}+\phi_{1} S_{1} \tag{3.76}
\end{equation*}
$$

which is characterised by the restrictions to the compact surfaces

$$
\left\{\begin{array}{l}
\left.F\right|_{S_{0}}=a_{0} e_{0}+b_{0} f_{0}  \tag{3.77}\\
\left.F\right|_{S_{1}}=a_{1} e_{1}+b_{1} f_{1}
\end{array}\right.
$$

The parameters $a_{i}, b_{i}$ are determined from the effective coupling (2.2). Imposing that

$$
\left(\begin{array}{cc}
\left.J\right|_{S_{0}} \cdot K_{S_{0}} & \left.\left.J\right|_{S_{0}} \cdot S_{1}\right|_{S_{0}}  \tag{3.78}\\
\left.\left.J\right|_{S_{1}} \cdot S_{0}\right|_{S_{1}} & \left.J\right|_{S_{1}} \cdot K_{S_{1}}
\end{array}\right) \stackrel{!}{=}\left(\frac{\partial^{2} \mathcal{F}_{5 d}}{\partial \phi_{i} \partial \phi_{i}}\right) \equiv\left(\tau_{\text {eff }}\right)
$$

is equivalent to imposing that the intersection numbers of $F$ with the $S_{i}$,

$$
\left(\left.\begin{array}{c}
\left.F\right|_{S_{0}} \cdot K_{S_{0}}  \tag{3.79}\\
\left.F\right|_{S_{0}} \cdot S_{1} \mid S_{S_{0}} \\
\left.F S_{0}\right|_{S_{1}}
\end{array} F\right|_{S_{1}} \cdot K_{S_{1}}, ~\left(\begin{array}{c}
8 a_{0}-2 b_{0}-10 a_{0}+b_{0} \\
a_{1}+4 b_{1}
\end{array}-2 a_{1}-2 b_{1}\right) \stackrel{!}{=}\left(\begin{array}{cc}
2 & -1 \\
-1 & 2
\end{array}\right) \equiv-h_{i j},\right.
$$

equal the negative inverse metric tensor $h_{i j}$ of the 5 d gauge group $\mathrm{SU}(3)$. For simply laced Lie algebras, $h_{i j}$ equals the Cartan matrix. To see that (3.78) is equivalent to (3.79), one recalls that, firstly, the cubic parts of prepotential (3.71) are reproduced by (3.74) and, secondly, there are no other deformation parameters or non-compact surfaces. Solving the system of linear equations (3.79), one finds

$$
\left\{\begin{array} { l l } 
{ a _ { 0 } = 0 , } & { b _ { 0 } = 1 , }  \tag{3.80}\\
{ a _ { 1 } = 1 , } & { b _ { 1 } = 0 , }
\end{array} \quad \longrightarrow \quad \left\{\begin{array}{l}
\left.F\right|_{S_{0}}=f_{0}, \\
\left.F\right|_{S_{1}}=e_{1}
\end{array}\right.\right.
$$

One observes that $\left.F\right|_{S_{i}}$ restricts to $f_{0}$ and $e_{1}$ respectively, because these act as simple roots (3.75). Moreover, the prefactor of 1 equals $\frac{2}{\left\langle\alpha_{i}, \alpha_{i}\right\rangle}$ for roots of $A_{2}$.

As a result, the Kähler form (3.76) leads to the following geometric prepotential:

$$
\begin{equation*}
6 \mathcal{F}_{\text {geom }} \equiv J^{3}=\frac{6}{g^{2}}\left(\phi_{0}^{2}-\phi_{0} \phi_{1}+\phi_{1}^{2}\right)+8 \phi_{0}^{3}+8 \phi_{1}^{3}+24 \phi_{0}^{2} \phi_{1}-30 \phi_{0} \phi_{1}^{2} \tag{3.81}
\end{equation*}
$$

which matches the field theory result (3.71). In addition, consider the elliptic fibre

$$
\begin{equation*}
f_{\mathrm{ell}}=\sum_{i=0}^{1} d_{i} f_{i}=2 f_{0}+f_{1} \tag{3.82}
\end{equation*}
$$

with $d_{i}=$ Coxeter numbers of $A_{2}^{(2)}$. Computing the volume with respect to (3.76) yields

$$
\left\{\begin{array}{l}
\operatorname{Vol}\left(f_{0}\right)=2 \phi_{0}-\phi_{1}  \tag{3.83}\\
\operatorname{Vol}\left(f_{1}\right)=\frac{1}{g^{2}}-4 \phi_{0}+2 \phi_{1}
\end{array} \quad \Rightarrow \quad \operatorname{Vol}\left(f_{\text {ell }}\right)=\frac{1}{g^{2}}\right.
$$

The geometry (3.72) displays the 6 d properties manifestly, like the fibre intersection (3.73) and the elliptic fibre (3.83). On the other hand, the 5 d description is based on a simple exchange of fibre and base in the $\mathbb{F}_{0}$ in (3.72), which becomes apparent in the intersection properties (3.75). Consequently, the 6 d and 5 d geometric frame are related by a fibrebase duality.

## $3.4-4$ curve: $\operatorname{SO}(8)$ with $\mathbb{Z}_{3}$ twist

The 6 d minimal $\mathrm{SO}(8) \mathrm{SCFT}$ is realised by a -4 curve supporting a $\mathfrak{s o}(8)$ gauge algebra. The circle compactification twisted by a $\mathbb{Z}_{3}$ outer automorphism on the $\mathfrak{s o}(8)$ algebra

$$
\begin{gather*}
\mathfrak{s o}(8)^{(3)}  \tag{3.84}\\
\mathbf{4}
\end{gather*}
$$

is known to be dual to a $5 \mathrm{~d} \mathrm{SU}(4)$ gauge theory with Chern-Simons level $\kappa=8$ [79].
5d description. The 5d gauge theory description is based on the prepotential (2.1), which can be evaluated using appendix A.1.2. Restricting to the Weyl-chamber of $A_{3}$, i.e. $\left\langle\phi, \alpha_{i}\right\rangle \geq 0$ for $i=1,2,3$, and specialising to $\kappa=8$ yields

$$
\begin{align*}
6 \mathcal{F}_{5 d}= & \frac{6}{g^{2}}\left(\phi_{0}^{2}-\phi_{0} \phi_{1}+\phi_{1}^{2}-\phi_{1} \phi_{2}+\phi_{2}^{2}\right)  \tag{3.85}\\
& +8 \phi_{0}^{3}+24 \phi_{1} \phi_{0}^{2}-30 \phi_{1}^{2} \phi_{0}+8 \phi_{1}^{3}+8 \phi_{2}^{3}-24 \phi_{1} \phi_{2}^{2}+18 \phi_{1}^{2} \phi_{2}
\end{align*}
$$

Geometry. As argued in [61], the geometry is described by

$$
\begin{equation*}
\mathbf{0}_{10} \stackrel{e_{0}}{h_{1}} \mathbf{1}_{8} \xrightarrow{e_{1}} 3 e_{2}+f_{2} \mathbf{2}_{0} \tag{3.86}
\end{equation*}
$$

and one verifies that

$$
-f_{i} \cdot S_{j}=\left(\begin{array}{ccc}
2 & -1 & 0  \tag{3.87}\\
-1 & 2 & -1 \\
0 & -3 & 2
\end{array}\right)
$$

which equals the affine Cartan matrix of $\widehat{D}_{4}^{(3)}$ up to similarity transformation. Again, this is consistent with the $6 \mathrm{~d} \mathfrak{s o}(8)$ gauge algebra together with the $\mathbb{Z}_{3}$ twist.

Based on the truncated Kähler form $J_{\phi}=\sum_{i=0}^{2} \phi_{i} S_{i}$, the triple intersection numbers of the compact surfaces $S_{i}$ induce the following cubic part of the prepotential

$$
\begin{equation*}
6 \mathcal{F}_{\text {trun }}=8 \phi_{0}^{3}+8 \phi_{1}^{3}+8 \phi_{2}^{3}+24 \phi_{0}^{2} \phi_{1}-30 \phi_{0} \phi_{1}^{2}+18 \phi_{1}^{2} \phi_{2}-24 \phi_{1} \phi_{2}^{2} \tag{3.88}
\end{equation*}
$$

In order to include the only gauge theory parameter, the coupling $g$, it is useful to notice

$$
\left(\begin{array}{cc}
f_{0} \cdot K_{S_{0}} & \left.f_{0} \cdot S_{1}\right|_{S_{0}}  \tag{3.89}\\
\left.f_{0} \cdot S_{2}\right|_{S_{0}} \\
\left.f_{1} \cdot S_{0}\right|_{S_{1}} & f_{1} \cdot K_{S_{1}} \\
\left.f_{1} \cdot S_{2}\right|_{S_{1}} \\
\left.e_{2} \cdot S_{0}\right|_{S_{2}} & \left.e_{2} \cdot S_{1}\right|_{S_{2}}
\end{array} e_{2} \cdot K_{S_{2}}\right)=\left(\begin{array}{ccc}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 2
\end{array}\right)=C_{A_{3}} \quad \Rightarrow \quad \begin{cases}-J_{\phi} \cdot f_{0} & =\left\langle\alpha_{1}, \phi\right\rangle \\
-J_{\phi} \cdot f_{1} & =\left\langle\alpha_{2}, \phi\right\rangle \\
-J_{\phi} \cdot e_{2} & =\left\langle\alpha_{3}, \phi\right\rangle\end{cases}
$$

which identifies $f_{0}, f_{1}$, and $e_{2}$ as acting as simple roots $\alpha_{i}$ of the 5 d gauge algebra $\mathfrak{s u}(4)$, see appendix A.1.2. Incorporating the gauge coupling contributions can be achieved by adding a non-compact surface $F$ to the Kähler form

$$
\begin{equation*}
J=-\frac{1}{g^{2}} F+\phi_{0} S_{0}+\phi_{1} S_{1}+\phi_{2} S_{2} \tag{3.90}
\end{equation*}
$$

which is characterised by its restrictions to the compact surfaces $S_{i}$, i.e.

$$
\begin{equation*}
\left.F\right|_{S_{i}}=a_{i} e_{i}+b_{i} f_{i} \quad \text { for } i=0,1,2 . \tag{3.91}
\end{equation*}
$$

The parameters $a_{i}, b_{i}$ are determined by matching the effective gauge coupling as follows:

$$
\begin{equation*}
\left(J \cdot S_{i} \cdot S_{j}\right) \stackrel{!}{=}\left(\frac{\partial^{2} \mathcal{F}_{5 d}}{\partial \phi_{i} \partial \phi_{j}}\right) \equiv\left(\tau_{\mathrm{eff}}\right) \tag{3.92}
\end{equation*}
$$

i.e. combining (2.2) and (2.25). Since the triple intersection numbers of the compact surfaces reproduce the cubic part of the prepotential (3.85), and as there are no other deformation parameters or non-compact surfaces, the set of conditions (3.92) is equivalent to

$$
\left(F \cdot S_{i} \cdot S_{j}\right)=\left(\begin{array}{ccc}
8 a_{0}-2 b_{0} & b_{0}-10 a_{0} & 0  \tag{3.93}\\
8 b_{1}+b_{1} & 6 a_{1}-2 b_{1} & b_{1}-8 a_{1} \\
0 & a_{2}+3 b_{2} & -2 a_{2}-2 b_{2}
\end{array}\right) \stackrel{!}{=}\left(\begin{array}{ccc}
-2 & 1 & 0 \\
1 & -2 & 1 \\
0 & 1 & -2
\end{array}\right) \equiv-h_{i j},
$$

where $h_{i j}$ is the inverse metric tensor of the $5 \mathrm{~d} \mathrm{SU}(4)$ gauge group, see appendix A.1.2. Again, for the simply laces $A_{3}$ algebra, $h_{i j}$ equals the Cartan matrix. The system of linear equations (3.93) is solved by

$$
\left\{\begin{array} { l l } 
{ a _ { 0 } = 0 , } & { b _ { 0 } = 1 , }  \tag{3.94}\\
{ a _ { 1 } = 0 , } & { b _ { 1 } = 1 , } \\
{ a _ { 2 } = 1 , } & { b _ { 2 } = 0 , }
\end{array} \quad \longrightarrow \quad \left\{\begin{array}{l}
\left.F\right|_{S_{0}}=f_{0}, \\
\left.F\right|_{S_{1}}=f_{1}, \\
\left.F\right|_{S_{2}}=e_{2} .
\end{array}\right.\right.
$$

One observes that $\left.F\right|_{S_{i}}$ restricts to $f_{0}, f_{1}$, and $e_{2}$ respectively, because these act as simple roots (3.89). Also, note that the prefactor of 1 equals $\frac{2}{\left\langle\alpha_{i}, \alpha_{i}\right\rangle}$ for roots of $A_{3}$.

The Kähler form (3.90) induces the following geometric prepotential:

$$
\begin{align*}
6 \mathcal{F}_{\text {geom }} \equiv J^{3}= & \frac{6}{g^{2}}\left(\phi_{0}^{2}-\phi_{0} \phi_{1}+\phi_{1}^{2}-\phi_{1} \phi_{2}+\phi_{2}^{2}\right)+8 \phi_{0}^{3}+8 \phi_{1}^{3}+8 \phi_{2}^{3}  \tag{3.95}\\
& +24 \phi_{0}^{2} \phi_{1}-30 \phi_{0} \phi_{1}^{2}+18 \phi_{1}^{2} \phi_{2}-24 \phi_{1} \phi_{2}^{2} .
\end{align*}
$$

Consequently, the geometric result (3.95) matches the field theory expectation (3.85). Moreover, consider the elliptic fibre

$$
\begin{equation*}
f_{\mathrm{ell}}=\sum_{i=0}^{2} d_{i} f_{i}=f_{0}+2 f_{1}+f_{2} \tag{3.96}
\end{equation*}
$$

with $d_{i}=$ Coxeter numbers of $D_{4}^{(3)}$. The fibre volumes of (3.86) with respect to (3.90) are

$$
\left\{\begin{array}{l}
\operatorname{Vol}\left(f_{0}\right)=2 \phi_{0}-\phi_{1},  \tag{3.97}\\
\operatorname{Vol}\left(f_{1}\right)=-\phi_{0}+2 \phi_{1}-\phi_{2}, \\
\operatorname{Vol}\left(f_{2}\right)=\frac{1}{g_{0}^{2}}-3 \phi_{1}+2 \phi_{2},
\end{array} \quad \Rightarrow \quad \operatorname{Vol}\left(f_{\mathrm{ell}}\right)=\frac{1}{g^{2}}\right.
$$

The geometry (3.86) had been derived to display the 6 d properties manifestly, like the fibre intersection (3.87) and the elliptic fibre (3.97). On the other hand, the 5 d gauge theory is more conveniently described by simply exchanging the fibre and base in the $\mathbb{F}_{0}$ in (3.86). As a result, the 5 d gauge algebra becomes manifest in the intersection properties (3.89). This is another example in which the 6 d and 5 d geometric frame are related by a fibrebase duality.

## 4 Fibre-base duality: 6d to 5d

The examples considered in section 3 followed a transparent pattern: the compactified 6 d theory is geometrically encoded in a collection of Hirzebruch surfaces such that their fibres encode the elliptic fibre from the F-theory construction. The geometry for the dual 5 d theory, on the other hand, has been obtained via fibre-base duality from the 6d frame. In this section, fibre-base duality between the 6 d frame and one particular 5 d frame is investigate for rank 1 and rank 2 KK theories of table 1. As it turns out, not all examples admit a $6 \mathrm{~d}-5 \mathrm{~d}$ fibre-base duality.

## $4.1 \quad 6 \mathrm{~d} \mathcal{N}=(2,0) A_{1}-5 \mathrm{~d} \mathrm{SU}(2)_{\theta=0}+1 \mathrm{Adj}$

To begin with, consider a -2 curve with an $\mathfrak{s u}(1)$ algebra

$$
\begin{gather*}
\mathfrak{s u ( 1 )}{ }_{\mathbf{2}}{ }^{(1)}  \tag{4.1}\\
n^{2}
\end{gather*}
$$

which is known as $\mathcal{N}=(2,0) A_{1}$ theory or 6 d rank- 1 M -string theory. The 5 d KK theory is known to be a $\mathrm{SU}(2)$ gauge theory with one adjoint hypermultiplet [94, 95].
$\mathbf{5 d}$ description. The prepotential (2.1) is computed by restricting to the Weyl chamber $\phi \geq 0$ and choosing, for instance, the phase

$$
\begin{equation*}
2 \phi+m \geq 0 \geq-2 \phi+m, \tag{4.2}
\end{equation*}
$$

where $m$ is the mass parameter of the adjoint hypermultiplet. The prepotential in this phase reads

$$
\begin{equation*}
6 \mathcal{F}_{5 d}=6 \frac{1}{g^{2}} \phi^{2}-6 m^{2} \phi \tag{4.3}
\end{equation*}
$$

and $g$ denotes the gauge coupling.
Geometry. As detailed in [61], the geometric description for (4.1) is given by

and one confirms that

$$
\begin{equation*}
J_{\phi}=\phi S \quad \Rightarrow \quad \mathcal{F}_{\text {trun }}=\frac{1}{3!} J_{\phi}^{3}=0 \quad \text { and } \quad f \cdot S=0 \tag{4.5}
\end{equation*}
$$

The last equation is compatible with the trivial 6d gauge algebra. In view of a 5 d gauge theory description, it is convenient to notice that

$$
\begin{equation*}
-J_{\phi} \cdot e=2 \phi=\langle\alpha, \phi\rangle \tag{4.6}
\end{equation*}
$$

such that the base $e$ is identified with the simple root $\alpha$ of $A_{1}$, see appendix A.1.2. To include the gauge theory parameter, one may parametrise the Kähler form as

$$
\begin{equation*}
\left.J\right|_{S}=\phi K_{S}+a e+b f+c x+d y \tag{4.7}
\end{equation*}
$$

and determine the coefficients as follows:

- The volume of curves with self-intersection -1 need to match the BPS masses. It is instructive to determine the truncated volume of the blowups

$$
\begin{equation*}
-J_{\phi} \cdot x=\langle 0, \phi\rangle=0 \tag{4.8}
\end{equation*}
$$

such that $x$ is identified as the trivial weight in the adjoint representation of $\operatorname{SU}(2)$. Recalling (4.6), one finds that the following -1 curves furnish the adjoint representation:

$$
\begin{align*}
& \alpha \downarrow \cdot \begin{array}{l}
\operatorname{Vol}(e+x)=\langle\alpha, \phi\rangle+m \\
\alpha \downarrow-\operatorname{Vol}(x)=\langle 0, \phi\rangle+m \\
-\operatorname{Vol}(e-x)=-(\langle-\alpha, \phi\rangle+m)
\end{array} \quad \text { and } \quad \operatorname{Vol}(e-x)=\operatorname{Vol}(e-y) \tag{4.9}
\end{align*}
$$

and the volume conditions are solved by

$$
\begin{equation*}
c=d=m, \quad b=0 . \tag{4.10}
\end{equation*}
$$

Note that these volumes motivate the choice of the phase (4.2).

- The effective gauge coupling is geometrically given by

$$
\begin{equation*}
J \cdot S \cdot S=-2 a \stackrel{!}{=} \frac{\partial^{2} \mathcal{F}}{\partial^{2} \phi}=\frac{2}{g^{2}} \quad \Rightarrow \quad a=-\frac{1}{g^{2}} \tag{4.11}
\end{equation*}
$$

and needs to match the field theory expression (2.2).
The Kähler form can now be written as

$$
\begin{equation*}
J=-\frac{1}{g^{2}} F+\phi S+m N \tag{4.12}
\end{equation*}
$$

where the non-compact surfaces $F$ and $N$ are glued to $S$ via

$$
\begin{equation*}
\left.F\right|_{S}=e \quad \text { and }\left.\quad N\right|_{S}=x+y . \tag{4.13}
\end{equation*}
$$

It is reassuring to see that the non-compact divisor $N$ associated to the mass $m$ of the adjoint hypermultiplet has gluing curves $x$ and $y$ with the compact surface $S$, as expected in [61]. In conclusion, the geometric prepotential $\mathcal{F}_{\text {geom }}=\frac{1}{3!} J^{3}$ agrees with the field theory expectation (4.3). Moreover, the basic curves have volumes given by

$$
\begin{equation*}
\operatorname{Vol}(e)=2 \phi, \quad \operatorname{Vol}(f)=\frac{1}{g^{2}}, \quad \operatorname{Vol}(x)=\operatorname{Vol}(y)=m . \tag{4.14}
\end{equation*}
$$

Here, fibre-base duality is clearly at play. The 6 d elliptic fibre $f$ has volume given by the coupling, while the fibre for the 5 d frame is $e$, because it acts as a root of $\mathfrak{s u}(2)$.

## $4.26 \mathrm{~d} \mathrm{SU}(3)+6 \mathrm{~F}$ with $\mathbb{Z}_{2}$ twist $-5 \mathrm{~d} \operatorname{Sp}(2)+3 \Lambda^{2}$

The $\mathbb{Z}_{2}$-twisted circle compactification of a -2 curve with an $\mathfrak{s u}(3)$ gauge algebra

$$
\begin{gather*}
\mathfrak{s u}(3)^{(2)}  \tag{4.15}\\
\mathbf{2}
\end{gather*}
$$

is known to be dual to a $5 \mathrm{~d} \operatorname{Sp}(2)$ gauge theory with three hypermultiplets transforming in the rank-2 anti-symmetric representation $\Lambda^{2} \equiv[0,1]_{C}$ of $\operatorname{Sp}(2)$ [28].

5d description. The 5d gauge theory description is based on the prepotential (2.1) and can be evaluated using appendix A.1.3. Since the 5 d theory has non-trivial matter content, restricting to the Weyl-chamber of $C_{2}$, i.e. $\left\langle\phi, \alpha_{i}\right\rangle \geq 0$ for $i=1,2$, is insufficient for reaching a single, well-defined 5d phase. A suitable choice of phase is given by
where $v_{i} \in[0,1]_{C}$. In the phase (4.16), the prepotential becomes

$$
\begin{align*}
6 \mathcal{F}_{5 d}= & \frac{6}{g^{2}}\left(2 \phi_{0}^{2}-2 \phi_{0} \phi_{1}+\phi_{1}^{2}\right)+8 \phi_{0}^{3}+12 \phi_{1} \phi_{0}^{2}-18 \phi_{1}^{2} \phi_{0}+5 \phi_{1}^{3} \\
& -3 \sum_{f=1}^{3} m_{f}\left(4 \phi_{0}^{2}-4 \phi_{0} \phi_{1}+\phi_{1}^{2}\right)-3 \sum_{f=1}^{3} m_{f}^{2} \phi_{1} \tag{4.17}
\end{align*}
$$

where $g$ denotes the gauge coupling and $m_{f}$ are the mass parameter of the three hypermultiplets.

Geometry. The $\mathbb{Z}_{2}$ twisted circle compactification of $\mathrm{SU}(3)$ on a -2 curve (4.15) is geometrically described by [47]

$$
\begin{equation*}
\mathbf{0}_{6} \xrightarrow{e_{0} \quad 4 e_{1}+2 f_{1}-2 \sum_{i} x_{i}} \mathbf{1}_{0}^{3} \tag{4.18}
\end{equation*}
$$

for which one computes that the fibre intersections

$$
-\left(\begin{array}{cc}
f_{0} \cdot K_{S_{0}} & \left.f_{0} \cdot S_{1}\right|_{S_{0}}  \tag{4.19}\\
\left.f_{1} \cdot S_{0}\right|_{S_{1}} & f_{1} \cdot K_{S_{1}}
\end{array}\right)=\left(\begin{array}{cc}
2 & -1 \\
-4 & 2
\end{array}\right) \equiv C_{\widehat{A}_{2}^{(2)}}
$$

yield the affine Cartan matrix of the twisted algebra $\widehat{A}_{2}^{(2)}$. Hence, (4.18) describes the 6 d duality frame of an $A_{2}$ algebra with a $\mathbb{Z}_{2}$ twist. The truncated Kähler form $J_{\phi}=\sum_{i} \phi_{i} S_{i}$ induces the following truncated prepotential

$$
\begin{equation*}
6 \mathcal{F}_{\text {trun }}=8 \phi_{0}^{3}+12 \phi_{0}^{2} \phi_{1}-18 \phi_{0} \phi_{1}^{2}+5 \phi_{1}^{3} \tag{4.20}
\end{equation*}
$$

which agrees with (4.17).
The parameters of the $5 \mathrm{~d} \operatorname{Sp}(2)$ gauge theory, one coupling $g$ and three mass parameters $m_{f}$, have to be incorporated into the Kähler form for a complete description. Before proceeding, it is useful to recall that the dual 5 d frame can be geometrically realised by an $\mathbb{F}_{0}$ isomorphism $e \leftrightarrow f$, see for instance [92, 96]. The immediate consequence is that

$$
-\left(\begin{array}{cc}
f_{0} \cdot K_{S_{0}} & \left.f_{0} \cdot S_{1}\right|_{S_{0}}  \tag{4.21}\\
\left.e_{1} \cdot S_{0}\right|_{S_{1}} & e_{1} \cdot K_{S_{1}}
\end{array}\right)=\left(\begin{array}{cc}
2 & -1 \\
-2 & 2
\end{array}\right) \equiv C_{C_{2}} \quad \Rightarrow \quad\left\{\begin{array}{l}
-J_{\phi} \cdot f_{0}=\left\langle\alpha_{1}, \phi\right\rangle \\
-J_{\phi} \cdot e_{1}=\left\langle\alpha_{2}, \phi\right\rangle
\end{array}\right.
$$

i.e. $f_{0}$ and $e_{1}$ correspond to the simple roots $\alpha_{1,2}$ of $C_{2}$, see appendix A.1.3. To begin with, the Kähler form can be extend to include all gauge theory parameters

$$
\begin{equation*}
J=-\frac{1}{g^{2}} F+\sum_{i=0}^{1} \phi_{i} S_{i}+\sum_{f=1}^{3} m_{f} N_{f} \tag{4.22}
\end{equation*}
$$

The gluing curves of the non-compact surfaces $F, N_{f}$ with the compact $S_{i}$ are determined by parametrising (4.22) restricted to the $S_{i}$ as follows:

$$
\begin{align*}
& \left.J\right|_{S_{0}}=\phi_{0} K_{S_{0}}+\left.\phi_{1} S_{1}\right|_{S_{0}}+a_{0} f_{0}+b_{0} e_{0} \\
& \left.J\right|_{S_{1}}=\left.\phi_{0} S_{0}\right|_{S_{1}}+\phi_{1} K_{S_{1}}+a_{1} f_{1}+b_{1} e_{1}+\sum_{f=1}^{3} c_{f} x_{f} \tag{4.23}
\end{align*}
$$

and the appearing parameters are fully fixed by physical requirements.

- The fundamental BPS particles need to be identified. It is helpful to evaluate the (truncated) volume of the blowups $x_{i}$

$$
\begin{equation*}
-J_{\phi} \cdot x_{f}=-2 \phi_{0}+\phi_{1}=\left\langle v_{4}, \phi\right\rangle \tag{4.24}
\end{equation*}
$$

where $v_{4}=(-2,1) \in[0,1]_{C}$, see appendix A.1.3. Recalling (4.21), the full set of $[0,1]_{C}$ weights is recovered via acting with the $C_{2}$ roots, i.e. adding $f_{0}$ or $e_{1}$ to the
blowups $x_{f}$. The resulting -1 curves have to have volumes given by the BPS masses as follows:

$$
\begin{align*}
& \alpha_{2} \downarrow \\
& \alpha_{1} \downarrow  \tag{4.25}\\
& \alpha_{1} \downarrow \\
& \left.\alpha_{2} \downarrow \downarrow \begin{array}{l}
\operatorname{Vol}\left(2 f_{0}+e_{1}+x_{f}\right) \stackrel{!}{=}\left\langle v_{1}, \phi\right\rangle+m_{f} \\
\operatorname{Vol}\left(2 f_{0}+x_{f}\right) \stackrel{!}{=}\left\langle v_{2}, \phi\right\rangle+m_{f} \\
\operatorname{Vol}\left(f_{0}+x_{f}\right) \stackrel{!}{=}\left\langle v_{3}, \phi\right\rangle+m_{f} \\
\operatorname{Vol}\left(x_{f}\right)!! \\
=
\end{array} v_{4}, \phi\right\rangle+m_{f} \\
& \\
& \operatorname{Vol}\left(e_{1}-x_{f}\right) \stackrel{!}{=}-\left(\left\langle v_{5}, \phi\right\rangle+m_{f}\right)
\end{align*} \Rightarrow\left\{\begin{array}{l}
c_{f}=m_{f}, \quad f=1,2,3, \\
a_{1}=b_{0}=0,
\end{array}\right.
$$

which in retrospect motivates the phase (4.16).

- The effective coupling from geometry (2.25) has to match the field theory expectation (2.2). The resulting linear equations are solved by

$$
\begin{equation*}
a_{0}=2 \sum_{f=1}^{3} m_{f}-2 \frac{1}{g^{2}}, \quad b_{1}=-\frac{1}{g^{2}} . \tag{4.26}
\end{equation*}
$$

As a result, the non-compact surfaces $F, N_{f}$ in (4.22) restrict to the compact surfaces as follows

$$
\left\{\begin{array} { l } 
{ F | _ { S _ { 0 } } = 2 f _ { 0 } }  \tag{4.27}\\
{ F | _ { S _ { 1 } } = e _ { 1 } }
\end{array} \quad \text { and } \quad \left\{\begin{array}{l}
N_{f} \mid S_{0}=2 f_{0} \\
\left.N_{f}\right|_{S_{1}}=x_{f}
\end{array} \quad \text { for } f=1,2,3 .\right.\right.
$$

One observes that $\left.F\right|_{S_{i}}$ restricts to $2 f_{0}$ and $e_{1}$ respectively, because $f_{0}, e_{1}$ act as simple roots (4.21). Also, the prefactors are equal to $\frac{2}{\left\langle\alpha_{i}, \alpha_{i}\right\rangle}=2,1$ for the roots of $C_{2}$.

Computing the volumes of the basic curves yields

$$
\begin{array}{ll}
\operatorname{Vol}\left(f_{0}\right)=2 \phi_{0}-\phi_{1} & \operatorname{Vol}\left(f_{1}\right)=-2 \phi_{0}+2 \phi_{1} \\
\operatorname{Vol}\left(e_{1}\right)=-4 \phi_{0}+2 \phi_{1}+\frac{1}{g^{2}} & \operatorname{Vol}\left(x_{j}\right)=-2 \phi_{0}+\phi_{1}+m_{j} \tag{4.28}
\end{array}
$$

Moreover, one can keep track of the volume of the elliptic fibre, which in the 6 d frame (4.18) is given by

$$
\begin{equation*}
f_{\text {ell }}=d_{0} f_{0}+d_{1} f_{1}=2 f_{0}+f_{1} \quad \operatorname{Vol}\left(f_{\text {ell }}\right)=\frac{1}{g^{2}} \tag{4.29}
\end{equation*}
$$

with $d_{i}=$ Coxeter numbers of $A_{2}^{(2)}$.

### 4.3 6d SU(3) +12 F with $\mathbb{Z}_{2}$ twist $-5 \mathrm{~d} G_{2}+6 \mathrm{~F}$

The 6 d theory given by a -1 curve supporting an $\mathfrak{s u}(3)$ gauge algebra has 12 fundamental hypermultiplets. Consider the $\mathbb{Z}_{2}$ twisted circle compactification

$$
\begin{equation*}
\stackrel{s u}{s u(3)}_{1}^{(2)} \tag{4.30}
\end{equation*}
$$

where $\mathbb{Z}_{2}$ acts as outer automorphism on $\mathfrak{s u}(3)$. The resulting KK theory is known to have three different 5d gauge theory descriptions; among them is a $G_{2}$ theory [47].

5d description. The $5 \mathrm{~d} G_{2}$ gauge theory with six fundamental hypermultiplets is based on the prepotential (2.1), which can be evaluated using appendix A.1.4. Since the 5 d theory has non-trivial matter content, restricting to the Weyl-chamber of $G_{2}$, i.e. $\left\langle\phi, \alpha_{i}\right\rangle \geq 0$ for $i=1,2$, is insufficient for reaching a single, well-defined 5 d phase. A suitable choice of phase is given by
where $w_{i} \in[1,0]_{G_{2}}$, see appendix A.1.4. In the phase (4.31), the prepotential becomes

$$
\begin{align*}
6 \mathcal{F}_{5 d}= & \frac{6}{g^{2}}\left(3 \phi_{1}^{2}-3 \phi_{1} \phi_{2}+\phi_{2}^{2}\right)+8 \phi_{1}^{3}-6 \phi_{2}^{2} \phi_{1}+2 \phi_{2}^{3} \\
& -3 \sum_{f=1}^{6} m_{f}\left(4 \phi_{1}^{2}-4 \phi_{1} \phi_{2}+\phi_{2}^{2}\right)-3 \sum_{f=1}^{6} m_{f}^{2} \phi_{2} \tag{4.32}
\end{align*}
$$

where $g$ is the gauge coupling and $m_{f}$ are the mass parameter.
Geometry. The geometric description for (4.30) is given by [60]

$$
\begin{equation*}
\mathbf{2}_{0}^{6} \xrightarrow{3 e+4 f-2 \sum_{i} x_{i}} \quad e \mathbf{1}_{2} \tag{4.33}
\end{equation*}
$$

and one verifies that

$$
-\left(\begin{array}{cc}
f_{1} \cdot K_{S_{1}} & \left.f_{1} \cdot S_{2}\right|_{S_{1}}  \tag{4.34}\\
\left.f_{2} \cdot S_{1}\right|_{S_{2}} & f_{2} \cdot K_{S_{2}}
\end{array}\right)=\left(\begin{array}{cc}
2 & -1 \\
-4 & 2
\end{array}\right) \equiv C_{\widehat{A}_{2}^{(2)}}
$$

yields the affine Cartan matrix of $A_{2}^{(2)}$, which is consistent with the 6 d gauge algebra $\mathfrak{s u}(3)$ accompanied by a $\mathbb{Z}_{2}$ twist.

The parameters of the $5 \mathrm{~d} G_{2}$ gauge theory, one coupling $g$ and six mass parameters $m_{f}$, have to be incorporated into the Kähler form

$$
\begin{equation*}
J=-\frac{1}{g^{2}} F+\sum_{i=1}^{2} \phi_{i} S_{i}+\sum_{f=1}^{6} m_{f} N_{f} \tag{4.35}
\end{equation*}
$$

for a complete description. Before proceeding, it is useful to recall that the dual 5 d frame can be geometrically realised by an $\mathbb{F}_{0}$ isomorphism $e \leftrightarrow f$ on $S_{2}$, see for instance [61, 92]. The immediate consequence is that

$$
-\left(\begin{array}{cc}
f_{1} \cdot K_{S_{1}} & \left.f_{1} \cdot S_{2}\right|_{S_{1}}  \tag{4.36}\\
\left.e_{2} \cdot S_{1}\right|_{S_{2}} & e_{2} \cdot K_{S_{2}}
\end{array}\right)=\left(\begin{array}{cc}
2 & -1 \\
-3 & 2
\end{array}\right) \equiv C_{G_{2}} \quad \Leftrightarrow \quad\left\{\begin{array}{l}
-J_{\phi} \cdot f_{1}=\left\langle\alpha_{1}, \phi\right\rangle \\
-J_{\phi} \cdot e_{2}=\left\langle\alpha_{2}, \phi\right\rangle
\end{array}\right.
$$

i.e. $f_{1}$ and $e_{2}$ correspond to the simple roots $\alpha_{1,2}$ of $G_{2}$, see appendix A.1.4.

The full Kähler form (4.35) restricted to the compact surfaces $S_{i}$ can be parametrised as

$$
\begin{align*}
& \left.J\right|_{S_{1}}=\phi_{1} K_{S_{1}}+\left.\phi_{2} S_{2}\right|_{S_{1}}+a_{1} e_{1}+b_{1} f_{1}, \\
& \left.J\right|_{S_{1}}=\left.\phi_{1} S_{1}\right|_{S_{2}}+\phi_{2} K_{S_{2}}+a_{2} e_{2}+b_{2} f_{2}+\sum_{f=1}^{6} c_{f} x_{f} \tag{4.37}
\end{align*}
$$

The parameters are determined as follows:

- The volume of -1 curves need to reproduce the masses of fundamental BPS particles. In detail, the truncated volume of the blowups are give by

$$
\begin{equation*}
-J_{\phi} \cdot x_{i}=-2 \phi_{1}+\phi_{2}=\left\langle w_{5}, \phi\right\rangle \tag{4.38}
\end{equation*}
$$

where $w_{5} \in[1,0]_{G_{2}}$, see appendix A.1.4. Recalling (4.36), the remaining weights can be constructed by acting with the simple roots of $G_{2}$, i.e. adding $f_{1}$ or $e_{2}$ to the blowups $x_{f}$. The resulting -1 curves need to have volumes that match the BPS masses as follows:

$$
\begin{align*}
& \alpha_{1} \downarrow \\
& \alpha_{2} \downarrow \\
& \alpha_{1} \downarrow \\
& \alpha_{1} \downarrow  \tag{4.39}\\
& \alpha_{2} \downarrow-\left[\begin{array}{l}
\operatorname{Vol}\left(3 f_{1}+e_{2}+x_{f}\right)=\left\langle w_{1}, \phi\right\rangle+m_{f} \\
\operatorname{Vol}\left(2 f_{1}+e_{2}+x_{f}\right)=\left\langle w_{2}, \phi\right\rangle+m_{f} \\
\operatorname{Vol}\left(2 f_{1}+x_{f}\right)=\left\langle w_{3}, \phi\right\rangle+m_{f} \\
\operatorname{Vol}\left(f_{1}+x_{f}\right)=\left\langle w_{4}, \phi\right\rangle+m_{f} \\
\operatorname{Vol}\left(x_{f}\right)=\left\langle w_{5}, \phi\right\rangle+m_{f} \\
\alpha_{1} \downarrow \\
\alpha_{1} \downarrow \\
\operatorname{Vol}\left(e_{2}-x_{f}\right)=-\left(\left\langle w_{6}, \phi\right\rangle+m_{f}\right) \\
\operatorname{Vol}\left(f_{1}+e_{2}-x_{f}\right)=-\left(\left\langle w_{7}, \phi\right\rangle+m_{f}\right)
\end{array}\right.
\end{align*} \quad \Rightarrow\left\{\begin{array}{l}
c_{f}=M_{f}, \quad f=1, \ldots, 6, \\
a_{1}=b_{2}=0 .
\end{array},\right.
$$

Note that this identification motivates the phase (4.31).

- The geometric effective coupling (2.25) needs to match the field theory expectation (2.2). The arising linear equations are solved by

$$
\begin{equation*}
b_{1}=-\frac{3}{g^{2}}+2 \sum_{f=1}^{6} m_{f}, \quad a_{2}=-\frac{1}{g^{2}} . \tag{4.40}
\end{equation*}
$$

As a result, the non-compact surfaces $F$ and $N_{f}$ in the Kähler form (4.35) are glued to the compact surfaces as follows:

$$
\left.F\right|_{S_{i}}=\left\{\left.\begin{array}{ll}
3 f_{1}, & i=1,  \tag{4.41}\\
e_{2}, & i=2,
\end{array} \quad N_{f} \right\rvert\, S_{i}= \begin{cases}2 f_{1}, & i=1, \\
x_{f}, & i=2 .\end{cases}\right.
$$

An immediate observation is that $\left.F\right|_{S_{i}}$ restricts to $3 f_{1}$ and $e_{2}$ respectively, because $f_{1}, e_{2}$ act as simple roots (4.36). Also, the prefactors are equal to $\frac{2}{\left\langle\alpha_{i}, \alpha_{i}\right\rangle}=3,1$ for the roots of $G_{2}$.

Moreover, one can keep track of the volume of the elliptic fibre, which in the 6 d frame (4.33) is given by

$$
\begin{equation*}
f_{\text {ell }}=d_{0} f_{1}+d_{1} f_{2}=2 f_{1}+f_{2} \quad \operatorname{Vol}\left(f_{\text {ell }}\right)=\frac{1}{g^{2}} \tag{4.42}
\end{equation*}
$$

with $d_{i}=$ Coxeter numbers of $A_{2}^{(2)}$.

## $4.46 \mathrm{dSp}(1)+10 \mathrm{~F}-5 \mathrm{~d} \operatorname{Sp}(2)+10 \mathrm{~F}$

The 6 d theory corresponding to a -1 curves with an $\mathfrak{s p}(1)$ gauge algebra has 10 fundamental hypermultiplets. The untwisted circle compactification of

$$
\begin{equation*}
{ }_{1}^{\mathfrak{s p}(1)^{(1)}} \tag{4.43}
\end{equation*}
$$

is known to have two 5 d gauge theory descriptions [25]: $\mathrm{SU}(3)_{0}$ with 10 fundamentals as well as $\operatorname{Sp}(2)$ with 10 fundamental hypermultiplets. In this section, the $5 \mathrm{~d} \operatorname{Sp}(2)$ theory is considered, because its geometric description is related to the 6d theory via fibre-base duality.
$\mathbf{5 d}$ description. The 5 d gauge theory description is based on the prepotential (2.1), which can be evaluated via appendix A.1.3. Besides restricting to the Weyl-chamber of $C_{2}$, one needs to choose a suitable phase, such as

$$
\begin{gather*}
\alpha_{1} \downarrow  \tag{4.44}\\
\alpha_{2} \downarrow- \\
\alpha_{1} \downarrow \\
\bullet
\end{gather*} \begin{aligned}
& \left\langle\phi, w_{1}\right\rangle+m_{f} \geq 0 \\
& \left\langle\phi, w_{2}\right\rangle+m_{f} \geq 0 \\
& \left\langle\phi, w_{3}\right\rangle+m_{f} \leq 0 \\
& \left\langle\phi, w_{4}\right\rangle+m_{f} \leq 0
\end{aligned}
$$

where $w_{i} \in[1,0]_{C}$, see appendix A.1.3. The prepotential becomes

$$
\begin{equation*}
6 \mathcal{F}_{5 d}=\frac{6}{g^{2}}\left(2 \phi_{1}^{2}-2 \phi_{1} \phi_{2}+\phi_{2}^{2}\right)+8 \phi_{1}^{3}-18 \phi_{1}^{2} \phi_{2}+12 \phi_{1} \phi_{2}^{2}-2 \phi_{2}^{3}-3 \sum_{f=1}^{10} m_{f} \phi_{1}^{2} \tag{4.45}
\end{equation*}
$$

with $g$ the gauge coupling, and $m_{f}$ the mass parameters of the fundamental hypermultiplets.

Geometry. Geometrically, the theory (4.43) is described by [61]

$$
\begin{equation*}
\mathbf{1}_{0} \xlongequal{2 e_{1}+f_{1}} 2 h_{2}-\sum_{i=1}^{10} x_{i} \mathbf{2}_{1}^{10} \tag{4.46}
\end{equation*}
$$

and one verifies straightforwardly that

$$
-\left(\begin{array}{cc}
f_{1} \cdot K_{S_{1}} & f_{1} \cdot S_{2} \mid S_{1}  \tag{4.47}\\
f_{2} \cdot S_{1} \mid S_{2} & f_{2} \cdot K_{S_{2}}
\end{array}\right)=\left(\begin{array}{cc}
2 & -2 \\
-2 & 2
\end{array}\right) \equiv C_{\widetilde{A}_{1}^{(1)}}
$$

which is consistent with the 6 d gauge algebra $A_{1} \cong C_{1}$ without any twist.
In view of the 5 d description, a useful observation on the geometry (4.46) is

$$
-\left(\begin{array}{cc}
e_{1} \cdot K_{S_{1}} & \left.e_{1} \cdot S_{2}\right|_{S_{1}}  \tag{4.48}\\
f_{2} \cdot S_{1} \mid S_{2} & f_{2} \cdot K_{S_{2}}
\end{array}\right)=\left(\begin{array}{cc}
2 & -1 \\
-2 & 2
\end{array}\right) \equiv C_{C_{2}} \quad \Rightarrow \quad \begin{cases}-J_{\phi} \cdot e_{1} & =\left\langle\alpha_{1}, \phi\right\rangle \\
-J_{\phi} \cdot f_{2} & =\left\langle\alpha_{2}, \phi\right\rangle\end{cases}
$$

which identifies $e_{1}, f_{2}$ as the simple roots $\alpha_{i}$ of the $5 \mathrm{~d} \operatorname{Sp}(2)$ gauge group, see appendix A.1.3.

In order the include all gauge theory parameters, one gauge coupling $g$ and ten mass parameters $m_{i}$, one may parametrise the Kähler form as

$$
\begin{equation*}
J=-\frac{1}{g^{2}} F+\sum_{i=1}^{2} \phi_{i} S_{i}+\sum_{f=1}^{10} \frac{m_{f}}{2} N_{f} \tag{4.49}
\end{equation*}
$$

The gluing curves of the non-compact $F, N_{f}$ with the compact $S_{i}$ are determined by parametrising the restrictions of (4.49) via

$$
\begin{align*}
& \left.J\right|_{S_{1}}=\phi_{1} K_{S_{1}}+\left.\phi_{2} S_{2}\right|_{S_{1}}+a_{1} e_{1}+b_{1} f_{1} \\
& \left.J\right|_{S_{2}}=\left.\phi_{1} S_{1}\right|_{S_{2}}+\phi_{2} K_{S_{2}}+a_{2} e_{2}+b_{2} f_{2}+\sum_{f=1}^{10} c_{f} x_{f} \tag{4.50}
\end{align*}
$$

The parameters are determined as follows:

- The volume of -1 curves should match the BPS masses. It is instructive to consider the blowups $x_{i}$

$$
\begin{equation*}
-J_{\phi} \cdot x_{f}=-\phi_{1}+\phi_{2}=\left\langle w_{2}, \phi\right\rangle, \tag{4.51}
\end{equation*}
$$

where $w_{2} \in[1,0]_{C}$, see appendix A.1.3. Recalling (4.48), the remaining weights are realised by adding $e_{1}$ or $f_{2}$ to the blowups. The volumes of the resulting -1 curves are required to match the BPS masses

$$
\begin{aligned}
& \alpha_{1} \downarrow \\
& \alpha_{2} \downarrow-\left[\begin{array} { l } 
{ \operatorname { V o l } ( e _ { 1 } + x _ { f } ) \stackrel { ! } { = } \langle w _ { 1 } , \phi \rangle + m _ { f } } \\
{ \operatorname { V o l } ( x _ { f } ) \stackrel { ! } { = } \langle w _ { 2 } , \phi \rangle + m _ { f } } \\
{ \alpha _ { 1 } \downarrow } \\
{ \operatorname { V o l } ( f _ { 2 } - x _ { f } ) \stackrel { ! } { = } - ( \langle w _ { 3 } , \phi \rangle + m _ { f } ) } \\
{ \operatorname { V o l } ( e _ { 1 } + f _ { 2 } - x _ { f } ) \stackrel { ! } { = } - ( \langle w _ { 4 } , \phi \rangle + m _ { f } ) }
\end{array} \quad \Rightarrow \left\{\begin{array}{l}
c_{f}=m_{f}, \quad f=1, \ldots, 10, \\
b_{1}=a_{2}=0 .
\end{array}, \quad\right.\right. \text { (4.52) }
\end{aligned}
$$

Again, the volume the -1 define the phase of the theory, which motivates (4.44).

- The geometric effective coupling (2.25) needs to match the field theory expectation (2.2). The arising linear equations are solved by

$$
\begin{equation*}
a_{1}=-\frac{2}{g^{2}}, \quad b_{2}=-\frac{1}{g^{2}}-\frac{1}{2} \sum_{f=1}^{10} m_{f} \tag{4.53}
\end{equation*}
$$

In summary, the non-compact surfaces $F$ and $N_{f}$ in (4.49) are glued to the $S_{i}$ as follows:

$$
\left.F\right|_{S_{i}}=\left\{\left.\begin{array}{ll}
2 e_{1}, & i=1,  \tag{4.54}\\
f_{2}, & i=2,
\end{array} \quad N_{f}\right|_{S_{i}}= \begin{cases}0, & i=1 \\
2 x_{f}-f_{2}, & i=2\end{cases}\right.
$$

Again, the $\left.F\right|_{S_{i}}$ restrict to $2 e_{1}$, and $f_{2}$ respectively, because the $e_{1}, f_{2}$ act as simple roots (4.48). The prefactors equal $\frac{2}{\left\langle\alpha_{i}, \alpha_{i}\right\rangle}=2,1$ for the roots of $C_{2}$. In addition, one may keep track of the elliptic fibre associated to (4.46) and compute its volume to be

$$
\begin{equation*}
\operatorname{Vol}\left(f_{\mathrm{ell}}\right)=\operatorname{Vol}\left(f_{1}\right)+\operatorname{Vol}\left(f_{2}\right)=\frac{2}{g^{2}} . \tag{4.55}
\end{equation*}
$$

### 4.5 6d Rank-2 E-string - 5d Sp(2) $+1 \Lambda^{2}+8 \mathrm{~F}$

The rank-2 E-string is given by a -1 curve with an $\mathfrak{s p}(0)$ algebra intersecting a -2 curve with a trivial $\mathfrak{s u}(1)$ algebra

$$
\begin{array}{cc}
\mathfrak{s p}(0)^{(1)} & \mathfrak{s u}(1)^{(1)}  \tag{4.56}\\
\mathbf{1} & \mathbf{2}
\end{array}
$$

The 5 d reduction has two gauge theory descriptions [16]: $\mathrm{SU}(3)_{\frac{3}{2}}$ with nine fundamentals and $\mathrm{Sp}(2)$ with one rank-2 anti-symmetric and eight fundamental hypermultiplets. In this section, the $\mathrm{Sp}(2)$ theory is consider because of the natural fibre-base duality to the 6d frame.

5d description. The 5d gauge theory description is based on the prepotential (2.1), which is evaluated via the details provided in appendix A.1.3. Since the 5 d theory has non-trivial matter content, restricting to the Weyl-chamber of $C_{2}$, i.e. $\left\langle\phi, \alpha_{i}\right\rangle \geq 0$ for $i=1,2$, is insufficient for reaching a single, well-defined 5 d phase. A suitable choice of phase is given by

$$
\begin{align*}
& \begin{array}{c}
\alpha_{1} \downarrow \\
\alpha_{2} \downarrow \downarrow-\begin{array}{l}
\left\langle\phi, w_{1}\right\rangle+m_{f} \geq 0 \\
\alpha_{1} \downarrow \\
\end{array} \cdot \begin{array}{l}
\left\langle\phi, w_{2}\right\rangle+m_{f} \geq 0 \\
\\
\left\langle\phi, w_{3}\right\rangle+m_{f} \leq 0 \\
\text { choice }
\end{array} \\
\left\langle\phi, w_{4}\right\rangle+m_{f} \leq 0
\end{array} \tag{4.57}
\end{align*}
$$

with $w_{i} \in[1,0]_{C}$ and $v_{i} \in[0,1]_{C}$, see appendix A.1.3. In the phase (4.57), the prepotential becomes

$$
\begin{equation*}
6 \mathcal{F}_{5 d}=\frac{6}{g^{2}}\left(2 \phi_{0}^{2}-2 \phi_{0} \phi_{1}+\phi_{1}^{2}\right)-3 \sum_{f=1}^{8} m_{f}^{2} \phi_{1}-6 M^{2} \phi_{0}, \tag{4.58}
\end{equation*}
$$

where $g$ denotes the gauge coupling, $m_{f}$ are the mass parameters of the fundamental hypermultiplets, and $M$ is the mass parameter of the $\Lambda^{2}$ hypermultiplet.

Geometry. Utilising [61], the geometry from 6 d setup (4.56) is given by

and one computes immediately

$$
-\left(\begin{array}{cc}
e_{0} \cdot K_{S_{1}} & e_{0} \cdot S_{2} \mid S_{1}  \tag{4.60}\\
f_{1} \cdot S_{1} \mid S_{2} & f_{1} \cdot K_{S_{2}}
\end{array}\right)=\left(\begin{array}{cc}
2 & -1 \\
-2 & 2
\end{array}\right) \equiv C_{C_{2}} \quad \Rightarrow \quad \begin{cases}-J_{\phi} \cdot e_{0} & =\left\langle\alpha_{1}, \phi\right\rangle \\
-J_{\phi} \cdot f_{1} & =\left\langle\alpha_{2}, \phi\right\rangle\end{cases}
$$

which identifies the Cartan matrix of the $5 \mathrm{~d} \operatorname{Sp}(2)$ gauge group. Moreover, $\left\{e_{0}, f_{1}\right\}$ act as simple roots $\alpha_{i}$ of $C_{2}$, see appendix A.1.3. On the other hand, The $6 d$ gauge algebras are trivial on both nodes of (4.56). This piece of information is encoded in the elliptic fibre $f_{\text {ell }}$ of the Calabi-Yau. One can readily confirm it by noticing that

$$
\begin{equation*}
-f_{0} \cdot S_{i}=K_{S_{1}} \cdot S_{i}=0, \tag{4.61}
\end{equation*}
$$

where $K_{S_{1}}=-2 e_{1}-3 f_{1}-\sum_{i=1}^{8} x_{i}$ is the canonical Kähler form of $\mathbb{F}_{1}^{8}$, as well as the gluing curve identified with $f_{0}$ in the geometry (4.59). Therefore the elliptic fibre is given by

$$
\begin{equation*}
f_{\mathrm{ell}}=f_{0}=-K_{S_{1}} . \tag{4.62}
\end{equation*}
$$

The full Kähler form

$$
\begin{equation*}
J=-\frac{1}{g^{2}} F+\sum_{i=0}^{1} \phi_{i} S_{i}+\sum_{f=1}^{8} \frac{m_{f}}{2} N_{f}^{F}+M N^{\Lambda} \tag{4.63}
\end{equation*}
$$

includes all gauge theory parameter $g, m_{f}$, and $M$. To derive the gluing curves of the non-compact surfaces with the compact surfaces, one parametrises (4.63) as follows:

$$
\begin{align*}
& \left.J\right|_{S_{0}}=\phi_{0} K_{S_{0}}+\left.\phi_{1} S_{1}\right|_{S_{0}}+a_{0} e_{0}+b_{0} f_{0}+d_{w} w+d_{z} z, \\
& \left.J\right|_{S_{1}}=\left.\phi_{0} S_{0}\right|_{S_{1}}+\phi_{1} K_{S_{1}}+a_{1} e_{1}+b_{1} f_{1}+\sum_{f=1}^{8} c_{f} x_{f} . \tag{4.64}
\end{align*}
$$

The parameters are determined as follows:

- The volumes of -1 curves should reproduce the masses of fundamental BPS particles. To get some insight, one computes the truncated volume of the blowups

$$
\begin{align*}
-J_{\phi} \cdot x_{f} & =-\phi_{0}+\phi_{1}=\left\langle w_{2}, \phi\right\rangle  \tag{4.65a}\\
-J_{\phi} \cdot w & =-J_{\phi} \cdot z=0=\left\langle v_{3}, \phi\right\rangle \tag{4.65b}
\end{align*}
$$

where $w_{2} \in[1,0]_{C}$ and $v_{3} \in[0,1]_{C}$, see appendix A.1.3. Recalling (4.60), one constructs the remaining -1 curves that furnish the representations $[1,0]_{C}$ and $[0,1]_{C}$ by adding the base $e_{0}$ and fibre $f_{1}$ to the blowups. The volumes of the resulting curves need to satisfy
such that one finds

$$
\begin{equation*}
b_{0}=a_{1}=0, \quad c_{f}=m_{f}, \quad f=1, \ldots 8, \quad d_{w}=d_{z}=M \tag{4.67}
\end{equation*}
$$

The volumes of these -1 curves define the phase of the theory, which motivates (4.57)

- The geometric effective coupling (2.25) needs to match the field theory expectation (2.2). The arising linear equations are solved by

$$
\begin{equation*}
a_{0}=-\frac{2}{g^{2}}, \quad b_{1}=-\frac{1}{g^{2}}-\frac{1}{2} \sum_{f=1}^{8} m_{f} \tag{4.68}
\end{equation*}
$$

As a result, the gluing curves of the non-compact surfaces $F, N_{f}^{F}$, and $N^{\Lambda}$, in (4.63) with the compact surfaces $S_{i}$ are as follows:

$$
\left.F\right|_{S_{i}}=\left\{\left.\begin{array}{ll}
2 e_{0}, & i=0,  \tag{4.69}\\
f_{1}, & i=1,
\end{array} \quad N_{f}^{F}\right|_{S_{i}}=\left\{\left.\begin{array}{ll}
0, & i=0, \\
2 x_{f}-f_{1}, & i=1,
\end{array} \quad N^{\Lambda}\right|_{S_{i}}= \begin{cases}w+z, & i=0 \\
0, & i=1\end{cases}\right.\right.
$$

An immediate observation is that $\left.F\right|_{S_{i}}$ restrict to $2 e_{0}, f_{1}$ respectively, because $\left\{e_{0}, f_{1}\right\}$ act as simple roots (4.60). The prefactors equal $\frac{2}{\left\langle\alpha_{i}, \alpha_{i}\right\rangle}=2,1$ for the roots of $C_{2}$. Moreover, for the $N_{f}^{F}$ divisors restricted to $S_{1}$, the volumes of curves $2 x_{f}-f_{1}$ have positive volumes $\operatorname{Vol}\left(2 x_{f}-f_{1}\right)=m_{f}$, implying the curves are effective. In addition, one may keep track of the elliptic fibre associated to (4.59) and compute its volume to be

$$
\begin{equation*}
\operatorname{Vol}\left(f_{\mathrm{ell}}\right)=\operatorname{Vol}\left(f_{0}\right)=\operatorname{Vol}\left(-K_{S_{1}}\right)=\frac{2}{g^{2}} \tag{4.70}
\end{equation*}
$$

The factor of 2 would be due to the node of -1 curve in (4.56), which has been also observed in rank-1 E-string, see (3.12).

## $4.6 \quad 6 \mathrm{~d} \mathcal{N}=(2,0) A_{\mathbf{2}}-5 \mathrm{~d} \mathrm{SU}(3)_{0}+1 \mathrm{Adj}$

The $6 \mathrm{~d} \mathcal{N}=(2,0) A_{2}$ theory is given by

$$
\begin{equation*}
\mathbf{2}^{\mathfrak{s u}(1)^{(1)}}-\mathbf{s u}^{(1)^{(1)}} \tag{4.71}
\end{equation*}
$$

and the untwisted circle reduction is known to be $\mathrm{SU}(3)_{0}$ with one hypermultiplet in the adjoint representation [94, 95].
$\mathbf{5 d}$ description. Since the 5 d KK descriptions is known, the prepotential (2.1) is derived using appendix A.1.2. In addition to restricting to the Weyl-chamber of $A_{2}$, i.e. $\left\langle\phi, \alpha_{i}\right\rangle \geq 0$ for $i=1,2$, the following phase is chosen
with $\alpha_{1,2}$ the simple roots of $A_{2}$, see appendix A.1.2. The prepotential becomes

$$
\begin{equation*}
6 \mathcal{F}_{5 d}=\frac{6}{g^{2}}\left(\phi_{0}^{2}-\phi_{0} \phi_{1}+\phi_{1}^{2}\right)-6 m^{2}\left(\phi_{0}+\phi_{1}\right), \tag{4.73}
\end{equation*}
$$

with $g$ the gauge coupling and $m$ the mass parameter of the adjoint hypermultiplet.
Geometry. Utilising [61], the geometry from 6 d setup (4.71) is given by

such that one readily confirms that

$$
\begin{equation*}
-f_{i} \cdot S_{j}=0, \tag{4.75}
\end{equation*}
$$

which is reflecting the fact that the 6 d gauge algebras are trivial. In order to describe 5 d $\mathrm{SU}(3)$ gauge theory via (4.74), it is useful to observe that

$$
-\left(\begin{array}{cc}
e_{0} \cdot K_{S_{0}} & e_{0} \cdot S_{1} \mid S_{0}  \tag{4.76}\\
e_{1} \cdot S_{0} \mid S_{1} & e_{1} \cdot K_{S_{1}}
\end{array}\right)=\left(\begin{array}{cc}
2 & -1 \\
-1 & 2
\end{array}\right) \equiv C_{A_{2}} \quad \Rightarrow \quad \begin{cases}-J_{\phi} \cdot e_{0} & =\left\langle\alpha_{1}, \phi\right\rangle \\
-J_{\phi} \cdot e_{1} & =\left\langle\alpha_{2}, \phi\right\rangle\end{cases}
$$

which identifies the $e_{0,1}$ as the simple roots $\alpha_{i}$ of the $5 \mathrm{~d} \mathfrak{s u}(3)$ gauge algebra, see appendix A.1.2. As above, the two gauge theory parameter, one gauge coupling $g$ and one mass parameter, are introduced by suitably parametrising the Kähler form:

$$
\begin{equation*}
J=-\frac{1}{g^{2}} F+\sum_{i=0}^{1} \phi_{i} S_{i}+m N . \tag{4.77}
\end{equation*}
$$

The intersections of the non-compact $F, N$ with the compact $S_{i}$ are determined by restricting (4.77) as

$$
\begin{align*}
& \left.J\right|_{S_{0}}=\phi_{0} K_{S_{0}}+\left.\phi_{1} S_{1}\right|_{S_{0}}+a_{0} e_{0}+b_{0} f_{0}+c_{x} x+c_{y} y,  \tag{4.78}\\
& \left.J\right|_{S_{1}}=\left.\phi_{0} S_{0}\right|_{S_{1}}+\phi_{1} K_{S_{1}}+a_{1} e_{1}+b_{1} f_{1}+c_{z} z+c_{w} w .
\end{align*}
$$

and determining the parameter as follows:

- Firstly, one identifies the -1 curves that give rise to the fundamental BPS particles. It is instructive to evaluate the volumes of the blowups; in detail,

$$
\begin{equation*}
-J_{\phi} \cdot x=-J_{\phi} \cdot y=-J_{\phi} \cdot z=-J_{\phi} \cdot w=0=\langle 0, \phi\rangle \tag{4.79}
\end{equation*}
$$

where 0 is a trivial weight in $[1,1]_{A}$, see appendix A.1.2. Recalling (4.76), the remaining weights are realised by the following -1 curves:
and volumes have to match the BPS masses. Additionally, the self-gluings in (4.74) impose

$$
\begin{equation*}
\operatorname{Vol}\left(e_{0}-x\right)=\operatorname{Vol}\left(e_{0}-y\right) \quad \text { and } \quad \operatorname{Vol}\left(e_{1}-z\right)=\operatorname{Vol}\left(e_{1}-w\right) \tag{4.81}
\end{equation*}
$$

such that one finds

$$
\begin{equation*}
c_{I}=m, \quad I=x, y, z, w, \quad b_{i}=0, \quad i=0,1 . \tag{4.82}
\end{equation*}
$$

One notes that the choices of volumes motivate the choice of phase (4.72).

- Secondly, the geometric effective gauge coupling (2.25) needs to match the field theory expectation (2.2). The arising linear equations are solved by

$$
\begin{equation*}
a_{0}=a_{1}=-\frac{1}{g^{2}} . \tag{4.83}
\end{equation*}
$$

Consequently, the non-compact surfaces $F, N$ in (4.77) restrict to the compact $S_{i}$ as follows:

$$
\left.F\right|_{S_{i}}=e_{i},\left.\quad N\right|_{S_{i}}= \begin{cases}x+y, & i=0,  \tag{4.84}\\ z+w, & i=1 .\end{cases}
$$

One may notice, as above, that the $\left.F\right|_{S_{i}}$ restrict to the $e_{i}$ respectively, because the $e_{i}$ act as simple roots (4.76). The prefactor of 1 equals $\frac{2}{\left\langle\alpha_{i}, \alpha_{i}\right\rangle}$ for the roots of $A_{2}$. Moreover, the volumes of the fibres are given by

$$
\begin{equation*}
\operatorname{Vol}\left(f_{i}\right)=\operatorname{Vol}\left(f_{\text {ell }}\right)=\frac{1}{g^{2}}, \quad i=0,1, \tag{4.85}
\end{equation*}
$$

which identifies the volume of the elliptic fibre.

## $4.76 \mathrm{~d} \mathcal{N}=(2,0) A_{3}$ with twist $-5 \mathrm{~d} \operatorname{Sp}(2)_{0}+1 \mathrm{Adj}$

Starting from the $6 \mathrm{~d} \mathcal{N}=(2,0) A_{3}$ theory, the twist by the permutation symmetry

leads to the 5 d KK theory described by $\mathrm{Sp}(2)_{0}$ with one adjoint hypermultiplet [97]. Although the theory is known to be non-geometric, it is nevertheless interesting to explore the possibility of fibre-base duality.
$\mathbf{5 d}$ description. Since the 5 d KK descriptions is known, the prepotential (2.1) is derived using appendix A.1.3. In addition to restricting to the Weyl-chamber of $C_{2}$, i.e. $\left\langle\phi, \alpha_{i}\right\rangle \geq 0$ for $i=1,2$, the following phase is chosen
with $\alpha_{1,2}$ the simple roots of $C_{2}$, see appendix A.1.3. The prepotential becomes

$$
\begin{equation*}
6 \mathcal{F}_{5 d}=\frac{6}{g^{2}}\left(2 \phi_{0}^{2}-2 \phi_{0} \phi_{1}+\phi_{1}^{2}\right)-6 m^{2}\left(\phi_{0}+\phi_{1}\right), \tag{4.88}
\end{equation*}
$$

with $g$ the gauge coupling and $m$ the mass parameter of the adjoint hypermultiplet.
Geometry. Following [61], the geometry derived from the 6 d setup (4.86) is given by

such that one readily confirms that

$$
\begin{equation*}
-f_{i} \cdot S_{j}=0 \tag{4.90}
\end{equation*}
$$

which is reflecting the fact that the 6 d gauge algebras are trivial.
For the $5 \mathrm{~d} \mathrm{Sp}(2)$ gauge theory, it is helpful to observe that

$$
-\left(\begin{array}{cc}
e_{0} \cdot K_{S_{0}} & e_{0} \cdot S_{1} \mid S_{0}  \tag{4.91}\\
e_{1} \cdot S_{0} \mid S_{1} & e_{1} \cdot K_{S_{1}}
\end{array}\right)=\left(\begin{array}{cc}
2 & -1 \\
-2 & 2
\end{array}\right) \equiv C_{C_{2}} \quad \Rightarrow \quad \begin{cases}-J_{\phi} \cdot e_{0} & =\left\langle\alpha_{1}, \phi\right\rangle \\
-J_{\phi} \cdot e_{1} & =\left\langle\alpha_{2}, \phi\right\rangle\end{cases}
$$

which identifies the $e_{0,1}$ as the simple roots $\alpha_{1,2}$ of the $5 \mathrm{~d} \mathfrak{s p}(2)$ gauge algebra, see appendix A.1.3. As above, the two gauge theory parameter, one gauge coupling $g$ and one mass parameter, can be included in the Kähler form:

$$
\begin{equation*}
J=-\frac{1}{g^{2}} F+\sum_{i=0}^{1} \phi_{i} S_{i}+m N . \tag{4.92}
\end{equation*}
$$

The gluing curves of the $F, N$ with the compact $S_{i}$ can be determined by parametrising (4.92) restricted to the $S_{i}$ as

$$
\begin{align*}
& \left.J\right|_{S_{0}}=\phi_{0} K_{S_{0}}+\phi_{1} S_{1} \mid S_{0}+a_{0} e_{0}+b_{0} f_{0}+c_{x} x+c_{y} y,  \tag{4.93}\\
& \left.J\right|_{S_{1}}=\left.\phi_{0} S_{0}\right|_{S_{1}}+\phi_{1} K_{S_{1}}+a_{1} e_{1}+b_{1} f_{1}+c_{z} z+c_{w} w .
\end{align*}
$$

The parameters are determined as follows:

- Firstly, one identifies the -1 curves that give rise to the fundamental BPS particles. It is instructive to evaluate the volumes of the blowups; in detail,

$$
\begin{equation*}
-J_{\phi} \cdot x=-J_{\phi} \cdot y=-J_{\phi} \cdot z=-J_{\phi} \cdot w=0=\langle 0, \phi\rangle \tag{4.94}
\end{equation*}
$$

where 0 is a trivial weight in $[2,0]_{C}$, see appendix A.1.3. Recalling (4.91), the remaining weights are realised by the following -1 curves:
and the volumes are required to match the BPS masses. The self-gluings in (4.89) impose

$$
\begin{equation*}
\operatorname{Vol}\left(e_{0}-x\right)=\operatorname{Vol}\left(e_{0}-y\right) \quad \text { and } \quad \operatorname{Vol}\left(e_{1}-z\right)=\operatorname{Vol}\left(e_{1}-w\right) \tag{4.96}
\end{equation*}
$$

such that one finds

$$
\begin{equation*}
c_{I}=m, \quad I=x, y, z, w, \quad b_{i}=0, \quad i=0,1 . \tag{4.97}
\end{equation*}
$$

This identification of volumes motivates the choice of phase (4.87).

- Secondly, the geometric effective gauge coupling (2.25) needs to match the field theory expectation (2.2). The arising linear equations are solved by

$$
\begin{equation*}
a_{0}=-\frac{2}{g^{2}}, \quad a_{1}=-\frac{1}{g^{2}} . \tag{4.98}
\end{equation*}
$$

As a result, the non-compact surfaces $F, N$ in (4.92) restrict to the compact $S_{i}$ ones as follows:

$$
\left.F\right|_{S_{i}}=\left\{\begin{array}{ll}
2 e_{0}, & i=0,  \tag{4.99}\\
e_{1}, & i=1,
\end{array},\left.\quad N\right|_{S_{i}}= \begin{cases}x+y, & i=0 \\
z+w, & i=1\end{cases}\right.
$$

One realises that $\left.F\right|_{S_{i}}$ restrict to $2 e_{0}$ and $e_{1}$ respectively, because the $e_{i}$ act as simple roots (4.91). The prefactors equal $\frac{2}{\left\langle\alpha_{i}, \alpha_{i}\right\rangle}=2,1$ for the roots of $C_{2}$. Moreover, the volumes of the fibres are given by

$$
\left\{\begin{array}{l}
\operatorname{Vol}\left(f_{0}\right)=\frac{2}{g^{2}},  \tag{4.100}\\
\operatorname{Vol}\left(f_{1}\right)=\frac{1}{g^{2}},
\end{array} \quad \text { such that } \quad \operatorname{Vol}\left(f_{\text {ell }}\right)=\frac{1}{g^{2}}\right.
$$

which yields the volume of the elliptic fibre. The factor 2 is due to the permutation twist in (4.86).

## $4.86 \mathrm{~d} \mathcal{N}=(2,0) D_{4}$ with twist $-5 d G_{2}+1 \mathrm{Adj}$

The $6 \mathrm{~d} \mathcal{N}=(2,0) D_{4}$ theory is geometrically realised by four -2 curves, supporting trivial $\mathfrak{s u}(1)$ algebras, which intersect in the pattern of the $D_{4}$ Dynkin diagram. The circle reduction twisted by permutations on the tensor multiplets

is known to admit a 5 d descriptions as $G_{2}$ with one adjoint hypermultiplet [97].

5d description. Since the 5 d KK descriptions is known, the prepotential (2.1) is derived using appendix A.1.4. In addition to restricting to the Weyl-chamber of $G_{2}$, i.e. $\left\langle\phi, \alpha_{i}\right\rangle \geq 0$ for $i=1,2$, the following phase is chosen

$$
\begin{equation*}
\left\langle\phi, \alpha_{2}\right\rangle+m_{f} \geq 0 \underbrace{}_{\downarrow} \tag{4.102}
\end{equation*}
$$

with $\alpha_{1,2}$ the simple roots of $G_{2}$, see appendix A.1.4. The prepotential becomes

$$
\begin{equation*}
6 \mathcal{F}_{5 d}=\frac{6}{g^{2}}\left(3 \phi_{0}^{2}-3 \phi_{0} \phi_{1}+\phi_{1}^{2}\right)-6 m^{2}\left(\phi_{0}+\phi_{1}\right) \tag{4.103}
\end{equation*}
$$

with $g$ the gauge coupling and $m$ the mass parameter of the adjoint hypermultiplet.
Geometry. The theory (4.101) is geometrically realised by [61]
and due to the self-gluing curves, the truncated prepotential is trivially zero. Moreover, one verifies

$$
\begin{equation*}
-f_{i} \cdot S_{j}=0 \tag{4.105}
\end{equation*}
$$

which is reflecting the fact that the 6d gauge algebras are trivial.
For the $5 \mathrm{~d} G_{2}$ gauge theory, it is instructive to note that

$$
-\left(\begin{array}{cc}
e_{0} \cdot K_{S_{0}} & \left.e_{0} \cdot S_{1}\right|_{S_{0}}  \tag{4.106}\\
\left.e_{1} \cdot S_{0}\right|_{S_{1}} & e_{1} \cdot K_{S_{1}}
\end{array}\right)=\left(\begin{array}{cc}
2 & -1 \\
-3 & 2
\end{array}\right) \equiv C_{G_{2}} \quad \Rightarrow \quad \begin{cases}-J_{\phi} \cdot e_{0} & =\left\langle\alpha_{1}, \phi\right\rangle \\
-J_{\phi} \cdot e_{1} & =\left\langle\alpha_{2}, \phi\right\rangle\end{cases}
$$

which identifies the $e_{0,1}$ as the simple roots $\alpha_{1,2}$ of the $5 \mathrm{~d} G_{2}$ gauge algebra, see appendix A.1.4. As above, the two gauge theory parameter, one gauge coupling $g$ and one
mass parameter $m$, are introduced by suitably parametrising the Kähler form:

$$
\begin{equation*}
J=-\frac{1}{g^{2}} F+\sum_{i=0}^{1} \phi_{i} S_{i}+m N . \tag{4.107}
\end{equation*}
$$

In order to determine the gluing curves of $F, N$ with the compact $S_{i}$, one restricts the Kähler form (4.107) to the $S_{i}$ and expands in a suitable base as follows:

$$
\begin{align*}
\left.J\right|_{S_{0}} & =\phi_{0} K_{S_{0}}+\left.\phi_{1} S_{1}\right|_{S_{0}}+a_{0} e_{0}+b_{0} f_{0}+c_{x} x+c_{y} y,  \tag{4.108}\\
\left.J\right|_{S_{1}} & =\left.\phi_{0} S_{0}\right|_{S_{1}}+\phi_{1} K_{S_{1}}+a_{1} e_{1}+b_{1} f_{1}+c_{z} z+c_{w} w .
\end{align*}
$$

The parameters are determined as follows:

- Firstly, one identifies the -1 curves that give rise to the fundamental BPS particles. It is instructive to evaluate the volumes of the blowups; in detail,

$$
\begin{equation*}
-J_{\phi} \cdot x=-J_{\phi} \cdot y=-J_{\phi} \cdot z=-J_{\phi} \cdot w=0=\langle 0, \phi\rangle \tag{4.109}
\end{equation*}
$$

where 0 is a trivial weight in $[0,1]_{G_{2}}$, see appendix A.1.4. Recalling (4.106), the remaining weights are realised by the following -1 curves:
and the self-gluing curves in (4.104) impose

$$
\begin{equation*}
\operatorname{Vol}\left(e_{0}-x\right)=\operatorname{Vol}\left(e_{0}-y\right) \quad \text { and } \quad \operatorname{Vol}\left(e_{1}-z\right)=\operatorname{Vol}\left(e_{1}-w\right) . \tag{4.111}
\end{equation*}
$$

These conditions are solved by

$$
\begin{equation*}
c_{I}=m, \quad I \in\{x,, y, z, w\}, \quad b_{i}=0, \quad i=0,1, \tag{4.112}
\end{equation*}
$$

One notes that these volumes motivate the choice of phase (4.102).

- Secondly, the geometric effective gauge coupling (2.25) needs to match the field theory expectation (2.2). The arising linear equations are solved by

$$
\begin{equation*}
a_{0}=-\frac{3}{g^{2}}, \quad a_{1}=-\frac{1}{g^{2}} . \tag{4.113}
\end{equation*}
$$

Consequently, the non-compact surfaces $F$ and $N$, introduced in (4.107), are glued to the compact $S_{i}$ as follows:

$$
\left.F\right|_{S_{i}}=\left\{\begin{array}{ll}
3 e_{0}, & i=0  \tag{4.114}\\
e_{1}, & i=1
\end{array},\left.\quad N\right|_{S_{i}}= \begin{cases}x+y, & i=0, \\
z+w, & i=1 .\end{cases}\right.
$$

An immediate observation is that the $\left.F\right|_{S_{i}}$ restrict to $3 e_{0}$ and $e_{1}$ respectively, because the $e_{i}$ act as simple roots (4.106). The prefactors equal $\frac{2}{\left\langle\alpha_{i}, \alpha_{i}\right\rangle}=3,1$ for the roots of $G_{2}$. Moreover, the volumes of the fibres are given by

$$
\left\{\begin{array}{l}
\operatorname{Vol}\left(f_{0}\right)=\frac{3}{g^{2}},  \tag{4.115}\\
\operatorname{Vol}\left(f_{1}\right)=\frac{1}{g^{2}},
\end{array} \quad \text { such that } \quad \operatorname{Vol}\left(f_{\mathrm{ell}}\right)=\frac{1}{g^{2}}\right.
$$

which yields the volume of the elliptic fibre. The factor 3 is due to the permutation twist in (4.101).

## 5 Fibre-base like duality: $5 d$ to $5 d$

Inspecting table 1, it is clear that some twisted compactifications of 6 d SCFTs may lead to several 5 d descriptions. In section 4 , only the 5 d theories that are related by fibre-base duality to the 6 d setup have been considered. Nevertheless, most of the 5 d theories admit an honest geometric description. As detailed, for instance in [60, 96], these geometric frames are in many cases related by some operations on the Hirzebruch surfaces and their gluing curves. In this section, two examples are considered for which the relation between two 5 d frames is simply realised in terms of the $\mathbb{F}_{0}$ isomorphism $e \leftrightarrow f$.

## 5.1 $\mathrm{SU}(3)_{\frac{3}{2}}+1 \mathrm{Sym}-\mathrm{Sp}(2)_{\pi}+1 \mathrm{Adj}$

Starting from the $6 \mathrm{~d} \mathcal{N}=(2,0) A_{4}$ theory, the twist by the $\mathbb{Z}_{2}$ permutation symmetry

leads to a 5 d KK theory with two known gauge theory descriptions: $\mathrm{SU}(3)_{\frac{3}{2}}$ with one rank-2 symmetric hypermultiplet [16] and $\mathrm{Sp}(2)_{\pi}$ with one adjoint hypermultiplet [97].

In this section, it is demonstrated that a suitable $5 d$ frame for $\operatorname{Sp}(2)$ is related via a $\mathbb{F}_{0}$ isomorphism $e \leftrightarrow f$ to a 5 d frame for the $\mathrm{SU}(3)$ description. In other words, this mimics a fibre-base like duality between the two dual 5 d theories.

5d $\mathbf{S p ( 2 )}$ description. Since the 5 d KK descriptions is known, the prepotential (2.1) is derived using appendix A.1.3. In addition to restricting to the Weyl-chamber of $C_{2}$, i.e. $\left\langle\phi, \alpha_{i}\right\rangle \geq 0$ for $i=1,2$, the following phase is chosen
with $\alpha_{1,2}$ the simple roots of $C_{2}$, see appendix A.1.3. The prepotential becomes

$$
\begin{align*}
6 \mathcal{F}_{5 d}= & 12 \phi_{1}^{2} \phi_{2}-18 \phi_{1} \phi_{2}^{2}+8 \phi_{2}^{3}+\frac{6}{g^{2}}\left(2 \phi_{1}^{2}-2 \phi_{1} \phi_{2}+\phi_{2}^{2}\right)  \tag{5.3}\\
& -6 m^{2} \phi_{1}-24 m \phi_{1}^{2}+36 m \phi_{1} \phi_{2}-18 m \phi_{2}^{2}
\end{align*}
$$

with $g$ the gauge coupling and $m$ the mass parameter of the adjoint hypermultiplet.
Geometry. The geometric proposal for (5.1) in [61] does not have a manifest fibre-base duality between the 6 d and 5 d frame. A convenient 5 d geometry is given by [96]

and one notices that

$$
-\left(\begin{array}{cc}
f_{1} \cdot K_{S_{1}} & \left.f_{1} \cdot S_{2}\right|_{S_{1}}  \tag{5.5}\\
\left.f_{2} \cdot S_{1}\right|_{S_{2}} & f_{2} \cdot K_{S_{2}}
\end{array}\right)=\left(\begin{array}{cc}
2 & -1 \\
-2 & 2
\end{array}\right) \equiv C_{C_{2}} \quad \Rightarrow \begin{cases}-J_{\phi} \cdot f_{1} & =\left\langle\alpha_{1}, \phi\right\rangle \\
-J_{\phi} \cdot f_{2} & =\left\langle\alpha_{2}, \phi\right\rangle\end{cases}
$$

which identifies the $f_{1,2}$ as the simple roots $\alpha_{1,2}$ of the $5 \mathrm{~d} \mathfrak{s p}(2)$ gauge algebra, see appendix A.1.3. The gauge theory parameters can be incorporated into the Kähler form via

$$
\begin{equation*}
J=-\frac{1}{g^{2}} F+\sum_{i=1}^{2} \phi_{i} S_{i}+m N \tag{5.6}
\end{equation*}
$$

where the gluing curves of the non-compact surfaces $F$ and $N$ with the compact $S_{i}$ are determined as follows:

- Firstly, one identifies the -1 curves that give rise to the fundamental BPS particles. It is instructive to evaluate the volumes of the blowups; in detail,

$$
\begin{equation*}
-J_{\phi} \cdot x=-J_{\phi} \cdot y=2 \phi_{1}=-\left\langle-\left(2 \alpha_{1}+\alpha_{2}\right), \phi\right\rangle \tag{5.7}
\end{equation*}
$$

where $2 \alpha_{1}+\alpha_{2}$ is the highest weight of $[2,0]_{C}$, see appendix A.1.3. Recalling (5.5), the volumes of the relevant -1 curves are required to satisfy:
and the self-gluing curves in (5.4) impose

$$
\begin{equation*}
\operatorname{Vol}(x)=\operatorname{Vol}(y) \tag{5.9}
\end{equation*}
$$

One notes that these volumes motivate the choice of phase (5.2).

- Secondly, the geometric effective coupling (2.25) needs to match the field theory result (2.2).

Consequently, one finds

$$
\left.F\right|_{S_{i}}=\left\{\begin{array}{ll}
2 f_{1}, & i=1  \tag{5.10}\\
f_{2}, & i=2
\end{array},\left.\quad N\right|_{S_{i}}= \begin{cases}6 f_{1}-x-y, & i=1 \\
3 f_{2}, & i=2\end{cases}\right.
$$

One may note that $\left.F\right|_{S_{i}}$ restricts to $2 f_{1}$ and $f_{2}$ respectively, because the $f_{i}$ act as simple roots (5.5). The prefactors equal $\frac{2}{\left\langle\alpha_{i}, \alpha_{i}\right\rangle}=2,1$ for the roots of $C_{2}$.

Fibre-base dual SU(3). Returning to the geometry (5.4), one aims to utilise the same geometry for the $\mathrm{SU}(3) \mathrm{KK}$ theory. As a first check, one computes

$$
-\left(\begin{array}{cc}
f_{1} \cdot K_{S_{1}} & \left.f_{1} \cdot S_{2}\right|_{S_{1}}  \tag{5.11}\\
\left.e_{2} \cdot S_{1}\right|_{S_{2}} & e_{2} \cdot K_{S_{2}}
\end{array}\right)=\left(\begin{array}{cc}
2 & -1 \\
-1 & 2
\end{array}\right)=C_{A_{2}} \Rightarrow \begin{cases}-J_{\phi} \cdot f_{1} & =\left\langle\alpha_{1}, \phi\right\rangle \\
-J_{\phi} \cdot e_{2} & =\left\langle\alpha_{2}, \phi\right\rangle\end{cases}
$$

which indicates the $5 \mathrm{~d} \mathfrak{s u}(3)$ gauge algebra and identifies $f_{1}, e_{2}$ as simple roots $\alpha_{i}$, see appendix A.1.2. Besides restricting to the Weyl-chamber of $A_{2}$, i.e. $\left\langle\phi, \alpha_{i}\right\rangle \geq 0$ for $i=1,2$,
the following phase is chosen
with $v_{i} \in[2,0]_{A}$, see appendix A.1.2. The prepotential becomes

$$
\begin{align*}
6 \mathcal{F}_{5 d}= & \frac{6}{g^{2}}\left(\phi_{1}^{2}-\phi_{1} \phi_{2}+\phi_{2}^{2}\right)+12 \phi_{1}^{2} \phi_{2}-18 \phi_{1} \phi_{2}^{2}+8 \phi_{2}^{3}  \tag{5.13}\\
& +3 m \phi_{1}^{2}-15 m \phi_{1} \phi_{2}+15 m \phi_{2}^{2}-6 m^{2} \phi_{1},
\end{align*}
$$

with $g$ the gauge coupling and $m$ the mass parameter of the hypermultiplet in the 2nd rank symmetric representation. The Kähler form is supplemented by non-compact surfaces as follows:

$$
\begin{equation*}
J=-\frac{1}{g^{2}} F+\sum_{i=1}^{2} \phi_{i} S_{i}+\frac{m}{2} N . \tag{5.14}
\end{equation*}
$$

The gluing curves of $F$ and $N$ with the compact surfaces $S_{i}$ are determined via two constraints.

- One identifies the -1 curves that give rise to the fundamental BPS particles. The blowups provide some insights

$$
\begin{equation*}
-J_{\phi} \cdot x=-J_{\phi} \cdot y=2 \phi_{1}=\left\langle v_{1}, \phi\right\rangle \tag{5.15}
\end{equation*}
$$

with the highest weight vector $v_{1} \in[2,0]_{A}$. The identical volumes for $x$ and $y$ is a consequence of the self-gluing in (5.4). Recalling (5.11), the set of -1 curves is given by

and the self-gluing imposes

$$
\begin{equation*}
\operatorname{Vol}(x) \stackrel{!}{=} \operatorname{Vol}(y) \tag{5.17}
\end{equation*}
$$

These conditions motivate the phase (5.12).

- The effective coupling (2.25) derived from the geometry has to agree with field theory (2.2).

These constraints lead to

$$
\left.F\right|_{S_{i}}=\left\{\begin{array}{ll}
f_{1}, & i=1  \tag{5.18}\\
e_{2}, & i=2
\end{array},\left.\quad N\right|_{S_{i}}=\left\{\begin{array}{ll}
2 x+2 y-5 f_{1}, & i=1 \\
-5 e_{2}, & i=2
\end{array} .\right.\right.
$$

As above, the $\left.F\right|_{S_{i}}$ restrict to $f_{1}$ and $e_{2}$ respectively, because the $f_{1}, e_{2}$ act as simple roots (5.11). The prefactor of 1 equals $\frac{2}{\left\langle\alpha_{i}, \alpha_{i}\right\rangle}$ for the roots of $A_{2}$. Moreover, one verifies that $6 \mathcal{F}_{\text {geom }}=J^{3}$ matches the field theory result (5.13).

## $5.25 \mathrm{~d} G_{2}+1 \mathrm{Adj}-5 \mathrm{~d} \mathrm{SU}(3)_{\frac{15}{2}}+1 \mathrm{~F}$

The twisted circle reduction of $6 \mathrm{~d} \mathcal{N}=(2,0) D_{4}$ (4.101) is known to admit two different 5 d descriptions: $G_{2}$ with one adjoint hypermultiplet [97] and $\mathrm{SU}(3)_{\frac{15}{2}}$ with one fundamental hypermultiplet [60].

The $G_{2}$ theory has already been discussed in section 4.8 , where the 5 d frame has been obtained via fibre-base duality from the 6 d frame. In this section, it is demonstrated that a suitable 5 d frame for $\mathrm{SU}(3)$ is related via a $\mathbb{F}_{0}$ isomorphism $e \leftrightarrow f$ to another 5 d frame for the $G_{2}$ description (in a different phase). In other words, this mimics a fibre-base like duality between the two dual 5 d theories.

5d SU(3) description. The $\operatorname{SU}(3)$ gauge theory with Chern-Simons level $\kappa=\frac{15}{2}$ has one fundamental hypermultiplet. The prepotential is derived from (2.1) via the data summarised in appendix A.1.2. In addition to restricting to the $A_{2}$ Weyl chamber, a phase needs to be choose. A suitable choice is

$$
\begin{gather*}
\alpha_{1} \downarrow  \tag{5.19}\\
\alpha_{2} \downarrow-\begin{array}{l}
\bullet\left\langle\phi, w_{1}\right\rangle+m_{f} \geq 0 \\
\bullet\left\langle\phi, w_{2}\right\rangle+m_{f} \geq 0
\end{array} \\
\cdot\left\langle\phi, w_{3}\right\rangle+m_{f} \leq 0
\end{gather*}
$$

with $w_{i} \in[1,0]_{A}$, see appendix A.1.2. In this phase, the prepotential becomes

$$
\begin{align*}
6 \mathcal{F}= & 8 \phi_{1}^{3}+18 \phi_{1}^{2} \phi_{2}-24 \phi_{1} \phi_{2}^{2}+7 \phi_{2}^{3}+\frac{6}{g^{2}}\left(\phi_{1}^{2}-\phi_{1} \phi_{2}+\phi_{2}^{2}\right)  \tag{5.20}\\
& -3 m \phi_{1}^{2}+3 m \phi_{1} \phi_{2}-3 m^{2} \phi_{2}
\end{align*}
$$

with $g$ the gauge coupling and $m$ the mass of the fundamental hypermultiplet.
Geometry. For the $\mathrm{SU}(3)$ frame, one may employ the following geometry [60]

$$
\begin{equation*}
\mathbf{1}_{8} \xrightarrow{e_{1}} e_{2}+3 f_{2} \mathbf{2}_{0}^{1} \tag{5.21}
\end{equation*}
$$

which is argued to be flop equivalent to (4.104). One straightforwardly verifies that

$$
-\left(\begin{array}{cc}
f_{1} \cdot K_{S_{1}} & f_{1} \cdot S_{2} \mid S_{1}  \tag{5.22}\\
f_{2} \cdot S_{1} \mid S_{2} & f_{2} \cdot K_{S_{2}}
\end{array}\right)=\left(\begin{array}{cc}
2 & -1 \\
-1 & 2
\end{array}\right) \equiv C_{A_{2}} \quad \Rightarrow \quad \begin{cases}-J_{\phi} \cdot f_{1} & =\left\langle\alpha_{i}, \phi\right\rangle \\
-J_{\phi} \cdot f_{2} & =\left\langle\alpha_{2}, \phi\right\rangle\end{cases}
$$

which identifies the Cartan matrix of the $5 \mathrm{~d} \mathfrak{s u}(3)$ gauge algebra. In addition, the fibres $f_{i}$ act as simple roots $\alpha_{i}$ of $A_{2}$, see appendix A.1.2.

The gauge theory parameter can be included as follows:

$$
\begin{equation*}
J=-\frac{1}{g^{2}} F+\sum_{i=1}^{2} \phi_{i} S_{i}+\frac{m}{2} N \tag{5.23}
\end{equation*}
$$

and the gluing curves of the non-compact surfaces with the compact $S_{i}$ are determined via two conditions:

- Firstly, the volume of the -1 curves match BPS masses. Based on (5.22) and

$$
\begin{equation*}
-J_{\phi} \cdot x=\phi_{2}=-\left\langle w_{3}, \phi\right\rangle, \tag{5.24}
\end{equation*}
$$

with $w_{2} \in[1,0]_{A}$, one concludes that the BPS mass are given by

$$
\left.\begin{array}{c}
\alpha_{1} \downarrow  \tag{5.25}\\
\alpha_{2} \downarrow-
\end{array}\right\} \begin{aligned}
& \operatorname{Vol}\left(f_{1}+f_{2}-x\right) \stackrel{!}{=}\left\langle w_{1}, \phi\right\rangle+m \\
& \operatorname{Vol}\left(f_{2}-x\right) \stackrel{!}{=}\left\langle w_{2}, \phi\right\rangle+m \\
& \operatorname{Vol}(x) \stackrel{!}{=}-\left(\left\langle w_{3}, \phi\right\rangle+m\right)
\end{aligned}
$$

which motivates the phase (5.19).

- The geometric effective gauge coupling (2.25) needs to match the field theory result (2.2).

As a result, the non-compact surfaces are glued as follows:

$$
\left.F\right|_{S_{i}}=f_{i} \quad \text { for } i=1,2,\left.\quad N\right|_{S_{i}}=\left\{\begin{array}{ll}
f_{1} & i=1  \tag{5.26}\\
f_{2}-2 x & i=2
\end{array} .\right.
$$

The $\left.F\right|_{S_{i}}$ restrict to the $f_{i}$ respectively, because the $f_{i}$ act as simple roots (5.22). The prefactor of 1 equals $\frac{2}{\left\langle\alpha_{i}, \alpha_{i}\right\rangle}$ for the roots of $A_{2}$.

Fibre-base dual $\boldsymbol{G}_{\mathbf{2}}$. Inspecting (5.21) shows an interesting instance of fibre-base duality. In detail,

$$
-\left(\begin{array}{cc}
f_{1} \cdot K_{S_{1}} & f_{1} \cdot S_{2} \mid S_{1}  \tag{5.27}\\
e_{2} \cdot S_{1} \mid S_{2} & e_{2} \cdot K_{S_{2}}
\end{array}\right)=\left(\begin{array}{cc}
2 & -1 \\
-3 & 2
\end{array}\right) \equiv C_{G_{2}} \quad \Rightarrow \quad \begin{cases}-J_{\phi} \cdot f_{1} & =\left\langle\alpha_{i}, \phi\right\rangle \\
-J_{\phi} \cdot e_{2} & =\left\langle\alpha_{2}, \phi\right\rangle\end{cases}
$$

such that $f_{1}, e_{2}$ act as simple roots $\alpha_{i}$ of $G_{2}$, see appendix A.1.4. Therefore, one is lead to another description of $G_{2}$ with one adjoint hypermulitplet, but this time in a different phase

$$
\begin{align*}
& \alpha_{2} \downarrow .\left\langle\phi, 3 \alpha_{1}+2 \alpha_{2}\right\rangle+m_{f} \geq 0 \\
& \alpha_{1} \swarrow\left\langle\phi, \alpha_{1}+\alpha_{2}\right\rangle+m_{f} \geq 0 \tag{5.28}
\end{align*}
$$

The prepotential in this phase equals

$$
\begin{align*}
6 \mathcal{F}= & 8 \phi_{1}^{3}+18 \phi_{1}^{2} \phi_{2}-24 \phi_{1} \phi_{2}^{2}+7 \phi_{2}^{3}+\frac{6}{g^{2}}\left(3 \phi_{1}^{2}-3 \phi_{1} \phi_{2}+\phi_{2}^{2}\right)  \tag{5.29}\\
& -72 m \phi_{1}^{2}+72 m \phi_{1} \phi_{2}-21 m \phi_{2}^{2}-3 m^{2} \phi_{2}
\end{align*}
$$

The Kähler form can be expressed as

$$
\begin{equation*}
J=-\frac{1}{g^{2}} F+\sum_{i=1}^{2} \phi_{i} S_{i}+m N \tag{5.30}
\end{equation*}
$$

and the non-compact surfaces $F$ and $N$ are glued to the $S_{i}$ via

$$
\left.F\right|_{S_{i}}=\left\{\begin{array}{ll}
3 f_{1} & i=1  \tag{5.31}\\
e_{2} & i=2
\end{array},\left.\quad N\right|_{S_{i}}= \begin{cases}12 f_{1} & i=1 \\
4 f_{2}-x & i=2\end{cases}\right.
$$

Analogously to the cases discussed above, the $\left.F\right|_{S_{i}}$ restrict to $3 f_{1}$ and $e_{2}$ respectively, because $f_{1}$ and $e_{2}$ act as simple roots (5.27). The prefactors equal $\frac{2}{\left\langle\alpha_{i}, \alpha_{i}\right\rangle}=3,1$ for the roots of $G_{2}$. Also, note that in this construction $-J_{\phi} \cdot x=-\left\langle-\left(3 \alpha_{1}+2 \alpha_{2}\right), \phi\right\rangle$, which justifies the phase (5.28).

## 6 Conclusions

In this work, a geometric description of twisted circle compactifications of $6 \mathrm{~d} \mathcal{N}=(1,0)$ SCFTs has been analysed with the aim to fully characterise the resulting 5d theory. For a number of such theories, we have shown how to recover the prepotential of the 5d gauge theory on the Coulomb branch from the geometric prepotential of M-theory on a Calabi-Yau
threefold $X_{S}$. We explicitly described the non-compact divisors which must be included in the Kähler potential of $X_{S}$ to reflect the contribution of mass parameters and gauge couplings. The strategy to find these divisors is composed of three steps: firstly, determine a geometric description such that there exist curves of self-intersection zero which act as roots of the 5 d gauge algebra. Secondly, identify the weights of the blowup modes under the gauge algebra and construct the -1 curves that correspond to the fundamental BPS particles. The volumes of these curves have to reproduce the BPS masses. Moreover, this step identifies the phase of the 5d theory on the Coulomb branch. Thirdly, the geometric expression for the matrix of effective gauge couplings has to reproduce the corresponding field theory expression. As a result, the non-compact surfaces in the Kähler form are determined by their intersection with the compact surfaces. This is sufficient information to reproduce the physical quantities and to compute the prepotential $\mathcal{F}_{\text {geom }}=\frac{1}{3!} J^{3}$ up to constant terms.

This description then allows us to identify the effective gauge coupling with the volume of the elliptic fibre of the F-Theory description of the parent 6d SCFTs. In particular, we made this identification for the 5 d reduction of the E-string theory, the untwisted 5 d reduction of $\mathfrak{s u}(n)$ on a -2 , the $\mathbb{Z}_{2}$ twisted compactification of the $6 \mathrm{~d} \mathcal{N}=(1,0)$ minimal $\mathrm{SU}(3)$ SCFT, and the $\mathbb{Z}_{3}$ twisted reduction of the minimal SO(8) SCFT.

In a number of examples of rank 1 and rank 2 KK theories, we observed a fibrebase duality between the 6 d and 5 d frame, which is realised via an exchange of fibre and base in $\mathbb{F}_{0}$. For non-gauge theoretic nodes and, in particular, non-geometric theories, the appearance of some form of fibre-base duality between the 6 d and 5 d frame is not guaranteed. In all cases considered in section 4, the geometry for the 6 d frame, which manifests the elliptic fibre, is fibre-dual to one of the 5 d descriptions, precisely by an $\mathbb{F}_{0}$ isomorphism $e \leftrightarrow f$.

For M-theory on a given non-compact Calabi-Yau threefold, compact surfaces in the geometry may exhibit different rulings, which corresponds to different effective gauge theory descriptions. In the limit in which we shrink the compact surfaces, all of these theories flow to the same 5d SCFT, so that such a phenomenon encodes a UV duality. A particular instance of this phenomenon is when the fibre of one ruling becomes a section of another. Seen from 6d, the circle reduction admits more than one 5 d description in this case, and not all 5 d frames are related to the 6 d frame via fibre-base duality. We have demonstrated that some of the dual 5 d theories are related among each other by a fibre-base like duality relating different 5 d frames.

The geometric description of 5 d theories by a collection of complex compact surfaces sitting inside a non-compact threefold is a resourceful approach to the understanding of quantum field theories. There are a number of open issues that would be interesting to address in future research. Starting from a specific twisted compactification, the resulting collection of surfaces and their gluing rules are far from unique. In this work, the geometric frame suitable for a gauge theory in one specific phase has been chosen. Hence, the transitions between different phases have not been addressed, although it is expected that these are straightforward. Moreover, one may also choose to follow the isomorphisms relating the different phases, and establish parameter maps between the different gauge
theory descriptions. Fibre-base like dualities relating different 5 d descriptions of the same SCFT have also been observed in [54, 59, 60, 96]. In the present work we have collected evidence for an elegant relation between the divisor $F$ and the fibres that establish the gauge algebra. One may wonder whether there is an analogous, systematic relation for the divisors $N_{f}$ associated to hypermultiplet mass parameters.

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## A Background material

## A. 1 Lie algebras and weight spaces

The evaluation of the prepotential (2.1) relies on the gauge group as well as the representations of the matter content. This appendix provides a brief summary. For more details, the reader is, for instance, referred to $[98,99]$ as well as $[16$, appendix A].

## A.1.1 Notation

Give a Lie algebra, the simple roots are $\alpha_{i}$, and the simple coroots are $\alpha_{i}^{\vee}$. The Cartan matrix is defined via

$$
\begin{equation*}
A_{i j}=\left\langle\alpha_{i}, \alpha_{j}^{\vee}\right\rangle=2 \frac{\left\langle\alpha_{i}, \alpha_{j}\right\rangle}{\left\langle\alpha_{i}, \alpha_{i}\right\rangle} . \tag{A.1}
\end{equation*}
$$

A convenient basis is the Dynkin basis, or weight basis, defined by the fundamental weights being the basis vectors. Then, the Cartan matrix is the transformation matrix that provides the expansion of the simple roots $\alpha_{i}$ in the basis of the fundamental weights, i.e.

$$
\begin{equation*}
\alpha_{i}=\sum_{k} A_{i k} \omega_{k} \quad \Leftrightarrow \quad \omega_{i}=\sum_{k}\left(A^{-1}\right)_{i k} \alpha_{k} \tag{A.2}
\end{equation*}
$$

Note that the $j$-th row of the Cartan matrix gives the simple root $\alpha_{j}$ in terms of the fundamental weights.

The vector $\phi$ of Coulomb moduli is an expanded in the basis of simple coroot

$$
\begin{equation*}
\phi=\sum_{i} \phi_{i} \alpha_{i}^{\vee}, \tag{A.3}
\end{equation*}
$$

which defines the components $\phi_{i}$ used throughout this paper. Next, consider the following expressions

$$
\begin{equation*}
\left\langle\phi, \alpha_{i}\right\rangle=\sum_{j, k} \phi_{j} A_{i k}\left\langle\alpha_{j}^{\vee}, \omega_{k}\right\rangle=\sum_{j, k} \phi_{j} A_{i k} \delta_{j k}=\sum_{k} A_{i k} \phi_{k} \tag{A.4}
\end{equation*}
$$

likewise

$$
\begin{equation*}
\langle\phi, \phi\rangle=\sum_{i, j} \phi_{i} \phi_{j}\left\langle\alpha_{i}^{\vee}, \alpha_{j}^{\vee}\right\rangle=\sum_{i, j} \phi_{j} D_{j}^{-1} A_{j i} \phi_{i} \equiv \sum_{i, j} \phi_{j} h_{j i} \phi_{i} \tag{A.5}
\end{equation*}
$$

where $h_{i j}$ is the inverse of the metric tensor defined as follows

$$
\begin{equation*}
\left(h^{-1}\right)_{i j}=\left(A^{-1}\right)_{i j} D_{j} \tag{A.6}
\end{equation*}
$$

where $D_{j}$ is defined via

$$
\begin{equation*}
A_{i j}=\left\langle\alpha_{i}, \alpha_{j}^{\vee}\right\rangle=2 \frac{\left\langle\alpha_{i}, \alpha_{j}\right\rangle}{\left\langle\alpha_{i}, \alpha_{i}\right\rangle}=D_{j}^{-1}\left\langle\alpha_{i}, \alpha_{j}\right\rangle \quad \Leftrightarrow \quad D^{-1}=\operatorname{diag}\left(\frac{2}{\left\langle\alpha_{1}, \alpha_{1}\right\rangle}, \ldots, \frac{2}{\left\langle\alpha_{r}, \alpha_{r}\right\rangle}\right) \tag{A.7}
\end{equation*}
$$

Recall that the factors $\frac{2}{\left\langle\alpha_{i}, \alpha_{i}\right\rangle}$ can only take three different values: 1 for all roots on $A, D$, $E$ algebras and the long roots in $B, C, F_{4} ; 2$ for the short roots of $B, C$, and $F_{4}$; and 3 for the short root of $G_{2}$. The Chern-Simons term in (2.1) can be expressed as follows:

$$
\begin{align*}
6 \mathcal{F}_{\text {tree-level }}^{\mathrm{CS}}=\kappa d_{i j k} \phi^{i} \phi^{j} \phi^{k} & \text { with } \tag{A.8}
\end{align*} \quad d_{i j k}=\frac{1}{2} \operatorname{Tr}_{\mathbf{F}}\left(t_{i}\left\{t_{j}, t_{k}\right\}\right), ~\left(\operatorname{Tr}_{\mathbf{R}_{f}}\left(t_{i}\left\{t_{j}, t_{k}\right\}\right) \phi^{i} \phi^{j} \phi^{k}=2 c_{\mathbf{R}_{f}}^{(3)} d_{i j k} \phi^{i} \phi^{j} \phi^{k}=2 c_{\mathbf{R}_{f}}^{(3)} \sum_{w \in \mathbf{R}_{f}}\langle\phi, w\rangle^{3}\right)
$$

with $c_{\mathbf{R}_{f}}^{(3)}$ the cubic Dynkin index. Since $c_{\mathbf{F}}^{(3)}=1$ for $\mathfrak{s u}(n)$, one finds

$$
\begin{equation*}
d_{i j k} \phi^{i} \phi^{j} \phi^{k}=\sum_{w \in \mathbf{F}}\langle\phi, w\rangle^{3} \quad \text { for } \mathfrak{s u}(n) . \tag{A.10}
\end{equation*}
$$

In the following subsection, the roots and weights for the Lie algebra relevant for this paper are summarised.

## A.1.2 $\mathrm{SU}(2), \mathrm{SU}(3)$, and $\mathrm{SU}(4)$

$\mathbf{S U ( 2 )}$. The simple root is $\alpha=2$ and the representations relevant for this work have weight systems given by

$$
\left.\begin{array}{cc}
{[1]_{A}}  \tag{A.11}\\
\alpha \downarrow \\
& \quad[2]_{A} \\
w_{2}=-1
\end{array} \begin{aligned}
& w_{1}=1 \\
& \alpha \downarrow
\end{aligned} \right\rvert\, \begin{aligned}
& 2 \\
& 0 \\
& -2
\end{aligned}
$$

where the arrows indicate the action of the simple roots. The Cartan matrix, which equals the inverse metric tensor, reads

$$
\begin{equation*}
C_{A_{1}}=2=h \tag{A.12}
\end{equation*}
$$

$\mathbf{S U ( 3 )}$. The simple roots are $\alpha_{1}=(2,-1), \alpha_{2}=(-1,2)$ and the weight systems for the relevant representation are as follows

where the arrows indicate the action of the simple roots. In addition, the Cartan matrix and the inverse metric tensor are given by

$$
C_{A_{2}}=\left(\begin{array}{cc}
2 & -1  \tag{A.14}\\
-1 & 2
\end{array}\right)=h_{i j}
$$

$\mathbf{S U ( 4 )}$. The simple roots are $\alpha_{1}=(2,-1,0), \alpha_{2}=(-1,2,-1), \alpha_{3}=(0,-1,2)$ and the Cartan matrix, which equals the inverse metric tensor, is given by

$$
C_{A_{3}}=\left(\begin{array}{ccc}
2 & -1 & 0  \tag{A.15}\\
-1 & 2 & -1 \\
0 & -1 & 2
\end{array}\right)=h_{i j}
$$

## A.1.3 $\operatorname{Sp}(2)$

The simple roots are $\alpha_{1}=(2,-1), \alpha_{2}=(-2,2)$ and $a_{1}$ is the short root. The representations relevant for this work have the weights systems given by

where the arrows indicate the action of the simple roots. In addition, the Cartan matrix and the inverse metric tensor $h_{i j}$ are given by

$$
C_{C_{2}}=\left(\begin{array}{cc}
2 & -1  \tag{A.17}\\
-2 & 2
\end{array}\right), \quad\left(h_{i j}\right)=\left(\begin{array}{cc}
4 & -2 \\
-2 & 2
\end{array}\right)
$$

## A.1. $4 \quad G_{2}$

The simple roots are given by $\alpha_{1}=(2,-1), \alpha_{2}=(-3,2)$ and $\alpha_{1}$ is the short root. The representations relevant for this paper have the following weight systems

where the arrows indicate the action of the simple roots. In addition, the Cartan matrix and the inverse metric tensor $h_{i j}$ are given by

$$
C_{G_{2}}=\left(\begin{array}{cc}
2 & -1  \tag{A.19}\\
-3 & 2
\end{array}\right), \quad\left(h_{i j}\right)=\left(\begin{array}{cc}
6 & -3 \\
-3 & 2
\end{array}\right)
$$

## A. 2 Geometry of Hirzebruch surfaces

A Hirzebruch surface is a $\mathbb{P}^{1}$ fibration over $\mathbb{P}^{1}$, and one denotes by $\mathbb{F}_{n}$ a Hirzebruch surface with a degree $-n$ fibration. The fibre $\mathbb{P}^{1}$ is denoted by $f$, while the base $\mathbb{P}^{1}$ is $e$. The intersections numbers are

$$
\begin{equation*}
e^{2}=-n, \quad f^{2}=0, \quad e \cdot f=1 \tag{A.20}
\end{equation*}
$$

Another curve inside $\mathbb{F}_{n}$ is defined by $h:=e+n f$, which for $\mathbb{F}_{0}$ becomes equal to $e$. For $n \geq 0$, the set of holomorphic curves, the Mori cone, is generated by $e$ and $f$.

One may also consider surfaces that arise from blowing up $\mathbb{F}_{n}$ is a number of points. Let $b$ be the number of blowups, such that the blowup of $\mathbb{F}_{n}$ at $b$ points is denoted by $\mathbb{F}_{n}^{b}$. The exceptional divisors created by the blowups are the curves $x_{i}$ for $i=1, \ldots, b$. The intersection number of $e, f, h$ with the $x_{i}$ are given by

$$
\begin{equation*}
x_{i} \cdot x_{j}=-\delta_{i, j}, \quad e \cdot x_{i}=0, \quad f \cdot x_{i}=0, \quad h \cdot x_{i}=0 . \tag{A.21}
\end{equation*}
$$

Following the conventions of [61], the total transforms of the curves $e, f, h$ are denoted by the same names $e, f, h$ in $\mathbb{F}_{n}^{b}$.

A single theory may enjoy many (isomorphic) geometric descriptions. To transition between them, it is useful to recall the $\mathbb{F}_{n}^{b} \rightarrow \mathbb{F}_{n+1}^{b}$ isomorphism, see for instance [62, eqs. (2.19)-(2.22)],

$$
\begin{align*}
e-x_{i} & \rightarrow e \\
f-x_{i} & \rightarrow x_{i}  \tag{A.22}\\
x_{i} & \rightarrow f-x_{i} \\
x_{j} & \rightarrow x_{j} \quad j \neq i
\end{align*}
$$

which comes into play in section 3.2.

## B Rulings on rational elliptic surfaces

In this appendix, we give an explicit realisation of a rational elliptic surface $d P_{9}$ [100] that shows that we can also view it as a blowup $\mathbb{F}_{0}^{8}$ of the Hirzebruch surface $\mathbb{F}_{0}=\mathbb{P}^{1} \times \mathbb{P}^{1}$ at eight points. Seen in this way, $d P_{9}$ inherits the two rulings on $\mathbb{F}_{0}$, and we can derive the inner form between divisors.

Consider a hypersurface $S$ of degree $(1,2,2)$ inside $\mathbb{P}_{w}^{1} \times \mathbb{P}_{y}^{1} \times \mathbb{P}_{z}^{1}$. Denoting the homogeneous coordinates of the three $\mathbb{P}^{1} \mathrm{~S}$ by $\left[w_{1}: w_{2}\right],\left[y_{1}: y_{2}\right]$ and $\left[z_{1}: z_{2}\right]$, such a hypersurface can be written as

$$
\begin{equation*}
P(y, z) w_{1}=Q(y, z) w_{2} \tag{B.1}
\end{equation*}
$$

for two homogeneous polynomials $P$ and $Q$ that have both degrees 2 in $\left[y_{1}: y_{2}\right]$ and $\left[z_{1}: z_{2}\right]$.
Projecting to $\mathbb{P}_{w}^{1}$, this surface has the structure of an elliptic fibration, with the fibre embedded in $\mathbb{P}_{y} \times \mathbb{P}_{z}$. Denoting the hyperplane divisors of the three $\mathbb{P}^{1}$ s by $H_{w}, H_{y}, H_{z}$, we find that $c_{1}(S)=H_{w}$, which is represented by the generic fibre of this elliptic fibration. We hence identify $S$ as a rational elliptic surface.

Let us now discuss the ruling. For a generic point on $\mathbb{P}_{y}^{1} \times \mathbb{P}_{z}^{1}$, (B.1) fixes a unique point on $\mathbb{P}_{w}^{1}$. However, over points $\left(q_{i}, p_{i}\right)$ on $\mathbb{P}_{y}^{1} \times \mathbb{P}_{z}^{1}$ where $Q=P=0$, there is no constraint on $\mathbb{P}_{w}^{1}$ and we find another $\mathbb{P}^{1}$ in $S$. As $Q$ and $P$ are in the class $2 H_{y}+2 H_{z}$, there are

$$
\begin{equation*}
\int_{\mathbb{P}_{y}^{1} \times \mathbb{P}_{z}^{1}}\left(2 H_{y}+2 H_{z}\right)^{2}=8 \tag{B.2}
\end{equation*}
$$

such points. We hence see that $d P_{9}$ is $\mathbb{F}_{0}$ blown up at 8 points ( $q_{i}, p_{i}$ ). We can now describe the inner form among divisors on $d P_{9}$ from this perspective. Let us denote $H_{y}$ restricted to $S$ by $e$ and $H_{z}$ restricted to $S$ by $f$. We then have that $e^{2}=f^{2}=0\left(\right.$ as $\left.H_{y} \cdot H_{y}=H_{z} \cdot H_{z}=0\right)$ and $e \cdot f=1$ (as $H_{y} \cdot H_{z}=1$ on S ). Let us denote the $\mathbb{P}^{1}$ s sitting at the 8 points $Q=P=0$ by $x_{i}$. As these are $\mathbb{P}^{1} \mathrm{~s}$ and $\int_{S} c_{1}(S) \cdot x_{i}=\int_{\mathbb{P}_{w}^{1} \times \mathbb{P}_{y}^{1} \times \mathbb{P}_{z}^{1}} H_{w} \cdot x_{i}=1$ we find using adjunction that $x_{i} \cdot x_{i}=-1$. All other intersections must vanish, $x_{i} \cdot x_{j}=-\delta_{i j}$ as these sit over different points of $\mathbb{P}_{y}^{1} \times \mathbb{P}_{z}^{1}$, and similarly $x_{i} \cdot e=x_{i} \cdot f=0$ as a generic hyperplane section misses any of the points $Q=P=0$. In summary, the only non-vanishing intersections of the inner form are

$$
\begin{equation*}
e \cdot f=1 \quad x_{i}^{2}=-1 \tag{B.3}
\end{equation*}
$$

which in particular shows that this lattice is unimodular, i.e. e, $f, x_{i}$ generate $H^{2}(S, \mathbb{Z})$. Note that the curves $x_{i}$ all become sections of the elliptic fibration on $S$ : for every $x_{i}$, we may assign every point on a copy of the base $\mathbb{P}_{w}^{1}$ to the point $\left(p_{i}, q_{i}\right)$ in the fibre.

We can also discuss the details of the (blown-up) ruling. As we want $f$ to be class represented by the fibre and $f=H_{z}$, i.e. the fibre class is given by e.g. $z_{1}=0$, the projection of the ruling acts on the level of the ambient space as $\pi: \mathbb{P}_{x}^{1} \times \mathbb{P}_{y}^{1} \times \mathbb{P}_{z}^{1} \rightarrow \mathbb{P}_{z}^{1}$, i.e. it projects $S$ down to $\mathbb{P}_{z}^{1}$. Over a generic point $z$, the fibre of the ruling is the $\mathbb{P}^{1}$ described by

$$
\begin{equation*}
Q(z, y) w_{1}=P(z, y) w_{2} \tag{B.4}
\end{equation*}
$$

which is smooth as $Q(z, y)$ and $P(z, y)$ do not vanish simultaneously for any $y$ for such a fixed $z$. The class of the fibre $f$. If we now take $\left[z_{1}: z_{2}\right]$ to correspond to the image of one of the $\left(q_{i}, p_{i}\right)$, the fibre is given by

$$
\begin{equation*}
Q\left(q_{i}, y\right) w_{1}=P\left(q_{i}, y\right) w_{2}, \tag{B.5}
\end{equation*}
$$

where now $P=Q=0$ for $y=p_{i}$. This implies that this set is the union of the $\mathbb{P}_{y}^{1}$ times the unique point in $\mathbb{P}_{w}^{1}$ given by the above equation, together with the $\mathbb{P}^{1}$ at $p_{i} \times \mathbb{P}_{w}^{1}=x_{i}$. If we denote the first curve by $\hat{x}_{i}$, linear equivalence gives

$$
\begin{equation*}
f=\hat{x}_{i}+x_{i} \tag{B.6}
\end{equation*}
$$

for all $i$. In other words, the fibre of the ruling splits into pairs of $\mathbb{P}^{1} \mathrm{~S}$ of classes $x_{i}$ and $\hat{x}_{i}=f-x_{i}$ over 8 points on $\mathbb{P}_{z}^{1}$.

As we may as well project down to $\mathbb{P}_{y}^{1}$, there is another ruling with fibre $e$ and base $f$. The fibre components of this ruling are seen to be $x_{i}$ and $\check{x}_{i}=e-x_{i}$ by repeating the same arguments as above.

As $h^{1,1}(S)=10$, the ten classes $e, f, x_{i}$ span all of $H^{1,1}(S, \mathbb{Q})$ and we can work out the class of $c_{1}(S)$ in terms of them by computing intersections. As

$$
\begin{align*}
c_{1}(S) \cdot e & =H_{w} \cdot H_{y}=2 \\
c_{1}(S) \cdot f & =H_{w} \cdot H_{z}=2  \tag{B.7}\\
c_{1}(S) \cdot x_{i} & =H_{w} \cdot x_{i}=1
\end{align*}
$$

we find that

$$
\begin{equation*}
c_{1}(S)=H_{w}=2 e+2 f-\sum_{i=1}^{8} x_{i} . \tag{B.8}
\end{equation*}
$$

This is of course precisely the result we are expecting for a surface that is reached from $\mathbb{F}_{0}$ by blowing up at 8 points.

By construction, the cone of effective curves contains $e, f$ and the $x_{i}$, as well as $c_{1}(S)$. Furthermore, we have seen that there are the additional effective curves $\hat{x}_{i}=f-x_{i}$ and $\check{x}_{i}=e-x_{i}$. All of these must have positive volume and due their negative selfintersection they are among the extremal generators of the Mori cone. Note that we can write $f=x_{i}+\hat{x}_{i}, e=x_{i}+\check{x}_{i}$, but that $c_{1}(S)$ is not a positive linear combination of $e, f, x_{i}, \hat{x}_{i}, \check{x}_{i}$, so that is also among the extremal rays of the cone of effective curves. Note that in the explicit realisation of $S$ we have given, the volumes of the $x_{i}$ are locked to all be identical.

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## References

[1] N. Seiberg, Five-dimensional SUSY field theories, nontrivial fixed points and string dynamics, Phys. Lett. B 388 (1996) 753 [hep-th/9608111] [InSPIRE].
[2] D.R. Morrison and N. Seiberg, Extremal transitions and five-dimensional supersymmetric field theories, Nucl. Phys. B 483 (1997) 229 [hep-th/9609070] [InSPIRE].
[3] K.A. Intriligator, D.R. Morrison and N. Seiberg, Five-dimensional supersymmetric gauge theories and degenerations of Calabi-Yau spaces, Nucl. Phys. B 497 (1997) 56 [hep-th/9702198] [INSPIRE].
[4] O. Aharony and A. Hanany, Branes, superpotentials and superconformal fixed points, Nucl. Phys. B 504 (1997) 239 [hep-th/9704170] [inSPIRE].
[5] O. Aharony, A. Hanany and B. Kol, Webs of $(p, q)$ five-branes, five-dimensional field theories and grid diagrams, JHEP 01 (1998) 002 [hep-th/9710116] [INSPIRE].
[6] O. DeWolfe, A. Hanany, A. Iqbal and E. Katz, Five-branes, seven-branes and five-dimensional $E(n)$ field theories, JHEP 03 (1999) 006 [hep-th/9902179] [INSPIRE].
[7] M.R. Douglas, S.H. Katz and C. Vafa, Small instantons, Del Pezzo surfaces and type-I-prime theory, Nucl. Phys. B 497 (1997) 155 [hep-th/9609071] [INSPIRE].
[8] H.-C. Kim, S.-S. Kim and K. Lee, 5-dim superconformal index with enhanced E(n) global symmetry, JHEP 10 (2012) 142 [arXiv:1206.6781] [INSPIRE].
[9] O. Bergman, D. Rodríguez-Gómez and G. Zafrir, Discrete $\theta$ and the $5 d$ superconformal index, JHEP 01 (2014) 079 [arXiv:1310.2150] [inSPIRE].
[10] O. Bergman, D. Rodríguez-Gómez and G. Zafrir, 5-brane webs, symmetry enhancement, and duality in 5 d supersymmetric gauge theory, JHEP 03 (2014) 112 [arXiv:1311.4199] [inSPIRE].
[11] G. Zafrir, Duality and enhancement of symmetry in 5d gauge theories, JHEP 12 (2014) 116 [arXiv:1408.4040] [INSPIRE].
[12] Y. Tachikawa, Instanton operators and symmetry enhancement in 5d supersymmetric gauge theories, PTEP 2015 (2015) 043B06 [arXiv:1501.01031] [inSPIRE].
[13] G. Zafrir, Instanton operators and symmetry enhancement in $5 d$ supersymmetric USp, SO and exceptional gauge theories, JHEP 07 (2015) 087 [arXiv:1503.08136] [INSPIRE].
[14] K. Yonekura, Instanton operators and symmetry enhancement in 5 d supersymmetric quiver gauge theories, JHEP 07 (2015) 167 [arXiv:1505.04743] [INSPIRE].
[15] S. Cremonesi, G. Ferlito, A. Hanany and N. Mekareeya, Instanton operators and the Higgs branch at infinite coupling, JHEP 04 (2017) 042 [arXiv:1505.06302] [INSPIRE].
[16] P. Jefferson, H.-C. Kim, C. Vafa and G. Zafrir, Towards classification of 5d SCFTs: single gauge node, arXiv:1705. 05836 [INSPIRE].
[17] G. Ferlito, A. Hanany, N. Mekareeya and G. Zafrir, 3d Coulomb branch and 5d Higgs branch at infinite coupling, JHEP 07 (2018) 061 [arXiv:1712.06604] [INSPIRE].
[18] L. Bao, E. Pomoni, M. Taki and F. Yagi, M5-branes, toric diagrams and gauge theory duality, JHEP 04 (2012) 105 [arXiv:1112.5228] [INSPIRE].
[19] O. Bergman and G. Zafrir, Lifting 4d dualities to 5d, JHEP 04 (2015) 141 [arXiv:1410.2806] [INSPIRE].
[20] S.-S. Kim, M. Taki and F. Yagi, Tao probing the end of the world, PTEP 2015 (2015) 083B02 [arXiv: 1504.03672] [INSPIRE].
[21] H. Hayashi, S.-S. Kim, K. Lee, M. Taki and F. Yagi, A new $5 d$ description of $6 d$ D-type minimal conformal matter, JHEP 08 (2015) 097 [arXiv:1505.04439] [INSPIRE].
[22] D. Gaiotto and H.-C. Kim, Duality walls and defects in $5 d \mathcal{N}=1$ theories, JHEP 01 (2017) 019 [arXiv: 1506.03871] [InSPIRE].
[23] O. Bergman and G. Zafrir, 5d fixed points from brane webs and O7-planes, JHEP 12 (2015) 163 [arXiv:1507.03860] [InSPIRE].
[24] G. Zafrir, Brane webs, 5d gauge theories and $6 d \mathcal{N}=(1,0)$ SCFT's, JHEP 12 (2015) 157 [arXiv:1509.02016] [INSPIRE].
[25] H. Hayashi, S.-S. Kim, K. Lee and F. Yagi, 6d SCFTs, $5 d$ dualities and tao web diagrams, JHEP 05 (2019) 203 [arXiv:1509.03300] [inSPIRE].
[26] K. Ohmori and H. Shimizu, $S^{1} / T^{2}$ compactifications of $6 d \mathcal{N}=(1,0)$ theories and brane webs, JHEP 03 (2016) 024 [arXiv:1509.03195] [inSPIRE].
[27] G. Zafrir, Brane webs and O5-planes, JHEP 03 (2016) 109 [arXiv:1512.08114] [InSPIRE].
[28] H. Hayashi, S.-S. Kim, K. Lee, M. Taki and F. Yagi, More on $5 d$ descriptions of $6 d$ SCFTs, JHEP 10 (2016) 126 [arXiv:1512.08239] [InSPIRE].
[29] G. Zafrir, Brane webs in the presence of an $O 5^{-}$-plane and $4 d$ class $S$ theories of type $D$, JHEP 07 (2016) 035 [arXiv:1602.00130] [inSPIRE].
[30] H. Hayashi, S.-S. Kim, K. Lee and F. Yagi, Equivalence of several descriptions for $6 d$ SCFT, JHEP 01 (2017) 093 [arXiv:1607.07786] [inSPIRE].
[31] H. Hayashi, S.-S. Kim, K. Lee and F. Yagi, Discrete theta angle from an O5-plane, JHEP 11 (2017) 041 [arXiv:1707.07181] [inSPIRE].
[32] H. Hayashi, S.-S. Kim, K. Lee and F. Yagi, 5-brane webs for $5 d \mathcal{N}=1 G_{2}$ gauge theories, JHEP 03 (2018) 125 [arXiv:1801.03916] [InSPIRE].
[33] H. Hayashi, S.-S. Kim, K. Lee and F. Yagi, Dualities and 5-brane webs for 5 d rank 2 SCFTs, JHEP 12 (2018) 016 [arXiv:1806. 10569] [INSPIRE].
[34] S. Cabrera, A. Hanany and F. Yagi, Tropical geometry and five dimensional Higgs branches at infinite coupling, JHEP 01 (2019) 068 [arXiv:1810.01379] [INSPIRE].
[35] H. Hayashi, S.-S. Kim, K. Lee and F. Yagi, Rank-3 antisymmetric matter on 5-brane webs, JHEP 05 (2019) 133 [arXiv:1902.04754] [inSPIRE].
[36] H. Hayashi, S.-S. Kim, K. Lee and F. Yagi, Complete prepotential for $5 d \mathcal{N}=1$ superconformal field theories, JHEP 02 (2020) 074 [arXiv:1912.10301] [INSPIRE].
[37] A. Bourget, J.F. Grimminger, A. Hanany, M. Sperling and Z. Zhong, Magnetic quivers from brane webs with O5 planes, JHEP 07 (2020) 204 [arXiv:2004.04082] [INSPIRE].
[38] M. Van Beest, A. Bourget, J. Eckhard and S. Schäfer-Nameki, (5d RG-flow) trees in the tropical rain forest, JHEP 03 (2021) 241 [arXiv:2011.07033] [InSPIRE].
[39] M. van Beest, A. Bourget, J. Eckhard and S. Schäfer-Nameki, (Symplectic) leaves and (5d Higgs) branches in the poly(go)nesian tropical rain forest, JHEP 11 (2020) 124 [arXiv:2008.05577] [INSPIRE].
[40] M. Akhond, F. Carta, S. Dwivedi, H. Hayashi, S.-S. Kim and F. Yagi, Five-brane webs, Higgs branches and unitary/orthosymplectic magnetic quivers, JHEP 12 (2020) 164 [arXiv:2008.01027] [INSPIRE].
[41] H. Hayashi, H.-C. Kim and K. Ohmori, $6 d / 5 d$ exceptional gauge theories from web diagrams, arXiv:2103.02799 [INSPIRE].
[42] M. Del Zotto, J.J. Heckman and D.R. Morrison, $6 D$ SCFTs and phases of $5 D$ theories, JHEP 09 (2017) 147 [arXiv:1703.02981] [inSPIRE].
[43] D. Xie and S.-T. Yau, Three dimensional canonical singularity and five dimensional $\mathcal{N}=1$ SCFT, JHEP 06 (2017) 134 [arXiv:1704.00799] [InSPIRE].
[44] M. Esole, P. Jefferson and M.J. Kang, The Geometry of $F_{4}$-Models, arXiv:1704.08251 [inSPIRE].
[45] M. Esole, R. Jagadeesan and M.J. Kang, The geometry of $G_{2}$, $\operatorname{Spin}(7)$, and $\operatorname{Spin}(8)$-models, arXiv:1709.04913 [INSPIRE].
[46] M. Esole, M.J. Kang and S.-T. Yau, Mordell-Weil torsion, anomalies, and phase transitions, arXiv:1712.02337 [INSPIRE].
[47] P. Jefferson, S. Katz, H.-C. Kim and C. Vafa, On geometric classification of $5 d$ SCFTs, JHEP 04 (2018) 103 [arXiv: 1801.04036] [inSPIRE].
[48] M. Esole and M.J. Kang, Flopping and slicing: SO(4) and Spin(4)-models, Adv. Theor. Math. Phys. 23 (2019) 1003 [arXiv:1802.04802] [inSPIRE].
[49] M. Esole and M.J. Kang, The geometry of the $\mathrm{SU}(2) \times G_{2}$-model, JHEP 02 (2019) 091 [arXiv:1805.03214] [INSPIRE].
[50] L. Bhardwaj and P. Jefferson, Classifying 5d SCFTs via 6d SCFTs: rank one, JHEP 07 (2019) 178 [Addendum ibid. 01 (2020) 153] [arXiv:1809.01650] [INSPIRE].
[51] L. Bhardwaj and P. Jefferson, Classifying 5d SCFTs via 6d SCFTs: arbitrary rank, JHEP 10 (2019) 282 [arXiv:1811.10616] [inSPIRE].
[52] F. Apruzzi, L. Lin and C. Mayrhofer, Phases of 5d SCFTs from M-/F-theory on non-flat fibrations, JHEP 05 (2019) 187 [arXiv:1811.12400] [INSPIRE].
[53] S. Banerjee, P. Longhi and M. Romo, Exploring 5d BPS spectra with exponential networks, Annales Henri Poincaré 20 (2019) 4055 [arXiv:1811.02875] [INSPIRE].
[54] C. Closset, M. Del Zotto and V. Saxena, Five-dimensional SCFTs and gauge theory phases: an M-theory/type IIA perspective, SciPost Phys. 6 (2019) 052 [arXiv:1812.10451] [INSPIRE].
[55] M. Esole and P. Jefferson, The Geometry of $\mathrm{SO}(3), \mathrm{SO}(5)$, and $\mathrm{SO}(6)$ models, arXiv:1905.12620 [InSPIRE].
[56] M. Esole, R. Jagadeesan and M.J. Kang, 48 crepant paths to $\operatorname{SU}(2) \times S U(3)$, arXiv:1905. 05174 [INSPIRE].
[57] F. Apruzzi, C. Lawrie, L. Lin, S. Schäfer-Nameki and Y.-N. Wang, 5d superconformal field theories and graphs, Phys. Lett. B 800 (2020) 135077 [arXiv:1906.11820] [inSPIRE].
[58] F. Apruzzi, C. Lawrie, L. Lin, S. Schäfer-Nameki and Y.-N. Wang, Fibers add flavor. Part I. Classification of 5d SCFTs, flavor symmetries and BPS states, JHEP 11 (2019) 068 [arXiv:1907.05404] [INSPIRE].
[59] F. Apruzzi, C. Lawrie, L. Lin, S. Schäfer-Nameki and Y.-N. Wang, Fibers add flavor. Part II. 5d SCFTs, gauge theories, and dualities, JHEP 03 (2020) 052 [arXiv:1909.09128] [INSPIRE].
[60] L. Bhardwaj, On the classification of 5d SCFTs, JHEP 09 (2020) 007 [arXiv:1909.09635] [INSPIRE].
[61] L. Bhardwaj, P. Jefferson, H.-C. Kim, H.-C. Tarazi and C. Vafa, Twisted circle compactifications of $6 d$ SCFTs, JHEP 12 (2020) 151 [arXiv:1909.11666] [INSPIRE].
[62] L. Bhardwaj, Dualities of 5d gauge theories from S-duality, JHEP 07 (2020) 012 [arXiv:1909.05250] [inSPIRE].
[63] V. Saxena, Rank-two 5d SCFTs from M-theory at isolated toric singularities: a systematic study, JHEP 04 (2020) 198 [arXiv:1911.09574] [InSPIRE].
[64] L. Bhardwaj, Do all 5d SCFTs descend from 6d SCFTs?, JHEP 04 (2021) 085 [arXiv:1912.00025] [INSPIRE].
[65] F. Apruzzi, S. Schäfer-Nameki and Y.-N. Wang, 5d SCFTs from decoupling and gluing, JHEP 08 (2020) 153 [arXiv:1912.04264] [inSPIRE].
[66] C. Closset and M. Del Zotto, On 5d SCFTs and their BPS quivers. Part I: B-branes and brane tilings, arXiv:1912.13502 [INSPIRE].
[67] A.-K. Kashani-Poor, Determining F-theory matter via Gromov-Witten invariants, arXiv:1912.10009 [INSPIRE].
[68] C. Closset, S. Schäfer-Nameki and Y.-N. Wang, Coulomb and Higgs branches from canonical singularities. Part 0, JHEP 02 (2021) 003 [arXiv:2007.15600] [INSPIRE].
[69] C. Closset, S. Giacomelli, S. Schäfer-Nameki and Y.-N. Wang, 5d and 4d SCFTs: canonical singularities, trinions and S-dualities, arXiv:2012.12827 [INSPIRE].
[70] Z. Duan, D.J. Duque and A.-K. Kashani-Poor, Weyl invariant Jacobi forms along Higgsing trees, JHEP 04 (2021) 224 [arXiv:2012.10427] [inSPIRE].
[71] D.R. Morrison, S. Schäfer-Nameki and B. Willett, Higher-form symmetries in 5d, JHEP 09 (2020) 024 [arXiv:2005.12296] [inSPIRE].
[72] L. Bhardwaj and S. Schäfer-Nameki, Higher-form symmetries of $6 d$ and $5 d$ theories, JHEP 02 (2021) 159 [arXiv:2008.09600] [INSPIRE].
[73] F. Albertini, M. Del Zotto, I. García Etxebarria and S.S. Hosseini, Higher form symmetries and M-theory, JHEP 12 (2020) 203 [arXiv:2005.12831] [inSPIRE].
[74] S.S. Razamat, C. Vafa and G. Zafrir, $4 d \mathcal{N}=1$ from $6 d(1,0)$, JHEP 04 (2017) 064 [arXiv:1610.09178] [INSPIRE].
[75] I. Bah, A. Hanany, K. Maruyoshi, S.S. Razamat, Y. Tachikawa and G. Zafrir, $4 d \mathcal{N}=1$ from $6 d \mathcal{N}=(1,0)$ on a torus with fluxes, JHEP 06 (2017) 022 [arXiv:1702.04740] [inSPIRE].
[76] H.-C. Kim, S.S. Razamat, C. Vafa and G. Zafrir, E-string theory on Riemann surfaces, Fortsch. Phys. 66 (2018) 1700074 [arXiv:1709.02496] [INSPIRE].
[77] H.-C. Kim, S.S. Razamat, C. Vafa and G. Zafrir, D-type conformal matter and SU/USp quivers, JHEP 06 (2018) 058 [arXiv:1802.00620] [InSPIRE].
[78] H.-C. Kim, S.S. Razamat, C. Vafa and G. Zafrir, Compactifications of ADE conformal matter on a torus, JHEP 09 (2018) 110 [arXiv:1806.07620] [inSPIRE].
[79] S.S. Razamat and G. Zafrir, Compactification of $6 d$ minimal SCFTs on Riemann surfaces, Phys. Rev. D 98 (2018) 066006 [arXiv:1806.09196] [INSPIRE].
[80] J. Chen, B. Haghighat, S. Liu and M. Sperling, $4 d N=1$ from $6 d$ D-type $N=(1,0)$, JHEP 01 (2020) 152 [arXiv:1907.00536] [inSPIRE].
[81] S.S. Razamat, E. Sabag and G. Zafrir, From 6d flows to 4 d flows, JHEP 12 (2019) 108 [arXiv:1907.04870] [INSPIRE].
[82] S. Pasquetti, S.S. Razamat, M. Sacchi and G. Zafrir, Rank $Q$ E-string on a torus with flux, SciPost Phys. 8 (2020) 014 [arXiv:1908.03278] [inSPIRE].
[83] B. Haghighat and S. Vandoren, Five-dimensional gauge theory and compactification on a torus, JHEP 09 (2011) 060 [arXiv:1107.2847] [inSPIRE].
[84] B. Haghighat, J. Manschot and S. Vandoren, A 5d/2d/4d correspondence, JHEP 03 (2013) 157 [arXiv:1211.0513] [inSPIRE].
[85] G. Beaujard, J. Manschot and B. Pioline, Vafa-Witten invariants from exceptional collections, arXiv:2004.14466 [inSPIRE].
[86] S.H. Katz, A. Klemm and C. Vafa, Geometric engineering of quantum field theories, Nucl. Phys. B 497 (1997) 173 [hep-th/9609239] [INSPIRE].
[87] S. Katz, P. Mayr and C. Vafa, Mirror symmetry and exact solution of $4-D N=2$ gauge theories: 1., Adv. Theor. Math. Phys. 1 (1998) 53 [hep-th/9706110] [inSPIRE].
[88] J. Gu, B. Haghighat, K. Sun and X. Wang, Blowup equations for $6 d$ SCFTs. Part I, JHEP 03 (2019) 002 [arXiv:1811.02577] [INSPIRE].
[89] J. Gu, A. Klemm, K. Sun and X. Wang, Elliptic blowup equations for $6 d$ SCFTs. Part II. Exceptional cases, JHEP 12 (2019) 039 [arXiv:1905.00864] [INSPIRE].
[90] J. Gu, B. Haghighat, A. Klemm, K. Sun and X. Wang, Elliptic blowup equations for $6 d$ SCFTs. Part III. E-strings, M-strings and chains, JHEP 07 (2020) 135 [arXiv:1911.11724] [inSPIRE].
[91] J. Gu, B. Haghighat, A. Klemm, K. Sun and X. Wang, Elliptic blowup equations for $6 d$ SCFTs. Part IV. matters, arXiv:2006.03030 [inSPIRE].
[92] H.-C. Kim, M. Kim, S.-S. Kim and K.-H. Lee, Bootstrapping BPS spectra of $5 d / 6 d$ field theories, JHEP 04 (2021) 161 [arXiv:2101.00023] [InSPIRE].
[93] J.J. Heckman, D.R. Morrison and C. Vafa, On the classification of $6 D$ SCFTs and Generalized ADE Orbifolds, JHEP 05 (2014) 028 [Erratum ibid. 06 (2015) 017] [arXiv:1312.5746] [INSPIRE].
[94] M.R. Douglas, On $D=5$ super Yang-Mills theory and (2,0) theory, JHEP 02 (2011) 011 [arXiv:1012.2880] [INSPIRE].
[95] N. Lambert, C. Papageorgakis and M. Schmidt-Sommerfeld, M5-branes, D4-branes and Quantum 5D super-Yang-Mills, JHEP 01 (2011) 083 [arXiv:1012.2882] [inSPIRE].
[96] L. Bhardwaj and G. Zafrir, Classification of $5 d \mathcal{N}=1$ gauge theories, JHEP 12 (2020) 099 [arXiv:2003.04333] [inSPIRE].
[97] Y. Tachikawa, On S-duality of $5 d$ super Yang-Mills on $S^{1}$, JHEP 11 (2011) 123 [arXiv:1110.0531] [INSPIRE].
[98] R. Feger and T.W. Kephart, LieART-A Mathematica application for Lie algebras and representation theory, Comput. Phys. Commun. 192 (2015) 166 [arXiv:1206.6379] [INSPIRE].
[99] R. Feger, T.W. Kephart and R.J. Saskowski, LieART 2.0 - A Mathematica application for Lie algebras and representation theory, Comput. Phys. Commun. 257 (2020) 107490 [arXiv:1912.10969] [INSPIRE].
[100] R. Miranda, The basic theory of elliptic surfaces: notes of lectures, ETS Editrice, Pisa Italy (1989).


[^0]:    ${ }^{1}$ The other section then has a positive self-intersection number.

