Fibring Non-Truth-Functional Logics: Completeness Preservation

C. Caleiro[1], W.A. Carnielli[2], M.E. Coniglio[2], A. Sernadas[1] and C. Sernadas[1]
[1] *CLC, Department of Mathematics, IST, UTL, Portugal*
[2] *CLE, Department of Philosophy, IFCH, UNICAMP, Brazil*

Abstract. Fibring has been shown to be useful for combining logics endowed with truth-functional semantics. However, the techniques used so far are unable to cope with fibring of logics endowed with non-truth-functional semantics as, for example, paraconsistent logics. The first main contribution of the paper is the development of a suitable abstract notion of logic, that may also encompass systems with non-truth-functional connectives, and where fibring can still be dealt with. Furthermore, it is shown that this extended notion of fibring preserves completeness under certain reasonable conditions. This completeness transfer result, the second main contribution of the paper, generalizes the one established in (Zanardo et al., 2001) but is obtained using new techniques that explore the properties of a suitable meta-logic (conditional equational logic) where the (possibly) non-truth-functional valuations are specified. The modal paraconsistent logic of (da Costa and Carnielli, 1988) is studied in the context of this novel notion of fibring and its completeness is so established.

Keywords: non-truth-functional logics, fibring, completeness.

1. Introduction

In recent years, the problem of combining logics has gained the attention of many researchers in mathematical logic. Besides leading to very interesting applications whenever it is necessary to work with different logics at the same time, combinations of logics are also of great interest on purely theoretical grounds (Blackburn and de Rijke, 1997).

The practical impact of the problem is clear, at least from the point of view of those working in knowledge representation (within artificial intelligence) and in formal specification and verification (within software engineering). Namely, in a knowledge representation problem it may be necessary to work with both temporal and deontic aspects. And in a software specification problem it may be necessary to work with both equational and temporal specifications. Indeed, in these fields, the need for working with several formalisms at the same time is the rule rather than the exception. We refer the reader, for instance, to (Finger and Gabbay, 1992; Goguen and Burstall, 1992; Sannella and Tarlecki, 1993; Astesiano and Cerioli, 1994) for a discussion of this and related problems, including some early attempts at their solution.

Obviously, an approach to the combination of logics will be of significance only if general preservation results are available. For example, if it had been established that completeness is preserved by the combination mechanism $\circ$ and it is known that logic $\mathcal{L}$ is given by $\mathcal{L}' \circ \mathcal{L}''$, then the completeness of the combination $\mathcal{L}$ would follow from the completeness of $\mathcal{L}'$ and $\mathcal{L}''$. No wonder that so much theoretical effort has been dedicated to establishing preservation results and/or finding preservation counterexamples within the community of logicians working in the problem of combining logics.

Among the different techniques for combining logics, *fibring* (Gabbay, 1996; Gabbay, 1998; Sernadas et al., 1999; Sernadas et al., 2000; Zanardo et al., 2001) deserves close attention. But what is fibring? The answer can be given in a few paragraphs for the special case of logics with a propositional base, that is, with propositional variables and connectives of arbitrary arity. Fibring is a mechanism that produces a new logic by mixing up two given logics. As mentioned above, ideally, the fibred logic would inherit the properties, namely completeness (soundness and adequacy), of its two component logics. Unfortunately, it is well known that it is not always the case. Still, it is sometimes possible to recover some lost property by further manipulation of the fibred logic. Let us first explain the mechanism of fibring by itself and delay the issue of preservation of properties for a few paragraphs.

The language of the fibring is obtained by the free use of the language constructors (atomic symbols and connectives) from the given logics. For example, when fibring a temporal logic and a deontic logic, mixed formulae like $((G\alpha) \supset (O(F\beta)))$ appear in the resulting logic. Naturally, in many cases, one wants to share some of the symbols. The previous example would involve the constrained form of fibring imposed by sharing a common propositional part.

At the deductive system level, provided that the two given logics are endowed with deductive systems of the same type, the deductive system of the fibring will be obtained by the free use of the inference rules from both. This approach will be of interest only if the two given deductive systems are schematic in the sense that their inference rules are open for application to formulae with foreign symbols. For instance, when one represents *Modus Ponens* by the rule MP, $\{(\xi_1 \supset \xi_2), \xi_1\} \vdash \xi_2$, in some Hilbert system, one may implicitly assume that the instantiation of the schema variables ξ_1, ξ_2 by any formulae, possibly with symbols from both logics, is allowed when applying MP in the fibring.

Although the most basic form of fibring is quite simple at the syntactic level as described above, the semantics of fibring is much more complex and it is advisable to consider only the special case where both logics have semantics with similar models. Following (Sernadas et al., 1999; Zanardo et al., 2001), a convenient, but quite general, model for a wide class of logics with propositional base is provided by a triple $\langle U, \mathcal{B}, \nu \rangle$ where U is a set (of points, worlds, states, whatever), $\mathcal{B} \subseteq \wp U$, and $\nu(c) : \mathcal{B}^n \to \mathcal{B}$ for each language constructor c of arity $n \geq 0$. We look at the pair $\langle \mathcal{B}, \nu \rangle$ as an algebra of truth-values. It is precisely in this sense that such models are said to be *truth-functional*. Given two logics $\mathcal{L}', \mathcal{L}''$ with models of this type, what is the semantics of their fibring? As first shown in (Sernadas et al., 1999), it is a class of models of the same type, such that at each point $u \in U$ it is possible to extract a model from $\mathcal{L}'$ and one from $\mathcal{L}''$. Clearly, if symbols are shared, the two extracted models should agree on them. In order to visualize the semantics of fibring, consider the fibring of a propositional linear temporal logic with a propositional linear space logic. Each model of the fibring will be a cloud U of points such that at each point one knows the time line and the space line crossing there. For instance, at the point $\langle$*Berlin, 10h15m 25 March 2000*$\rangle$ one knows the time line (of past, present and future) of Berlin and the space line (the universe taken as a line for the sake of the example) at that time.

It is well known that, contrarily to soundness, adequacy is in general not preserved by fibring. Still, adequacy can sometimes be recovered by adding further interaction axioms and/or inference rules to the fibred logic (see for instance the *modal fibring rule* in (Gabbay, 1998), Chapter 3). Another approach to adequacy preservation consists in imposing reasonable extra conditions to the given logics that may be sufficient to guarantee that the fibred logic turns out to be adequate (Zanardo et al., 2001).

In this paper we aim at broadening fibred semantics in order to cope with *non-truth-functional* logics like paraconsistent logics. Paraconsistent logics were introduced in (da Costa, 1963) and since then have been the object of continued attention, because of their theoretical and practical significance. In particular, the paraconsistent systems $\mathcal{C}_n$ of (da Costa, 1963) are subsystems of propositional classical logic in which the principle of *Pseudo Scotus* $\gamma, \neg\gamma \vdash \delta$ does not hold. It is well known that, in all the $\mathcal{C}_n$ systems, negation cannot be given a truth-functional semantics (Mortensen, 1980).

The first main contribution of this paper is the definition of a general notion of logic that also encompasses non-truth-functional logics. In previous work on the semantics of fibring this kind of logics has never been considered. In fact, non-truth-functional logics could not even be represented using those approaches. In order to overcome this limitation we consider a broader notion of logic system that accommodates this novelty. The main ingredient is the use of a suitable auxiliary logic, that we call the meta-logic, where the (possibly) non-truth-functional valuations are defined. Since it is enough for the present purposes, we choose conditional equational logic (CEQ, (Goguen and Meseguer, 1985; Meseguer, 1998)) as the meta-logic. Furthermore, we manage to recover fibring in this wider context and also to prove that this extended notion of fibring preserves completeness under reasonable conditions. This completeness transfer result, the second main contribution of this paper, generalizes the one established in (Zanardo et al., 2001) and is obtained using a new adequacy preservation technique exploiting the properties of the meta-logic, in this case CEQ. We should stress that the present approach is not just an adaptation of previous work but it involves the conceptual breakthrough of dropping the widely accepted principle of truth-functionality.

As an example of application we analyze the system $\mathcal{C}_1^D$ of paraconsistent modal logic of (da Costa and Carnielli, 1988). One might wonder if we could recover such a mixed logic by fibring the underlying modal and paraconsistent logics. In fact, it turns out that by simply fibring the two, using the method we propose, the fibred logic obtained is a little weaker than the original paraconsistent modal logic $\mathcal{C}_1^D$. This problem has to do with the simple fact that $\mathcal{C}_1^D$ contains an essential interaction axiom that cannot even be expressed in either of the logics being fibred. As a consequence, the target paraconsistent modal logic can be easily recovered by adding that axiom to the obtained fibred logic (similarly to the above mentioned technique for completeness preservation), together with a corresponding semantic restriction. This process is part of the essential idea of fibring as proposed in (Gabbay, 1998), Chapter 1.

Following the same methodology used in previous work, namely (Sernadas et al., 1999), we advocate that the basic form of fibring must be characterized, in precise terms, by means of a categorial construction with a universal property. Still, in this paper, we shall keep the categorial apparatus to the minimum in order to keep the focus of attention on the issue of non-truth-functionality, rather than on the category theoretic details. Therefore, we believe that the paper can still be fully assessed by the reader not conversant with the elementary language of categories (MacLane, 1971; Barr and Wells, 1990).

The paper is organized as follows. In Section 2, the notion of *interpretation system presentation* as a specification of the intended valuations within the meta-logic is introduced. The *interpretation structures* appear as models of the specification. Section 3 defines the notions of *unconstrained* and *constrained fibring* of interpretation system presentations. The main example, fibring the paraconsistent system $\mathcal{C}_1$ and the modal system KD, is also discussed in Section 3 at the semantic level. Section 4 contains a brief account of the appropriate proof-theoretic notions and returns to the main example at the deductive level. Section 5 establishes the completeness preservation theorem and applies it for proving the completeness of the modal paraconsistent logic $\mathcal{C}_1^D$ of (da Costa and Carnielli, 1988). Section 6 discusses applications of self-fibring, namely in the context of the $\mathcal{C}_n$ hierarchy of paraconsistent systems. Section 7 concludes with an assessment of what was achieved and what lays ahead.

2. Specifying valuation semantics

Observe that, when setting-up an algebraic semantics for a truth-functional logic, we endow it with models that are algebras (of truth-values) over the signature of the logic and evaluate formulae homomorphically. This approach does not work when the logic is not truth-functional. But still within the spirit of "algebraic semantics", there is a solution: work instead with two-sorted algebras of formulae and truth-values and include the valuation map as an operation between the two sorts! This new approach, first sketched in (Coniglio et al., 2000), captures, as a special case, truth-functional logics by imposing the homomorphism conditions on the valuation map which can be done with equations. Looking at examples of non-truth-functional logics we find that the envisaged requirements on the valuation map could also be imposed by, albeit conditional, equations. Therefore, we are led to the following algebraic notion of possibly non-truth-functional semantics: each model is a two-sorted algebra (of formulae and truth-values) including a valuation operation that satisfies some requirements written in a suitable conditional equational meta-logic. As mentioned in the introduction, we adopt CEQ (Goguen and Meseguer, 1985; Meseguer, 1998) as the meta-logic.

Let us start by setting up the syntax that we need to use. Since the object logics under investigation are propositional-based, the following notion of signature suffices for our purposes:

DEFINITION 1. An *object signature* is a family $C = \{C_k\}_{k\in\mathbb{N}}$ where each C_k is a set (of *connectives* of arity k).

In particular, the set of propositional symbols is included in C_0.

We assume given once and for all the set $\Xi = \{\xi_1, \xi_2, \ldots\}$ of *propositional schema variables*, to be used in *inference rules*, such that $\Xi \cap C_0 = \emptyset$.

We denote by $L(C, \Xi)$ the set of *schema formulae* inductively built from C and Ξ. For example, $\xi_1 \supset (p \vee \neg \xi_2)$ is a schema formula, if $p \in C_0$, $\neg \in C_1$ and $\supset, \vee \in C_2$.

Our next step will be to define an equational signature induced by a given object signature C as a meta-linguistic device which permits to talk about the semantics of logics based on C. For this purpose it is convenient to consider two sorts, sort ϕ (for formulae) and sort τ (for truth-values). As usual, given a set of sorts $\mathbf{S}$, we write the Kleene closure $\mathbf{S}^*$ to denote the set of all strings over $\mathbf{S}$ and ϵ to denote the empty string. In the following definition, if $w \in \mathbf{S}^*$ and $s \in \mathbf{S}$ then $O_{w\,s}$ denotes the set of operations with domain w and codomain s.

DEFINITION 2. Given an object signature C, the *induced meta-signature* is the 2-sorted equational signature $\Sigma(C, \Xi) = \langle \mathbf{S}, O \rangle$ where $\mathbf{S} = \{\phi, \tau\}$ and:

- $O_{\epsilon\,\phi} = C_0 \cup \Xi$;
- $O_{\phi^k\,\phi} = C_k$ for $k > 0$;
- $O_{\phi\,\tau} = \{v\}$;
- $O_{\epsilon\,\tau} = \{\top, \bot\}$;
- $O_{\tau\,\tau} = \{-\}$;
- $O_{\tau\tau\,\tau} = \{\sqcap, \sqcup, \Rightarrow\}$;
- $O_{\omega\,s} = \emptyset$ in the other cases.

We shall use $\Sigma(C)$ to denote the subsignature $\Sigma(C, \emptyset)$, that is, where $O_{\epsilon\,\phi} = C_0$.

The symbols $\top$, $\bot$, $-$, $\sqcap$, $\sqcup$ and $\Rightarrow$ are used as generators of truth-values. The symbol v will be interpreted as a valuation map.

We consider the following sets of variables for $\Sigma(C)$ and $\Sigma(C, \Xi)$: $X_\phi = \{y_1, y_2, \ldots\}$ and $X_\tau = \{x_1, x_2, \ldots\}$. For ease of notation we simply use X to denote the two-sorted family $\{X_\phi, X_\tau\}$. Recall that a term t is called a ground term if it does not contain variables, and that a substitution θ is said to be ground if it replaces every variable by a ground term.

We want to write valuation specifications (within the adopted meta-logic CEQ) over $\Sigma(C)$ and X. Recall that a CEQ-specification is composed of conditional equations of the general form:

$$(\text{equation}_1 \ \& \ \ldots \ \& \ \text{equation}_n \ \rightarrow \ \text{equation})$$

with $n \geq 0$. Each equation is of the form $t = t'$ where t, t' are terms of the same sort built over $\Sigma(C)$ and X. The sort of each equation is defined to be the sort of its terms. A conditional equation that only involves equations of a given sort is said to be a conditional equation of that sort. Conditional equations are universally quantified, although, for the sake of simplicity, we omit the quantifier, contrarily to the notation used in (Meseguer, 1998). For example, $(\to v(y_1 \wedge y_2) = \sqcap(v(y_1), v(y_2)))$ is a conditional equation of sort τ, supposing that $\wedge \in C_2$. It is clear, from this and also the forthcoming examples, that we only need to consider specifications containing exclusively conditional equations (or *meta-axioms*) of sort τ. Such specifications are called τ-specifications in the sequel. The deductive system of CEQ (Meseguer, 1998) is a system for deriving equations from a given specification of conditional equations. It consists of the usual rules for reflexivity, symmetry, transitivity and congruence of equality, plus a form of *Modus Ponens* that allows us to obtain an equation $\mathrm{eq}\,\theta$ from already obtained equations $\mathrm{eq}_1\theta, \ldots, \mathrm{eq}_n\theta$, given a conditional equation $(\mathrm{eq}_1 \ \& \ \ldots \ \& \ \mathrm{eq}_n \ \to \ \mathrm{eq})$ in the specification and a substitution θ. In the sequel, we use $\vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)}$ to denote the corresponding consequence relation.

An important remark is that, in the context of the meta-signature $\Sigma(C, \Xi)$, it might seem that we have two different ways to represent arbitrary formulae: by means of propositional schema variables (i.e., ξ_1, ξ_2, etc.) and by means of variables of sort ϕ (i.e., y_1, y_2, etc.). The former shall indeed represent arbitrary formulae but only in the context of Hilbert calculi (to be defined in Section 4). The latter represent arbitrary formulae in the meta-language of CEQ. In this meta-language, propositional schema variables appear as constants.

DEFINITION 3. An *interpretation system presentation* (isp) is a pair $\mathcal{S} = \langle C, S \rangle$ where C is an object signature and S is a τ-specification over $\Sigma(C)$.

DEFINITION 4. Given an isp $\mathcal{S}$, the class $\mathrm{Int}(\mathcal{S})$ of *interpretation structures* presented by $\mathcal{S}$ is the class of all Heyting algebras over $\Sigma(C, \Xi)$ satisfying the specification S.

We denote by $S^\bullet$ the specification composed of the meta-axioms in S plus τ-equations over $\Sigma(C)$ specifying the class of all Heyting algebras. Note that $\mathrm{Int}(\mathcal{S})$, that is, the class of all algebras over $\Sigma(C, \Xi)$ satisfying $S^\bullet$, is always non-empty. Indeed, the trivial algebra with singleton carrier sets for all sorts satisfies any set of conditional equations.

In the sequel, we need to refer to the denotation $[\![t]\!]^\rho_\mathcal{A}$ of a meta-term t given an assignment ρ over an algebra $\mathcal{A}$. As expected, an assignment maps each variable to an element in the carrier set of the sort of the variable. In the case of a ground term t, as usual, we just write $[\![t]\!]_\mathcal{A}$ for its denotation in $\mathcal{A}$.

For the sake of economy of presentation, we introduce the following abbreviations: $x_1 \leq x_2$ for $\sqcap(x_1, x_2) = x_1$, and $\Leftrightarrow(x_1, x_2)$ for $\sqcap(\Rightarrow(x_1, x_2), \Rightarrow(x_2, x_1))$. The relation symbol $\leq$ denotes a partial order on truth-values. Furthermore, the partial order is a bounded lattice with meet $\sqcap$, join $\sqcup$, top $\top$ and bottom $\bot$ (cf. (Birkhoff, 1967)). As expected, given an algebra $\mathcal{A}$, $a_1 \leq_\mathcal{A} a_2$ and $\Leftrightarrow_\mathcal{A}(a_1, a_2)$ are abbreviations of $\sqcap_\mathcal{A}(a_1, a_2) = a_1$ and $\sqcap_\mathcal{A}(\Rightarrow_\mathcal{A}(a_1, a_2), \Rightarrow_\mathcal{A}(a_2, a_1))$, respectively. It is also well known that the Heyting algebra axioms further entail the following result:

PROPOSITION 1. Let $\mathcal{S}$ be an isp, t_1 and t_2 terms of sort τ and $\mathcal{A} \in \mathrm{Int}(\mathcal{S})$. Then, for every assignment ρ over $\mathcal{A}$:

$$[\![t_1]\!]^\rho_\mathcal{A} \leq_\mathcal{A} [\![t_2]\!]^\rho_\mathcal{A} \quad \text{iff} \quad \Rightarrow_\mathcal{A} ([\![t_1]\!]^\rho_\mathcal{A}, [\![t_2]\!]^\rho_\mathcal{A}) = \top_\mathcal{A}$$

and

$$[\![t_1]\!]^\rho_\mathcal{A} = [\![t_2]\!]^\rho_\mathcal{A} \quad \text{iff} \quad \Leftrightarrow_\mathcal{A} ([\![t_1]\!]^\rho_\mathcal{A}, [\![t_2]\!]^\rho_\mathcal{A}) = \top_\mathcal{A}.$$

As explained, our framework is intended to study properties of fibring of non-truth-functional logics in general. We now illustrate the notion of isp with two examples that will be used throughout the rest of the paper.

EXAMPLE 1. *Paraconsistent system* $\mathcal{C}_1$ (da Costa, 1963):

– Object signature - C:

- $C_0 = \{p_n : n \in \mathbb{N}\} \cup \{\mathbf{t}, \mathbf{f}\}$;
- $C_1 = \{\neg\}$;
- $C_2 = \{\wedge, \vee, \supset\}$.

– Meta-axioms - S:

- Truth-values axioms – further axioms in order to obtain a specification of the class of all Boolean algebras, e.g., adding the equation:
  - $(\ \rightarrow -(-(x_1)) = x_1)$.
- Valuation axioms:
  - $(\ \rightarrow v(\mathbf{t}) = \top)$;
  - $(\ \rightarrow v(\mathbf{f}) = \bot)$;
  - $(\ \rightarrow v(y_1 \wedge y_2) = \sqcap(v(y_1), v(y_2)))$;
  - $(\ \rightarrow v(y_1 \vee y_2) = \sqcup(v(y_1), v(y_2)))$;
  - $(\ \rightarrow v(y_1 \supset y_2) = \Rightarrow(v(y_1), v(y_2)))$;
  - $(\ \rightarrow -(v(y_1)) \leq v(\neg\, y_1))$;
  - $(\ \rightarrow v(\neg\neg\, y_1) \leq v(y_1))$;
  - $(\ \rightarrow \sqcap(v(y_1^\circ), \sqcap(v(y_1), v(\neg\, y_1))) = \bot)$;
  - $(\ \rightarrow \sqcap(v(y_1^\circ), v(y_2^\circ)) \leq v((y_1 \wedge y_2)^\circ))$;
  - $(\ \rightarrow \sqcap(v(y_1^\circ), v(y_2^\circ)) \leq v((y_1 \vee y_2)^\circ))$;
  - $(\ \rightarrow \sqcap(v(y_1^\circ), v(y_2^\circ)) \leq v((y_1 \supset y_2)^\circ))$.

As usual in the $\mathcal{C}_n$ systems, γ° is an abbreviation of $\neg(\gamma \wedge \neg\,\gamma)$.

The reader should be warned that we are using Boolean algebras here as a metamathematical environment sufficient to carry out the computations of truth-values for the formulae in $\mathcal{C}_1$. Specifically we are not introducing any unary operator in the Boolean algebras corresponding to paraconsistent negation, but we are computing the values of formulae of the form $\neg\,\gamma$ by means of conditional equations in the algebras. In other words, $\neg$ does not correspond to the Boolean algebra complement $-$. Therefore we are not attempting to algebraize $\mathcal{C}_1$ in any usual way.[1]

It is straightforward to verify that every paraconsistent bivaluation introduced in (da Costa and Alves, 1977) has a counterpart in $\mathrm{Int}(\mathcal{S})$. Furthermore, the additional interpretation structures do not change the semantic entailment (as defined below). Note that it is easy to extend this example in order to set up the isp's for the whole hierarchy $\mathcal{C}_n$ by specifying the paraconsistent n-valuations introduced in (Loparić and Alves, 1980). △

After this example, we can now clarify the meaning of non-truth-functional semantics. To be as general as possible we shall not only consider primitive connectives (as given by the object signature) but also derived ones. As usual, a *derived connective* of arity k is a λ-term $\lambda y_1 \dots y_k \,.\, \delta$, where the variables occurring in the schema formula δ are taken from $y_1, \dots, y_k$. Of course, if $c \in C_k$ is a primitive connective it can also be considered as the derived connective $\lambda y_1 \dots y_k \,.\, c(y_1, \dots, y_k)$.

DEFINITION 5. A derived connective $\lambda y_1 \dots y_k \,.\, \delta$ is said to be *truth-functional* in a given isp $\mathcal{S}$ if

$$S^{\bullet} \vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)} v(\delta) = t\, \theta^{v(y)}_{x}$$

for some τ-term t written only on the variables $x_1, ..., x_k$, where $\theta^{v(y)}_{x}$ is the substitution such that $\theta^{v(y)}_{x}(x_n) = v(y_n)$ for every $n \geq 1$.

If it is not possible to fulfill the above requirement, the connective is said to be *non-truth-functional* in $\mathcal{S}$.

For obvious reasons, showing that a certain connective is non-truth-functional can be a very hard task. In $\mathcal{C}_1$, classical negation $\sim := \lambda y_1 \,.\, \neg y_1 \wedge y_1^{\circ}$ (take t as $-(x_1)$) and equivalence $\equiv := \lambda y_1 y_2 \,.\, (y_1 \supset y_2) \wedge (y_2 \supset y_1)$ (take t as $\Leftrightarrow(x_1, x_2)$) are both truth-functional. And, of course, so are the primitive conjunction $\lambda y_1 y_2 \,.\, y_1 \wedge y_2$, disjunction $\lambda y_1 y_2 \,.\, y_1 \vee y_2$, and implication $\lambda y_1 y_2 \,.\, y_1 \supset y_2$. On the other hand, paraconsistent negation $\lambda y_1 \,.\, \neg y_1$ is known to be non-truth-functional. We refer the reader to (Mortensen, 1980) for a proof of this fact.

EXAMPLE 2. *Modal system KD* (Hughes and Cresswell, 1996; Lemmon and Scott, 1977):

- Object signature - C:
  - $C_0 = \{p_n : n \in \mathbb{N}\} \cup \{\mathbf{t}, \mathbf{f}\}$;
  - $C_1 = \{\neg, L\}$;
  - $C_2 = \{\wedge, \vee, \supset\}$.
- Meta-axioms - S:

[1] The question of algebraizing paraconsistent logic is a separate issue and we refer the interested reader to (Mortensen, 1980; Lewin et al., 1991).

- Truth-values axioms:
  - $*$ Further axioms in order to obtain a specification of the class of all Boolean algebras as in the previous example.
- Valuation axioms:
  - $*$ $(\;\to v(\mathbf{t}) = \top)$;
  - $*$ $(\;\to v(\mathbf{f}) = \bot)$;
  - $*$ $(\;\to v(\neg\, y_1) = -(v(y_1)))$;
  - $*$ $(\;\to v(y_1 \wedge y_2) = \sqcap(v(y_1), v(y_2)))$;
  - $*$ $(\;\to v(y_1 \vee y_2) = \sqcup(v(y_1), v(y_2)))$;
  - $*$ $(\;\to v(y_1 \supset y_2) = \Rightarrow(v(y_1), v(y_2)))$;
  - $*$ $(\;\to v(L\, \mathbf{t}) = \top)$;
  - $*$ $(\;\to v(L(y_1 \wedge y_2)) = \sqcap(v(L\, y_1), v(L\, y_2)))$;
  - $*$ $(\;\to \sqcap(v(L\, y_1), v(\neg\, L \neg\, y_1)) = v(L\, y_1)$;
  - $*$ $(v(y_1) = v(y_2) \to v(L\, y_1) = v(L\, y_2))$.

It is straightforward to verify that every Kripke model has a counterpart in $\text{Int}(\mathcal{S})$: consider the algebra of truth-values given by the power set of the set of worlds. Furthermore, every general model in (Zanardo et al., 2001) also has a counterpart in $\text{Int}(\mathcal{S})$: take $\langle \mathcal{B}, \nu \rangle$ as the algebra of the truth-values. Again, the extra interpretation structures do not change the semantic entailment. △

In the isp above, all derived connectives are truth-functional, but the modality $\lambda y_1 \,.\, L\, y_1$ would require in $\Sigma(C)$ the extra generator $\Box$ in $O_{\tau\,\tau}$ satisfying:

- $(\;\to \Box(\top) = \top)$;
- $(\;\to \Box(\sqcap(x_1, x_2)) = \sqcap(\Box(x_1), \Box(x_2)))$;
- $(\;\to \sqcap(\Box(x_1), -(\Box(-(x_1))) = \Box(x_1))$;
- $(\;\to v(L\, y_1) = \Box(v(y_1)))$.

Note that these axioms on $\Box$ are very closely related to the last four valuation axioms used in Example 2, which allowed us to specify the intended modal algebras and still avoid the use of $\Box$. Although such an operation $\Box$ can be easily defined over the set of truth-values according to the axioms above, our definition does not comply with its inclusion in the signature $\Sigma(C)$.

We are now ready to define the (global and local) semantic entailments.

DEFINITION 6. Given an isp $\mathcal{S}$, a set Γ of schema formulae and a schema formula δ, we say that:

- $\Gamma \vDash_g^{\mathcal{S}} \delta$ (Γ *globally entails* δ) if, for every $\mathcal{A} \in \text{Int}(\mathcal{S})$, $v_{\mathcal{A}}([\![\gamma]\!]_{\mathcal{A}}) = \top_{\mathcal{A}}$ for each $\gamma \in \Gamma$ implies $v_{\mathcal{A}}([\![\delta]\!]_{\mathcal{A}}) = \top_{\mathcal{A}}$;
- $\Gamma \vDash_l^{\mathcal{S}} \delta$ (Γ *locally entails* δ) if, for every $\mathcal{A} \in \text{Int}(\mathcal{S})$ and every $b \in \mathcal{A}_\phi$, $v_{\mathcal{A}}(b) \leq_{\mathcal{A}} v_{\mathcal{A}}([\![\gamma]\!]_{\mathcal{A}})$ for each $\gamma \in \Gamma$ implies $v_{\mathcal{A}}(b) \leq_{\mathcal{A}} v_{\mathcal{A}}([\![\delta]\!]_{\mathcal{A}})$.

Observe that $\Gamma \vDash_l^{\mathcal{S}} \delta$ implies $\Gamma \vDash_g^{\mathcal{S}} \delta$ provided that for every $\mathcal{A} \in \text{Int}(\mathcal{S})$ there exists $b \in \mathcal{A}_\phi$ such that $v_{\mathcal{A}}(b) = \top_{\mathcal{A}}$. On the other hand, if $\Gamma = \emptyset$ then $\Gamma \vDash_g^{\mathcal{S}} \delta$ implies $\Gamma \vDash_l^{\mathcal{S}} \delta$.

Before turning our attention to the problem of fibring isp's, we first define the categories **Sig** and **Isp**. The objects of the category **Sig** are object signatures. A morphism $h : C \to C'$ in **Sig** is of the form $h = \{h_k : C_k \to C'_k\}_{k\in\mathbb{N}}$, each h_k being a map. The objects of the category **Isp** are isp's. The appropriate notion of morphism in the category **Isp** is as follows: each $h : \langle C, S\rangle \to \langle C', S'\rangle$ is a morphism $h : C \to C'$ in **Sig** such that for each $s \in S$, $\widehat{h}(s)$ is in $S'^{\bullet}$. Here, $\widehat{h}$ is the free extension of h to a map from the meta-language over $\Sigma(C)$ to the meta-language over $\Sigma(C')$. Note that such a morphism imposes the condition that for every $\mathcal{A}' \in \text{Int}(\mathcal{S}')$ its reduct to $\Sigma(C)$ via h is in $\text{Int}(\mathcal{S})$. As usual, $\mathcal{A}'|^h_{\Sigma(C)}$ denotes the corresponding reduct algebra, that is characterized up to isomorphism by the following property:

$$[\![t]\!]_{\mathcal{A}'|^h_{\Sigma(C)}} = [\![\widehat{h}(t)]\!]_{\mathcal{A}'}, \text{ for every term } t \text{ over } \Sigma(C).$$

In what follows, we shall also use the forgetful functor N from **Isp** to the category **Sig** of object signatures. This functor maps each $\mathcal{S}$ to the underlying object signature C and each morphism h to the underlying object signature morphism.

3. Fibring non-truth-functional logics

Fibring, as originally proposed by (Gabbay, 1996; Gabbay, 1998), may be a rather complex form of combining given logics. Here, we consider only the most basic forms of fibring seen as "operations" between logics as in (Sernadas et al., 1999; Zanardo et al., 2001): *unconstrained fibring* where two logics are combined by putting together their signatures and rules, and by picking up as models all structures over the new signature whose reducts are models in the two given logics; *constrained fibring* where two logics are combined as for the unconstrained fibring but requiring that some symbols are to be shared. These basic forms of fibring lead to new logics that sometimes need fine tuning for the application at hand, namely by adding further interaction rules (axioms). This idea of agreement on the reducts when fibring logics endowed with homomorphic algebraic semantics is already present in (Gabbay, 1998), Chapter 20.

Here we face the novel problem of defining these two basic forms of fibring as operations on logics endowed with non-truth-functional semantics as defined in the previous section. Recall that models are now two-sorted algebras (of formulae and truth-values).

In the fibring, like in the truth-functional case, we still expect to find two-sorted algebras over the new signature whose reducts are models of the logics being fibred. Therefore, when fibring two isp's, we expect to put together the signatures and the requirements on the valuation map.

Assume that we are given two isp's $\mathcal{S}'$ and $\mathcal{S}''$. We start by considering the notion of *unconstrained fibring* that corresponds to combining the two isp's without sharing any of the symbols of the object signatures C' and C''. That is, if we so combine $\mathcal{C}_1$ and KD we shall obtain in the result of the fibring two different symbols for conjunction, disjunction, etc. This construction appears as the coproduct of $\mathcal{S}'$ and $\mathcal{S}''$ in the category **Isp**. Therefore,

$$\mathcal{S}' \oplus \mathcal{S}'' = \langle C, S\rangle$$

where:

– $C = C' \oplus C''$ is a coproduct within **Sig** with the injections i' and i'';

– $S = \widehat{i'}(S') \cup \widehat{i''}(S'')$.

The following result confirms the intuitions that guided the definition:

PROPOSITION 2. Given $\mathcal{S}'$ and $\mathcal{S}''$ as above, a $\Sigma(C' \oplus C'', \Xi)$-algebra $\mathcal{A}$ belongs to $\mathrm{Int}(\mathcal{S}' \oplus \mathcal{S}'')$ if and only if:

– $\mathcal{A}|^{i'}_{\Sigma(C',\Xi)} \in \mathrm{Int}(\mathcal{S}')$;

– $\mathcal{A}|^{i''}_{\Sigma(C'',\Xi)} \in \mathrm{Int}(\mathcal{S}'')$;

where $\mathcal{A}|^{i'}_{\Sigma(C',\Xi)}$ and $\mathcal{A}|^{i''}_{\Sigma(C'',\Xi)}$ are the reducts of $\mathcal{A}$ to the signatures $\Sigma(C', \Xi)$ and $\Sigma(C'', \Xi)$, respectively, via the indicated inclusions.

It is now easy to introduce the notion of *constrained fibring by sharing connectives and/or propositional symbols* that corresponds to combining the two isp's while sharing some of the symbols of the object signatures C' and C''. The construction appears as a co-Cartesian lifting by the functor $N : \mathbf{Isp} \to \mathbf{Sig}$ along the signature coequalizer for the envisaged pushout of the signatures. We refrain from dwelling further on the details of this construction since it does not bring any insight to the main issue of this paper (that is, the fibring of logics possibly with non-truth-functional semantics). For illustration, consider the following example of constrained fibring.

EXAMPLE 3. *Modal paraconsistent logic:*
In (da Costa and Carnielli, 1988), a paraconsistent deontic logic called $\mathcal{C}_1^D$ is introduced including the paraconsistent system $\mathcal{C}_1$ and the modal system KD (interpreting the modal operator L as "obligatory"). Let us see if we can recover $\mathcal{C}_1^D$ as a fibring.

The idea is to combine $\mathcal{C}_1$ and KD by fibring them while sharing the propositional symbols, conjunction, disjunction, implication, true and false. Let $\mathcal{S}' = \langle C', S'\rangle$ be the isp for $\mathcal{C}_1$ as described in Example 1 and $\mathcal{S}'' = \langle C'', S''\rangle$ the isp for KD as described in Example 2.

We work first in the category **Sig** in order to set up the desired sharing of symbols. Consider the following propositional-based signature B of shared symbols:

– $B_0 = \{p_n : n \in \mathbb{N}\} \cup \{\mathbf{t}, \mathbf{f}\}$;

– $B_2 = \{\wedge, \vee, \supset\}$;

– $B_k = \emptyset$ for the other values of k.

The matching signature inclusions $f' : B \to C'$ and $f'' : B \to C''$ both map the symbols in B to the corresponding symbols in C' and C''.

The propositional-based signature $C' \overset{f'Bf''}{\oplus} C''$ of the envisaged constrained fibring is now obtained by the pushout of $C' \overset{f'}{\leftarrow} B \overset{f''}{\rightarrow} C''$. To compute it, we first have to obtain the coproduct $C' \oplus C''$, corresponding to the signature of the unconstrained fibring of $\mathcal{C}_1$ and KD, as follows:

– $(C' \oplus C'')_0 = \{p'_n : n \in \mathbb{N}\} \cup \{p''_n : n \in \mathbb{N}\} \cup \{\mathbf{t}', \mathbf{t}'', \mathbf{f}', \mathbf{f}''\}$;

– $(C' \oplus C'')_1 = \{\neg', \neg'', L''\}$;

– $(C' \oplus C'')_2 = \{\wedge', \wedge'', \vee', \vee'', \supset', \supset''\}$.

The corresponding injections $i' : C' \to C' \oplus C''$ and $i'' : C'' \to C' \oplus C''$ are such that i' maps each constructor $\sharp$ of C' to $\sharp'$ and i'' maps each constructor $\sharp$ of C'' to $\sharp''$.

Finally, the envisaged signature $C' \overset{f'Bf''}{\oplus} C''$ is obtained by identifying in $C' \oplus C''$ all the constructors obtained via f' and f'' from the same constructor of the shared signature B:

– $(C' \overset{f'Bf''}{\oplus} C'')_0 = \{p_n : n \in \mathbb{N}\} \cup \{\mathbf{t}, \mathbf{f}\}$;

– $(C' \overset{f'Bf''}{\oplus} C'')_1 = \{\neg', \neg'', L\}$;

– $(C' \overset{f'Bf''}{\oplus} C'')_2 = \{\wedge, \vee, \supset\}$.

The unique compatible morphism z from $C' \oplus C''$ to $C' \overset{f'Bf''}{\oplus} C''$ is defined by $z(p'_n) = z(p''_n) = p_n$ for each $n \in \mathbb{N}$, $z(\mathbf{t}') = z(\mathbf{t}'') = \mathbf{t}$, $z(\mathbf{f}') = z(\mathbf{f}'') = \mathbf{f}$, $z(\neg') = \neg'$, $z(\neg'') = \neg''$, $z(L'') = L$, $z(\wedge') = z(\wedge'') = \wedge$, $z(\vee') = z(\vee'') = \vee$ and $z(\supset') = z(\supset'') = \supset$. To be precise, we should have written $z_0(p'_n)$, $z_1(\neg')$ and so on, but we have omitted the arity subscripts to improve the readability.

This signature morphism $z : C' \oplus C'' \to C' \overset{f'Bf''}{\oplus} C''$ is finally used to obtain the envisaged fibred isp

$$\mathcal{S}' \overset{f'Bf''}{\oplus} \mathcal{S}'' = \langle C' \overset{f'Bf''}{\oplus} C'', \widehat{z}(\widehat{i'}(S') \cup \widehat{i''}(S''))\rangle$$

corresponding to the respective co-Cartesian lifting briefly described at the end of the previous section. Expectedly, since z is a morphism, we have that $\mathcal{A}|^z_{\Sigma(C' \oplus C'', \Xi)} \in \text{Int}(\widehat{i'}(S') \cup \widehat{i''}(S''))$ for every $\mathcal{A} \in \text{Int}(\widehat{z}(\widehat{i'}(S') \cup \widehat{i''}(S'')))$, and therefore the interpretation structures presented by this isp are precisely those algebras in the unconstrained fibring that agree on the shared symbols.

Note that we end up having two negations: $\neg'$ coming from C' and $\neg''$ coming from C''. The former is a paraconsistent negation and the latter is the classical negation inherited from KD. Clearly, the derived (classical) strong negation $\lambda y_1 . \neg' y_1 \wedge y_1^\circ$ inherited from $\mathcal{C}_1$ collapses into $\neg''$. Note that now γ° is an abbreviation of $\neg'(\gamma \wedge \neg' \gamma)$.

In order to recover $\mathcal{C}_1^D$, we have to add one additional meta-axiom on valuations to the previously obtained fibred isp:

– $(\;\to v(y_1^\circ) \leq v((L\, y_1)^\circ))$.

Using the terminology introduced in (Carnielli and Coniglio, 1999), this procedure can be seen as a *splitting* of $\mathcal{C}_1^D$ in the components KD and $\mathcal{C}_1$. This idea is also in the spirit of Gabbay's proposal on the broad meaning of fibring, as described in (Gabbay, 1998), Chapter 1. △

There are other interesting examples of combination of modal and paraconsistent reasoning that would deserve to be analyzed from this point of view, namely those in (Deutsch, 1979; Deutsch, 1984; Puga et al., 1988) that, using paraconsistent techniques, deal with problems of deontic logic having to do with deontic paradoxes and moral dilemmas.

4. Logic systems

This section is devoted to extending fibring to the proof-theoretical counterpart of isp's. For their simplicity and ubiquity we use a suitable notion of *Hilbert calculus*. As we have hinted before, we shall use the propositional schema variables in $\Xi = \{\xi_1, \xi_2, \ldots\}$ to write *inference rules*.

DEFINITION 7. A *Hilbert calculus* over Ξ is a triple $\langle C, P, D\rangle$ in which (1) C is an object signature, (2) P is a subset of $\wp_{fin} L(C, \Xi) \times L(C, \Xi)$, (3) D is a subset of $(\wp_{fin} L(C, \Xi) \setminus \emptyset) \times L(C, \Xi)$, and (4) $D \subseteq P$.

Given any $r = \langle \Gamma, \gamma\rangle$ in P, the (finite) set Γ is the set of premises of r and γ is the conclusion; we will often write $r = \langle Prem(r), Conc(r)\rangle$. If $Prem(r) = \emptyset$, then r is said to be an *axiom schema*; otherwise, it is said to be a *proof rule schema*. Each r in D is said to be a *derivation rule schema*.

EXAMPLE 4. *Paraconsistent system $\mathcal{C}_1$ revisited:*

Adapting the well known axiomatics presented in (da Costa, 1963; da Costa and Alves, 1977), a Hilbert calculus for $\mathcal{C}_1$ is easily defined:

– $P = \{\langle \emptyset, \xi_1 \supset (\xi_2 \supset \xi_1)\rangle,$
$\langle \emptyset, (\xi_1 \supset (\xi_2 \supset \xi_3)) \supset ((\xi_1 \supset \xi_2) \supset (\xi_1 \supset \xi_3))\rangle,$
$\langle \emptyset, (\xi_1 \wedge \xi_2) \supset \xi_1\rangle,$
$\langle \emptyset, (\xi_1 \wedge \xi_2) \supset \xi_2\rangle,$
$\langle \emptyset, \xi_1 \supset (\xi_2 \supset (\xi_1 \wedge \xi_2))\rangle,$

$\langle\emptyset, \xi_1 \supset (\xi_1 \vee \xi_2)\rangle$,
$\langle\emptyset, \xi_2 \supset (\xi_1 \vee \xi_2)\rangle$,
$\langle\emptyset, (\xi_1 \supset \xi_3) \supset ((\xi_2 \supset \xi_3) \supset ((\xi_1 \vee \xi_2) \supset \xi_3))\rangle$,
$\langle\emptyset, \neg\neg\,\xi_1 \supset \xi_1\rangle$,
$\langle\emptyset, \xi_1 \vee \neg\,\xi_1\rangle$,
$\langle\emptyset, \xi_1^\circ \supset (\xi_1 \supset (\neg\,\xi_1 \supset \xi_2))\rangle$,
$\langle\emptyset, (\xi_1^\circ \wedge \xi_2^\circ) \supset (\xi_1 \wedge \xi_2)^\circ\rangle$,
$\langle\emptyset, (\xi_1^\circ \wedge \xi_2^\circ) \supset (\xi_1 \vee \xi_2)^\circ\rangle$,
$\langle\emptyset, (\xi_1^\circ \wedge \xi_2^\circ) \supset (\xi_1 \supset \xi_2)^\circ\rangle$,
$\langle\emptyset, \mathbf{t} \equiv (\xi_1 \supset \xi_1)\rangle$,
$\langle\emptyset, \mathbf{f} \equiv (\xi_1^\circ \wedge (\xi_1 \wedge \neg\,\xi_1))\rangle$,
$\langle\{\xi_1, \xi_1 \supset \xi_2\}, \xi_2\rangle\}$;

– $D = \{\langle\{\xi_1, \xi_1 \supset \xi_2\}, \xi_2\rangle\}$. △

EXAMPLE 5. *Modal system KD revisited:*

Adapting from (Hughes and Cresswell, 1996; Lemmon and Scott, 1977), in the modal Hilbert calculus for KD we have:

– $P = \{\langle\emptyset, \xi_1 \supset (\xi_2 \supset \xi_1)\rangle$,
$\langle\emptyset, (\xi_1 \supset (\xi_2 \supset \xi_3)) \supset ((\xi_1 \supset \xi_2) \supset (\xi_1 \supset \xi_3))\rangle$,
$\langle\emptyset, (\neg\,\xi_1 \supset \neg\,\xi_2) \supset (\xi_2 \supset \xi_1)\rangle$,
$\langle\emptyset, L(\xi_1 \supset \xi_2) \supset (L\,\xi_1 \supset L\,\xi_2)\rangle$,
$\langle\emptyset, L\,\xi_1 \supset \neg\,L\,\neg\,\xi_1\rangle$,
$\langle\emptyset, (\xi_1 \vee \xi_2) \equiv (\neg\,\xi_1 \supset \xi_2)\rangle$,
$\langle\emptyset, (\xi_1 \wedge \xi_2) \equiv \neg(\neg\,\xi_1 \vee \neg\,\xi_2)\rangle$,
$\langle\emptyset, \mathbf{t} \equiv (\xi_1 \supset \xi_1)\rangle$,
$\langle\emptyset, \mathbf{f} \equiv (\xi_1 \wedge \neg\,\xi_1)\rangle$,
$\langle\{\xi_1, \xi_1 \supset \xi_2\}, \xi_2\rangle$,
$\langle\{\xi_1\}, L\,\xi_1\rangle\}$;

– $D = \{\langle\{\xi_1, \xi_1 \supset \xi_2\}, \xi_2\rangle\}$. △

DEFINITION 8. A schema formula $\delta \in L(C,\Xi)$ is *provable* from the set of schema formulae $\Gamma \subseteq L(C,\Xi)$ in the Hilbert calculus $\langle C, P, D\rangle$, denoted by $\Gamma \vdash_p^{PD} \delta$, if there is a sequence $\gamma_1, \ldots, \gamma_m \in L(C,\Xi)^+$ such that $\gamma_m = \delta$ and, for $i = 1$ to m, either

(1) $\gamma_i \in \Gamma$, or

(2) there exist a rule $r \in P$ and a schema variable substitution $\sigma : \Xi \to L(C,\Xi)$ such that $Conc(r)\sigma = \gamma_i$ and $Prem(r)\sigma \subseteq \{\gamma_1, \ldots, \gamma_{i-1}\}$.

DEFINITION 9. A schema formula $\delta \in L(C,\Xi)$ is *derivable* from the set of schema formulae $\Gamma \subseteq L(C,\Xi)$ in the Hilbert calculus $\langle C, P, D\rangle$, denoted by $\Gamma \vdash_d^{PD} \delta$, if there is a sequence $\gamma_1, \ldots, \gamma_m \in L(C,\Xi)^+$ such that $\gamma_m = \delta$ and, for $i = 1$ to m, either

(1) $\gamma_i \in \Gamma$, or

(2) γ_i is provable from the empty set of formulae, or

(3) there exist a rule $r \in D$ and a schema variable substitution σ such that $Conc(r)\sigma = \gamma_i$ and $Prem(r)\sigma \subseteq \{\gamma_1, \ldots, \gamma_{i-1}\}$.

Clearly, if $\Gamma \vdash_d^{PD} \delta$ then also $\Gamma \vdash_p^{PD} \delta$. Furthermore, if $\emptyset \vdash_p^{PD} \delta$ then $\emptyset \vdash_d^{PD} \delta$. As usual, we say that δ is a *theorem schema* whenever $\emptyset \vdash_p^{PD} \delta$ (iff $\emptyset \vdash_d^{PD} \delta$), and simply write $\vdash_p^{PD} \delta$ and $\vdash_d^{PD} \delta$. The following *structurality* properties are also immediate: for every schema variable substitution σ, if $\Gamma \vdash_p^{PD} \delta$ then $\Gamma\sigma \vdash_p^{PD} \delta\sigma$, and if $\Gamma \vdash_d^{PD} \delta$ then $\Gamma\sigma \vdash_d^{PD} \delta\sigma$.

DEFINITION 10. The *unconstrained fibring* of the Hilbert calculi $\langle C', P', D'\rangle$ and $\langle C'', P'', D''\rangle$ is the Hilbert calculus

$$\langle C', P', D'\rangle \oplus \langle C'', P'', D''\rangle = \langle C' \oplus C'', \widehat{i'}(P') \cup \widehat{i''}(P''), \widehat{i'}(D') \cup \widehat{i''}(D'')\rangle.$$

DEFINITION 11. The *constrained fibring* of the Hilbert calculi $\langle C', P', D'\rangle$ and $\langle C'', P'', D''\rangle$ sharing C according to the injective morphisms $f' : C \to C'$ and $f'' : C \to C''$ is the Hilbert calculus $\langle C', P', D'\rangle \overset{f'Cf''}{\oplus} \langle C'', P'', D''\rangle$ defined as follows:

$$\langle C' \overset{f'Cf''}{\oplus} C'', \widehat{z}(\widehat{i'}(P')) \cup \widehat{z}(\widehat{i''}(P'')), \widehat{z}(\widehat{i'}(D')) \cup \widehat{z}(\widehat{i''}(D''))\rangle.$$

As a matter of fact, by adopting the notion of Hilbert calculus morphism proposed in (Sernadas et al., 1999), both forms of fibring appear again as universal categorial constructions (coproduct and co-Cartesian lifting, respectively). It follows that there is a morphism from each given Hilbert calculus to the fibring, e.g., h' from $\langle C', P', D'\rangle$ to $\langle C, P, D\rangle$ and therefore:

- if $\Gamma \vdash_p^{P'D'} \delta$ then $\widehat{h}'(\Gamma) \vdash_p^{PD} \widehat{h}'(\delta)$;
- if $\Gamma \vdash_d^{P'D'} \delta$ then $\widehat{h}'(\Gamma) \vdash_d^{PD} \widehat{h}'(\delta)$.

EXAMPLE 6. *Modal paraconsistent logic revisited:*
The fibring of Hilbert calculi for $\mathcal{C}_1$ and KD, sharing the propositional symbols, conjunction, disjunction, implication, true and false, is the Hilbert calculus where we have all the proof and derivation rules for both $\mathcal{C}_1$ and KD. In order to get the deontic paraconsistent system $\mathcal{C}_1^D$ of (da Costa and Carnielli, 1988), at the proof-theoretic level, we need to introduce the following proof rule:

- $\langle \emptyset, \xi_1^\circ \supset (L\,\xi_1)^\circ\rangle$.

This interaction axiom is already present in $\mathcal{C}_1^D$ and could never be obtained using the basic fibring operation since it makes full use of the mixed language. Note that the semantic counterpart of this axiom was also added to the corresponding fibred isp in Example 3. △

DEFINITION 12. A *logic system* is a tuple $\mathcal{L} = \langle C, S, P, D\rangle$ where the pair $\langle C, S\rangle$ constitutes an isp and the triple $\langle C, P, D\rangle$ constitutes a Hilbert calculus.

As expected, the (unconstrained and constrained) fibring of logic systems is obtained by the corresponding fibring of the underlying isp's and Hilbert calculi.

EXAMPLE 7. The logic systems for $\mathcal{C}_1$ and KD will be denoted by $\mathcal{L}_{\mathcal{C}_1}$ and $\mathcal{L}_{\mathrm{KD}}$, respectively, and their fibring while sharing B will be denoted by $\mathcal{L}_{\mathcal{C}_1 \oplus \mathrm{KD}}$.

DEFINITION 13. Given a logic system $\mathcal{L} = \langle C, S, P, D\rangle$, we say that the deductive system $\langle C, P, D\rangle$ is *sound w.r.t. the isp* $\langle C, S\rangle$, or simply that $\mathcal{L}$ is *sound*, if for every set Γ of schema formulae and every schema formula δ:

- $\Gamma \vdash_p^{PD} \delta$ implies $\Gamma \vDash_g^{\mathcal{S}} \delta$;
- $\Gamma \vdash_d^{PD} \delta$ implies $\Gamma \vDash_l^{\mathcal{S}} \delta$.

We say that $\langle C, P, D\rangle$ is *adequate w.r.t.* $\langle C, S\rangle$, or simply that $\mathcal{L}$ is *adequate*, if for every set Γ of schema formulae and every schema formula δ:

- $\Gamma \vDash_g^{\mathcal{S}} \delta$ implies $\Gamma \vdash_p^{PD} \delta$;
- $\Gamma \vDash_l^{\mathcal{S}} \delta$ implies $\Gamma \vdash_d^{PD} \delta$.

Furthermore, we say that $\langle C, P, D\rangle$ is *complete w.r.t.* $\langle C, S\rangle$, or simply that $\mathcal{L}$ is *complete*, if it is both sound and adequate.

EXAMPLE 8. The logic systems $\mathcal{L}_{\mathcal{C}_1}$ and $\mathcal{L}_{\mathrm{KD}}$ are complete.

5. Preservation results

The main goal of this section is to establish sufficient conditions for the preservation of completeness by fibring. To this end, it is convenient to take advantage of the completeness of the meta-logic CEQ, as proved for instance in (Goguen and Meseguer, 1985; Meseguer, 1998), by encoding the relevant part of the deductive system of CEQ in the object Hilbert calculus.

In order to deal with local reasoning at the meta-level, we shall take advantage of the following two schema variable substitutions:

- σ^{+1} such that $\sigma^{+1}(\xi_i) = \xi_{i+1}$ for every $i \geq 1$;
- σ^{-1} such that $\sigma^{-1}(\xi_1) = \xi_1$ and $\sigma^{-1}(\xi_i) = \xi_{i-1}$ for every $i \geq 2$.

Note that if γ is a schema formula then $\gamma\sigma^{+1}$ is a variant of γ where ξ_1 does not occur. Furthermore, easily, $\gamma\sigma^{+1}\sigma^{-1} = \gamma$.

5.1. Encoding

First we analyze what can be obtained proof-theoretically within CEQ. Given an isp $\mathcal{S} = \langle C, S\rangle$, we adopt the following abbreviations, where $\Gamma \cup \{\delta\} \subseteq L(C, \Xi)$:

– $\Gamma \vdash_g^{\mathcal{S}} \delta$ for $S^\bullet \cup \{(\to v(\gamma) = \top) : \gamma \in \Gamma\} \vdash_{\Sigma(C,\Xi)}^{\mathrm{CEQ}} v(\delta) = \top$;

– $\Gamma \vdash_l^{\mathcal{S}} \delta$ for $S^\bullet \cup \{(\to v(\xi_1) \leq v(\gamma\sigma^{+1})) : \gamma \in \Gamma\} \vdash_{\Sigma(C,\Xi)}^{\mathrm{CEQ}} v(\xi_1) \leq v(\delta\sigma^{+1})$.

THEOREM 1. Given an isp $\mathcal{S} = \langle C, S\rangle$ and $\Gamma \cup \{\delta\} \subseteq L(C,\Xi)$, we have:

– $\Gamma \vDash_g^{\mathcal{S}} \delta$ iff $\Gamma \vdash_g^{\mathcal{S}} \delta$;

– $\Gamma \vDash_l^{\mathcal{S}} \delta$ iff $\Gamma \vdash_l^{\mathcal{S}} \delta$.

Proof: This is an immediate consequence of the completeness of CEQ. In the local case it is essential to note that, since schema variables cannot occur in $S^\bullet$, we can freely change the denotation of schema variables given by an algebra $\mathcal{A} \in \mathrm{Int}(\mathcal{S})$ (namely according to σ^{+1} or σ^{-1}) and still obtain an algebra in $\mathrm{Int}(\mathcal{S})$. The fact that ξ_1 cannot occur in schema formulae instantiated by σ^{+1} does the rest. QED

For the envisaged encoding, it is necessary to assume that the logic system at hand is sufficiently expressive:

DEFINITION 14. A logic system $\mathcal{L} = \langle C, S, P, D\rangle$ is said to be *rich* if:

1. $\mathbf{t}, \mathbf{f} \in C_0$ and $\wedge, \vee, \supset \in C_2$;
2. $S^\bullet \vdash_{\Sigma(C,\Xi)}^{\mathrm{CEQ}} v(\mathbf{t}) = \top$;
3. $S^\bullet \vdash_{\Sigma(C,\Xi)}^{\mathrm{CEQ}} v(\mathbf{f}) = \bot$;
4. $S^\bullet \vdash_{\Sigma(C,\Xi)}^{\mathrm{CEQ}} v(y_1 \wedge y_2) = \sqcap(v(y_1), v(y_2))$;
5. $S^\bullet \vdash_{\Sigma(C,\Xi)}^{\mathrm{CEQ}} v(y_1 \vee y_2) = \sqcup(v(y_1), v(y_2))$;
6. $S^\bullet \vdash_{\Sigma(C,\Xi)}^{\mathrm{CEQ}} v(y_1 \supset y_2) = \Rightarrow(v(y_1), v(y_2))$;
7. $\langle\{\xi_1, \xi_1 \supset \xi_2\}, \xi_2\rangle \in D$.

EXAMPLE 9. Both logic systems $\mathcal{L}_{\mathcal{C}_1}$ and $\mathcal{L}_{\mathrm{KD}}$, as well as many other common logics, are rich.

Within a rich logic system it is possible to translate from the meta-logic level to the object logic level. A ground term of sort τ over $\Sigma(C,\Xi)$ is mapped to a formula in $L(C,\Xi)$ according to the following rules:

$v(\gamma)^*$ is γ;

$\top^*$ is $\mathbf{t}$;

$\bot^*$ is $\mathbf{f}$;

$-(t)^*$ is $t^* \supset \mathbf{f}$;

$\sqcap(t_1,t_2)^*$ is $t_1^* \wedge t_2^*$;

$\sqcup(t_1,t_2)^*$ is $t_1^* \vee t_2^*$;

$\Rightarrow(t_1,t_2)^*$ is $t_1^* \supset t_2^*$.

Moreover, a ground τ-equation $(t_1 = t_2)$ is translated to $(t_1 = t_2)^*$ given by $t_1^* \equiv t_2^*$. Finally, if E is a set of ground τ-equations, then E^* will denote the set $\{\text{eq}^* : \text{eq} \in E\}$.

LEMMA 1. Let $\mathcal{L}$ be a rich logic system and t a ground τ-term over $\Sigma(C,\Xi)$. Then:

$$S^\bullet \vdash^{\text{CEQ}}_{\Sigma(C,\Xi)} v(t^*) = t.$$

Proof: Immediate by definition, taking into account the requirements of richness and the completeness of CEQ. QED

LEMMA 2. Let $\mathcal{L}$ be a rich logic system, t_1 and t_2 ground τ-terms over $\Sigma(C,\Xi)$ and $\mathcal{A} \in \text{Int}(\mathcal{S})$. Then:

$$[\![t_1]\!]_\mathcal{A} \leq_\mathcal{A} [\![t_2]\!]_\mathcal{A} \quad \text{iff} \quad v_\mathcal{A}([\![t_1^* \supset t_2^*]\!]_\mathcal{A}) = \top_\mathcal{A}$$

and

$$[\![t_1]\!]_\mathcal{A} = [\![t_2]\!]_\mathcal{A} \quad \text{iff} \quad v_\mathcal{A}([\![t_1^* \equiv t_2^*]\!]_\mathcal{A}) = \top_\mathcal{A}.$$

Proof: Direct corollary of Proposition 1 using the previous lemma and taking into account the completeness of CEQ. QED

In a rich logic system, under certain conditions (cf. Definition 15 below), one can encode the relevant part of the meta-reasoning into the object calculus.

DEFINITION 15. A rich logic system $\mathcal{L}$ is said to be *equationally appropriate* if for every conditional equation $(\text{eq}_1 \ \& \ \ldots \ \& \ \text{eq}_n \ \rightarrow \ \text{eq})$ in $S^\bullet$ and every ground substitution θ:

$$\{(\text{eq}_1\theta)^*, \ldots, (\text{eq}_n\theta)^*\} \vdash^{PD}_{p} (\text{eq}\,\theta)^*.$$

Finally, we obtain the main results of this section relating adequacy to equational appropriateness. Such results are important because it is much easier to analyze the preservation by fibring of equational appropriateness than of adequacy directly.

THEOREM 2. Every rich and adequate logic system is equationally appropriate.

Proof: Assume that $\mathcal{L}$ is a rich and adequate logic system and $\mathcal{A} \in \text{Int}(\mathcal{S})$, and let $(t_1 = s_1 \ \& \ \ldots \ \& \ t_n = s_n \ \rightarrow \ t = s)$ be a conditional equation in $S^\bullet$ and θ a ground substitution.

If it is the case that $v_{\mathcal{A}}([\![(t_i\theta)^* \equiv (s_i\theta)^*]\!]_{\mathcal{A}}) = \top_{\mathcal{A}}$ for $i = 1, \ldots, n$, then, according to the previous lemma, this means precisely that $[\![t_i\theta]\!]_{\mathcal{A}} = [\![s_i\theta]\!]_{\mathcal{A}}$ for $i = 1, \ldots, n$. Consider the assignment $\rho = [\![\_]\!]_{\mathcal{A}} \circ \theta$. It is straightforward to verify that $[\![r\theta]\!]_{\mathcal{A}} = [\![r]\!]^{\rho}_{\mathcal{A}}$, for every τ-term r over $\Sigma(C, \Xi)$. So, since by definition of $\mathrm{Int}(\mathcal{S})$ we know that $\mathcal{A}$ is a model of the conditional equation, it immediately follows that also $[\![t\theta]\!]_{\mathcal{A}} = [\![s\theta]\!]_{\mathcal{A}}$, or equivalently, $v_{\mathcal{A}}([\![(t\theta)^* \equiv (s\theta)^*]\!]_{\mathcal{A}}) = \top_{\mathcal{A}}$.

Therefore, we have $\{(t_1\theta)^* \equiv (s_1\theta)^*, \ldots, (t_n\theta)^* \equiv (s_n\theta)^*\} \vDash_g^{\mathcal{S}} (t\theta)^* \equiv (s\theta)^*$. Now, equational appropriateness follows easily since, from adequacy, we must have $\{(t_1\theta)^* \equiv (s_1\theta)^*, \ldots, (t_n\theta)^* \equiv (s_n\theta)^*\} \vdash_p^{PD} (t\theta)^* \equiv (s\theta)^*$. QED

Before proving the converse of this theorem, we need to establish some technical lemmas.

LEMMA 3. Let $\mathcal{L}$ be an equationally appropriate logic system and $\Gamma \cup \{\delta\}$ be a set of schema formulae where ξ_1 does not occur. If $\{\xi_1 \supset \gamma : \gamma \in \Gamma\} \vdash_p^{PD} \xi_1 \supset \delta$ then $\Gamma \vdash_d^{PD} \delta$.

Proof: First of all we note that, since $S^\bullet$ must contain a specification of the class of all Heyting algebras, equational appropriateness implies that every intuitionistic theorem written with $\mathbf{t}, \mathbf{f}, \wedge, \vee, \supset$ must be provable in the Hilbert calculus. In the sequel, we shall use this fact without further notice.

Let us assume that $\{\xi_1 \supset \gamma : \gamma \in \Gamma\} \vdash_p^{PD} \xi_1 \supset \delta$. Immediately, by the finite character of derivations, $\{\xi_1 \supset \gamma_1, \ldots, \xi_1 \supset \gamma_n\} \vdash_p^{PD} \xi_1 \supset \delta$, where each $\gamma_i \in \Gamma$. Let now γ be the schema formula $\gamma_1 \wedge \ldots \wedge \gamma_n$ and take the schema variable substitution σ such that $\sigma(\xi_1) = \gamma$ and $\sigma(\xi_i) = \xi_i$ for every $i \geq 2$. Since ξ_1 does not occur in Γ or δ, the structurality of proofs easily implies that we must also have $\{\gamma \supset \gamma_1, \ldots, \gamma \supset \gamma_n\} \vdash_p^{PD} \gamma \supset \delta$. But it is clear by easy intuitionistic reasoning that $\vdash_p^{PD} \gamma \supset \gamma_i$ for $i = 1, \ldots, n$. So, it follows that $\vdash_p^{PD} \gamma \supset \delta$, and since by further intuitionistic reasoning we have $\vdash_p^{PD} \gamma_1 \supset (\ldots \supset (\gamma_n \supset \gamma) \ldots)$, the derivation rule of *Modus Ponens* immediately implies that $\Gamma \vdash_d^{PD} \delta$. QED

LEMMA 4. Let $\mathcal{L}$ be an equationally appropriate logic system, E a set of ground τ-equations over $\Sigma(C, \Xi)$ and θ a ground substitution. If

$$S^\bullet \cup \{(\,\rightarrow \mathrm{eq}\,) : \mathrm{eq} \in E\} \vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)} t_1 = t_2$$

then either t_1, t_2 are the same term of sort ϕ, or t_1, t_2 are of sort τ and

$$E^* \vdash_p^{PD} (t_1\theta)^* \equiv (t_2\theta)^*.$$

Proof: Recall that the deduction rules of CEQ are Reflexivity, Symmetry, Transitivity, Congruence and *Modus Ponens*.

Given a ground substitution θ, we proceed by induction on the length n of a proof of $S^\bullet \cup \{(\,\rightarrow \mathrm{eq}\,) : \mathrm{eq} \in E\} \vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)} t_1 = t_2$.

Base: $n = 1$.

(i) t_1 is $s_1\theta'$, t_2 is $s_2\theta'$ and $t_1 = t_2$ is obtained by CEQ *Modus Ponens* from $(\,\rightarrow s_1 = s_2) \in S^\bullet$.

Obviously t_1 and t_2 are τ-terms. Immediately, by equational appropriateness, we have $\vdash_p^{PD} (s_1\theta'\theta)^* \equiv (s_2\theta'\theta)^*$ and therefore $E^* \vdash_p^{PD} (t_1\theta)^* \equiv (t_2\theta)^*$ by the monotonicity of provability.

(ii) t_1 is $s_1\theta'$, t_2 is $s_2\theta'$ and $t_1 = t_2$ is obtained by CEQ *Modus Ponens* from $(\to s_1 = s_2)$ with $s_1 = s_2 \in E$.

Obviously t_1 and t_2 are closed τ-terms, $s_1\theta'$ is s_1 and $s_2\theta'$ is s_2. Thus, $(t_1\theta)^* \equiv (t_2\theta)^* \in E^*$, i.e., $t_1^* \equiv t_2^* \in E^*$ and $E^* \vdash_p^{PD} t_1^* \equiv t_2^*$ by the extensiveness of provability.

(iii) t_1 and t_2 are the same term, of either sort ϕ or τ, and $t_1 = t_2$ is obtained by Reflexivity.

If the sort is ϕ we are done. Otherwise, obviously, $(t_1\theta)^*$ and $(t_2\theta)^*$ are the same formula and trivial intuitionistic reasoning allows us to conclude that $\vdash_p^{PD} \xi_1 \equiv \xi_1$. Therefore, $E^* \vdash_p^{PD} (t_1\theta)^* \equiv (t_2\theta)^*$ by the structurality and monotonicity of provability.

Step: $n > 1$.

(i) $t_1 = t_2$ is obtained from $S^\bullet \cup \{(\to \mathrm{eq}) : \mathrm{eq} \in E\} \vdash_{\Sigma(C,\Xi)}^{\mathrm{CEQ}} t_2 = t_1$ by Symmetry.

If the terms have sort ϕ, by induction hypothesis, they coincide. Otherwise, also by induction hypothesis, we know that $E^* \vdash_p^{PD} (t_2\theta)^* \equiv (t_1\theta)^*$. Elementary intuitionistic reasoning allows us to conclude that $\vdash_p^{PD} (\xi_1 \equiv \xi_2) \supset (\xi_2 \equiv \xi_1)$, and therefore also $E^* \vdash_p^{PD} (t_1\theta)^* \equiv (t_2\theta)^*$.

(ii) $t_1 = t_2$ is obtained from $S^\bullet \cup \{(\to \mathrm{eq}) : \mathrm{eq} \in E\} \vdash_{\Sigma(C,\Xi)}^{\mathrm{CEQ}} t_1 = t_3, t_3 = t_2$ by Transitivity.

If the terms have sort ϕ by induction hypothesis they coincide. Otherwise, also by induction hypothesis, we know that $E^* \vdash_p^{PD} (t_1\theta)^* \equiv (t_3\theta)^*, (t_3\theta)^* \equiv (t_2\theta)^*$. Simple intuitionistic reasoning allows us to conclude that $\{\xi_1 \equiv \xi_2, \xi_2 \equiv \xi_3\} \vdash_p^{PD} \xi_1 \equiv \xi_3$, and therefore also $E^* \vdash_p^{PD} (t_1\theta)^* \equiv (t_2\theta)^*$.

(iii) t_1 is $f(t_{11}, \ldots, t_{1k})$, t_2 is $f(t_{21}, \ldots, t_{2k})$ and $t_1 = t_2$ is obtained from $S^\bullet \cup \{(\to \mathrm{eq}) : \mathrm{eq} \in E\} \vdash_{\Sigma(C,\Xi)}^{\mathrm{CEQ}} t_{11} = t_{21}, \ldots, t_{1k} = t_{2k}$ by Congruence.

It t_1 and t_2 have sort ϕ then $f \in C_k$ and therefore all the terms t_{ij} are also of sort ϕ. By induction hypothesis, then, t_{1j} and t_{2j} must be identical and t_1 coincides with t_2. Otherwise, f can either be v or a generator among $-, \sqcap, \sqcup, \Rightarrow$. In the first case $k = 1$ and by induction hypothesis t_{11} coincides with t_{21} since they must have sort ϕ. Therefore, t_1 and t_2 also coincide and we repeat step (iii) of the Base to obtain $E^* \vdash_p^{PD} (t_1\theta)^* \equiv (t_2\theta)^*$. Finally, if f is a generator then all the terms t_{ij} are of sort τ. Thus, by induction hypothesis, $E^* \vdash_p^{PD} (t_{11}\theta)^* \equiv (t_{21}\theta)^*, \ldots, (t_{1k}\theta)^* \equiv (t_{2k}\theta)^*$. Using again intuitionistic reasoning we have that $\{\xi_1 \equiv \xi_2, \xi_3 \equiv \xi_4\} \vdash_p^{PD} (\xi_1 \wedge \xi_3) \equiv (\xi_2 \wedge \xi_4)$, $(\xi_1 \vee \xi_3) \equiv (\xi_2 \vee \xi_4)$, $(\xi_1 \supset \xi_3) \equiv (\xi_2 \supset \xi_4)$. Given that $-$ is translated using $\mathbf{f}$ and $\supset$ this is enough to guarantee that $E^* \vdash_p^{PD} (t_1\theta)^* \equiv (t_2\theta)^*$.

(iv) t_1 is $s_1\theta'$, t_2 is $s_2\theta'$ and $t_1 = t_2$ is obtained using the conditional equation $(s_{11} = s_{21} \;\&\; \ldots \;\&\; s_{1k} = s_{2k} \to s_1 = s_2) \in S^\bullet$ from $S^\bullet \cup \{(\to \mathrm{eq}) : \mathrm{eq} \in E\} \vdash_{\Sigma(C,\Xi)}^{\mathrm{CEQ}} s_{11}\theta' = s_{21}\theta', \ldots, s_{1k}\theta' = s_{2k}\theta'$ by CEQ *Modus Ponens*.

Obviously, all the terms are τ-terms and the induction hypothesis implies that $E^* \vdash_p^{PD} (s_{11}\theta'\theta)^* \equiv (s_{21}\theta'\theta)^*, \ldots, (s_{1k}\theta'\theta)^* \equiv (s_{2k}\theta'\theta)^*$. Moreover, by equational appropriateness, we have $\{(s_{11}\theta'\theta)^* \equiv (s_{21}\theta'\theta)^*, \ldots, (s_{1k}\theta'\theta)^* \equiv (s_{2k}\theta'\theta)^*\} \vdash_p^{PD} (s_1\theta'\theta)^* \equiv (s_2\theta'\theta)^*$. Therefore, $E^* \vdash_p^{PD} (t_1\theta)^* \equiv (t_2\theta)^*$. QED

THEOREM 3. Every equationally appropriate logic system is adequate.

Proof: Let $\mathcal{L}$ be an equationally appropriate logic system and $\Gamma \cup \{\delta\} \subseteq L(C, \Xi)$.

If $\Gamma \vDash_g^{\mathcal{S}} \delta$ then also $\Gamma \vdash_g^{\mathcal{S}} \delta$, i.e., $S^\bullet \cup \{(\ \rightarrow v(\gamma) = \top) : \gamma \in \Gamma\} \vdash_{\Sigma(C,\Xi)}^{\text{CEQ}} v(\delta) = \top$, by Theorem 1. Therefore, using the previous lemma, we have that $\{\gamma \equiv \mathbf{t} : \gamma \in \Gamma\} \vdash_p^{PD} \delta \equiv \mathbf{t}$. Trivial intuitionistic reasoning allows us to conclude that $\vdash_p^{PD} \xi_1 \equiv (\xi_1 \equiv \mathbf{t})$, and therefore it follows that $\Gamma \vdash_p^{PD} \delta$.

If $\Gamma \vDash_l^{\mathcal{S}} \delta$ then also $\Gamma \vdash_l^{\mathcal{S}} \delta$, i.e., $S^\bullet \cup \{(\ \rightarrow v(\xi_1) \leq v(\gamma\sigma^{+1})) : \gamma \in \Gamma\} \vdash_{\Sigma(C,\Xi)}^{\text{CEQ}} v(\xi_1) \leq v(\delta\sigma^{+1})$, by Theorem 1. Therefore, using the previous lemma, we have that $\{(\xi_1 \wedge \gamma\sigma^{+1}) \equiv \xi_1 : \gamma \in \Gamma\} \vdash_p^{PD} (\xi_1 \wedge \delta\sigma^{+1}) \equiv \xi_1$. Trivial intuitionistic reasoning allows us to conclude that $\vdash_p^{PD} (\xi_1 \supset \xi_2) \equiv ((\xi_1 \wedge \xi_2) \equiv \xi_1)$, and therefore it follows that $\{\xi_1 \supset \gamma\sigma^{+1} : \gamma \in \Gamma\} \vdash_p^{PD} \xi_1 \supset \delta\sigma^{+1}$. Thus, by Lemma 3, we already know that $\Gamma\sigma^{+1} \vdash_d^{PD} \delta\sigma^{+1}$ and by structurality, using σ^{-1}, we have $\Gamma \vdash_d^{PD} \delta$. QED

The equivalence between adequacy and equational appropriateness for rich systems will be used below for showing that adequacy is preserved by fibring rich systems, but this equivalence may also be useful for establishing the adequacy of logics endowed with a semantics presented by conditional equations. Indeed, it is a much easier task to verify equational appropriateness than to establish adequacy directly.

5.2. Preservation of completeness by fibring

We consider in turn the preservation of soundness and of adequacy by fibring.

THEOREM 4. Soundness is preserved by fibring.

Proof: Let $\mathcal{L}$ be the fibring of two sound logic systems $\mathcal{L}'$ and $\mathcal{L}''$. It is enough to prove the following: $\Gamma \vdash_p^{PD} \delta$ implies that $\Gamma \vdash_g^{\mathcal{S}} \delta$, and $\Gamma \vdash_d^{PD} \delta$ implies that $\Gamma \vdash_l^{\mathcal{S}} \delta$, by Theorem 1. Moreover, it is enough to prove that $\mathit{Prem}(r) \vdash_g^{\mathcal{S}} \mathit{Conc}(r)$ for every $r \in P$, and $\mathit{Prem}(r) \vdash_l^{\mathcal{S}} \mathit{Conc}(r)$ for every $r \in D$. Let $r \in P$. Assume, without loss of generality, that $r = \widehat{h}'(r')$. Then, by definition of proof, $\mathit{Prem}(r') \vdash_p^{P'D'} \mathit{Conc}(r')$ and, so, $\mathit{Prem}(r') \vdash_g^{\mathcal{S}'} \mathit{Conc}(r')$, by the soundness of $\mathcal{L}'$. This means that

$$S'^\bullet \cup \{(\ \rightarrow v(\gamma') = \top) : \gamma' \in \mathit{Prem}(r')\} \vdash_{\Sigma(C',\Xi)}^{CEQ} v(\mathit{Conc}(r')) = \top$$

and then, by the uniformness of CEQ under change of notation by $\widehat{h}'$ (cf. (Meseguer, 1998)), we obtain

$$\widehat{h}'(S')^\bullet \cup \{(\ \rightarrow v(\widehat{h}'(\gamma')) = \top) : \gamma' \in \mathit{Prem}(r')\} \vdash_{\Sigma(C,\Xi)}^{CEQ} v(\widehat{h}'(\mathit{Conc}(r'))) = \top.$$

This immediately implies $\widehat{h}'(\mathit{Prem}(r')) \vdash_g^{\mathcal{S}} \widehat{h}'(\mathit{Conc}(r'))$, that is, $\mathit{Prem}(r) \vdash_g^{\mathcal{S}} \mathit{Conc}(r)$. The proof for derivations is similar. QED

Consider two sound logic systems $\mathcal{L}'$ and $\mathcal{L}''$ that are both consistent, in the sense that both contain formulae that are not theorems. It may happen that the corresponding fibred logic system $\mathcal{L}$ is no longer consistent. For instance, assume that $\mathcal{L}'$ corresponds to classical propositional logic plus an additional axiom A, for some proposition A, and that $\mathcal{L}''$ also corresponds to classical propositional logic but with additional axiom $\neg A$. Obviously, $\mathcal{L}$ is not consistent. In order to have both $\mathcal{L}'$ and $\mathcal{L}''$ sound, it is clear that the corresponding isp's must imply $v(A) = \top$ and $-(v(A)) = \top$, respectively. Thus, the isp of $\mathcal{L}$ must imply $\top = \bot$. Therefore, the only interpretation structure presented by the isp of $\mathcal{L}$ corresponds to the trivial Boolean algebra, which, as mentioned earlier (just after Definition 4), satisfies any set of conditional equations. This example makes clear that Theorem 4 just states the preservation of soundness by fibring and clearly does not imply the preservation of consistency. Preservation of soundness is nevertheless very important in itself. Finding sufficient conditions under which consistency might be preserved by fibring is beyond the scope of this paper.

Finally, we consider the problem of preservation of adequacy by fibring, taking advantage of the technical machinery presented before on the encoding of the meta-logic in the object Hilbert calculus.

LEMMA 5. Richness is preserved by fibring provided that conjunction, disjunction, implication, true and false are shared.

Proof: It is trivial that the signature and valuation requirements are preserved since we are sharing conjunction, disjunction, implication, true and false. Moreover, it is clear that *Modus Ponens* is still a derivation rule in the fibring. QED

LEMMA 6. Equational appropriateness is preserved by fibring provided that conjunction, disjunction, implication, true and false are shared.

Proof: Let $\mathcal{L}'$ and $\mathcal{L}''$ be equationally appropriate logic systems, and $\mathcal{L}$ their fibring by sharing conjunction, disjunction, implication, true and false. From the previous lemma we already know that $\mathcal{L}$ is rich.

Now, let ceq be $(t_1 = s_1 \; \& \; \ldots \; \& \; t_n = s_n \to t = s) \in S^\bullet$, and θ a ground substitution. Clearly, by definition of fibring, ceq must be the translation of a conditional equation in some of the components. Let us assume, without loss of generality, that ceq comes from $\mathcal{L}'$, i.e., ceq is $\widehat{h}'(\text{ceq}')$, where ceq$'$ is the conditional equation $(t_1' = s_1' \; \& \; \ldots \; \& \; t_n' = s_n' \to t' = s') \in S'^\bullet$. Since we know that $\mathcal{L}'$ is equationally appropriate, it follows that

$$\{(t_1'\theta')^* \equiv (s_1'\theta')^*, \ldots, (t_n'\theta')^* \equiv (s_n'\theta')^*\} \vdash_p^{P'D'} (t'\theta')^* \equiv (s'\theta')^*,$$

where θ' is the following ground substitution:

- for every $i \geq 1$, $\theta'(x_i) = v(\xi_{2i-1})$ and $\theta'(y_i) = \xi_{2i}$.

By definition of fibring of Hilbert calculi, then, it must be also the case that

$$\{\widehat{h}'((t_1'\theta')^* \equiv (s_1'\theta')^*), \ldots, \widehat{h}'((t_n'\theta')^* \equiv (s_n'\theta')^*)\} \vdash_p^{PD} \widehat{h}'((t'\theta')^* \equiv (s'\theta')^*).$$

Consider now the substitution σ on schema variables defined by:

– $\sigma(\xi_{2i-1}) = \theta(x_i)^*$;

– $\sigma(\xi_{2i}) = \theta(y_i)$.

Using σ and the structurality of the Hilbert calculus, now, we must also have

$$\{\widehat{h}'((t'_1\theta')^* \equiv (s'_1\theta')^*), \ldots, \widehat{h}'((t'_n\theta')^* \equiv (s'_n\theta')^*)\}\sigma \vdash_p^{PD} \widehat{h}'((t'\theta')^* \equiv (s'\theta')^*)\sigma.$$

But, in fact, a straightforward inductive proof allows us to conclude that $\widehat{h}'((u'\theta')^*)\sigma = (\widehat{h}'(u')\theta)^*$ for every term u' of sort τ over $\Sigma(C', \Xi)$, and therefore

$$\{(t_1\theta)^* \equiv (s_1\theta)^*, \ldots, (t_n\theta)^* \equiv (s_n\theta)^*\} \vdash_p^{PD} (t\theta)^* \equiv (s\theta)^*.$$

Thus, $\mathcal{L}$ is equationally appropriate. QED

THEOREM 5. Given two rich and complete logic systems, their fibring while sharing conjunction, disjunction, implication, true and false is also complete.

Proof: The preservation of soundness is immediate consequence of Theorem 4. The preservation of adequacy is a consequence of the previous lemma and the equivalence between equational appropriateness and adequacy for rich systems. In fact, let $\mathcal{L}'$ and $\mathcal{L}''$ be two rich and complete logic systems and let $\mathcal{L}$ be their fibring while sharing conjunction, disjunction, implication, true and false. By Theorem 2, $\mathcal{L}'$ and $\mathcal{L}''$ are also equationally appropriate and thus so is $\mathcal{L}$, by Lemma 6. Finally, by Theorem 3, $\mathcal{L}$ is adequate. QED

EXAMPLE 10. By fibring while sharing conjunction, disjunction, implication, true and false the logic systems $\mathcal{L}_{\mathcal{C}_1}$ and $\mathcal{L}_{\mathrm{KD}}$ we obtain a new modal paraconsistent logic system $\mathcal{L}_{\mathcal{C}_1 \oplus \mathrm{KD}}$ that is complete. Observe that if we add to $\mathcal{L}_{\mathcal{C}_1 \oplus \mathrm{KD}}$:

– $(\ \to v(y_1^\circ) \leq v((L\, y_1)^\circ))$ as a valuation axiom;

– $\langle \emptyset, \xi_1^\circ \supset (L\, \xi_1)^\circ \rangle$ as an axiom in the Hilbert calculus;

we still obtain a complete logic system that is equivalent to the system $\mathcal{C}_1^D$ of (da Costa and Carnielli, 1988) both at the proof-theoretic and the semantic levels. △

6. Self-fibring versus truth-functionality

In this section we address the problem of fibring two copies of the same logic (self-fibring) and show that, contrarily to what happens in the case of a truth-functional logic, in the case of a logic with non-truth-functional semantics there is no collapse of unshared symbols. This construction is illustrated within the context of the $\mathcal{C}_n$ hierarchy of paraconsistent systems (da Costa, 1963).

It is not difficult to see that, if $\mathcal{S}$ is a truth-functional isp (that is, all the connectives are truth-functional derived connectives) then the self-fibring $\mathcal{S} \oplus \mathcal{S}$

of $\mathcal{S}$ with itself, without sharing of connectives (just sharing the propositional symbols in C_0), produces a copy of $\mathcal{S}$ where each connective appears duplicate. In fact, if $c \in C_k$ $(k > 0)$ and c' is its duplicate then $v_{\mathcal{A}}([\![c(t_1, ..., t_k)]\!]^{\rho}_{\mathcal{A}})$ is equal to $v_{\mathcal{A}}([\![c'(t_1, ..., t_k)]\!]^{\rho}_{\mathcal{A}})$ for every interpretation $\mathcal{A}$, assignment ρ over $\mathcal{A}$ and terms $t_1, .., t_k$ of sort ϕ. Additionally, if $c \in C_0$ is a constant symbol, such as $\mathbf{t}$ or $\mathbf{f}$, then $v_{\mathcal{A}}([\![c]\!]_{\mathcal{A}})$ is also equal to $v_{\mathcal{A}}([\![c']\!]_{\mathcal{A}})$. Of course, this property can be extended to every rich and complete logic system $\mathcal{L}$ while sharing C_0, conjunction, disjunction and implication in the self-fibring $\mathcal{L} \oplus \mathcal{L}$. In such cases the formulae $c(t_1, ..., t_k)$ and $c'(t_1, ..., t_k)$ will be equivalent in $\mathcal{L} \oplus \mathcal{L}$, even if c and c' are not explicitly shared.

On the other hand, if $\mathcal{S}$ is an isp with some non-truth-functional connective c, then $v_{\mathcal{A}}([\![c(t_1, ..., t_k)]\!]^{\rho}_{\mathcal{A}})$ and $v_{\mathcal{A}}([\![c'(t_1, ..., t_k)]\!]^{\rho}_{\mathcal{A}})$ do not necessarily coincide in the models of $\mathcal{S} \oplus \mathcal{S}$ and, consequently, the formulae $c(t_1, ..., t_k)$ and $c'(t_1, ..., t_k)$ are not necessarily equivalent in $\mathcal{L} \oplus \mathcal{L}$, unless c and c' are explicitly shared.

As a concrete instance, consider the isp $\mathcal{S}$ of Example 1, representing the semantics of the paraconsistent calculus $\mathcal{C}_1$ (cf. (da Costa, 1963)). If we perform the fibring $\mathcal{S} \oplus \mathcal{S}$ of $\mathcal{S}$ with itself, while sharing the symbols in C_0, then we obtain two families of connectives: $\{\wedge, \vee, \supset, \neg\}$ and $\{\wedge', \vee', \supset', \neg'\}$. A model for $\mathcal{S} \oplus \mathcal{S}$ gives a valuation map $v_{\mathcal{A}}$ such that

$$v_{\mathcal{A}}([\![t_1 \sharp t_2]\!]^{\rho}_{\mathcal{A}}) = v_{\mathcal{A}}([\![t_1 \sharp' t_2]\!]^{\rho}_{\mathcal{A}}) \qquad \text{for } \sharp \in \{\wedge, \vee, \supset\},$$

because those connectives are truth-functional. On the other hand, $v_{\mathcal{A}}([\![\neg t]\!]^{\rho}_{\mathcal{A}})$ does not coincide necessarily with $v_{\mathcal{A}}([\![\neg' t]\!]^{\rho}_{\mathcal{A}})$. For example, let v_1 and v_2 be two $\mathcal{C}_1$-bivaluations such that $v_1(p_1) = v_1(\neg p_1) = 1$ and $v_2(p_1) = 1$, $v_2(\neg p_1) = 0$. Moreover, assume that in fact $v_1(p) = v_2(p)$ for every propositional symbol $p \in C_0$. We can thus define an interpretation $\mathcal{A}$ of $\mathcal{S} \oplus \mathcal{S}$ as follows:

- $\mathcal{A}_\tau = \{0, 1\}$ (with its usual Boolean algebra structure);
- $v_{\mathcal{A}}$ restricted to the fragment $\{\wedge, \vee, \supset, \neg\}$ coincides with v_1;
- $v_{\mathcal{A}}$ restricted to the fragment $\{\wedge', \vee', \supset', \neg'\}$ coincides with v_2;
- $v_{\mathcal{A}}$ extended to the mixed language $C \oplus C'$ is obtained from v_1 and v_2 by using the same techniques used in the proof of Proposition 6.1 in (Carnielli and Coniglio, 1999), namely, $v_{\mathcal{A}}([\![\neg\, t]\!]^{\rho}_{\mathcal{A}}) = 1$ iff $v_{\mathcal{A}}([\![t]\!]^{\rho}_{\mathcal{A}}) = 0$ iff $v_{\mathcal{A}}([\![\neg'\, t]\!]^{\rho}_{\mathcal{A}}) = 1$ for all assignments ρ and all mixed terms t of sort ϕ.

The interpretation $\mathcal{A}$ satisfies: $v_{\mathcal{A}}([\![\neg p_1]\!]_{\mathcal{A}}) \neq v_{\mathcal{A}}([\![\neg' p_1]\!]_{\mathcal{A}})$, showing that $\neg$ and $\neg'$ do not collapse. Considering the fibring at the logic system level, we obtain, by the completeness preservation Theorem 5, that $\neg t_1$ and $\neg' t_1$ are not equivalent formulae (unless they are both theorems).

The example above shows that the fibring of $\mathcal{C}_n$ with itself produces, for every $n \geq 1$, two disjoint copies of $\mathcal{C}_n$ (as the same argument can be applied to the whole hierarchy of paraconsistent calculi $\mathcal{C}_n$). We exclude $\mathcal{C}_0$(which is just the Classical Propositional Calculus) since in this case the self -fibring will collapse with $\mathcal{C}_0$, because this system is truth-functional. On the other hand, the fibring of $\mathcal{C}_n$ with $\mathcal{C}_m$, with $m > n$, produces a new paraconsistent system with two paraconsistent negations, $\neg_n$ and $\neg_m$, whose axioms correspond to adding the axioms of $\mathcal{C}_n$ and of $\mathcal{C}_m$, and whose interpretations are given by maps which are simultaneously $\mathcal{C}_n$ and $\mathcal{C}_m$ valuations. It is an open question whether or not there exists a formula (in the language of $\mathcal{C}_m$) which encodes in $\mathcal{C}_m$ the paraconsistent negation $\neg_n$, for $m > n$ (this, in fact, happens with the classical negation, which is representable in every calculi $\mathcal{C}_n$). If this question has a positive answer, then the fibring of $\mathcal{C}_n$ and $\mathcal{C}_m$ (without sharing the negation) will be equivalent to $\mathcal{C}_m$, for $m > n$. Of course, the fibring of $\mathcal{C}_n$ and $\mathcal{C}_m$ (while sharing the negation) will be equivalent to $\mathcal{C}_n$. If we generalize this argument, including in the object signature C an unary symbol $\neg_n$ for every $n \geq 0$, then the infinite fibring of the whole hierarchy $\mathcal{C}_n$, without sharing the negations $\neg_n$, will produce a new paraconsistent system with infinitely many paraconsistent negations. If negations are shared in the fibring, the result coincides with the Classical Propositional Calculus $\mathcal{C}_0$. It is worth remarking that, although we concentrate most of the time on paraconsistent calculi because these are excellent examples of interesting non-truth-functional logic systems, it is clear that our treatment is fully general.

7. Concluding remarks

The first main contribution of this paper is a general semantics for two basic forms of fibring propositional-based logics encompassing systems with possibly non-truth-functional valuations. We should stress that the present approach is not just an adaptation of previous work but it involves the conceptual breakthrough of dropping the widely accepted principle of truth-functionality. This goal is achieved by recognizing that such valuations can be represented in some appropriate meta-logic and by developing new techniques based on this representation. In this case, since it suited our needs, we have used conditional equational logic as the meta-logic. However, it must be clear that we could have adopted, instead, any other meta-logic where non-truth-functional valuation semantics could be defined. Although restricted to systems with a finitary propositional base, the proposed semantics deals with a wide variety of logics from paraconsistent to modal, many-valued and intuitionistic systems. In this setting, fibring appears as a universal construction within the underlying category, generalizing previous results for truth-functional systems (Sernadas et al., 1999).

It should be stressed that the two basic forms of fibring we consider (unconstrained fibring and constrained fibring by sharing some symbols) appear as “operations” on the class of logics at hand. Usually, in applications, these operations are not enough to obtain the envisaged logic: a fine tuning of the resulting logic may be necessary, namely by adding interaction axioms, like it is illustrated in Example 6.

The second main contribution of this paper is the completeness preservation theorem that generalizes to possibly non-truth-functional logics the result established in (Zanardo et al., 2001). This new result is obtained using a different technique exploiting the properties of conditional equational logic where the requirements on the valuations are specified.

As an example of application of our techniques, the use of fibring for combining paraconsistent logics with other logics is illustrated by recovering the modal paraconsistent logic $\mathcal{C}_1^D$ of da Costa and Carnielli (da Costa and Carnielli, 1988) as a fibring plus an additional interaction axiom. We believe that fibring is a very natural way of establishing new combined systems involving non-truth-functional logics. This approach is more widely applicable than it appears to be: a large family of logics (including many-valued and intuitionistic) admits bivalued non-truth-functional semantics (cf. (Béziau and da Costa, 1994)). Whenever that happens, the completeness preservation theorem can then be used for establishing the completeness of the result as long as the given logics are complete and fulfill the requirements of richness. As another example, the fibring of a logic with itself is examined within the context of the $\mathcal{C}_n$ hierarchy of paraconsistent systems (da Costa, 1963).

In short, our approach offers a framework formalizing the minimal metamathematical requirements that are sufficient to express a large variety of logic systems, possibly non-truth-functional; such representations of logic systems constitute a category and they can be combined by means of fibring (that is, through universal constructions in the category). We have also proved that the completeness of such logic systems is preserved by fibring under certain reasonable assumptions on the logic systems, guaranteeing the important property of non-destructiveness of our fibring constructions.

Note that, although we have adopted Hilbert calculi as the proof-theoretic notion of logic, it would have been possible to consider other well known formalisms. Namely tableaux systems, as in (Beckert and Gabbay, 1998), or systems of natural deduction, as in (Rasga et al., 2002).

Other lines of research are obvious, towards relaxing the assumptions of this paper. For instance, we may want to work with more general object logics (e.g., predicate logics), or with a more general meta-logic (e.g., disjunctive conditional equational logic), or with an even more general universe of truth-values (e.g., involving less or extra generators), or with weaker richness requirements and still obtain completeness preservation by fibring. This line of work is even more important as it should help to solve a small but annoying technicality related to our notion of truth-functional connective. In fact, as mentioned with respect to Example 2, the modality L is not truth-functional according to our definition. As we have explained, it would be very easy to make it truth-functional (as it should) by adding a modal operator $\Box$ to the meta-signature. It is important that we solve this lack of expressiveness in order to raise the distinction between this modality L and the paraconsistent negation $\neg$ of Example 1 that, on the contrary, is well known to be non-truth-functional in an essential way.

Still other lines of research are related to more general forms of fibring, namely heterogeneous forms of fibring where we want to combine two (or more) logics that are defined in quite different forms (either at the deductive system level or at the semantic level).

Acknowledgments

The authors are grateful to João Rasga and João Marcos for many valuable discussions on these and related topics. Thanks are also due to the anonymous referees for suggesting several improvements on an earlier version of the paper. This work is a natural sequel of the work presented in (Coniglio et al., 2000), which started during a visit by the third author to CMA at IST, entirely supported by *Fundação para a Ciência e a Tecnologia*, Portugal. A subsequent visit to CMA by the second and third authors was also supported by *Coordenação de Aperfeiçoamento do Pessoal do Ensino Superior*, Brazil. The third author also acknowledges financial support of his postdoc at IST under the grant 01/1045-0 of *Fundação de Amparo à Pesquisa do Estado de São Paulo*, Brazil. The work of the first, fourth and fifth authors was partially supported by *Fundação para a Ciência e a Tecnologia*, Portugal, namely via the ProbLog (PRAXIS/P/MAT/10002/1998) and FibLog (POCTI/2001/MAT/37239) Projects, also supported by EU FEDER.

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