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FIBRED SPACES WITH PROJECTABLE RIEMANNIAN METRIC

KENTARO YANO & SHIGERU ISHIHARA

Introduction

In our previous papers [8] and [9], we studied fibred spaces with invariant affine connection and those with invariant Riemannian metric, the fibres being 1-dimensional in both cases.

The idea of fibred spaces with invariant affine connection goes back to the representation of spaces with projective connection. To represent an n-dimensional manifold with projective connection, Princeton School used an $(n+1)$-dimensional manifold with affine connection admitting a concurrent vector field with respect to which the affine connection is invariant (See for example [5]), and Dutch School used a slightly general manifold with affine connection (See for example, [4]). They all identified a point in the manifold with projective connection with a trajectory of the vector field with respect to which the affine connection is invariant.

The idea of fibred spaces with invariant Riemannian metric goes back to the five dimensional Riemannian space considered by Th. Kaluza [1] and O. Klein [2] for getting a unified field theory of gravitation and electromagnetism. To represent the space-time, they used a 5-dimensional Riemannian space admitting a unit vector field with respect to which the Riemannian metric is invariant, and identified a point in the space-time with a trajectory of the unit vector field with respect to which the 5-dimensional Riemannian metric is invariant.

In the present paper, we study fibred spaces with Riemannian metric under the assumption that the Riemannian metric is projectable instead of being invariant (See [3], [7]). In §1, we state definitions and study some properties of a fibred space with projectable Riemannian metric, and in §2 we develop the tensor calculus in the space. §3 is devoted to the discussions on the Riemannian connection and the induced connection. We discuss geodesics in §4, and structure equations and curvatures in §5. In the last §6, we assume that the Riemannian metric is *invariant* with respect to a not necessarily unit vector field tangent to the fibre, and the manifold is then slightly more general than that we studied in [9].

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1. Fibred space with projectable Riemannian metric

Let $\tilde{M}$ and M be two differentiable manifolds of dimensions $n+1$ and n respectively, and assume that there exists a differentiable mapping $\pi : \tilde{M} \to M$, which is onto and of the maximum rank n. (The manifolds, objects and mappings we discuss in the sequel are supposed to be of differentiability class C^∞, and the manifolds are assumed to be connected.) Then, for each point P of M, the inverse image $\pi^{-1}(P)$ of P is a 1-dimensional submanifold of $\tilde{M}$. We denote $\pi^{-1}(P)$ by F_P, and call F_P the *fibre* over the point P of M. We suppose that *every fibre F_P is connected*, and moreover that there are given in $\tilde{M}$ a vector field $\tilde{C}$ tangent to the fibre and a positive definite Riemannian metric $\tilde{g}$ satisfying the condition

$$\tilde{g}(\tilde{C},\tilde{C}) = 1. \tag{1.1}$$

If we introduce in $\tilde{M}$ a 1-form $\tilde{\eta}$ defined by the equation

$$\tilde{\eta}(\tilde{X}) = \tilde{g}(\tilde{C},\tilde{X}), \tag{1.2}$$

$\tilde{X}$ being an arbitrary vector field in $\tilde{M}$, we have

$$\tilde{\eta}(\tilde{C}) = 1. \tag{1.3}$$

The set $(\tilde{M}, M, \pi; \tilde{C}, \tilde{g})$ satisfying the conditions above is called a *fibred space with Riemannian metric $\tilde{g}$*. We suppose moreover that the condition

$$(\mathcal{L}\,\tilde{g})(\tilde{X},\tilde{Y}) = 0 \tag{1.4}$$

is satisfied for any two vector fields $\tilde{X}$ and $\tilde{Y}$ in $\tilde{M}$ such that $\tilde{\eta}(\tilde{X}) = \tilde{\eta}(\tilde{Y}) = 0$, where $\mathcal{L}$ denotes the operator of Lie derivation with respect to $\tilde{C}$. If this is the case, the fibred space $(\tilde{M}, M, \pi; \tilde{C}, \tilde{g})$ is called a *fibred space with projectable Riemannian metric $\tilde{g}$*. We call $\tilde{M}$ and M the *total space* and the *base space* respectively. The vector field $\tilde{C}$ and the 1-form $\tilde{\eta}$ are called the *structure field* and the *structure form* respectively. The mapping $\pi : \tilde{M} \to M$ is called the *projection*. If there is no fear of confusion, we call, for the sake of simplicity, a fibred space $(\tilde{M}, M, \pi; \tilde{C}, \tilde{g})$ with projectable Riemannian metric simply a *fibred space $\tilde{M}$*. The n-dimensional distribution defined in $\tilde{M}$ by the equation $\tilde{\eta} = 0$ is called the *field of horizontal planes* and its value at a point of $\tilde{M}$ is called the *horizontal plane* at that point.

We shall introduce some notations and terminologies for fibred space $(\tilde{M}, M, \pi; \tilde{C}, \tilde{g})$ (Cf. [8], [9]). $T(\tilde{M})$ is the tangent bundle of $\tilde{M}$. $\mathcal{T}_s^r(\tilde{M})$ is the space of all tensor fields of type (r,s) in $\tilde{M}$. We put $\mathcal{T}(\tilde{M}) = \Sigma_{r,s}\mathcal{T}_s^r(\tilde{M})$. The notations $T(M), \mathcal{T}_s^r(M)$ and $\mathcal{T}(M)$ denote the respective spaces with respect to M corresponding to $T(\tilde{M}), \mathcal{T}_s^r(\tilde{M})$ and $\mathcal{T}(\tilde{M})$ respectively.

Horizontal parts. A linear endomorphism $T \to T^H$ of $\mathcal{T}(\tilde{M})$ is defined by the following properties:

(H.1) $$\tilde{f}^H = \tilde{f} \quad \text{for} \quad \tilde{f} \in \mathcal{T}^0_0(\tilde{M}),$$

(H.2) $$\tilde{X}^H = \tilde{X} - \tilde{\eta}(\tilde{X})\tilde{C} \quad \text{for} \quad \tilde{X} \in \mathcal{T}^1_0(\tilde{M}),$$

(H.3) $$\tilde{\omega}^H = \tilde{\omega} - \tilde{\omega}(\tilde{C})\tilde{\eta} \quad \text{for} \quad \tilde{\omega} \in \mathcal{T}^0_1(\tilde{M}),$$

(H.4) $$(\tilde{S} \otimes \tilde{T})^H = (\tilde{S}^H) \otimes (\tilde{T}^H) \quad \text{for} \quad \tilde{S}, \tilde{T} \in \mathcal{T}(\tilde{M}).$$

The tensor field $\tilde{T}^H$ is called the *horizontal part* of $\tilde{T}$ for any element $\tilde{T}$ of $\mathcal{T}(\tilde{M})$. If a tensor field $\tilde{T}$ in $\tilde{M}$ satisfies the condition $\tilde{T} = \tilde{T}^H$, then $\tilde{T}$ is said to be *horizontal.* On putting $\tilde{T}^V = \tilde{T} - \tilde{T}^H$, we call $\tilde{T}^V$ the *non-horizontal part* of $\tilde{T}$ for any element $\tilde{T}$ of $\mathcal{T}(\tilde{M})$. In particular, for any element $\tilde{X}$ of $\mathcal{T}^1_0(\tilde{M})$ and any element $\tilde{w}$ of $\mathcal{T}^0_1(\tilde{M}), \tilde{X}^V$ and $\tilde{w}^V$ are called the *vertical parts* of $\tilde{X}$ and $\tilde{w}$ respectively. The space of all horizontal tensor fields is denoted by $\mathcal{T}^H(\tilde{M})$. If we put $\mathcal{T}^{H}{}^r_s(\tilde{M}) = \mathcal{T}^H(\tilde{M}) \cap \mathcal{T}^r_s(\tilde{M})$, we have $\mathcal{T}^H(\tilde{M}) = \Sigma_{r,s}\mathcal{T}^{H}{}^r_s(\tilde{M})$.

Projectable tensor fields. When an element $\tilde{T}$ of $\mathcal{T}(\tilde{M})$ satisfies the condition $(\mathcal{L}\ (\tilde{T}^H))^H = 0$, we say that $\tilde{T}$ is *projectable.* The space of all projectable tensor fields in $\tilde{M}$ is denoted by $\mathcal{P}(\tilde{M})$. If we put $\mathcal{P}^r_s(\tilde{M}) = \mathcal{P}(\tilde{M}) \cap \mathcal{T}^r_s(\tilde{M})$, $\mathcal{P}^{H}{}^r_s(\tilde{M}) = \mathcal{P}(\tilde{M}) \cap \mathcal{T}^{H}{}^r_s(\tilde{M})$, $\mathcal{P}^H(\tilde{M}) = \mathcal{P}(\tilde{M}) \cap \mathcal{T}^H(\tilde{M})$, then we have

$$\mathcal{P}(\tilde{M}) = \sum_{r,s} \mathcal{P}^r_s(\tilde{M}), \quad \mathcal{P}^H(\tilde{M}) = \sum_{r,s} \mathcal{P}^{H}{}^r_s(\tilde{M}).$$

We see that the Riemannian metric $\tilde{g}$ given in $\tilde{M}$ which satisfies (1.4) is *projectable*, i.e., $\tilde{g} \in \mathcal{P}^0_2(\tilde{M})$. In fact, from (1.4) we have $(\mathcal{L}\,\tilde{g})^H = (\mathcal{L}\ \tilde{g}^H + \mathcal{L}\ \tilde{g}^V)^H = (\mathcal{L}\ \tilde{g}^H)^H$, since $\tilde{g}^V = \tilde{\eta} \otimes \tilde{\eta}$, and consequently $(\mathcal{L}\ \tilde{g}^V)^H = (\mathcal{L}(\tilde{\eta} \otimes \tilde{\eta}))^H = 0$. If we take an arbitrary element $\tilde{\omega}$ of $\mathcal{P}^{H}{}^0_1(\tilde{M})$, we find $(\mathcal{L}\ \tilde{\omega})(\tilde{C}) = \mathcal{L}\ (\tilde{\omega}(\tilde{C})) - \tilde{\omega}(\mathcal{L}\ \tilde{C}) = 0$, which implies $\mathcal{L}\ \tilde{\omega} = 0$ because of $(\mathcal{L}\ \tilde{\omega})^H = 0$. Thus we see that, *for any element* $\tilde{T}$ of $\mathcal{P}^{H}{}^0_8(\tilde{M})$, *the condition* $\mathcal{L}\ \tilde{T} = 0$ *holds.*

Lifts. We shall introduce the operation of taking lifts (Cf. [8], [9]). The operation of taking lifts is a linear homomorphism $T \to T^L$ of $T(M)$ into $T(\tilde{M})$ characterized by the following properties:

(L.1) $$f^L = f \circ \pi \quad \text{for} \quad f \in \mathcal{T}^0_0(M);$$

(L.2)
For any element $X$of $\mathcal{T}^1_0(M)$, there exists a unique element $X^L$of $\mathcal{T}^{H}{}^1_0(\tilde{M})$ such that $\pi X^L = X$;

(L.3) $$\omega^L =^* \pi(\omega) \quad \text{for} \quad \omega \in \mathcal{T}^0_1(M);$$

(L.4) $$(S \otimes T)^L = (S^L) \otimes (T^L) \quad \text{for} \quad S, T \in \mathcal{T}(M);$$

where the differential mapping of the projection $\pi : \tilde{M} \to M$ is denoted also by π, and the dual mapping of the differential mapping π by ${}^*\pi$.

Taking an arbitrary element X of $\mathcal{T}_0^1(M)$, we find $X^{L f L} = (Xf)^L$ and $(\mathcal{L}X^L)^{H f L} = \mathcal{L}((Xf)^L) = 0$, f being an arbitrary element of $\mathcal{T}_0^0(M)$, i.e., $(\mathcal{L}X^L)^H = 0$. Thus X^L belongs to $\mathcal{P}^H{}_0^1(\tilde{M})$. If we take an arbitrary element ω of $\mathcal{T}_1^0(M)$, we find $\omega^L(X^L) = (\omega(X))^L$ and $(\mathcal{L}\,\omega^L)^H(X^L) = 0$, X being an arbitrary element of $\mathcal{T}_0^1(M)$, i.e., $(\mathcal{L}\,\omega^L)^H = 0$. Thus ω^L belongs to $\mathcal{P}^H{}_1^0(\tilde{M})$. Therefore, taking account of (L.4), we see that T^L *belongs to* $\mathcal{P}^H(\tilde{M})$ *for any element* T *of* $\mathcal{T}(M)$. The element T^L of $\mathcal{P}^H(\tilde{M})$ is called the *lift* of T.

Projections. Let $\tilde{f}$ be an element of $\mathcal{P}_0^0(\tilde{M})$. Then we have $\mathcal{L}\,\tilde{f} = 0$. Taking an arbitrary element $\tilde{X}$ of $\mathcal{P}_0^1(\tilde{M})$, we find $\mathcal{L}\,(\tilde{X}^{fL}) = 0$ for any element f of $\mathcal{T}_0^0(M)$. If $\tilde{\omega}$ is an arbitrary element of $\mathcal{P}_1^0(\tilde{M})$, then we find $\mathcal{L}\,(\tilde{\omega}(x^L)) = 0$ for any element X of $\mathcal{T}_0^1(M)$. Therefore, we can define a linear homomorphism p: $\mathcal{P}(\tilde{M}) \to \mathcal{T}(M)$ by the following properties:

$$(p\tilde{f})(P) = \tilde{f}(\tilde{P}) \quad \text{for} \quad \tilde{f} \in \mathcal{P}_0^0(\tilde{M}), \tag{P.1}$$

where $\tilde{P}$ is an arbitrary point such that $\pi(\tilde{P}) = P$, an arbitrary point of M.

$$(p\tilde{X})f = p(\tilde{X}(f^L)) \quad \text{for} \quad \tilde{X} \in \mathcal{P}_0^1(M), \tag{P.2}$$

f being an arbitrary element of $\mathcal{T}_0^0(M)$.

$$(p\tilde{\omega})(X) = p(\tilde{\omega}(X^L)) \quad \text{for} \quad \tilde{\omega} \in \mathcal{P}_1^0(\tilde{M}), \tag{P.3}$$

X being an arbitrary element of $\mathcal{T}_0^1(M)$.

$$p(\tilde{S} \otimes \tilde{T}) = (p\tilde{S}) \otimes (p\tilde{T}) \quad \text{for} \quad \tilde{S}, \tilde{T} \in \mathcal{P}(\tilde{M}). \tag{P.4}$$

The tensor field $p\tilde{T}$ is called the *projection* of $\tilde{T}$ for any element $\tilde{T}$ of $\mathcal{P}(\tilde{M})$.

Taking account of (L.1) $\sim$ (L.4) and (P.1) $\sim$ (P.4), we easily find

$$p(T^L) = T \quad \text{for} \quad T \in \mathcal{T}(M); \qquad (p\tilde{T})^L = \tilde{T}^H \quad \text{for} \quad \tilde{T} \in \mathcal{P}(\tilde{M}).$$

Thus the two spaces $\mathcal{P}^H(M)$ and $\mathcal{T}(M)$ are isomorphic to each other and $p : \mathcal{P}^H(\tilde{M}) \to \mathcal{T}(M)$ is the isomorphism between them. The operation of taking lifts is the inverse of the projection p restricted to $\mathcal{P}^H(\tilde{M})$. We have now

Proposition 1.1. *In a fibred space with projectable Riemannian metric, we have, for any elements* $\tilde{X}$, $\tilde{Y}$ *of* $P_0^1(\tilde{M})$,

$$[\tilde{X}, \tilde{Y}] \in \mathcal{P}_0^1(\tilde{M}), [\tilde{X}, \tilde{Y}]^H = [\tilde{X}^H, \tilde{Y}^H]^H, [\tilde{X}^H, \tilde{Y}^H]^V = -2\tilde{\Omega}(\tilde{X}, \tilde{Y})\tilde{C}, \tag{1.5}$$

and $p[\tilde{X}, \tilde{Y}] = [p\tilde{X}, p\tilde{Y}]$, *where* $\tilde{\Omega}$ *is a* 2*-form defined by the equation*

$$\tilde{\Omega} = (d\tilde{\eta})^H \tag{1.6}$$

in $\tilde{M}$ (Cf. [8], [9]).

Formal tensor products. We denote by $\mathcal{T}(\tilde{M}) \sharp \mathcal{T}^H(\tilde{M})$ the *formal tensor product*, i.e., the tensor product of the two spaces $\mathcal{T}(\tilde{M})$ and

$\mathcal{T}^H(\tilde{M})$ regarded as two abstract tensor spaces over $\tilde{M}$. Since $\mathcal{T}^H(\tilde{M})$ is a subspace of $\mathcal{T}(\tilde{M})$, we denote by $j : \mathcal{T}^H(\tilde{M}) \to \mathcal{T}(\tilde{M})$ the injection. We shall now introduce a linear homomorphism $i : \mathcal{T}(\tilde{M}) \sharp \mathcal{T}^H(\tilde{M}) \to \mathcal{T}(\tilde{M})$ by the following property:

$$\text{(I.1)} \qquad i(\tilde{T} \sharp \dot{S}) = \tilde{T} \otimes j(\dot{S}) \quad \text{for} \quad \tilde{T} \in \mathcal{T}(\tilde{M}), \dot{S} \in \mathcal{T}^H(\tilde{M}).$$

The induced metric. Since the Riemannian metric $\tilde{g}$ given in $\tilde{M}$ is projectable, its projection $g = p\tilde{g}$ is a positive definite Riemannian metric in the base space M. We call g the *induced metric* of M. We have the equation

$$\text{(1.7)} \qquad (g(X,Y))^L = \tilde{g}(X^L, Y^L) \quad \text{for} \quad X, Y \in \mathcal{T}^1_0(M),$$

or equivalently (See [3])

$$\text{(1.8)} \qquad g(p\tilde{X}, p\tilde{Y}) = p(\tilde{g}(\tilde{X}, \tilde{Y})) \quad \text{for} \quad \tilde{X}, \tilde{Y} \in \mathcal{P}^H{}^1_0(\tilde{M}).$$

The induced connection. Let there be given an affine connection $\tilde{\nabla}$ in the total space $\tilde{M}$, and assume that the vector field $\tilde{\nabla}_{\tilde{Y}}\tilde{X}$ is projectable for any two elements $\tilde{X}$ and $\tilde{Y}$ of $\mathcal{P}^H{}^1_0(\tilde{M})$. Then the given affine connection $\tilde{\nabla}$ is said to be *projectable.* When an affine connection $\tilde{\nabla}$ is projectable, we can introduce an affine connection ∇ in the base space M by the equation

$$\text{(1.9)} \qquad \nabla_Y X = p(\tilde{\nabla}_{Y^L} X^L),$$

X and Y being arbitrary elements of $\mathcal{T}^1_0(M)$. The affine connection ∇ thus introduced is called the *projection* of $\tilde{\nabla}$, or the *induced connection* in M. It is easily verified that *if the given projectable affine connection* $\tilde{\nabla}$ *is torsionless, so is the induced connection* ∇. We have now the following formulas:

$$\text{(1.10)} \qquad \begin{aligned} \nabla_Y T &= p(\tilde{\nabla}_{Y^L} T^L) \quad \text{for} \quad T \in \mathcal{T}(M), Y \in \mathcal{T}^1_0(M), \\ p(\tilde{\nabla}_{\tilde{Y}} \tilde{T}) &= \nabla_Y T \quad \text{for} \quad \tilde{T} \in \mathcal{P}^H(\tilde{M}), \tilde{Y} \in \mathcal{P}^H{}^1_0(\tilde{M}), \end{aligned}$$

Y and T being defined by $Y = p\tilde{Y}$ and $T = p\tilde{T}$. We shall now state

Proposition 1.2. *In a fibred space with projectable Riemannian metric* $\tilde{g}$, *the Riemannian connection* $\tilde{\nabla}$ *determined by* $\tilde{g}$ *is also projectable, and the projection* ∇ *of* $\tilde{\nabla}$ *coincides with the Riemannian connection determined by the induced metric* $g = p\tilde{g}$ *in the base space.*

Proposition 1.2 will be proved in §3 (Cf. Proposition 3.1). In the sequel, we always denote by $\tilde{\nabla}$ the Riemannian connection determined by the projectable Riemannian metric $\tilde{g}$ in the total space $\tilde{M}$.

Van der Waerden-Bortolotti covariant derivatives. Given an element $\tilde{Y}$ of $\mathcal{T}^1_0(\tilde{M})$, we define a derivation $\overset{*}{\nabla}_{\tilde{Y}}$ in the formal tensor product $\mathcal{T}(\tilde{M}) \sharp \mathcal{T}^H(\tilde{M})$ by the following properties:

(W.1) $$\overset{*}{\nabla}_{\tilde{Y}}\tilde{T} = \tilde{\nabla}_{\tilde{Y}}\tilde{T} \quad \text{for} \quad \tilde{T} \in \mathcal{T}(\tilde{M}).$$

(W.2) $$\overset{*}{\nabla}_{\tilde{Y}}\overset{*}{S} = (\tilde{\nabla}_{\tilde{Y}^H}\overset{*}{S} + \tilde{\eta}(\tilde{Y})\mathcal{L}\ \overset{*}{S})^H \quad \text{for} \quad \overset{*}{S} \in \mathcal{T}^H(\tilde{M}).$$

(W.3)
$$\overset{*}{\nabla}_{\tilde{Y}}(\tilde{T} \sharp \overset{*}{S} = (\overset{*}{\nabla}_{\tilde{Y}}\tilde{T}) \sharp \overset{*}{S} + \tilde{T} \sharp \overset{*}{\nabla}_{\tilde{Y}}\overset{*}{S}) \quad \text{for} \quad \tilde{T} \in \mathcal{T}(\tilde{M}), \overset{*}{S} \in \mathcal{T}^H(\tilde{M}).$$

For any element $\overset{*}{W}$ of $\mathcal{T}(\tilde{M}) \sharp \mathcal{T}^H(\tilde{M})$, the correspondence $\tilde{Y} \to \overset{*}{\nabla}_{\tilde{Y}}\overset{*}{W}$ defines an element $\overset{*}{\nabla}\overset{*}{W}$ of $\mathcal{T}(\tilde{M}) \sharp \mathcal{T}^H(\tilde{M})$, which is called the *van der Waerden-Bortolotti covariant derivative* of $\overset{*}{W}$. When $\overset{*}{W}$ and $\tilde{Y}$ belong respectively to $\mathcal{P}^H(\tilde{M})$ and $\mathcal{P}^{H}{}^1_0(\tilde{M})$, $\overset{*}{\nabla}_{\tilde{Y}}\overset{*}{W}$ is an element of $\mathcal{P}^H(\tilde{M})$.

The second fundamental tensors. We define an element $\tilde{h}$ of $\mathcal{T}^{H}{}^0_2(\tilde{M})$ by equation

(1.11) $$\tilde{h}(\tilde{Y}, \tilde{X})\tilde{C} = (\tilde{\nabla}_{\tilde{Y}^H}\tilde{X}^H)^V,$$

$\tilde{X}$ and $\tilde{Y}$ being arbitrary elements of $\mathcal{T}^1_0(\tilde{M})$, and define an element $\tilde{H}$ of $\mathcal{T}^{H}{}^1_1(\tilde{M})$ by the equation

(1.12) $$\tilde{H}\tilde{X} = -\tilde{\nabla}_{\tilde{X}^H}\tilde{C},$$

$\tilde{X}$ being an arbitrary element of $\mathcal{T}^1_0(\tilde{M})$. The tensor fields $\tilde{h}$ and $\tilde{H}$ are called the *second fundamental tensors* of the given fibred space.

Applying the operator $\tilde{\nabla}_{\tilde{Y}^H}$ on both sides of the identity $\tilde{g}(\tilde{C}, \tilde{X}^H) = 0$, we find

(1.13) $$\tilde{h}(\tilde{Y}, \tilde{X}) = \tilde{g}(\tilde{H}\tilde{Y}, \tilde{X}),$$

$\tilde{X}$ and $\tilde{Y}$ being two arbitrary elements of $\mathcal{T}^1_0(\tilde{M})$. If we take an element $\tilde{X}$ of $\mathcal{P}^1_0(\tilde{M})$, then $\mathcal{L}$
$\tilde{X}^H = [\tilde{C}, \tilde{X}^H]$ is vertical, and hence $\tilde{\nabla}_{\tilde{X}^H}\tilde{C} = (\tilde{\nabla}_{\tilde{C}}X^H)^H$, which implies

(1.14) $$\tilde{H}\tilde{X} = -(\tilde{\nabla}_{\tilde{C}}\tilde{X}^H)^H,$$

$\tilde{X}$ being an arbitrary element of $P^1_0(\tilde{M})$. On the other hand, we have $\tilde{\nabla}_{\tilde{C}}\tilde{g}(\tilde{X}^H, \tilde{Y}^H) = 0$ for any two elements $\tilde{X}$ and $\tilde{Y}$ of $P^1_0(\tilde{M})$. Thus we obtain

(1.15) $$\tilde{g}(\tilde{H}\tilde{X}, \tilde{Y}) = -\ \tilde{g}(\tilde{X}, \tilde{H}\tilde{Y}),$$

because of (1.14). From (1.13) and (1.15), we have

Proposition 1.3. *In a fibred space with projectable Riemannian metric $\tilde{g}$, the second fundamental tensors $\tilde{h}$ and $\tilde{H}$ are horizontal, and have the properties*

$$\tilde{h}(\tilde{X}, \tilde{Y}) + \tilde{h}(\tilde{Y}, \tilde{X}) = 0, \quad \tilde{h}(\tilde{X}, \tilde{Y}) = \tilde{g}(\tilde{H}\tilde{X}, \tilde{Y}),$$

$\tilde{X}$ and $\tilde{Y}$ being arbitrary elements of $\mathcal{T}^1_0(\tilde{M})$.

Since $\mathcal{L}\,\tilde{X}^H$ is vertical for any element $\tilde{X}$ of $\mathcal{P}_0^1(\tilde{M})$, we can put

$$\tilde{\lambda}(\tilde{X})\tilde{C} = -\mathcal{L}\,\tilde{X}^H \quad \text{for} \quad \tilde{X} \in \mathcal{P}_0^1(\tilde{M}), \tag{1.16}$$

and easily see that (1.16) defines in $\tilde{M}$ a 1-form $\tilde{\lambda}$, which is horizontal. Taking account of (1.11), (1.12), (1.14) and (1.16), we have the following formulas:

$$\begin{aligned} &\tilde{\nabla}_{\tilde{Y}}\tilde{X} = (\tilde{\nabla}_{\tilde{Y}}\tilde{X})^H + \tilde{h}(\tilde{Y},\ \tilde{X})\tilde{C}, \quad \tilde{\nabla}_{\tilde{Y}}\tilde{C} = -\tilde{H}\tilde{Y}, \\ &\tilde{\nabla}_{\tilde{C}}\tilde{Y} \ = -\tilde{H}\tilde{Y} - \tilde{\lambda}(\tilde{Y})\tilde{C}, \qquad \text{for} \qquad \tilde{X}, \tilde{Y} \in \mathcal{P}^{H}{}_0^1(\tilde{M}), \\ &\tilde{\nabla}_{\tilde{C}}\tilde{C} = \tilde{P}, \end{aligned} \tag{1.17}$$

where $\tilde{P}$ is a horizontal vector field defined by

$$\tilde{g}(\tilde{P}, \tilde{Z}) = \tilde{\lambda}(\tilde{Z}) \quad \text{for} \quad \tilde{Z} \in \mathcal{T}_0^1(\tilde{M}). \tag{1.18}$$

The first equation of (1.17) reduces to

$$\tilde{\nabla}_{\tilde{Y}}\tilde{X} = (\nabla_Y X)^L + \tilde{h}(\tilde{Y},\ \tilde{X})\tilde{C} \quad \text{for} \quad \tilde{X}, \tilde{Y} \in P^{H}{}_0^1(\tilde{M}), \tag{1.19}$$

where $X = p\tilde{X}$ and $Y = p\tilde{Y}$. Equation (1.19) is called the *co-Gauss equation,* and the second equation of (1.17) is called the *co-Weingarten equation.* As a consequence of (1.5) and (1.17), we have the equation

$$\tilde{h} = -\ \tilde{\Omega} = -\ (d\tilde{\eta})^H. \tag{1.20}$$

When the 1-form $\tilde{\lambda}$ and one of the second fundamental tensors $\tilde{h}$ and $\tilde{H}$ vanish identically in $\tilde{M}$, equations (1.17) reduce to

$$\tilde{\nabla}_Y\tilde{X} = (\nabla_{\tilde{Y}}X)^L,\ \tilde{\nabla}_{\tilde{Y}}\tilde{C} = 0,\ \tilde{\nabla}_{\tilde{C}}\tilde{X} = 0,\ \tilde{\nabla}_{\tilde{C}}\tilde{C} = 0$$

for any two elements $\tilde{X}$ and $\tilde{Y}$ of $\mathcal{P}^{H}{}_0^1(\tilde{M})$, where $X = p\tilde{X}$ and $Y = p\tilde{Y}$. As is well known, when the above equations are satisfied, the Riemannian manifold $\tilde{M}$ is locally a Pythagorean product of a Riemannian space and a straight line. In such a case, we say that the fibred space is *locally trivial.*

The field of horizontal planes defined by $\tilde{\eta} = 0$ is integrable if and only if $d\tilde{\eta} \equiv 0 \pmod{\tilde{\eta}}$, i.e., if and only if $\tilde{h} = -\tilde{\Omega} = -\ (d\tilde{\eta})^H = 0$ (Cf. (1.20)). Thus we have

Proposition 1.4. *In a fibred space $\tilde{M}$ with projectable Riemannian metric, the field of horizontal planes defined by $\tilde{\eta} = 0$ is integrable, if and only if the second fundamental tensor $\tilde{h}$ or $\tilde{H}$ vanishes identically in $\tilde{M}$.*

The vector field $\tilde{P}$ appearing in the last equation of (1.17) is the first curvature vector of the fibre. Thus the vector field $\tilde{P}$ vanishes identically in $\tilde{M}$ if and only if each of the fibres of $\tilde{M}$ is a geodesic. Thus, taking account of Proposition 1.4, we have

Proposition 1.5. *A fibred space with projectable Riemannian metric is locally trivial if and only if each of its fibres is a geodesic and the field of horizontal planes is integrable.*

When the curvature vector field $\tilde{P}$ of fibre vanishes identically in $\tilde{M}$, the given fibred space $\tilde{M}$ with projectable Riemannian metric $\tilde{g}$ reduces to that with invariant Riemannian metric $\tilde{g}$ in the sense of [9].

2. The tensor calculus in a fibred space with projectable Riemannian metric

Let $(\tilde{M}, M, \pi; \tilde{C}, \tilde{g})$ be a fibred space with projectable Riemannian metric $\tilde{g}$. Since the projection $\pi : \tilde{M} \to M$ is differentiable and of the maximum rank n everywhere, there exists, for any point $\tilde{P}$ of $\tilde{M}$, a coordinate neighborhood $\tilde{U}$ containing $\tilde{P}$ such that $U = \pi(\tilde{U})$ is a coordinate neighborhood of the point $P = \pi(\tilde{P})$ in M, and the intersection $F_Q \cap \tilde{U}$ is expressed in $\tilde{U}$ by equations $y^1 = a^1, \cdots, y^n = a^n$ with constants $a^1, \cdots, a^n$ with respect to certain coordinate system $(y^1, \cdots, y^{n+1})$ defined in $\tilde{U}$, where Q is an arbitrary point of U. We call such a neighborhood $\tilde{U}$ a *cylindrical neighborhood* of $\tilde{M}$. Since we restrict ourselves to cylindrical neighborhoods in $\tilde{M}$, we call them simply *neighborhoods* of $\tilde{M}$. Given a neighborhood $\tilde{U}$ in $\tilde{M}$, the set $(\tilde{U}, U, \pi; \tilde{C}, \tilde{g})$ is a fibred space with projectable Riemannian metric $\tilde{g}$, where $U = \pi(\tilde{U})$, and $\pi, \tilde{C}$ and $\tilde{g}$ denote respectively the restrictions of $\pi, \tilde{C}$ and $\tilde{g}$ given in $\tilde{M}$ to $\tilde{U}$. In the sequel, we shall identify the operations of taking horizontal parts, lifts, projections, etc. in $(\tilde{U}, U, \pi; \tilde{C}, \tilde{g})$ with the corresponding operations in $(\tilde{M}, M, \pi, \tilde{C}, \tilde{g})$ respectively.

Let (x^h) be coordinates defined in $\tilde{U}$ of $\tilde{M}$, and (ξ^a) coordinates defined in $U = \pi(\tilde{U}) \subset M$. (The indices h, i, j, k, m, s, t run over the range $\{1, 2, \cdots, n+1\}$, the indices a, b, c, d, e, f the range $\{1, 2, \cdots, n\}$, and the so-called Einstein summation convention is used with respect to these two systems of indices.) We denote by E^h and g_{ji} the components of the structure field $\tilde{C}$ and the projectable Riemannian metric $\tilde{g}$ with respect to (x^h) in $\tilde{U}$. Then the structure 1-form $\tilde{\eta}$ defined by (1.2) has in $\tilde{U}$ components of the form

$$E_i = g_{ih} E^h, \text{ i.e., } \tilde{\eta} = E_i dx^i. \tag{2.1}$$

Taking a point P with coordinates (ξ^a) arbitrarily in U, we may assume that $F_p \cap \tilde{U}$ be expressed by n equations

$$\xi^a = \xi^a(x^h) \tag{2.2}$$

in $\tilde{U}$, where n functions $\xi^a(x^h)$ are differentiable in $\tilde{U}$, and their Jacobian matrix $(\partial \xi^a / \partial x^i)$ is of the maximum rank n. Putting

$$E_i{}^a = \partial_i \xi^a, \tag{2.3}$$

where ∂_i denotes the operator $\partial_i = \partial / \partial x^i$, we see that n local covector fields $\tilde{\zeta}^a$ with components $E_i{}^a$ are linearly independent in $\tilde{U}$. Taking account of (1.1) and (1.2), we obtain

$$(2.4)\qquad E_iE^i = g_{ji}E^jE^i = 1, \quad E^iE_i{}^a = 0,$$

because the structure field $\tilde{C}$ is tangent to fibres. Thus the $n+1$ local covector fields $\tilde{\eta}$ and $\tilde{\zeta}^a$ are linearly independent.

Taking account of (2.4), we see that the inverse of the matrix $(E_i{}^a, E_i)$ has the form

$$(2.5)\qquad (E_i^a, E_i)^{-1} = \begin{pmatrix} E^h{}_b \\ E^h \end{pmatrix}.$$

where E_b^h, for each fixed index b, are components of a local vector field $\tilde{B}_b$ in $\tilde{U}$. Then the $n+1$ local vector fields $\tilde{B}_b$ and $\tilde{C}$ are linearly independent in $\tilde{U}$. The equation (2.5) is equivalent to the conditions

$$(2.6)\qquad \begin{aligned} E^i{}_bE_i{}^a &= \delta^a{}_b, &\quad E^i{}_bE_i &= 0,\\ E^iE_i{}^a &= 0, &\quad E^iE_i &= 1, \end{aligned}$$

that is,

$$(2.6')\qquad \tilde{\zeta}^a(\tilde{B}_b) = \delta_b^a, \quad \tilde{\eta}(\tilde{B}_b) = 0, \quad \tilde{\zeta}^a(\tilde{C}) = 0, \quad \tilde{\eta}(\tilde{C}) = 1,$$

or to the condition

$$(2.7)\qquad E_i{}^aE^h{}_a + E_iE^h = \delta_i^h.$$

The first and the second equations of (2.6) or (2.6)′ show that n local vector fields $\tilde{B}_b$ span the horizontal plane at each point of $\tilde{U}$.

Since the given Riemannian metric $\tilde{g}$ is projectable, taking account of the equations $\mathcal{L}\ g_{ji} = \tilde{\nabla}_jE_i + \tilde{\nabla}_iE_j$ and $(\tilde{\nabla}_jE_i)E^i = 0$, we can put

$$(2.8)\qquad \mathcal{L}\ g_{ji} = P_b(E_j{}^bE_i + E_jE_i{}^b)$$

in $\tilde{U}$, where P_b are certain n functions in $\tilde{U}$. Thus, as a consequence of the definition (1.2) of $\tilde{\eta}$, we obtain

$$(2.9)\qquad \mathcal{L}\ E_i = P_bE_i{}^b.$$

As was proved in [8], we find $\mathcal{L}\ E_i^a = 0$ and $\mathcal{L}\ E^h = 0$. Thus, taking account of (2.6), we find

$$(2.10)\qquad \begin{aligned} \mathcal{L}\ E^h{}_b &= -P_bE^h, &\quad \mathcal{L}\ E^h &= 0,\\ \mathcal{L}\ E_i{}^a &= 0, &\quad \mathcal{L}\ E_i &= P_bE_i{}^b. \end{aligned}$$

Horizontal parts. Let there be given a tensor field, say, $\tilde{T}$ of type (1,1) in the total space $\tilde{M}$. Then $\tilde{T}$ has components of the form

$$(2.11)\qquad \tilde{T}_i{}^h = T_b{}^aE_i{}^bE^h{}_a + T_b{}^0E_i{}^bE^h + T_0{}^aE_iE^h{}_a + T_0{}^0E_iE^h$$

in each neighborhood $\tilde{U}$ of $\tilde{M}$, where $T_b{}^a, T_b{}^0, T_0{}^a$ and $T_0{}^0$ are all functions in $\tilde{U}$. Then, taking account of (H.1) $\sim$ (H.4), we see that the horizontal part $\tilde{T}^H$ of $\tilde{T}$ has in $\tilde{U}$ components $\tilde{T}^H{}_i{}^h = T_b{}^a E_i{}^b E^h{}_a$.

Invariant functions. Let $\tilde{f}$ be an element of $\mathcal{P}^0_0(\tilde{M})$. We have, by definition, $\mathcal{L}\,\tilde{f} = 0$, which implies that $\tilde{f}$ is expressed as $\tilde{f} = f(\xi^a(x^h))$ in each neighborhood $\tilde{U}$ of $\tilde{M}$, where $\xi^a(x^h)$ are the functions appearing in equation (2.2) defining fibres. Thus we have $\partial_i\tilde{f} = E_i{}^a\partial_a\tilde{f}$ (Cf. [8]), where ∂_a denotes the operator $\partial_a = \partial/\partial\xi^a$. Any element $\tilde{f}$ of $\mathcal{P}^0_0(\tilde{M})$ is called an *invariant function* in $\tilde{M}$. We shall identify any invariant function $\tilde{f}$ with its projection $f = p\tilde{f}$ and denote the invariant function $\tilde{f}$ by the same symbol f as its projection.

Projectable tensor fields, projections and lifts. Let there be given a tensor field, say, $\tilde{T}$ of type (1,1) in the total space $\tilde{M}$. Then $\tilde{T}$ *is projectable if and only if it has in each neighborhood* $\tilde{U}$ *components* $\tilde{T}_i{}^h$ *of the form* (2.11) *with invariant functions* $T_b{}^a$, *i.e., if and only if* $\mathcal{L}\ T_b{}^a = 0$. Thus, taking account of (P.1) $\sim$ (P.4), we easily see that for any projectable tensor field, say, $\tilde{T}$ of type (1,1), its projection $T = p\tilde{T}$ has components $T_b{}^a(\xi)$ with respect to coordinates (ξ^a) defined in $U = \pi(\tilde{U})$.

Let there be given a tensor field, say, T of type (1, 1) in the base space M, and $T_b{}^a$ its components in $U = \pi(\tilde{U})$. Then, taking account of (L.1) $\sim$ (L.4), we easily see that the *lift* T^L *of* T *has components of the form*

$$T_i{}^h = T_b{}^a E_i{}^b E^h{}_a \tag{2.12}$$

with respect to coordinates (x^h) defined in $\tilde{U}$, where $T_i{}^h$ appearing in (2.12) denotes the lift of $T_b{}^a$.

Projectable Riemannian metric. If we put

$$g_{cb} = g_{ji}E^j{}_c E^i{}_b, \tag{2.13}$$

then g_{cb} are invariant functions in $\tilde{U}$ by virture of (2.6) and (2.8). Thus, the projection $g = p\tilde{g}$ has components g_{cb} in $U = \pi(\tilde{U})$. Taking account of (2.6) and (2.7), we have the formula

$$g_{ji} = g_{cb}E_j{}^c E_i{}^b + E_j E_i. \tag{2.14}$$

If we define g^{ih} by the equation

$$(g^{ih}) = (g_{ji})^{-1}, \text{ i.e., } g_{ji}g^{ih} = \delta^h_j, \tag{2.15}$$

then g^{ih} are components of an element $\tilde{G}$ of $\mathcal{P}^2_0(\tilde{M})$ in $\tilde{U}$. If we define g^{ba} by the equation

$$(g^{ba}) = (g_{cb})^{-1}, \text{ i.e., } g_{cb}g^{ba} = \delta^a_c, \tag{2.16}$$

then g^{ba} are invariant functions in $\tilde{U}$, and hence the projection $G = p\tilde{G}$ has components g^{ba} in $U = \pi(\tilde{U})$. We have the formulas

$$g^{ba} = g^{ih} E_i{}^b E_h{}^b, \quad g^{ih} = g^{ba} E^i{}_b\, E^h{}_a + E^i E^h. \tag{2.17}$$

Moreover, taking account of (2.6), we find the following formulas:

$$\begin{aligned} E_i^{\ a} &= g_{ih} g^{ba} E^h{}_a, & E_i &= g_{ih} E^h, \\ E^h{}_b &= g^{hi} g_{ba} E_i^{\ a}, & E^h &= g^{hi} E_i. \end{aligned} \tag{2.18}$$

The curvature vector field of fibres. The 1-from $\tilde{\lambda}$ defined by (1.16) being horizontal, $\tilde{\lambda}$ has components of the form

$$P_i = P_b E_i^{\ b}, \text{ i.e., } \tilde{\lambda} = (P_b E_i^{\ b}) dx^i \tag{2.19}$$

in each neighborhood $\tilde{U}$ of $\tilde{M}$, and the curvature vector field $\tilde{P}$ of fibres, which is defined by (1.18), has components of the form

$$\tilde{P}^h = P^a E_h{}^a, \quad P^a = g^{ab} P_b, \tag{2.20}$$

where P_b are functions appearing in (2.10).

3. The Riemannian connection and the induced connection

The Riemannian connection $\tilde{\nabla}$ determined by the projectable Riemannian metric $\tilde{g}$ has the Christoffel's symbols $\left\{ {h \atop j\ i} \right\}$ constructed from g_{ji} as its coefficients in each neighborhood $\tilde{U}$ of the total space $\tilde{M}$. For any vector field $\tilde{X}$ in $\tilde{M}$, its covariant derivative $\tilde{\nabla}\tilde{X}$ has components of the form $\tilde{\nabla}_j \tilde{X}^h = \partial_j \tilde{X}^h + \left\{ {h \atop j\ i} \right\} \tilde{X}^i$ in $\tilde{U}$, $\tilde{X}^h$ being the components of $\tilde{X}$ in $\tilde{U}$.

If we take account of (1.17), (2.19) and (2.20), we can put in $\tilde{U}$

$$\begin{aligned} \tilde{\nabla}_j E^h{}_b &= \Gamma^a_{cb} E_j^{\ c} E^h_a + h_{cb} E_j^{\ c} E^h - h_b{}^a E_j E^h_a - P_b E_j E^h, \\ \tilde{\nabla}_j E^h &= -h_c{}^b E_j^{\ c} E^h_b + P^b E_j E^h_b, \end{aligned} \tag{3.1}$$

where P_b are the functions appearing in (2.10), P^a are defined by $P^a = g^{ab} P_b$ and Γ^a_{cb}, h_{cb} and $h_b{}^a$ are certain functions in $\tilde{U}$. We note here that the functions Γ^a_{cb} has the symmetry property $\Gamma^a_{cb} = \Gamma^a_{bc}$. Comparing (1.17) with (3.1), we see that the second fundamental tensors $\tilde{h}$ and $\tilde{H}$ have respectively the components of the form

$$\tilde{h}_{ji} = h_{cb} E_j{}^c E_i{}^b, \quad \tilde{h}_i{}^h = h_b{}^a E_j{}^b E^h_a. \tag{3.2}$$

Therefore, as consequences of Proposition 1.3, we have the following equations:

(3.3) $$\tilde{h}_{ji} + \tilde{h}_{ij} = 0, \qquad h_{cb} + h_{bc} = 0,$$

(3.4) $$\tilde{h}_{ji} = \tilde{h}_j{}^h g_{hi}, \qquad h_{cb} = h_c{}^a g_{ba}.$$

Equations (3.1) are called the *co-Gauss equation* and the *co-Weingarteu equation* respectively. Differentiating covariantly both sides of equations (2.6), we have, as consequences of (2.6) and (3.1),

(3.5)
$$\tilde{\nabla}_j E_i{}^a = -\Gamma^a{}_{cb} E_j{}^c E_i{}^b + h_c{}^a E_j{}^c E_i + h_b{}^a E_j E_i{}^b - P^a E_j E_i,$$
$$\tilde{\nabla}_j E_i = -h_{cb} E_j{}^c E_i{}^b + P_b E_j E_i{}^b.$$

Let $\tilde{X}$ and $\tilde{Y}$ be two vector fields with components $\tilde{X}^h$ and $\tilde{Y}^h$ in $\tilde{M}$ respectively. Then, in each neighborhood $\tilde{U}$ of $\tilde{M}$, we have $\tilde{X}^h = X^a E^h{}_a + X^0 E^h$ and $\tilde{Y}^h = Y^a E^h{}_a + Y^0 E^h$, where X^a, X^0 and Y^a, Y^0 are certain functions in $\tilde{U}$. Taking account of (3.1), we see that $\tilde{\nabla}_{\tilde{Y}} \tilde{X}$ has components of the form

(3.6)
$$(\tilde{\nabla}_{\tilde{Y}} \tilde{X})^h = Y^c(\nabla_c X^a - h_c{}^a X^0) E^h{}_a + Y^0(\partial_0 X^a - h_b{}^a X^b + P_b X^b) E^h{}_a$$
$$+ Y^c(\partial_c X^0 + h_{cb} X^b) E^h + Y^0(\partial_0 X^0 - P_b X^b) E^h,$$

∂_a and ∂_0 being defined by $\partial_a = E^h{}_a \partial_h$ and $\partial_0 = E^h \partial_h$, and $\nabla_c X^a$ by

(3.7) $$\nabla_c X^a = \partial_c X^a + \Gamma^a_{cb} X^b.$$

We shall prove that the functions Γ^a_{cb} are invariant in $\tilde{U}$. To do this, we shall first find the Lie derivative $\mathcal{L} \left\{ {h \atop j\ i} \right\}$ of the Christoffel's symbols $\left\{ {h \atop j\ i} \right\}$. As is well known, the Lie derivative of $\left\{ {h \atop j\ i} \right\}$ is given by the equation

(3.8) $$\mathcal{L} \left\{ {h \atop j\ i} \right\} = \frac{1}{2} g^{hk} [\tilde{\nabla}_j(\mathcal{L}\, g_{ik}) + \tilde{\nabla}_i(\mathcal{L}\, g_{jk}) - \tilde{\nabla}_k(\mathcal{L}\, g_{ji})]$$

in each neighborhood $\tilde{U}$ (Cf. [6]). On the other hand, (2.8) reduces to $\mathcal{L}\, g_{ji} = \tilde{P}_j E_i + \tilde{P}_i E_j$, $\tilde{P}_j$ being defined by $\tilde{P}_j = P_b E_j{}^b$. Thus we have

(3.9)
$$\mathcal{L} \left\{ {h \atop j\ i} \right\} = \frac{1}{2}(\tilde{\nabla}_j \tilde{P}_i + \tilde{\nabla}_i \tilde{P}_j) E^h + \tilde{Q}_j{}^h E_i + \tilde{Q}_i{}^h E_j$$
$$- (h_c{}^a P_b + h_b{}^a P_c) E_j{}^c E_i{}^b E^h{}_a + (E_j \tilde{P}_i + E_i \tilde{P}_j) P^h - \tilde{P}_j \tilde{P}_i E^h,$$

by virtue of the second equation of (3.5), where we have put

(3.10) $$\tilde{Q}_{ji} = \frac{1}{2}(\tilde{\nabla}_j \tilde{P}_i - \tilde{\nabla}_i \tilde{P}_j), \quad \tilde{Q}_j^{\ h} = \tilde{Q}_{ji} g^{ih}, \quad \tilde{P}^h = \tilde{P}_i g^{ih}.$$

As is well known, the identities

$$\mathcal{L}(\tilde{\nabla}_j\tilde{X}^h)-\tilde{\nabla}_j(\mathcal{L}\tilde{X}^h) = \left(\mathcal{L}\left\{ {h \atop j\, i} \right\}\right)\tilde{X}^i,\ \mathcal{L}(\tilde{\nabla}_j\tilde{\omega}_i)-\tilde{\nabla}_j(\mathcal{L}\tilde{\omega}_i) = -\left(\mathcal{L}\left\{ {h \atop ji} \right\}\right)\tilde{\omega}_h$$

hold for any vector field $\tilde{X}^h$ and any covector field $\tilde{\omega}_i$ in $\tilde{M}$ (Cf. [6]). Thus we have, by virtue of (2.10) and (3.9),

$$\text{(3.11)}\quad \mathcal{L}(\tilde{\nabla}_j E_i{}^a) = (h_c{}^a P_b + h_b{}^a P_c)E_j{}^c E_i{}^b - \tilde{Q}_j{}^h E_i E_h{}^a - \tilde{Q}_i{}^h E_j E_h{}^a - (E_j\tilde{P}_i + E_i\tilde{P}_j)P^a,$$

$$\text{(3.12)}\quad \mathcal{L}(\tilde{\nabla}_j E^h) = \frac{1}{2}(\tilde{\nabla}_j\tilde{P}_i + \tilde{\nabla}_i\tilde{P}_j) + \tilde{Q}_j{}^h + E_j(E^i\tilde{Q}_i{}^h) + \tilde{P}_j\tilde{P}^h.$$

Applying the operator $\mathcal{L}$ to the two equations of (3.5), we find respectively

$$\text{(3.13)}\quad \begin{aligned}\mathcal{L}(\tilde{\nabla}_j E_i{}^a) = &-(\mathcal{L}\,\Gamma^a_{cb})E_j{}^c E_i{}^b + (h_c{}^a P_b + h_b{}^a P_c)E_j{}^c E_i{}^b \\ &+(\mathcal{L}\,h_c{}^a)E_j{}^c E_i + (\mathcal{L}\,h_b{}^a)E_j E_i{}^b - (\mathcal{L}\,P^a)E_j E_i \\ &-P^a P_b(E_j{}^b E_i + E_j E_i{}^b),\end{aligned}$$

$$\text{(3.14)}\quad \begin{aligned}\mathcal{L}(\tilde{\nabla}_j E^h) = &-(\mathcal{L}\,h_c{}^a)E_j{}^c E^h{}_a + P_j P^h + (h_c{}^a P_a)E_j{}^c E^h \\ &+(\mathcal{L}\,P^a)E_j E^h{}_a - (P_a P^a)E_j E^h.\end{aligned}$$

If we compare the right-hand sides of (3.11), (3.12) with those of (3.13), (3.14) respectively, we obtain the following equations:

$$\text{(3.15)}\quad \mathcal{L}\,\Gamma^a_{cb} = 0,$$

$$\text{(3.16)}\quad \begin{aligned}&\mathcal{L}\,h_{cb} = -\tilde{Q}_{ji}E^j{}_c E^i{}_b, &&\mathcal{L}\,P_b = 2\tilde{Q}_{ji}E^j E^i{}_b\\ &h^a_c P_a = (\tilde{\nabla}_j\tilde{P}_i)E^j{}_c E^i, &&P_a P^a = -(\tilde{\nabla}_j\tilde{P}_i)E^j E^i.\end{aligned}$$

Equation (3.15) shows that the functions Γ^a_{cb} are invariant in each neighborhood $\tilde{U}$ of $\tilde{M}$.

Let X and Y be two arbitrary vector fields in the base space M. Then, because of (3.6), $\tilde{\nabla}_{Y^L}X^L$ has in $\tilde{U}$ components of the form

$$(\tilde{\nabla}_{Y^L}X^L)^h = (Y^c\nabla_c X^a)E^h{}_a + (h_{cb}Y^c X^b)E^h,$$

X^a and Y^a being the components of X and Y in $U = \pi(\tilde{U})$. If we take account of (3.7) and (3.15), we find $\mathcal{L}(Y^c\nabla_c X^a) = 0$, which is equivalent to the condition $(\mathcal{L}(\tilde{\nabla}_{Y^L}X^L)^H)^H = 0$. Therefore the Riemannian connection $\tilde{\nabla}$ is projectable. Thus, as a consequence of the definition (1.9) of the induced connection ∇, we see that the vector field $\nabla_Y X$ has components of the form

$$Y^c\nabla_c X^a = Y^c(\partial_c X^a + \Gamma^a_{cb}X^b)$$

in U, where Γ^a_{cb} appearing in the equation above are the projections of the invariant function Γ^a_{cb} appearing in (3.1) and (3.5). Consequently, the induced connection ∇ has Γ_{cb}^a as its coefficients in U. On the other hand, the equation $\nabla g = 0$ is a direct consequence of $\tilde{\nabla}\tilde{g} = 0$. Therefore, we obtain the equation

$$\Gamma_{cb}^a = \left\{ {a \atop cb} \right\}, \tag{3.17}$$

because of $\Gamma_{cb}^a = \Gamma_{bc}^a$ and $\nabla g = 0$, $\left\{ {a \atop cb} \right\}$ being the Christoffel's symbols determined by g_{cb}. Thus we have

Proposition 3.1. *In a fibred space with projectable Riemannian metric $\tilde{g}$, the Riemannian connection $\tilde{\nabla}$ determined by $\tilde{g}$ is projectable and the induced connection ∇ coincides with the Riemannian connection determined by the induced metric $g = p\tilde{g}$.*

Taking account of (3.10) and (3.16), we have

Proposition 3.2. *In a fibred space with projectable Riemannian metric, the second fundamental tensors $\tilde{h}$ and $\tilde{H}$ are projectable if and only if $(d\tilde{\lambda})^H = 0$, and the curvature vector field $\tilde{P}$ of fibres is projectable if and only if $(d\tilde{\lambda})^V = 0$, where $\tilde{\lambda}$ and $\tilde{P}$ are defined respectively by (1.16) and (1.18). Both $\tilde{h}$ and $\tilde{P}$ are projectable if and only if the 1-form $\tilde{\lambda}$ is closed.*

Proposition 3.3. *In a fibred space with projectable Riemannian metric, $\tilde{P} = 0$ holds if and only if $(\tilde{V}_{\tilde{C}}\tilde{P})^V = 0$; $\tilde{h}(\tilde{X}, \tilde{P}) = 0$, or, equivalently, $\tilde{\lambda}(\tilde{H}\tilde{X}) = 0$ holds for an element $\tilde{X}$ of $\mathcal{T}_0^1(\tilde{M})$ if and only if $(\tilde{\nabla}_{\tilde{X}^H}\tilde{P})^V = 0$.*

Van der Waerden-Bortolotti covariant derivatives. Let there be given an element of the formal tensor product $\mathcal{T}(\tilde{M}) \sharp \mathcal{T}^H(\tilde{M})$, say, $\overset{*}{T}$ belonging to $\mathcal{T}_1^1(\tilde{M}) \sharp \mathcal{T}^H{}_1^1(\tilde{M})$. Then $\overset{*}{T}$ is expressed as follows: $\overset{*}{T} = \overset{*}{T}{}_k{}^j{}_b{}^a \tilde{e}^k \sharp \tilde{e}_j \sharp \tilde{\zeta}^b \sharp \tilde{B}_a$ in each neighborhood $\tilde{U}$ of $\tilde{M}$, $\overset{*}{T}{}_k{}^j{}_b{}^a$ being certain functions in $\tilde{U}$, where $\{\tilde{e}_j\} = \{\partial/\partial x^j\}$ is the natural frame of coordinates (x^h) defined in $\tilde{U}$, $\{\tilde{e}^k\}$ the dual base to $\{\tilde{e}_j\}$, $\tilde{B}_a$ local vector fields with components $E^h{}_a$, and $\tilde{\zeta}^b$ the local covector fields with components $E_i{}^b$, all in $\tilde{U}$. We call $\overset{*}{T}{}_k{}^j{}_b{}^a$ the *components* of $\overset{*}{T}$ in $(U, \tilde{U})$, where $U = \pi(\tilde{U})$. Let $i : \mathcal{T}(\tilde{M}) \sharp \mathcal{T}^H(\tilde{M}) \to \mathcal{T}(\tilde{M})$ be the linear homomorphism defined by (I.1) in §1. Then the image $\tilde{T} = i(\overset{*}{T})$ has in $\tilde{U}$ components of the form

$$\tilde{T}_k{}^j{}_i{}^h = \overset{*}{T}{}_k{}^j{}_b{}^a E_i{}^b E^h{}_a, \tag{3.18}$$

and the van der Waerden-Bortolotti covariant derivative $\overset{*}{\nabla}\overset{*}{T}$ of $\overset{*}{T}$ has, in $(U, \tilde{U})$, components of the form

$$\begin{aligned} \overset{*}{\nabla}_l \overset{*}{T}{}_k{}^j{}_b{}^a = \partial_l \overset{*}{T}{}_k{}^j{}_b{}^a + \left\{ {j \atop lm} \right\} \overset{*}{T}{}_k{}^m{}_b{}^a - \left\{ {m \atop lk} \right\} \overset{*}{T}{}_m{}^j{}_b{}^a \\ + E_k{}^d \left(\left\{ {a \atop de} \right\} \overset{*}{T}{}_k{}^j{}_b{}^e - \left\{ {e \atop db} \right\} \overset{*}{T}{}_k{}^j{}_e{}^a \right), \end{aligned} \tag{3.19}$$

by virtue of (W.1) $\sim$ (W.4) given in §1 (Cf. [8], [9]). We put conventionally in $\tilde{U}$

$$\overset{*}{\nabla}_j E^h{}_b = \partial_j E^h{}_b + \left\{ {h \atop j\ i} \right\} E^i{}_b - \left\{ {a \atop cb} \right\} E_j{}^c E^h{}_a, \tag{3.20}$$
$$\overset{*}{\nabla}_j E_i{}^a = \partial_j E^i{}_a - \left\{ {h \atop j\ i} \right\} E_h{}^a + \left\{ {a \atop cb} \right\} E_j{}^c E_i{}^b,$$

which are the components of $(\tilde{\nabla}\tilde{B}_b)^V and (\tilde{\nabla}\tilde{\zeta}^a)^V$ respectively. If we take account of (3.18), (3.19) and (3.20), we have the formula

(3.21)
$$\tilde{\nabla}_l \tilde{T}_k{}^j{}_i{}^h = \tilde{\nabla}_l (\overset{*}{T}{}_k{}^j{}_b{}^a E_i{}^b E^h{}_a)$$
$$= (\overset{*}{\nabla}_l \overset{*}{T}{}_k{}^j{}_b{}^a) E_i{}^b E^h{}_a + \overset{*}{T}{}_k{}^j{}_b{}^a (\overset{*}{\nabla}_l E_i{}^b) E^h{}_a + \overset{*}{T}{}_k{}^j{}_b{}^a E_i{}^b (\overset{*}{\nabla}_l E^h{}_a),$$

which are the components of $\tilde{\nabla}\tilde{T} = \tilde{\nabla}_i(\overset{*}{T})$ in $\tilde{U}$. The first equations of (3.1) and (3.5) reduce respectively to

$$\overset{*}{\nabla}_j E^h{}_b = h_{cb} E_j{}^c E^h - h_b{}^a E_j E^h{}_a - P_b E_j E^h, \tag{3.22}$$
$$\overset{*}{\nabla}_j E_i{}^a = h_c{}^a E_j{}^c E_i + h_b{}^a E_j E_i{}^b - P^a E_j E_i,$$

which are the co-Gauss equations.

Let there be given an element of $\mathcal{T}^H(\tilde{M})$, say, $\overset{*}{S}$ belonging to $\mathcal{T}^H{}^1_1(\tilde{M})$. Let $S_b{}^a$ be the components of $\overset{*}{S}$. Then $\overset{*}{\nabla}\overset{*}{S}$ has, in $(U, \tilde{U})$, components of the form

$$\overset{*}{\nabla}_j \overset{*}{S}{}_b{}^a = E_j{}^c \nabla_c S_b{}^a + E_j \partial_0 S_b{}^a, \tag{3.23}$$

because of (3.19), where we have put

$$\nabla_c S_b{}^a = \partial_c S_b{}^a + \left\{ {a \atop ce} \right\} S_b{}^e - \left\{ {e \atop cb} \right\} S_e{}^a,$$

the operators ∂_c and ∂_0 being defined by $\partial_c = E^j{}_c \partial_j$ and $\partial_0 = E^j \partial_j$. On putting $\tilde{S} = i(\overset{*}{S})$, we have $(\tilde{\nabla}\tilde{S})^H = (i(\overset{*}{\nabla}\overset{*}{S}))^H$, which shows that $(\tilde{\nabla}\tilde{S})^H$ has components of the form $(\nabla_c S_b{}^a) E_j{}^c E_i{}^b E^h{}_a$. Therefore we see that for an element $\overset{*}{S}$ of $\mathcal{P}^H{}^1_1(\tilde{M})$, the projection $p(\tilde{\nabla} i(\overset{*}{S})) = \nabla(p\overset{*}{S})$ has components of the form $\nabla_c S_b{}^a$ in U.

The Ricci formulas. As is well known, we have the Ricci formula

$$\tilde{\nabla}_k \tilde{\nabla}_j \tilde{X}^h - \tilde{\nabla}_j \tilde{\nabla}_k \tilde{X}^h = \tilde{K}_{kji}{}^h \tilde{X}^i \tag{3.24}$$

for any element $\tilde{X}$ of $\mathcal{T}^1_0(\tilde{M})$, $\tilde{X}^h$ being the components of $\tilde{X}$, where $\tilde{K}_{kji}{}^h$ denote the components of the curvature tensor $\tilde{K}$ of the projectable Riemannian metric $\tilde{g}$ given in $\tilde{M}$.

For any element of $\mathcal{T}(\tilde{M}) \sharp \mathcal{T}^H(\tilde{M})$, say, $\overset{*}{T}$ belonging to $\mathcal{T}^1_0(\tilde{M}) \sharp \mathcal{T}^H{}^0_1(\tilde{M})$, we have the formula, by virtue of (3.19),

$$\overset{*}{\nabla}_k \overset{*}{\nabla}_j T^h{}_b - \overset{*}{\nabla}_j \overset{*}{\nabla}_k T^h{}_b = \tilde{K}_{kji}{}^h T^i{}_b - E^d_k E^c_j K_{dcb}{}^a T^h{}_a, \tag{3.25}$$

where $T^h{}_b$ are the components of $\overset{*}{T}$ in $(U, \tilde{U})$, and

$$(3.26) \qquad K_{dcb}{}^a = \partial_d \left\{ {a \atop cb} \right\} - \partial_c \left\{ {a \atop db} \right\} + \left\{ {a \atop de} \right\} \left\{ {e \atop cb} \right\} - \left\{ {a \atop ce} \right\} \left\{ {e \atop db} \right\}$$

are invariant functions in $\tilde{U}$, the components of the curvature tensor K in U. We denote by $K_{kji}{}^h = K_{dcb}{}^a E_k{}^d E_j{}^c E_i{}^b E^h{}_a$ the components of the lift K^L of K. The formula (3.25) is the *Ricci formula for the van der Waerden-Bortolotti covariant differentiation.*

4. Geodesics

We have studied in [9] the behaviour of geodesics in a fibred space with invariant Riemannian metric $\tilde{g}$, and proved six Propositions 4.1~4.6, where we assumed the condition $\mathcal{L}\tilde{g} = 0$. In our present case, we can show that these six Propositions 4.1~4.6, except Proposition 4.2, are all valid for any fibred space with projectable Riemannian metric. We can prove now the following proposition instead of Proposition 4.2 stated in [9].

Proposition. *In a fibred space $\tilde{M}$ with projectable Riemannian metric, the projection of a geodesic given arbitrarily in $\tilde{M}$ is also a geodesic in the base space M with respect to the induced metric, if and only if the second fundamental tensor $\tilde{h}$ or $\tilde{H}$ vanishes identically.*

5. Structure equations and curvatures

Let there be given a fibred space with projectable Riemannian metric $\tilde{g}$. Then, by a similar device as given in §4 of [9], taking account of (3.20) and (3.22), we can prove the following *structure equations*:

$$(5.1) \qquad \tilde{K}_{dcb}{}^a - K_{dcb}{}^a = (h_{db}h_c{}^a - h_{cb}h_d{}^a) + 2h_{dc}h_b{}^a,$$

$$(5.2) \qquad \tilde{K}_{dcb}{}^0 = (\nabla_d h_{cb} - \nabla_c h_{db}) + 2h_{dc}P_b,$$

$$(5.3) \qquad \tilde{K}_{0cb}{}^0 = \partial_0 h_{cb} + h_{ce}h_b{}^e + \nabla_c P_b - P_c P_b$$

by virtue of the Ricci formula (3.24) and (3.25), where we have put

$$\tilde{K}_{dcb}{}^a = \tilde{K}_{kji}{}^h E^k{}_d E^j{}_c E^i{}_b E_h{}^a, \quad \tilde{K}_{dcb}{}^0 = \tilde{K}_{kji}{}^h E^k{}_d E^j{}_c E^i{}_b E_h,$$
$$\tilde{K}_{0cb}{}^0 = \tilde{K}_{kji}{}^h E^k E^j{}_c E^i_b E_h,$$

$\tilde{K}_{kji}{}^h$ being the components of the curvature tensor $\tilde{K}$ of $\tilde{g}$. Equations (5.1), (5.2) and (5.3) are called the *co-Gauss equation*, the *co-Codazzi equation* and the *co-Ricci equation* respectively.

If we take account of the well known identity $\tilde{K}_{kji}{}^h + \tilde{K}_{jik}{}^h + \tilde{K}_{ikj}{}^h = 0$, we find the identities

$$(5.4) \qquad \begin{aligned} &\nabla_d h_{cb} + \nabla_c h_{bd} + \nabla_b h_{dc} + h_{dc}P_b + h_{cb}P_d + h_{bd}P_c = 0, \\ &\partial_0 h_{cb} + \frac{1}{2}(\nabla_c P_b - \nabla_b P_c) = 0 \end{aligned}$$

because of (5.2) and (5.3) respectively. The identities (5.4) are equivalent to the identity $d(d\tilde{\eta}) = 0$, where $\tilde{\eta}$ is the structure 1-form. Equations (5.2) and (5.3) reduce respectively to

$$(5.5)\qquad \begin{aligned} \tilde{K}_{dcb}{}^0 &= -\nabla_b h_{dc} + h_{dc}P_b + h_{bc}P_d - h_{bd}P_c, \\ \tilde{K}_{0cb}{}^0 &= h_{ce}h_b{}^e + \frac{1}{2}(\nabla_c P_b + \nabla_c P_b) - P_c P_b \end{aligned}$$

by virtue of (5.4). Taking account of (5.1), we have

Proposition 5.1. *If, in a fibred space with projectable Riemannian metric $\tilde{g}$, the second fundamental tensors are projectable, then the curvature tensor $\tilde{K}$ of $\tilde{g}$ is also projectable. When $\tilde{K}$ is projectable, the equality $p\tilde{K} = K$ holds if and only if the second fundamental tensors vanish identically, where K denotes the curvature tensor of the induced metric $g = p\tilde{g}$.* (Cf. Proposition 5.1 in [9]).

Denote by $\tilde{\gamma}(\tilde{X}, \tilde{Y})$ the sectional curvature with respect to the section determined by two vectors $\tilde{X}$ and $\tilde{Y}$ in $\tilde{M}$, and by $\gamma(X, Y)$ the sectional curvature with respect to the section determined by two vectors X and Y in M. Then, taking account of (5.1), we find

$$(5.6)\qquad (\gamma(X,Y))^L - \tilde{\gamma}(X^L, Y^L) = 3\{h(X^L, Y^L)\}^2(|X \wedge Y|^2)^L \geq 0$$

for any two vector fields X and Y in M, where $|X \wedge Y|$ denotes the magnitude of the bivector $X \wedge Y$ in M. Therefore we have

Proposition 5.2. *In a fibred space $\tilde{M}$ with projectable Riemannian metric, $\tilde{g}$, the inequality*

$$(\gamma(X,Y))^L \geq \tilde{\gamma}(X^L, Y^L) \quad \text{for} \quad X, Y \in \mathcal{T}_0^1(\tilde{M})$$

holds. The equality $(\gamma(X,Y))^L = \tilde{\gamma}(X^L, Y^L)$ holds for any two elements X and Y of $\mathcal{T}_0^1(\tilde{M})$ if and only if the second fundamental tensor vanishes identically in $\tilde{M}$ (Cf. [3]).

6. Fibred spaces with invariant Riemannian metric

Let there be given a fibred space $(\tilde{M}, M, \pi; \tilde{C}, \tilde{g})$ with Riemannian metric $\tilde{g}$. When the condition

$$(6.1)\qquad \mathcal{L}_{\rho\tilde{C}}\tilde{g} = 0$$

is satisfied, ρ being a certain function positive everywhere in $\tilde{M}$, the fibred space is called a *fibred space with invariant Riemannian metric* $\tilde{g}$. (In a previous paper [9], we meant by a fibred space with invariant Riemannian metric $\tilde{g}$ a fibred space satisfying the condition $\mathcal{L}_{\tilde{C}}\tilde{g} = 0$.) The condition (6.1) reduces to

$$\mathcal{L}\, g_{ji} = \tilde{P}_j E_i + \tilde{P}_i E_j, \quad \tilde{P}_j = -\partial_j \log \rho, \tag{6.2}$$

$\mathcal{L}$ denoting the Lie derivation with respect to $\tilde{C}$. Taking account of (2.8), we get $\tilde{P}_j = P_c E_j{}^c$, which implies $\mathcal{L}\rho = 0$, i.e., that the function ρ is invariant in $\tilde{M}$. Thus we have

Proposition 6.1. *Any fibred space with invariant Riemannian metric $\tilde{g}$ is a fibred space with projectable Riemannian metric $\tilde{g}$. In a fibred space with invariant Riemannian metric, the curvature vector field $\tilde{P}$ of fibres is projectable, and its components are given by $\tilde{P}_j = -\partial_j \log \rho$, where ρ is the function appearing in* (6.1), *and is an invariant function in $\tilde{M}$.*

In our case, $\tilde{P}_j$ is a gradient. Thus, taking account of Propositions 3.2 and 5.1, we have

Proposition 6.2. *In a fibred space with invariant Riemannian metric $\tilde{g}$, the second fundamental tensors $\tilde{h}$ and $\tilde{H}$ are projectable and the curvature tensor $\tilde{K}$ of $\tilde{g}$ is also projectable.*

As a consequence of (3.5), we have $d\tilde{\eta} = -\tilde{h} - \tilde{\eta} \wedge d(\log \rho)$, which implies $d(\rho^{-1}\tilde{\eta}) = -\rho^{-1}\tilde{h}$, and consequently

$$d(\rho^{-1}\tilde{h}) = 0, \text{ i.e., } \nabla_d(\rho^{-1}h_{cb}) + \nabla_c(\rho^{-1}h_{bd}) + \nabla_b(\rho^{-1}h_{dc}) = 0.$$

The last equation is a consequence of (5.4), and $P_c = -\partial_c \log \rho$. The cohomology class determined in the base space M by the closed form $\rho^{-1}h_{cb}d\xi^c \wedge d\xi^b$ is called the *characteristic class* of the given fibred space with invariant Riemannian metric.

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