FIELD-THEORETIC TECHNIQUE
FOR IRREVERSIBLE REACTION PROCESSES

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The single-species annihilation reaction $A + A \rightarrow \varnothing$ is studied in the presence of random advecting field. In order to determine possible infrared behavior of the system, all stable fixed points are presented to two-loop approximation in double (ϵ, Δ) expansion with the corresponding regions of stability. The main result of this paper is the calculation of all the renormalization constants and the decay exponent to the second-order precision as well as calculation of scaling function for the mean particle number to the first order. Effects of random sources and sinks on reaction kinetics in the master-equation description have been investigated in the framework of a field-theoretic model, obtained by the «second quantization» a la Doi of the corresponding master equation. It has been demonstrated that random sources and sinks have a significant effect on the asymptotic behaviour of the model, and two universality classes for their description have been identified by the scaling analysis. Results are compared with the Langevin-equation description of the same process.

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INTRODUCTION

The irreversible annihilation reaction $A + A \rightarrow \varnothing$, also known as the mutually annihilating random walk, is a fundamental model of nonequilibrium physics. The A particles perform chaotic motion due to diffusion, and after the mutual collision they may react with constant microscopic probability K_0 per unit time. Usually it is assumed that resulting molecule $\varnothing$ is inert, i.e., chemically inactive and without any backward influence on the motion of reacting A particles. Many reactions of this type are observed in diverse chemical, biological, or physical systems. For instance, various models, such as formation of domain in some magnetic

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materials [1], annihilation of excitons in crystals [2], or model for spreading of opinion of voters in one dimension [3], can be described in terms of annihilation process of this type.

The usual approach to the problems dealing with chemical reactions is based on the use of the kinetic rate equation [4] for concentration field $n(t, \mathbf{x})$. It leads to a self-consistent description analogous to the mean-field approximation in the theory of critical phenomena in the sense that fluctuations in the concentration are neglected. Equivalently, one can assume that the particle concentration is spatially homogeneous $n = n(t)$. This homogeneity can be thought as a consequence of either very high mobility of the reactants or of a very small probability that a reaction actually occurs after mutual collision of reacting particles. For the annihilation process $A + A \rightarrow \varnothing$, kinetic rate equation can be formulated as

$$\partial_t n(t) = -K_0 n^2(t), \tag{1}$$

which can be easily integrated and the obtained solution is

$$n(t) = \frac{n_0}{1 + n_0 K_0 t}, \tag{2}$$

where $n_0 \equiv n(0)$ is initial number of particles. For a long-time ($t \rightarrow \infty$) asymptotic decay, equation (1) predicts power law behaviour for concentration $n(t) \sim t^{-1}$ without any dependence on the value of space dimension. This is a common situation observed in the mean field-like theories. In what follows, we will refer to the value of the time exponent α in power-law dependence for concentration $n(t) \propto t^{-\alpha}$ as the *decay exponent*. Note also that the long-time behaviour does not depend on the initial number n_0 of reacting particles. In the other case, when the particle mobility is sufficiently small, or equivalently, if the microscopic reaction probability K_0 becomes large enough (so particles react immediately after mutual collision), there is a possible transition to a new regime. In which it is more probable that the given particle reacts rather with particles in its neighbourhood than with distant ones. This behaviour is known as the diffusion-controlled regime [4,5]. To gain physical insight, let us consider diffusion process (also known as continuous random walk) to be responsible for the motion of particles. A well-known property of diffusion [6] is the re-entrancy of the visited sites in low space dimensions. In particular, for $d = 1$ and $d = 2$, the probability that the diffusing particle will ever return ($t \rightarrow \infty$) to the starting point is equal to 1. Physically it means that the diffusing particle sweeps thoroughly its local neighbourhood, and thus, it is highly probable that it will react with another particle in its vicinity. Hence, it is reasonable to expect that, after short period of time, the system will be in a state, where there is a lot of isolated particles, that need effectively longer time to traverse to each other and hence to annihilate. This mechanism can effectively slow down the time

evolution of the process and thus lower the decay exponent to other value than 1 predicted by Eq. (2). The approximate value of the decay exponent can be guessed according to the following scaling argument. The re-entrancy property leads to the scaling relation

$$V(t) \sim r^d(t), \tag{3}$$

where $\sim$ denotes corresponding scaling relation between physical quantities. The root-mean-square distance for the diffusing particle scales as $r(t) \sim (Dt)^{1/2}$ and therefore the mean particle number should behave as

$$n(t) \propto \frac{1}{V(t)} \sim \frac{1}{t^{d/2}} = \frac{1}{t^{1+\Delta}}, \tag{4}$$

where the exponent Δ denotes the deviation from the space dimension 2 via the relation

$$d = 2 + 2\Delta. \tag{5}$$

For the space dimension $d = 3$, we have $V(t) \sim t$, because now the diffusing particle effectively explores always new volume, and the re-entrancy property can be neglected. Therefore, the same behaviour as the one described by (1) would be observed. From this simple analysis it could be estimated that space dimension $d_c = 2$ is the upper critical dimension for the annihilation process, above which the mean field approximation is valid. A more rigorous treatment [7] based on renormalization group proves this conclusion and also produces a logarithmic correction for $n(t)$ at the critical dimension, which could not be determined by the simple scaling analysis. In the preceding discussion we have considered only the diffusive motion of reacting particles. However, a typical reaction usually occurs in liquid or gaseous environment. Thermal fluctuations of this environment or some external advection field such as atmospheric eddies could have additional influence on motion of the reacting particles. Therefore, it is interesting to study what effect the external velocity field can have on the annihilation process.

The most flexible approach to the theoretical analysis of the effects of fluctuations in reaction kinetics seems to be the second-quantization method due to Doi [8]. Most of the renormalization-group studies of the effect of random drift on the annihilation reaction $A + A \to \varnothing$ in the framework of the Doi approach have been carried out for the case of a quenched random drift field. Potential random drift with long-range [9, 10] and short-range correlations [11] has been studied as well as «turbulent» flow (i.e., quenched solenoidal random field), with potential disorder [12, 13]. For a more realistic description of a turbulent flow, time-dependent velocity field would be more appropriate. In [14], dynamic disorder with a given Gaussian distribution has been considered, whereas the most ambitious approach on the basis of a velocity field generated by the stochastic Navier–Stokes equation has been introduced here by two of the present authors [15]. From the point of view of the Navier–Stokes equation, the situation

near the critical dimension $d_c = 2$ of the pure reaction model is even more intriguing due to the properties of the Navier–Stokes equation. It is a well-known fact [16] that in the case of space dimension $d = 2$, there is inviscid conservation law of enstropy absent in the three-dimensional case. Calculations in [15] were performed in the one-loop approximation. As may be readily seen from examination of the Feynman graphs, in the one-loop approximation there is no influence of the velocity fluctuations on the renormalization of the interaction vertices. However, the influence of higher-order terms of the perturbation series can have significant effect on the critical properties.

To this end, the most suitable choice for generation of random velocity field is the stochastic Navier–Stokes equation, which can be used to produce a velocity field corresponding to thermal fluctuations [17] and a turbulent velocity field with the Kolmogorov scaling behaviour [18].

A powerful tool for analyzing asymptotic behaviour of stochastic systems is provided by the renormalization-group (RG) method. It allows one to determine long-time — or infrared (IR) — asymptotic regimes of the system and also it is very efficient tool for calculation of various universal physical quantities, e.g., critical exponents. The aim of this study is to examine the IR behaviour of the annihilation process under the influence of advecting velocity fluctuations and to determine its stability. Using the mapping procedure based on the Doi formalism [8], an effective field-theoretic model for the annihilation process will be described in detail in Sec. 1. Consequently, the RG method is applied to this model, and within the two-parameter expansion the renormalization constants and fixed points of the renormalization group are determined in the two-loop approximation. The nonlinear integro-differential equation, which includes the first nontrivial corrections to the (1), is obtained for the mean particle number and it is shown how the information about IR asymptotics can be extracted from it.

Another aspect of the annihilation problem is connected with its theoretical description in the form of a functional integral with a given action. This action resembles actions for field-theoretic models of critical dynamics obtained from the Langevin equation [19] within the Martin–Siggia–Rose approach [20].

In the Langevin equation, the random field compensates for dissipative and reactive losses, hence bringing about a steady state of dynamics of the system. In reaction kinetics, the random sources and sinks, in fact, reflect the real physical situation, in which, during the chemical reaction of follow-up species, particles can appear or disappear due to uncontrolled random interaction with a particle bath, e.g., due to active chemical radicals. The interpretation of random fields in the Langevin equation as physical sources and sinks is rather problematic. Therefore, we propose to analyze the alternative approach provided by the master equation with terms corresponding to interactions with the bath.

In the present paper, we follow the ideas of [4] and describe the random sources and sinks in terms of new birth and death reactions in the master equation

for the single-species annihilation reaction. The simplest choice does not conserve the particle number and we have no possibility to compare it with the standard Langevin approach. A slightly more involved set of birth-death reactions allows one to conserve the particle number and the result may be compared with that of the standard multiplicative noise in the Langevin equation.

The aim of this paper is to give a detailed account of how the field-theoretic approach can be applied to the analysis of the large-scale asymptotic behaviour of annihilation process $A + A \to \varnothing$. We give an elaborate derivation of the action functional for this reaction and show how the velocity fluctuations and effects of sinks and sources can be described. For the analysis of the large-scale behaviour, methods of the renormalization group are employed.

This paper is organized as follows. In Sec. 1, a detailed description of the Doi approach for construction of the field-theoretic model for the annihilation process $A + A \to \varnothing$ is summarized. In Sec. 2, it is shown how both the Kolmogorov scaling and thermal fluctuations can be included into the model and the ultraviolet (UV) renormalization of the model, and elaborated algorithm for the calculation of the renormalization constants is described. Fixed points of the RG are classified together with their stability regions, and possible scaling regimes are presented, and the integro-differential equation for the mean particle number is derived, and analysis of its solution is given. Section 3 is devoted to the introduction of random sources and sinks into the field-theoretic model of reaction-diffusion processes. We recall the basic features of the Doi formalism and construct the basic field-theoretic dynamic action functional together with scaling analysis of the dynamic actions.

1. FIELD-THEORETIC REPRESENTATION OF THE MASTER EQUATION

1.1. Introduction. In our work, we are mainly interested in the annihilation process $A + A \xrightarrow{K_0} \varnothing$, therefore in this section, we describe a derivation of the field-theoretic model for such a process, which allows one to take into account spatial inhomogeneities and randomness in individual reaction events. Probably, the most fundamental description of reaction processes is based on the use of master equation [4]. We summarize general principles of how such a master equation can be mapped onto a suitable Fock space, which makes it possible to use very powerful methods of quantum field theory. The resulting action functional can be treated systematically by such methods as the Feynman diagrammatic technique, renormalization group and operator product expansion [21]. Here we focus mainly on the field-theoretic description of diffusion and reaction process, but it is possible [22] to generalize this approach to include effects as multiple reaction schemes, disorder effects, influence of spatial boundaries, etc.

Let us start with the particles on a regular, infinite, hypercubic lattice with the lattice spacing a in d-dimensional space. The sites of lattice can be labeled by natural numbers $(i = 1, 2, \ldots)$. The A particles are performing continuous random walk on this lattice (random hopping betweeen adjoint sites) with the diffusion constant D/a^2 (the factor a^2 will be eliminated in the continuum limit). It is assumed that reaction process can happen only for particles that occupy the same lattice site with the probability rate K_0. A complete (microscopic) description of such a stochastic problem can be given in terms of evolution equations for probabilities $P(t; \{n\})$, where $\{n\}$ is a given microstate $\{n_1, n_2, \ldots\}$ characterized by n_1 particles at site 1, n_2 at site 2 and so on. These equations (known as master equations) express the balance between incoming and outcoming probabilities [4] for the state $\{n\}$ and in a compact notation they can be written as

$$\frac{dP(t; \{n\})}{dt} = \sum_{\{m\}} R_{m \to n} P(t; \{n\}) - \sum_{\{m\}} R_{n \to m} P(t; \{n\}), \qquad (6)$$

where $R_{m \to n}$ is the transition probability rate from the state m to the state n. According to the work of Doi [8] (see also [23]), such a system of coupled differential equations (6) can be rewritten in terms of creation and annihilation operators well known from quantum mechanics. If there is no site occupation restriction, bosonic operators for each lattice site i can be introduced with the following commutation relations:

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, \quad [\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0. \qquad (7)$$

The ground state $|0\rangle$ is defined as

$$\hat{a}_i |0\rangle = 0 \quad \text{for all sites } i, \qquad (8)$$

which corresponds to the empty lattice (without any A particle). From the bosonic commutation relations (7) important relations follow:

$$\hat{a}_i^n \hat{a}_i^\dagger = n \hat{a}_i^{n-1} + \hat{a}_i^\dagger \hat{a}^n, \quad \hat{a}_i \hat{a}_i^{n\dagger} = n \hat{a}_i^{(n-1)\dagger} + \hat{a}^{n\dagger} \hat{a}_i. \qquad (9)$$

The state $|\{n\}\rangle$ with the given lattice configuration $\{n\} = \{n_1, n_2, \ldots\}$ is introduced with a normalization different from that used in the second quantization method in quantum field theory

$$|\{n\}\rangle = \hat{a}_1^{\dagger n_1} \hat{a}_2^{\dagger n_2} \cdots |0\rangle. \qquad (10)$$

Using relations (9), it can be directly shown that

$$\hat{a}_i |\{n\}\rangle = n_i |\{n_1, n_2, \ldots, n_i - 1, \ldots\}\rangle,$$
$$\hat{a}_i^\dagger |\{n\}\rangle = |\{n_1, n_2, \ldots, n_i + 1, \ldots\}\rangle, \qquad (11)$$
$$\hat{a}_i^\dagger \hat{a}_i |\{n\}\rangle = n_i |\{n\}\rangle,$$

where the last relation legitimizes the identification

$$\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i \tag{12}$$

for the number operator at site i. The scalar product between two states $|\{n\}\rangle$ and $|\{m\}\rangle$ can be obtained:

$$\langle\{n\}|\{m\}\rangle = \prod_{i=1} \delta_{n_i,m_i} n_i!, \tag{13}$$

where $\delta_{i,j}$ stands for Kronecker symbol, and $n! = 1 \times 2 \times \ldots \times n$ is the factorial function. The complete information about the stochastic system is embodied in the probabilities $P(t; \{n\})$, and in the Doi formalism it is incorporated into the state vector $|\Phi\rangle$, which is defined as follows:

$$|\Phi(t)\rangle \equiv \sum_{\{n\}} P(t; \{n\})|\{n\}\rangle = \sum_{\{n\}} P(t; \{n\})\, \hat{a}_1^{\dagger n_1} \hat{a}_2^{\dagger n_2} \cdots |0\rangle, \tag{14}$$

where the sum runs over all possible lattice occupations. Now the task is to rewrite master equation (6) into the Schrödinger-like form for the state vector $|\Phi\rangle$

$$\frac{d}{dt}|\Phi(t)\rangle = -\hat{H}|\Phi(t)\rangle, \tag{15}$$

with some «Hamiltonian» $\hat{H}$, whose exact form depends on the system under consideration. Then Eq. (15) can be formally integrated to obtain $|\Phi(t)\rangle = \mathrm{e}^{-\hat{H}t}|\Phi(0)\rangle$. The initial state $|\Phi(0)\rangle$ has to be specified for the full description. In the case of chemical reactions, initial distribution of particles $P(0; \{n\})$ is usually prescribed and initial state follows from the definition (14). From the technical point of view, the most convenient choice for the singe-species annihilation reaction is the Poisson distribution. Unlike the bimolecular reaction [24], the long-time behaviour is independent of the concrete form of initial conditions.

Let us derive the diffusion part $\hat{H}_D$ of the Hamiltonian H, which corresponds to the diffusive (random walk) movement of A particles. First consider a two-site system with n_1 particles at site 1 and n_2 particles at site 2 with the one-directional hopping process $1 \to 2$ at the rate D_0/a^2. For such a process the master equation (6) is

$$\frac{dP(n_1, n_2)}{dt} = \frac{D_0}{a^2}(n_1 + 1)P(n_1 + 1, n_2 - 1) - \frac{D_0}{a^2} n_1 P(n_1, n_2), \tag{16}$$

where $n_1 + 1$ and n_1 are combinatorial factors resulting from the fact that A particles jump independently of each other. Multiplying both sides of Eq. (16) by

the term $a_1^{\dagger n_1} a_2^{\dagger n_2}$ and performing sum $\sum\limits_{\{n_1, n_2\}}$ over all possible occupations of sites, we arrive at

$$\frac{d|\Phi\rangle}{dt} = \frac{D_0}{\hat{a}^2}(\hat{a}_2^{\dagger}\hat{a}_1 - \hat{a}_1^{\dagger}\hat{a}_1)|\Phi\rangle. \tag{17}$$

This result is easily generalized to the two-directional case $1 \leftrightarrow 2$ described by the following master equation:

$$\frac{d|\Phi\rangle}{dt} = -\frac{D_0}{a^2}(\hat{a}_2^{\dagger} - \hat{a}_1^{\dagger})(\hat{a}_2 - \hat{a}_1)|\Phi\rangle, \tag{18}$$

and hence the diffusion part H_D between two given sites can be written as

$$\hat{H}_D = \frac{D_0}{a^2}(\hat{a}_2^{\dagger} - \hat{a}_1^{\dagger})(\hat{a}_2 - \hat{a}_1). \tag{19}$$

Now consider the annihilation process at given site 1 and derive the corresponding part H_R of the Hamiltonian. Because any two particles can react together, we can write

$$\frac{dP(n)}{dt} = K_0(n+2)(n+1)P(n+2) - K_0 n(n-1)P(n), \tag{20}$$

where again the combinatorial factors are taken into account. After short algebraic manipulations and the use of the relations (7) and (12), relations (20) can be rewritten in the Doi formalism as

$$\frac{d|\phi\rangle}{dt} = K_0(\hat{a}^2 - \hat{a}^{\dagger 2}\hat{a}^2)|\phi\rangle, \tag{21}$$

from which we deduce the reaction part of the Hamiltonian

$$\hat{H}_R = -K_0(\hat{a}^2 - \hat{a}^{\dagger 2}\hat{a}^2). \tag{22}$$

Results (19) and (22) are easily generalized to include all the lattice sites. Consequently, the total Hamiltonian that accounts for diffusion and annihilation process on the hypercubic lattice has the following form:

$$\hat{H}_D + \hat{H}_R = \frac{D_0}{a^2}\sum_{\langle ij\rangle}(\hat{a}_i^{\dagger} - \hat{a}_j^{\dagger})(\hat{a}_i - \hat{a}_j) - K_0\sum_i(\hat{a}_i^2 - \hat{a}_i^{\dagger 2}\hat{a}_i^2), \tag{23}$$

where the first sum runs over the neighbouring sites i and j. Non-Hermitian Hamiltonians such as the one given in (23) are often observed in the case of systems out of equilibrium [3], which cannot be obtained as dynamical counterparts of some static models. Non-Hermiticity also means that the reaction rates in (6)

do not satisfy the detailed balance condition and thus the equilibrium state cannot be characterized by the Gibbs distribution. The Doi formalism also exhibits other differences from the usual quantum mechanics. They are caused by the fact that physical observables cannot be given as bilinear products $\langle\Phi|A|\Phi\rangle$, since according to (14) this would imply expressions bilinear in probability $P(t;\{n\})$.

Let us now take a closer look at derivation of the ensemble average value for an observable quantity A within the Doi approach. It is physically reasonable to assume that A can be expressed as a function of the occupation numbers $A = A(\{n\})$. Examples of such quantities interesting for the case of chemical reactions are:

(a) mean particle number (concentration)

$$n \longleftrightarrow \sum_i \hat{a}_i^\dagger \hat{a}_i, \tag{24}$$

(b) two-point correlation function (between sites i and j)

$$C(i,j) \longleftrightarrow \hat{a}_i^\dagger \hat{a}_i \hat{a}_j^\dagger \hat{a}_j. \tag{25}$$

The ensemble average of A is then clearly given by the expression

$$\langle A(t)\rangle = \sum_{\{n\}} P(t;\{n\})A(\{n\}), \tag{26}$$

and, from the technical point of view, it would be very convenient to have a projection state $\langle\mathcal{P}|$ such that following identity is valid:

$$\langle A(t)\rangle = \sum_{\{n\}} P(t;\{n\})\langle\mathcal{P}|\hat{A}(\{\hat{a}^\dagger\hat{a}\})\hat{a}_1^{\dagger n_1}\hat{a}_2^{\dagger n_2}\dots|0\rangle = \langle\mathcal{P}|\hat{A}|\Phi(t)\rangle. \tag{27}$$

Substitution of the formal solution of the «Schrödinger» equation (15) leads to yet another form of the last expression on the right side of (27)

$$\langle A(t)\rangle = \langle\mathcal{P}|\hat{A}\exp\left(-\hat{H}t\right)|\Phi(0)\rangle. \tag{28}$$

Here, the operator $\hat{A}(\{\hat{a}^\dagger\hat{a}\})$ is obtained from the classical function $A(\{n\})$ with the substitution $n_i \to \hat{a}_i^\dagger\hat{a}_i$ at every site i, what is justified by (12). Hence, by comparing (26) and (27) it is easy to guess that the projection state $\langle\mathcal{P}|$ should satisfy relation

$$\langle\mathcal{P}|\hat{A}(\{\hat{a}^\dagger\hat{a}\})\hat{a}_1^{\dagger n_1}\hat{a}_2^{\dagger n_2}\dots|0\rangle = A(\{n\})\langle\mathcal{P}|\hat{a}_1^{\dagger n_1}\hat{a}_2^{\dagger n_2}\dots|0\rangle = A(\{n\}). \tag{29}$$

It implies that the following two conditions:

$$\langle\mathcal{P}|\hat{a}_i^\dagger = \langle\mathcal{P}| \quad \text{for every site } i, \quad \langle\mathcal{P}|0\rangle = 1 \tag{30}$$

have to be valid for the operator $\mathcal{P}$. By direct use of relations (7), we can conclude that the following choice of the projection state

$$\langle \mathcal{P}| = \langle 0| \exp\left(\sum_i \hat{a}_i\right) \tag{31}$$

serves our purpose. Note that if the operator $\hat{A}$ is written in the normal-ordered form (all creation operators are commuted to the left), it can then be written in terms of the annihilation operators a_i only using properties (30) of the projection state $\langle \mathcal{P}|$, i.e.,

$$\langle \mathcal{P}|\hat{A}(\{\hat{a}^\dagger \hat{a}\}) = \langle \mathcal{P}|N\left[\hat{A}_N(\{\hat{a}^\dagger, \hat{a}\})\right] = \langle \mathcal{P}|\hat{A}_N(\{1, \hat{a}\}). \tag{32}$$

The normal form $\hat{A}_N$ of the operator $\hat{A}$ is defined by relation $\hat{A} = N\left[\hat{A}_N(\{\hat{a}^\dagger, \hat{a}\})\right]$, where N denotes the normal product (creation operators are put to the left of annihilation operators). The $\hat{a}$-dependent operator $\hat{A}_N(\{1, \hat{a}\})$ corresponding to (24) is then simply $\hat{a}_i$, while the correlation function (25) corresponds to the operator $\hat{a}_i \hat{a}_j + \delta_{ij}\hat{a}_j$. For technical reasons it is reasonable to commute the factor $\exp\left(\sum_i \hat{a}_i\right)$ to the right in Eq. (27). In order to do this, we employ the following formula:

$$e^{\hat{a}_i}\hat{a}_i^\dagger = (\hat{a}_i^\dagger + 1)\, e^{\hat{a}_i}, \tag{33}$$

which is derived from relations (9), and thus equation (28) can be cast into the form

$$\langle A(t)\rangle = \left\langle 0 \left| \hat{A}_N(\{1, \hat{a}\}) \exp\left(-\hat{H}(\{\hat{a}^\dagger + 1, \hat{a}\})t\right) \right| \exp\left(\sum_i \hat{a}_i\right) \Phi(0) \right\rangle, \tag{34}$$

where the substitution $\hat{a}_i^\dagger \to \hat{a}_i^\dagger + 1$ was performed both in the expression for the operator $\hat{A}$ (with the subsequent substitution $\hat{a}_i^\dagger \to 0$) and also in the original Hamiltonian $\hat{H}(\{a^\dagger, a\})$. For further use, let us write expression for the mean particle number (24)

$$n(t) = \left\langle 0 \left| \sum_i \hat{a}_i \exp\left(-\hat{H}(\{\hat{a}^\dagger + 1, \hat{a}\})t\right) \right| \exp\left(\sum_i \hat{a}_i\right) \Phi(0) \right\rangle. \tag{35}$$

Thus, in the case of annihilation process described by Eq. (23) we obtain

$$\hat{H}_D + \hat{H}_R = \frac{D_0}{a^2}\sum_{\langle ij\rangle}(\hat{a}_i^\dagger - \hat{a}_j^\dagger)(\hat{a}_i - \hat{a}_j) + K_0\sum_i(2\hat{a}_i^\dagger \hat{a}_i^2 + \hat{a}_i^{\dagger 2}\hat{a}_i^2). \tag{36}$$

As was mentioned above, the convenient choice for the initial condition is the Poisson distribution, which for a given site i corresponds to

$$p(n_i) = \mathrm{e}^{-n_0} \frac{n_0^{n_i}}{n_i!}, \tag{37}$$

where n_0 stands for the mean particle number. Using definition (14), we arrive at the initial state vector in the form

$$|\Phi(0)\rangle = \sum_{\{n_1\}} \mathrm{e}^{-n_0} \frac{n_0^{n_1}}{n_1!} \hat{a}_1^{\dagger n_1} \sum_{\{n_2\}} \mathrm{e}^{-n_0} \frac{n_0^{n_2}}{n_2!} \hat{a}_2^{\dagger n_2} \ldots |0\rangle =$$

$$= \prod_i \mathrm{e}^{-n_0} \exp\left(n_0 \sum_i \hat{a}_i^\dagger\right) |0\rangle. \tag{38}$$

We see that after the substitution $\hat{a}^\dagger \to \hat{a}^\dagger + 1$, the term e^{-n_0} drops out.

1.2. Continuum Limit. In the field of critical phenomena emphasis often lies on the analysis and determination of possible behaviour of the studied system. It turns out that near athe second-order phase transition [25] a whole set of different models behave in the same way. Despite the fact that they describe different physical systems, they can exhibit the same behaviour of the so-called universal quantities. They are model-independent, but can depend on universal parameters such as space dimension, number of components of the order parameter, or symmetries of the system. Examples of such universal quantities are critical exponents [26], describing singular behaviour of various functions.

Now we summarize the main points of the derivation of the continuum limit for the Hamiltonian (23), which allows us to study universal properties of the annihilation process $A + A \to \varnothing$ around its critical dimension. In fact, here we just briefly outline a few important steps, following [22], of the introduction of the functional integral of the continuum limit by the popular interpolation procedure. The alternative operator approach will be sketched later. The main task consists of evaluation of matrix elements for the evolution operator $\exp(-\hat{H}(\{a^\dagger, a\})t)$. In order to do this, we apply the Trotter formula [27], according to which the exponential $\mathrm{e}^{-\hat{H}t}$ can be written as the infinite product

$$\exp\left(-\hat{H}t\right) = \lim_{\Delta t \to 0} (1 - \hat{H}\Delta t)^{t/\Delta t} = (1 - \hat{H}\Delta t)(1 - \hat{H}\Delta t)\ldots \tag{39}$$

Here we assume that $N\Delta t = t$, and at the end of our derivation we let the number of time slices $N \to \infty$ (or equivalently $\Delta t \to 0$). Now into each time slice complete set of coherent states is inserted, which results into mapping of operators $\hat{a}_i^\dagger, \hat{a}_i$ onto complex numbers. Coherent states are explicitly defined as [27]

$$|\psi\rangle = \exp\left(-\frac{1}{2}|\psi|^2 + \psi\hat{a}^\dagger\right)|0\rangle, \tag{40}$$

and they form the eigenstate basis of the annihilation operator

$$\hat{a}|\psi\rangle = \psi|\psi\rangle, \quad \langle\psi|\hat{a}^\dagger = \psi^*\langle\psi|, \tag{41}$$

where the star stands for complex conjugation. The important property of coherent states is their overlap function between different eigenstates

$$\langle\psi_1|\psi_2\rangle = \exp\left(-\frac{1}{2}|\psi_1|^2 - \frac{1}{2}|\psi_2|^2 + \psi_1^*\psi_2\right). \tag{42}$$

For a single site we can write an identity resolution in the form

$$1 = \sum_n \frac{1}{n!}|n\rangle\langle n| = \sum_{m,n} \frac{1}{n!}|n\rangle\langle m|\delta_{mn} = \int \frac{d\psi^* \, d\psi}{\pi}|\psi\rangle\langle\psi|, \tag{43}$$

where the orthogonality relation

$$\delta_{mn} = \frac{1}{\pi m!} \int d\psi^* \, d\psi \, \exp\left(-|\psi|^2\right)\psi^{*m}\psi^n \tag{44}$$

has been used (note the appearance of the weight function $e^{-|\psi|^2}$, the measure adopted here is $d\psi^* d\psi = \operatorname{Re}\psi \operatorname{Im}\psi$). Generalization of (43) to the whole lattice is straightforward

$$1 = \int \prod_i \frac{d\psi^*_{(i,j)} \, d\psi_{(i,j)}}{\pi}|\{\psi\}_j\rangle\langle\{\psi\}_j|, \tag{45}$$

where now $\{\psi\}_j = (\psi_{(1,j)}, \psi_{(2,j)}, \ldots)$ denotes the set of all eigenvalues corresponding to the annihilation operators $\hat{a}_i$ at each lattice site at the time instant $j\Delta t$ $(j = 0, \ldots, N)$. Inserting (45) into each time slice in (39) we get

$$\exp\left(-\hat{H}t\right) = \frac{1}{\mathcal{N}} \lim_{\Delta t \to 0} \int [d\psi^*][d\psi] \, |\{\psi\}_N\rangle \times$$

$$\times \prod_{j=1}^N \langle\{\psi\}_j| \exp\left(-\hat{H}(\{\hat{a}^\dagger, \hat{a}\}) \, \Delta t\right) |\{\psi\}_{j-1}\rangle\langle\{\psi\}_0|, \tag{46}$$

where $\mathcal{N}$ is the normalization constant, and we have introduced the notation

$$[d\psi^*][d\psi] \equiv \prod_{i,j} d\psi^*_{(i,j)} \, d\psi_{(i,j)} \tag{47}$$

for the functional measure. Note that if we deal with normal ordered Hamiltonian (which we shall assume and which explicitly is the case for (36)), then using relations (41) we can immediately write

$$\langle\{\psi\}_j| \exp\left(-H(\{a^\dagger, a\})\Delta t\right)|\{\psi\}_{j-1}\rangle =$$

$$= \langle\{\psi\}_j|\{\psi\}_{j-1}\rangle \exp\left(-H(\{\psi^*\}_j, \{\psi\}_{j-1}) \, \Delta t\right), \tag{48}$$

where $H(\{\psi^*\}_j, \{\psi\}_{j-1})$ is obtained by the replacement of operators by their eigenvalues $\hat{a}_i \to \psi_i, \hat{a}_i^\dagger \to \psi_i^*$. The remaining term in (48) more precisely stands for the expression

$$\langle \{\psi\}_j | \{\psi\}_{j-1} \rangle = \prod_i \langle \psi_{(i,j)} | \psi_{(i,j-1)} \rangle, \tag{49}$$

which, with the overlap relation (42), can be rewritten as

$$\langle \psi_{(i,j)} | \psi_{(i,j-1)} \rangle = \exp\left(-\psi_{(i,j)}^*[\psi_{(i,j)} - \psi_{(i,j-1)}]\right) \times$$
$$\times \exp\left(\frac{1}{2}|\psi_{(i,j)}|^2 - \frac{1}{2}|\psi_{(i,j-1)}|^2\right). \tag{50}$$

The whole scalar product from (46) can be expressed to the first order in the time increment as

$$\prod_j \langle \psi_{(i,j)} | \psi_{(i,j-1)} \rangle = \exp\left(-\sum_j \psi_{(i,j)}^* \frac{d\psi_{(i,j)}}{dt} \Delta t + O(\Delta t)\right) \times$$
$$\times \exp\left(\frac{1}{2}|\psi_{(i,N)}|^2 - \frac{1}{2}|\psi_{(i,0)}|^2\right), \tag{51}$$

so that in the continuum time limit $\Delta t \to 0$ we obtain the functional integral representation for the evolution operator in the form

$$\exp\left(-\hat{H}t\right) = \int \frac{[d\psi^*][d\psi]}{\mathcal{N}} |\{\psi\}_N\rangle \times$$
$$\times \exp\left(\frac{1}{2}|\psi_i(t)|^2 - \frac{1}{2}|\psi_i(0)|^2 - \int_0^t dt[\psi_i^* \partial_t \psi_i + H(\{\psi^*, \psi\})]\right) \langle\{\psi\}_0|. \tag{52}$$

In calculation of expectation values such as (24) and (25) we have to act on the expression (52) from the left by the projection state $\langle \mathcal{P}|$ and from the right by the initial state $|\Phi(0)\rangle$. It can be done in the following manner: first, we note that the following relation holds:

$$\langle\{\psi\}_0 | \Phi(0)\rangle \prod_i \exp\left(-\frac{1}{2}|\psi_i(0)|^2\right) = \exp\left(\sum_i [n_0\psi_i^*(0) - n_0 - |\psi_i(0)^2|]\right), \tag{53}$$

where we have used the initial state in the form (38) and relations (40), (41).

Using equation (33), we proceed as follows:

$$
\langle \mathcal{P} \,|\{\psi\}_N\rangle \prod_i \exp\left(\frac{1}{2}|\psi_i(t)|^2\right) =
$$

$$
= \langle 0| \,\exp\left(\sum_i \hat{a}_i\right) \exp\left[\sum_i \left(-\frac{1}{2}|\psi_i(t)|^2 + \psi_i(t)\hat{a}_i^\dagger\right)\right] |0\rangle \prod_i \exp\left(\frac{1}{2}|\psi_i(t)|^2\right) =
$$

$$
= \langle 0| \prod_i \exp\left(\hat{a}_i\right) \exp\left(\psi_i(t)\hat{a}_i^\dagger\right) |0\rangle =
$$

$$
= \langle 0| \prod_i \sum_{k=0} \frac{\psi_i^k(t)}{k!} \,\exp\left(\hat{a}_i\right) \hat{a}_i^{\dagger k} |0\rangle =
$$

$$
= \langle 0| \prod_i \exp\left(\psi_i(t)(\hat{a}_i^\dagger + 1)\right) \exp\left(\hat{a}_i\right) |0\rangle = \exp\left(\sum_i \psi_i(t)\right). \quad (54)
$$

Putting together terms from (52)–(54) and inserting them into expression (28) for the expectation value of the quantity A, we arrive at the important expression

$$
\langle A(t)\rangle = \mathcal{N}^{-1} \int [d\psi^*][d\psi] A_N(\{1,\psi\}) \exp\left[S(\{\psi^*,\psi\})\right], \quad (55)
$$

where the functional $A_N(\{1,\psi\})$ is obtained from the normal form $\hat{A}_N(\{1,\hat{a}\})$ of the operator $\hat{A}$ by replacing the operators $\hat{a}_i$ by their eigenvalues ψ_i, and the action functional S is given as

$$
S(\{\psi^*,\psi\}) = \sum_i \left(\psi_i(t) + n_0\psi_i^*(0) - n_0 - |\psi_i(0)|^2 - \right.
$$

$$
\left. - \int_0^t dt\,[\psi_i^*\partial_t\psi_i + H(\{\psi^*\},\{\psi\})]\right). \quad (56)
$$

The normalization constant $\mathcal{N}$ is now fixed by the condition

$$
\mathcal{N} = \int [d\psi^*][d\psi] \exp\left[S(\{\psi^*,\psi\})\right]. \quad (57)
$$

Before taking the continuum limit in space, let us note that the initial term $-|\psi_i(0)|^2$ in (56) can actually be dropped in calculations within perturbation theory. In fact, we will define the functional integral in (55) in terms of perturbation expansion and use the bilinear part of the dynamic action to generate propagators, since otherwise the convergence of the functional integral is somewhat problematic [22]. Thus, we arrive at perturbation theory with a retarded propagator,

which we choose such that its value at coinciding time arguments is zero by definition. In the traditional field-theoretic parlance this is tantamount to defining the time-ordered product at coinciding time arguments as the normal-ordered product. Since we are dealing with expectation values of functions of the annihilation operators only, a short reflection of the perturbation expansion reveals that all graphs containing vertices brought about by the initial term $-|\psi_i(0)|^2$ either are proportional to the equal-time value of the propagator, which we have chosen to vanish, or contain closed loops of propagators and therefore vanish as well.

Continuum limit then can be performed in traditional manner according to the substitution

$$\sum_i \rightarrow \frac{\int d\mathbf{x}}{a^d}, \quad \psi_i \rightarrow \psi(t, \mathbf{x})\,a^d, \quad \psi_i^*(t) \rightarrow \psi^\dagger(t, \mathbf{x}), \quad n_0 \rightarrow n_0 a^d, \quad (58)$$

that leads to the field-theoretic action for the scalar fields $\psi^\dagger(\mathbf{x}, t)$ and $\psi(\mathbf{x}, t)$

$$S = -\int_0^t dt \int d\mathbf{x}\{\psi^\dagger \partial_t \psi - D_0 \psi^\dagger \nabla^2 \psi - \lambda_0[1 - \psi^{\dagger 2}]\psi^2\}+$$

$$+ \int d\mathbf{x}\big[\psi(t, \mathbf{x}) + n_0 \psi^\dagger(0, \mathbf{x}) - n_0\big]. \quad (59)$$

It corresponds to the lattice Hamiltonian (23). After the shift $\psi^\dagger \rightarrow \psi^\dagger + 1$, we arrive at the desired result, which is the field-theoretic action S for the annihilation reaction $A + A \rightarrow \varnothing$

$$S = -\int_0^t dt \int d\mathbf{x}\bigg\{\psi^\dagger \partial_t \psi - D_0 \psi^\dagger \nabla^2 \psi+$$

$$+ \lambda_0 D_0[2\psi^\dagger + (\psi^\dagger)^2]\psi^2 + n_0 \int d\mathbf{x}\,\psi^\dagger(\mathbf{x}, 0)\bigg\}, \quad (60)$$

which corresponds to the continuum limit of the action used in (36). It should be noted that the shift $\psi^\dagger \rightarrow \psi^\dagger + 1$ allows one to replace the final term $\int d\mathbf{x}\,\psi(t, \mathbf{x})$ in (59) by the initial term $\int d\mathbf{x}\,\psi(0, \mathbf{x})$ and the contribution of the latter to the perturbation expansion vanishes by the same one taken as the contribution of the quadratic initial term. Therefore, we have not included this linear initial term in the dynamic action either.

The action functional (60) will be used in calculation of physical quantities such as the mean value (34) that can be expressed as the functional integral

$$\langle A(t) \rangle = \int \mathcal{D}\psi^\dagger \mathcal{D}\psi A_N \{1, \psi(t)\}\, \mathrm{e}^S. \quad (61)$$

Here $\mathcal{D}\psi^\dagger\,\mathcal{D}\psi$ is the continuum functional measure. The continuum limit of the expression (35) for mean particle number becomes

$$n(t,\mathbf{x}) = \langle 0|\,\psi(\mathbf{x})\exp\big(-H\{\psi^\dagger(t)+1,\psi(t)\}t\big)\exp\Big(n_0\int d\mathbf{x}\,\psi^\dagger\Big)|0\rangle. \quad (62)$$

1.3. Langevin Equation and Fokker–Planck Equation. Many equations describing evolution of physical, chemical, biological, and social processes are written as mean-field equations for averages of quantities, which intrinsically are random processes to some extent. To take fluctuations around the averages into account, a straightforward way to proceed is to introduce a source of randomness directly in the mean-field equation. Then the quantities solved from the mean-field equations become *stochastic processes* depending on coordinate variables, i.e., *stochastic fields*.

The paradigmatic example of this procedure is the Langevin equation for random walk, which describes the position $\mathbf{r}$ of a test particle subject to random force $\boldsymbol{\eta}$

$$\frac{d\mathbf{r}}{dt} = \boldsymbol{\eta}.$$

Here, the random force is of zero mean and uncorrelated in time (white noise), i.e.,

$$\langle\eta_i(t)\,\eta_j(t')\rangle = D\delta_{ij}\,\delta(t-t').$$

Strictly speaking, this standard physical formulation is mathematically inconsistent, which gives rise to inevitable ambiguities in the case of multiplicative noise (when the noise term is multiplied by a function of the random position).

Consider, for instance, the average distance between two points of the path of the random walk, i.e., (d is the dimension of space) a hint to the property that the Brownian path, although continuous, is not differentiable anywhere.

Increments of the Brownian path

$$\mathbf{W}(t) - \mathbf{W}(t_0) = \int\limits_{t_0}^{t} dt\,\boldsymbol{\eta}(t)$$

constitute an extremely important random process, the *Wiener process*, whose conditional probability density is the Gaussian

$$p\big(\mathbf{W},t\,|\,\mathbf{W}_0,t_0\big) = \frac{1}{[4\pi(t-t_0)]^{d/2}}\,e^{-(\mathbf{W}-\mathbf{W}_0)^2/2(t-t_0)}.$$

In particular, the Wiener process is the basis of the mathematically consistent definition of the Langevin equation (the stochastic differential equation, SDE).

Critical dynamics. In the Landau theory of phase transitions, the dynamics of the order parameter φ near equilibrium are described by the kinetic equation

(time-dependent Ginzburg–Landau (TDGL) equation)

$$\frac{\partial \varphi}{\partial t} = -\Gamma \left(-\nabla^2 \varphi + a\varphi + \frac{\lambda}{6}\varphi^3 \right). \tag{63}$$

In linear response theory, dynamics of fluctuations near equilibrium [28] are often described by kinetic equations similar to (63), but with a random noise term added to the right-hand side.

In a more generic setup, standard models of critical dynamics are based on nonlinear Langevin equations

$$\frac{\partial \varphi}{\partial t} = -\Gamma \frac{\delta H}{\delta \varphi} + f := V(\varphi) + f, \tag{64}$$

where H is the effective equilibrium Hamiltonian. For the random source, a suitable Gaussian distribution is assumed in which the correlation function is determined through the connection to the static equilibrium (fluctuation-dissipation theorem). For instance, model A for the nonconserved order parameter [29] is described by the SDE obtained from 63 by the addition of a white-noise field to the right-hand side.

In reaction kinetics and population dynamics, the simplest kinetic description of the dynamics of the average particle numbers is given by the *rate equation*. The rate equation is a deterministic differential equation for average particle numbers in a homogeneous system, therefore it does not take into account boundary conditions, spatial inhomogeneities, and randomness in the individual reaction events. Spatial dependence is often accounted for by a diffusion term, which gives rise to models of *diffusion-limited reactions* (DLR).

As a simple example, consider the coagulation reaction $A + A \to A$. The diffusion-limited rate equation for the concentration φ of the compound A is

$$\frac{\partial \varphi}{\partial t} = D\nabla^2 \varphi - k\varphi^2,$$

where k is the *rate constant*.

The most straightforward way to take into account various effects of randomness is to add a random source and sink term to the rate equation:

$$\frac{\partial \varphi}{\partial t} = D\nabla^2 \varphi - k\varphi^2 + f. \tag{65}$$

This is a nonlinear Langevin equation for the field φ; physically, in the case of concentration $\varphi \geqslant 0$.

There is an important difference between the reaction models and the critical dynamics: in the latter, deviations of the fluctuating order parameter from the (usually zero) mean may physically be of any sign (or direction). In particular,

deviations from the equilibrium value are always allowed. In the reaction there is often an absorbing steady state, which does not permit fluctuations therefrom: once the system arrives at the absorbing state, it stays there forever. In particular, if the empty state is an absorbing state of the reaction, then the random source should be introduced multiplied by a factor vanishing in the limit $\varphi \to 0$ to prevent the system returning from the absorbing state by the noise. The simplest choice yields

$$\frac{\partial \varphi}{\partial t} = D\nabla^2\varphi - k\varphi^2 + f\varphi$$

instead of (65). This is an equation with a *multiplicative noise*.

1.4. Multiplicative Noise. Consider the Langevin equation with the multiplicative noise of generic form

$$\frac{\partial \varphi}{\partial t} = V(\varphi) + fb(\varphi) := -K\varphi + U(\varphi) + fb(\varphi), \tag{66}$$

where f is (usually) a Gaussian random field with zero mean and the white-in-time correlation function

$$\langle f(t, \mathbf{x}) f(t', \mathbf{x}')\rangle = \overline{D}(x - x') = \delta(t - t')D(\mathbf{x} - \mathbf{x}'), \tag{67}$$

where the shorthand notation $x = (t, \mathbf{x})$ has been used. In (66), $b(\varphi)$ is a functional of φ and $U(\varphi)$ is a nonlinear functional of φ. Both functionals are time-local, i.e., depend only on the current times instant of the SDE.

The Langevin equation with white-in-time noise f is mathematically inconsistent, because the time integral of the noise $\int f \, dt$ is a Wiener process which is not differentiable anywhere as a function of time.

This problem may be approached by starting with the set of correlation functions consisting of a δ sequence in time, i.e.,

$$\langle f(t, \mathbf{x})f(t', \mathbf{x}')\rangle = \overline{D}(t, \mathbf{x}; t', \mathbf{x}') \xrightarrow[t' \to t]{} \delta(t - t')D(\mathbf{x}, \mathbf{x}') \tag{68}$$

and passing to the white-noise limit at a later stage. From the mathematical point of view, this treatment gives rise to the solution of the stochastic differential equation (66) in the Stratonovich sense [30]. Physically, this is often the most natural way to approach the white-noise case. However, technically the Stratonovich interpretation gives rise to a rather complicated treatment and the SDE is most often used in the Ito interpretation in mathematical analyses.

1.5. Fokker–Planck Equation. Recall that the point of introducing of the SDE with white noise is to avoid dealing with the limit (68) explicitly. To this end, instead of using the mathematically problematic, although physically transparent, Langevin equation, the stochastic problem (66), (67) may be equivalently stated in terms of the Fokker–Planck equation (FPE), which is an equation for both the

conditional probability density $p(\varphi, t | \varphi_0, t_0)$ and the probability density $p(\varphi, t)$ of the variable φ. Recall that both the master equation and the Fokker–Planck equation are special cases of the generic (forward) Kolmogorov equation and thus the two problems discussed here are closely related. The simple way to demonstrate this equivalence uses rules of Ito calculus [30], which is beyond the scope of the present treatment. Therefore, only the correspondence between the quantities specifying the stochastic problem in both approaches will be quoted here. The main advantage of the Fokker–Planck equation is that the equation itself is completely well-defined partial differential (or functional-differential for field variables) equation. The ambiguity of the Langevin problem shows that the FPE is different for different interpretations of the SDE.

For simplicity of notation, consider zero-dimensional field theory. The Fokker–Planck equation for the conditional probability density $p(\varphi, t | \varphi_0, t_0)$ in the case of the Ito equation is

$$\frac{\partial}{\partial t} p(\varphi, t | \varphi_0, t_0) = -\frac{\partial}{\partial \varphi} \{[-K\varphi + U(\varphi)] p(\varphi, t | \varphi_0, t_0)\} +$$
$$+ \frac{1}{2} \frac{\partial^2}{\partial \varphi^2} [b(\varphi) D b(\varphi) p(\varphi, t | \varphi_0, t_0)]. \quad (69)$$

If the SDE (66) is interpreted in the Stratonovich sense, the FPE is

$$\frac{\partial}{\partial t} p(\varphi, t | \varphi_0, t_0) = -\frac{\partial}{\partial \varphi} \{[-K\varphi + U(\varphi)] p(\varphi, t | \varphi_0, t_0)\} +$$
$$+ \frac{1}{2} \frac{\partial}{\partial \varphi} \left\{ b(\varphi) \frac{\partial}{\partial \varphi} [D b(\varphi) p(\varphi, t | \varphi_0, t_0)] \right\}. \quad (70)$$

The conditional probability density $p(\varphi, t | \varphi_0, t_0)$ is the fundamental solution of the FPE (69) or (70), i.e.,

$$p(\varphi, t_0 | \varphi_0, t_0) = \delta(\varphi - \varphi_0).$$

Contractions are not quite obvious, when the random variable has several components. For instance, the Fokker–Planck equation in the Ito form becomes

$$\frac{\partial}{\partial t} p(\varphi, t | \varphi_0, t_0) = -\frac{\partial}{\partial \varphi_i} \{[-K_{ij}\varphi_j + U_i(\varphi)] p(\varphi, t | \varphi_0, t_0)\} +$$
$$+ \frac{1}{2} \frac{\partial^2}{\partial \varphi_i \partial \varphi_j} [b_{ik}(\varphi) D_{kl} b_{jl}(\varphi) p(\varphi, t | \varphi_0, t_0)]$$

for a multicomponent variable φ_i.

The Fokker–Planck equation may be regarded as the Schrödinger equation with imaginary time. Using this analogy, the solution of the FPE as well as calculation of expectation values may be represented in a way analogous to

quantum field theory [31]. Construction with the FPE as the starting point gives rise to the famous Martin–Siggia–Rose solution of the SDE [20], but avoids ambiguities inherent in the SDE (they have been fixed by the choice of the FPE).

Consider, for definiteness, the Fokker–Planck equation (69) corresponding to the Ito interpretation of the Langevin equation (66). Introduce — in analogy with Dirac's notation in quantum mechanics — the state vector $|p_t\rangle$ according to the following representation of the PDF:

$$p(\varphi, t) = \langle \varphi | p_t \rangle,$$

which is the solution of the FPE (69) with the initial condition $p(\varphi, 0) = p_0(\varphi)$. To construct the evolution operator for the state vector, introduce momentum and coordinate operators in the manner of quantum mechanics by relations

$$\hat{\pi} f(\varphi) = -\frac{\partial}{\partial \varphi} f(\varphi), \quad \hat{\varphi} f(\varphi) = \varphi f(\varphi), \quad [\hat{\varphi}, \hat{\pi}] = 1.$$

In these terms, the FPE for the PDF gives rise to the evolution equation for the state vector in the form

$$\frac{\partial}{\partial t} |p_t\rangle = -\hat{H} |p_t\rangle,$$

where the «Hamilton» operator for the FPE corresponding to the Ito interpretation of the SDE assumes, according to (69), the form

$$\hat{H} = -\hat{\pi} \left[-K\hat{\varphi} + U(\hat{\varphi}) \right] - \frac{1}{2} \hat{\pi}^2 b(\hat{\varphi}) D b(\hat{\varphi}). \tag{71}$$

Note that, contrary to quantum mechanics, there is no ordering ambiguity in the construction of the Hamilton operator here. In this notation, the conditional PDF may be expressed as the matrix element

$$p(\varphi, t \,|\, \varphi_0, t_0) = \langle \varphi \,|\, \mathrm{e}^{-\hat{H}(t-t_0)} \,|\, \varphi_0 \rangle, \tag{72}$$

and the functional-integral representation may be constructed in the same fashion as in the master-equation case.

2. FIELD-THEORETIC STUDY OF REACTION PROCESS $A + A \to \varnothing$

2.1. Field-Theoretic Model of Annihilation Process. Let us study anomalous kinetics of the general type of the irreversible single-species annihilation reaction

$$A + A \xrightarrow{K_0} \varnothing, \tag{73}$$

with the unrenormalized (mean field) rate constant K_0. The first step of the Doi approach [8] (see also [23]) consists of the introduction of the creation and annihilation operators $\psi^\dagger$ and ψ and the vacuum state $|0\rangle$ satisfying the usual bosonic commutation relations

$$[\psi(\mathbf{x}), \psi^\dagger(\mathbf{x}')] = \delta(\mathbf{x} - \mathbf{x}'),$$
$$[\psi(\mathbf{x}), \psi(\mathbf{x}')] = [\psi^\dagger(\mathbf{x}), \psi^\dagger(\mathbf{x}')] = 0, \tag{74}$$
$$\psi(\mathbf{x})|0\rangle = 0, \quad \langle 0|\psi^\dagger(\mathbf{x}) = 0, \quad \langle 0|0\rangle = 1.$$

Let $P(\{n_i\}, t)$ be the joint probability density function (PDF) for observing n_i particles at positions $\mathbf{x}_i$. The information about the macroscopic state of the classical many-particle system may be transferred into the state vector $|\Phi(t)\rangle$ defined as the sum over all occupation numbers

$$|\Phi(t)\rangle = \sum_{\{n_i\}} P(\{n_i\}, t)|\{n_i\}\rangle, \tag{75}$$

where the basis vectors are defined as

$$|\{n_i\}\rangle = \prod_i [\psi^\dagger(\mathbf{x}_i)]^{n_i}|0\rangle. \tag{76}$$

The whole set of coupled partial differential equations for the PDFs may be rewritten in the compact form of a master equation [7,8]

$$\frac{\partial}{\partial t}|\Phi(t)\rangle = -\hat{H}|\Phi(t)\rangle, \tag{77}$$

where $\hat{H} = \hat{H}_A + \hat{H}_D + \hat{H}_R$ and for the annihilation process $A + A \to \varnothing$ under consideration

$$\hat{H}_A = \int d\mathbf{x}\, \psi^\dagger \nabla[\mathbf{v}(\mathbf{x}, t)\psi(\mathbf{x})],$$
$$\hat{H}_D = -D_0 \int d\mathbf{x}\, \psi^\dagger \nabla^2 \psi(\mathbf{x}), \tag{78}$$
$$\hat{H}_R = \lambda_0 D_0 \int d\mathbf{x}\, (\psi^\dagger)^2 \psi^2,$$

corresponding to the advection, diffusion and reaction part [15]. Due to dimensional reasons, we have extracted the diffusion constant D_0 from the rate constant $K_0 = \lambda_0 D_0$. The mean of a physical quantity $A(t)$ may be expressed [15] – with the use of the notation of Sec. 1 — as the vacuum expectation value

$$\langle A(t)\rangle = \langle 0|T\left[\hat{A}_N\left(\{1, \psi(t)\}\right)\exp\left[-\int_0^\infty \hat{H}_I'\, dt + n_0 \int d\mathbf{x}\, \psi^\dagger(\mathbf{x}, 0)\right]\right]|0\rangle. \tag{79}$$

Here, the interaction operator is defined as $\hat{H}'_I = \hat{H}' - \hat{H}'_0$ and the substitution $\psi^\dagger \to \psi^\dagger + 1 : \hat{H}' \equiv \hat{H}(\psi^\dagger + 1, \psi)$ is understood. The field operators (74) have been replaced by the time-dependent operators of the interaction representation

$$\psi^\dagger(t, \mathbf{x}) = e^{\hat{H}'_0 t}\, \psi^\dagger \mathbf{x}\, e^{-\hat{H}'_0 t}, \quad \psi(t, \mathbf{x}) = e^{\hat{H}'_0 t}\, \psi(\mathbf{x})\, e^{-\hat{H}'_0 t}.$$

In this formulation — assuming the Poisson distribution as the initial condition — the average number density can be computed via the expression

$$n(t, \mathbf{x}) = \langle 0|\, \psi(\mathbf{x})\, e^{-\hat{H}' t}\, \exp\left(n_0 \int d\mathbf{x}\, \psi^\dagger \right) |0\rangle. \tag{80}$$

The expectation value of the time-ordered product in (79) can be cast [38] into the form of a functional integral over scalar fields $\psi^\dagger(\mathbf{x}, t)$ and $\psi(\mathbf{x}, t)$:

$$\langle A(t) \rangle = \int \mathcal{D}\psi^\dagger\, \mathcal{D}\psi\, A_N(\{1, \psi(t)\})\, e^{S_1}, \tag{81}$$

where the action S_1 for the annihilation reaction $A + A \to \varnothing$ is

$$S_1 = -\int\limits_0^\infty dt \int d\mathbf{x} \bigg\{ \psi^\dagger \partial_t \psi + \psi^\dagger \nabla(\mathbf{v}\psi) - D_0 \psi^\dagger \nabla^2 \psi +$$

$$+ \lambda_0 D_0 [2\psi^\dagger + (\psi^\dagger)^2]\psi^2 + n_0 \int d\mathbf{x}\, \psi^\dagger(\mathbf{x}, 0) \bigg\}. \tag{82}$$

In order to analyze the effect of velocity fluctuations on the reaction process, we average the expectation value (81) over the random velocity field $\mathbf{v}$. The most realistic description of the velocity field $\mathbf{v}(x)$ is based on the use of the stochastic Navier–Stokes equation. Due to the incompressibility conditions $\nabla \cdot \mathbf{v} = 0$ and $\nabla \cdot \mathbf{f}^v = 0$ imposed on the velocity field $\mathbf{v}$ and the random-force field $\mathbf{f}^v$, it is possible to eliminate pressure from the Navier–Stokes equation [32] and hence it is sufficient to consider only the transverse components

$$\partial_t \mathbf{v} + P(\mathbf{v} \cdot \nabla)\mathbf{v} - \nu_0 \nabla^2 \mathbf{v} = \mathbf{f}^v. \tag{83}$$

Here, ν_0 is the molecular kinematic viscosity; $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$ is the transverse projection operator, and $k = |\mathbf{k}|$ is the norm of the wave vector $\mathbf{k}$. Here and below we use the subscript «0» for all «bare» parameters to distinguish them from their renormalized counterparts, which will appear during the renormalization procedure.

The large-scale random force per unit mass $\mathbf{f}$ is assumed to be a Gaussian random variable with zero mean and the following correlation function:

$$\langle f_m(\mathbf{x}_1, t_1) f_n(\mathbf{x}_2, t_2) \rangle = \delta(t_1 - t_2) \int \frac{d\mathbf{k}}{(2\pi)^d} P_{mn}(\mathbf{k}) d_f(k)\, e^{i\mathbf{k} \cdot (\mathbf{x}_1 - \mathbf{x}_2)}, \tag{84}$$

where the kernel function is chosen in the form

$$d_f(k) = g_{10}\nu_0^3 k^{4-d-2\epsilon} + g_{20}\nu_0^3 k^2. \tag{85}$$

The nonlocal term is often used to generate the turbulent velocity field with Kolmogorov's scaling [18,19,33,39]. This case is achieved by setting $\epsilon = 2$. The local term $g_{20}\nu_0^3 k^2$ has been added not only because of renormalization reasons but has also an important physical meaning. Such a term in the force correlation function describes generation of thermal fluctuations of the velocity field near equilibrium [17] and thus can mimic the usual environment in which chemical reactions take place.

Averaging (61) over the random velocity field $\mathbf{v}$ is done with the «weight» functional $\mathcal{W}_2 = \mathrm{e}^{S_2}$, where S_2 is the effective action for the advecting velocity field

$$S_2 = \frac{1}{2} \int dt\, d\mathbf{x}\, d\mathbf{x}'\, \tilde{\mathbf{v}}(\mathbf{x}, t) \cdot \tilde{\mathbf{v}}(\mathbf{x}', t)\, d_f(|\mathbf{x} - \mathbf{x}'|) +$$
$$+ \int dt\, d\mathbf{x}\, \tilde{\mathbf{v}} \cdot [-\partial_t \mathbf{v} - (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} + \nu_0 \nabla^2 \mathbf{v}]. \tag{86}$$

With the use of the complete weight functional

$$\mathcal{W} = \mathrm{e}^{S_1 + S_2}, \tag{87}$$

the expectation value of any desirable physical quantity is possible to be calculated.

Actions (82) and (86) for the studied model are written in the form convenient for the use of the standard Feynman diagrammatic technique. There is, however, one delicate point which should be noted. In the pure reaction case we were dealing with the Cauchy problem and, consequently, all time integrals in the dynamic action were written over the positive time axis. When the random drift is included, it is rather natural and technically much simpler to regard the drift as a stationary random process given on the whole time axis. Then, the integration over the fields ψ and $\psi^\dagger$ should include negative time arguments as well for consistency. In the pure reaction part, this does not make any difference, because the retarded propagators render the time integrations at interaction vertices insensitive to the lower limit of integration. In the velocity part this leads to immense technical simplification, because the translation invariance with respect to time is preserved in construction of the perturbation expansion with the only exception of the initial conditions for the fields ψ and $\psi^\dagger$. In the case of Poisson initial distribution, the initial condition is expressed as a linear term in the action and, in particular, does not affect renormalization of the model.

Thus, henceforth we assume all time integrals in the dynamic action to be taken over the whole time axis and may use the standard Fourier representation to express the Feynman rules for the model. The inverse matrix of the quadratic part of the actions determines the form of the bare propagators. It is easily seen that the studied model contains three different types of propagators in Fig. 1. In the momentum-frequency representation they are given as

$$\Delta^{\psi\psi^\dagger}(w_k, \mathbf{k}) = \frac{1}{-iw_k + D_0 i k^2} \qquad (88)$$

and in the momentum-time representation as

$$v_i \quad\quad v_j \quad = \langle v_i v_j \rangle_0 \equiv \Delta^{vv}_{ij}(\omega_k, \mathbf{k})$$

$$v_i \quad\quad \tilde{v}_j \quad = \langle v_i \tilde{v}_j \rangle_0 \equiv \Delta^{v\tilde{v}}_{ij}(\omega_k, \mathbf{k})$$

$$\psi \quad\quad \psi^\dagger \quad = \langle \psi\psi^\dagger \rangle_0 \equiv \Delta^{\psi\psi^\dagger}(\omega_k, \mathbf{k})$$

Fig. 1. The propagators of the model

$$\Delta^{\psi\psi^\dagger}(t, \mathbf{k}) = \theta(t) \exp\left(-D_0 k^2 t\right). \qquad (89)$$

The vertex factor

$$V_m(x_1, x_2, \ldots, x_m; \Phi) = \frac{\delta^m V(\Phi)}{\delta\Phi(x_1)\delta\Phi(x_2)\cdots\delta\Phi(x_m)} \qquad (90)$$

is associated to each interaction vertex of a Feynman graph. Here, Φ could be any member from the set of all fields $\{\psi^\dagger, \psi, \tilde{v}, v\}$. The interaction vertices from action (86) describe interactions between and it may be rewritten in a technically more convenient form

$$-\int dt\, d\mathbf{x}\, \tilde{\mathbf{v}}(\mathbf{v}\partial)\mathbf{v} = -\int dt\, d\mathbf{x}\, \tilde{v}_i v_k\, \partial_k v_i = \int dt\, d\mathbf{x}\, (\partial_k \tilde{v}_i)\, v_k v_i,$$

where the incompressibility condition $\partial_i v_i = 0$ and partial integration method have been used. We assume that the velocity fields fall off rapidly for $|\mathbf{x}| \to \infty$. Rewriting this functional in the symmetric form $v_i V_{ijl} v_j v_l / 2$, it is easy to find the explicit form for the corresponding vertex factor in the momentum space

$$V_{ijl} = i(k_j \delta_{il} + k_l \delta_{ij}). \qquad (91)$$

Here, the momentum $\mathbf{k}$ is flowing into the vertex through the field $\tilde{v}$. The advecting term from the action (82) can be similarly presented as

$$-\int dt\, d\mathbf{x}\, \psi^\dagger \boldsymbol{\nabla}(\mathbf{v}\psi) = \int dt\, d\mathbf{x}\, \psi^\dagger\, \partial_i(v_i\psi) =$$

$$-\int dt\, d\mathbf{x}\, \psi^\dagger v_i\, \partial_i\psi = \int dt\, d\mathbf{x}\, (\partial_i\psi^\dagger)\, v_i\psi. \qquad (92)$$

Rewriting this expression in the form $\psi^\dagger V_j v_j \psi$, we immediately obtain the vertex factor in the momentum space

$$V_j = ik_j, \tag{93}$$

where the momentum $\mathbf{k}$ represents the momentum flowing into the vertex through the field $\psi^\dagger$. The vertices $\tilde{v}vv$ and $\psi^\dagger\psi v$ are depicted graphically in Fig. 2. The two reaction vertices derived from the functional (82) according to the definition (90) are depicted in Fig. 3 and physically describe the density fluctuations of the reactant particles.
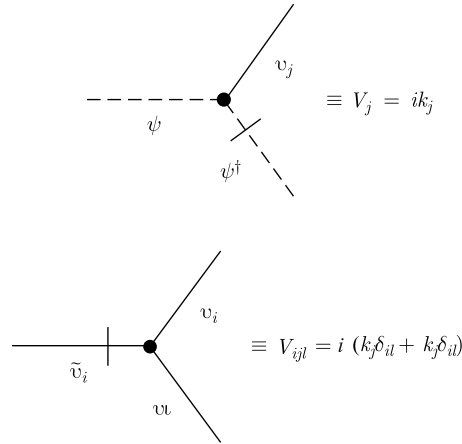

Fig. 2. Interaction vertices describing velocity fluctuation and advection with corresponding vertex factors
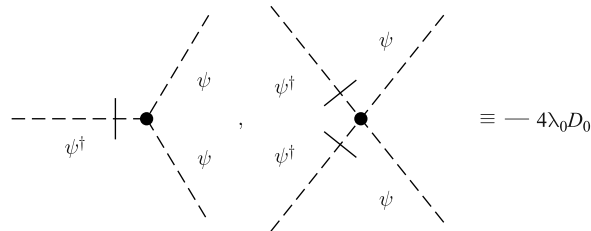

Fig. 3. Interaction vertices responsible for density fluctuations with corresponding vertex factor

It should be stressed that in our model there is no influence of the reactants on the velocity field itself. Therefore, the model given by actions (82) and (86) may be characterized as a model for the advection of the «passive» chemically active admixture.

2.2. UV Renormalization. The functional formulation provides a theoretical framework suitable for applying methods of quantum field theory. Using RG

methods it is possible to determine the IR asymptotic (large spatial and time scales) behaviour of the correlation functions. First of all, a proper renormalization procedure is needed for the elimination of ultraviolet (UV) divergences. There are various renormalization prescriptions applicable for such a task, each with its own advantages. To most popular belong the Pauli–Villars lattice and dimensional regularization [21]. In what follows, we will employ the modified minimal subtraction ($\overline{\text{MS}}$) scheme. Strictly speaking, in the analytic renormalization there is no consistent MS scheme. What we mean here, is the ray scheme [36], in which the two regularizing parameters ϵ, Δ (ϵ has been introduced in (85) and $2\Delta = d - 2$ was introduced in (5)) are taken proportional to each other: $\Delta = \xi\epsilon$, where the coefficient ξ is arbitrary but fixed. In this case, only one independent small parameter, say ϵ, remains and the notion of minimal subtraction becomes meaningful. UV divergences manifest themselves in the form of poles in the small expansion parameter and the minimal subtraction scheme is characterized by discarding all finite parts of the Feynman graphs in the calculation of the renormalization constants. In the modified scheme, as usual, certain geometric factors are not expanded in ϵ, however. This is the content of the $\overline{\text{MS}}$ scheme used in our analysis.

In order to apply the dimensional regularization for the evaluation of renormalization constants, an analysis of possible superficial divergences has to be performed. For the power counting in the actions (82) and (86) we use the scheme [18], in which to each quantity Q two canonical dimensions are assigned, one with respect to the wave number d_Q^k and the other to the frequency d_Q^ω. The normalization for these dimensions is

$$d_\omega^\omega = -d_t^\omega = 1, \quad d_k^k = -d_x^k = 1, \quad d_k^\omega = d_\omega^k = 0. \tag{94}$$

The canonical (engineering) dimensions for fields and parameters of the model are derived from the condition for action to be a scale-invariant quantity, i.e., to have a zero canonical dimension.

The quadratic part of the action (82) determines only the canonical dimension of the quadratic product $\psi^\dagger\psi$. In order to keep both terms in the nonlinear part of the action

$$\lambda_0 D_0 \int dt \, d\mathbf{x} [2\psi^\dagger + (\psi^\dagger)^2]\psi^2, \tag{95}$$

the field $\psi^\dagger$ must be dimensionless. If the field $\psi^\dagger$ has a positive canonical dimension, which is the case for $d > 2$, then the quartic term should be discarded as irrelevant by the power counting. The action with the cubic term only, however, does not generate any loop integrals corresponding to the density fluctuations and thus is uninteresting for the analysis of fluctuation effects in the space dimension $d = 2$.

Using the normalization choice (94) we are able to obtain the canonical dimensions for all the fields and parameters in the d-dimensional space. The results are summarized in Table 1.

Table 1. **Canonical dimensions for the parameters and the fields of the model**

Q	ψ	$\psi^\dagger$	v	$\tilde{v}$	ν_0	D_0	λ_0	g_{10}	g_{20}
d_Q^k	d	0	-1	$d+1$	-2	-2	-2Δ	2ϵ	-2Δ
d_Q^ω	0	0	1	-1	1	1	0	0	0
d_Q	d	0	1	$d-1$	0	0	-2Δ	2ϵ	-2Δ

Here, $d_Q = d_Q^k + 2d_Q^\omega$ is the total canonical dimension and it is determined from the condition that the parabolic differential operator of the diffusion and Navier–Stokes equation scale uniformly under the simultaneous momentum and frequency dilatation $k \to \mu k, \omega \to \mu^2 \omega$.

The model is logarithmic when all coupling constants $g_{10}, g_{20}, \lambda_0$ vanish simultaneously. From Table 1 it follows that this situation occurs for the choice $\epsilon = \Delta = 0$. The UV divergences have the form of poles in various linear combinations of ϵ and Δ. The total canonical dimension of an arbitrary one-particle irreducible Green (1PI) function $\Gamma = \langle \Phi \cdots \Phi \rangle_{1-\mathrm{ir}}$ is given by the relation $d_\Gamma = d + 2 - N_\Phi d_\Phi$, where $N_\Phi = \{N_{\psi^\dagger}, N_\psi, N_v, N_{\tilde{v}}\}$ are the numbers of corresponding external fields. The statistical averaging $\langle \ldots \rangle$ means averaging over all possible realizations of fields $\tilde{\mathbf{v}}, \mathbf{v}, \psi^\dagger, \psi$ satisfying appropriate boundary conditions with the use of the complete weight functional (87). Superficial UV divergences may be present only in those Γ functions for which d_Γ is a non-negative integer. The superficial degree of divergence for the 1PI Green function Γ is

$$d_\Gamma = 4 - N_v - N_{\tilde{v}} - 2N_\psi. \tag{96}$$

However, the real degree of divergence δ_Γ is smaller, because of the structure of the interaction vertex (91),which allows for factoring out the differential operator ∂ to each external line $\tilde{v}$. The real divergence exponent δ_Γ may then be expressed as

$$\delta_\Gamma \equiv d_\Gamma - N_{\tilde{v}} = 4 - N_v - 2N_{\tilde{v}} - 2N_\psi. \tag{97}$$

Although the canonical dimension for the field $\psi^\dagger$ is zero, there is no proliferation of superficial divergent graphs with arbitrary number of external $\psi^\dagger$ legs. This is due to the fact that $n_{\psi^\dagger} \leqslant n_\psi$, which may be established by a straightforward analysis of the graphs [7]. Brief analysis shows that the UV divergences are expected only for the 1PI Green functions listed in Table 2.

Table 2. **Canonical dimensions for the (1PI) divergent Green functions of the model**

$\Gamma_{1-\mathrm{ir}}$	$\langle\psi_\dagger\psi\rangle$	$\langle\psi_\dagger\psi v\rangle$	$\langle\tilde{v}v\rangle$	$\langle\tilde{v}vv\rangle$	$\langle\tilde{v}\tilde{v}\rangle$	$\langle\psi^\dagger\psi^2\rangle$	$\langle(\psi^\dagger)^2\psi^2\rangle$
d_Γ	2	1	2	1	2	0	0
δ_Γ	2	1	1	0	0	0	0

This theoretical analysis leads to the following renormalization of parameters g_0, D_0, and u_0:

$$g_1 = g_{10}\mu^{-2\epsilon}Z_1^3, \quad g_2 = g_{20}\mu^{2\Delta}Z_1^3 Z_3^{-1}, \quad u = u_0 Z_1 Z_2^{-1},$$

$$\lambda = \lambda_0 \mu^{2\Delta} Z_2 Z_4^{-1}, \quad \nu = \nu_0 Z_1^{-1}, \quad D = D_0 Z_2^{-1}, \tag{98}$$

where μ is the reference mass scale in the MS scheme [21] and we have introduced the inverse Prandtl number $u = D/\nu$ for convenience. It represents the ratio between diffusion and viscosity forces in a liquid. In terms of introduced renormalized parameters, the total renormalized action for the annihilation reaction in a fluctuating velocity field is

$$S_R = \int d\mathbf{x}\, dt \left\{ \psi^\dagger \partial_t \psi + \psi^\dagger \boldsymbol{\nabla}(\mathbf{v}\psi) - u\nu Z_2 \nabla^2 \psi + \lambda u\nu\mu^{-2\Delta} Z_4 [2\psi^\dagger + (\psi^\dagger)^2]\psi^2 + \right.$$

$$+ n_0 \int d\mathbf{x}\, \psi^\dagger(\mathbf{x},0) - \frac{1}{2}\tilde{\mathbf{v}}[g_1\nu^3\mu^{2\epsilon}(-\nabla^2)^{1-\Delta-\epsilon} - g_2\nu^3\mu^{-2\Delta} Z_3\nabla^2]\tilde{\mathbf{v}} +$$

$$\left. + \tilde{\mathbf{v}}\cdot[\partial_t\mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} - \nu Z_1\nabla^2\mathbf{v}] \right\}. \tag{99}$$

The renormalization constants $Z_i, i = 1,2,3,4$ are to be calculated perturbatively through the calculation of the UV divergent parts of the 1PI functions $\Gamma_{\psi^\dagger\psi}, \Gamma_{\psi^\dagger\psi^2}, \Gamma_{(\psi^\dagger)^2\psi^2}, \Gamma_{\tilde{v}v}$, and $\Gamma_{\tilde{v}\tilde{v}}$. Interaction terms corresponding to these functions have to be added to the original action $S = S_1 + S_2$ with the aim to ensure UV finiteness of all Green functions generated by the renormalized action S_R. At this stage, the main goal is to calculate the renormalization constants Z_i, $i = 1,2,3,4$.

The singularities in various Green functions will be realized in the form of poles in ϵ and Δ and their linear combinations such as $2\epsilon + \Delta$ or $\epsilon - \Delta$. Recall that for the consistency of the $\overline{\mathrm{MS}}$ scheme it is necessary that the ratio

$$\xi = \frac{\Delta}{\epsilon} \tag{100}$$

is a finite real number. It should be noted that the graphs corresponding to $\Gamma_{\psi^\dagger\psi^2}$ and $\Gamma_{(\psi^\dagger)^2\psi^2}$ differ only by one external vertex and thus give rise to equal renormalization of the rate constant $\lambda_0 D_0$. Therefore, in what follows, we will always consider the function $\Gamma_{\psi^\dagger\psi^2}$. In order to calculate the renormalization

constants Z_2 and Z_4, we proceed according to the general scheme suggested in [35, 36]. We require the fulfillment of UV finiteness (i.e., finite limit when $\epsilon, \Delta \to 0$) of the 1PI functions $\Gamma_{\psi^\dagger\psi}|_{\omega=0}$ and $\Gamma_{\psi^\dagger\psi^2}|_{\omega=0}$. Because the divergent part of the Feynman graphs should not depend on the value of ω, we have adopted the simplest choice $\omega = 0$. It is convenient to introduce the dimensionless expansion variables of the perturbation theory as

$$\alpha_{10} \equiv \frac{g_{10}\overline{S_d}}{p^{2\epsilon}}, \quad \alpha_{20} \equiv \frac{g_{20}\overline{S_d}}{p^{-2\Delta}}, \quad \alpha_{30} \equiv \frac{\lambda_0\overline{S_d}}{p^{-2\Delta}}, \tag{101}$$

where S_d is the surface area of the unit sphere in d-dimensional space; p is the total momentum flowing into the Feynman diagram and $\overline{S_d} = S_d/(2\pi)^d$. For brevity, in the following we use the abbreviation $\overline{g}_0 \equiv g_0\overline{S_d}$ for the parameters $\{g_{10}, g_{20}, \lambda_0\}$ or their renormalized counterparts, respectively. Next, we demonstrate the perturbation series for the 1PI Green functions to the second-order approximation. The perturbative expansion for $\Gamma_{\psi^\dagger\psi}$ may be written as

$$\Gamma_{\psi^\dagger\psi}|_{\omega=0} = D_0 p^2 \left[-1 + \sum_{\substack{n_1,n_2 \geqslant 0, \\ n_1+n_2 \geqslant 1}}^{n_1+n_2=2} \alpha_{10}^{n_1} \alpha_{20}^{n_2} \gamma_{\psi^\dagger\psi}^{(n_1,n_2)}(d, u_0) \right], \tag{102}$$

where $\gamma_{\psi^\dagger\psi}$ are dimensionless coefficients which contain poles in ϵ and Δ. Explicit dependence on the space dimension d and inverse Prandtl number u_0 is emphasized. It is important to note that there are no terms in this sequence proportional to the expansion parameter α_{30}. In terms of the renormalized parameters perturbative expansion for the Green function is (102)

$$\frac{\Gamma_{\psi^\dagger\psi}|_{\omega=0}}{Dp^2} = Z_2 \left[-1 + \sum_{\substack{n_1,n_2 \geqslant 0, \\ n_1+n_2 \geqslant 1}}^{n_1+n_2=2} \alpha_1^{n_1} \alpha_2^{n_2} \gamma_{\psi^\dagger\psi}^{(n_1,n_2)}(d, u) \right], \tag{103}$$

with the renormalized parameters $\alpha_1 = \overline{g_1}s^{2\epsilon}Z_1^{-3}$ and $\alpha_2 = \overline{g_2}s^{-2\Delta}Z_3Z_1^{-3}$ in accordance with the relations (98) and (101), where $s \equiv \mu/p$. Here we would like to stress that in order to get the correct expansion in ϵ and Δ, one has to make replacement

$$d \to 2 + 2\Delta, \qquad u_0 \to Z_1^{-1}Z_2 u \tag{104}$$

in the arguments of $\gamma_{\psi^\dagger\psi}^{(n_1,n_2)}$. In the same way, the perturbation expansion series for the Green function $\Gamma_{\psi^\dagger\psi^2}$ is

$$\Gamma_{\psi^\dagger\psi^2}|_{\omega=0} = -4D_0\lambda_0 \left[1 + \sum_{\substack{n_1,n_2,n_3 \geqslant 0, \\ n_1+n_2+n_3 \geqslant 1}}^{n_1+n_2+n_3=2} \alpha_{10}^{n_1} \alpha_{20}^{n_2} \alpha_{30}^{n_3} \gamma_{\psi^\dagger\psi^2}^{(n_1,n_2,n_3)}(d, u_0) \right], \tag{105}$$

where $\gamma_{\psi^\dagger\psi^2}$ are dimensionless coefficients resulting from calculation of Feynman graphs. Again by replacing the bare parameters with the renormalized counterparts, the following series is obtained:

$$\frac{\Gamma_{\psi^\dagger\psi^2}|_{\omega=0}}{4\lambda D\mu^{-2\Delta}} = -Z_4\left[1 + \sum_{\substack{n_1,n_2,n_3\geqslant 0, \\ n_1+n_2+n_3\geqslant 1}}^{n_1+n_2+n_3=2} \alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}\gamma_{\psi^\dagger\psi^2}^{(n_1,n_2,n_3)}(d,u)\right], \quad (106)$$

where the dimensionless parameter $\alpha_3 = \overline{\lambda}s^{-2\Delta}Z_2^{-1}Z_4$ is introduced and the change (104) is understood. The perturbation series for the Green function $\Gamma_{(\psi^\dagger)^2\psi^2}$ has the same form, so we do not present it.

Denoting by $Z^{(n)}$ the contribution of the order g^n, $g = \{g_1, g_2, \lambda\}$, the first order of renormalization constants Z_2 and Z_4 may be calculated via equations

$$Z_2^{(1)} = \mathcal{L}\left[\overline{g_1}s^{2\epsilon}\gamma_{\psi^\dagger\psi}^{(1,0)} + \overline{g_2}s^{-2\Delta}\gamma_{\psi^\dagger\psi}^{(0,1)}\right], \quad (107)$$

$$Z_4^{(1)} = -\mathcal{L}\left[\overline{g_1}s^{2\epsilon}\gamma_{\psi^\dagger\psi^2}^{(1,0,0)} + \overline{g_2}s^{-2\Delta}\gamma_{\psi^\dagger\psi^2}^{(0,1,0)} + \overline{\lambda}s^{-2\Delta}\gamma_{\psi^\dagger\psi^2}^{(0,0,1)}\right], \quad (108)$$

where $\mathcal{L}$ stands for the operation of extraction of the UV-divergent part (poles in ϵ and Δ or their linear combination). In the $\overline{\text{MS}}$ scheme finite terms are discarded, so we do not need to take care of them. At the second order the term for Z_2 can be schematically written as

$$Z_2^{(2)} = \mathcal{L}\left[-\frac{\overline{g_1}s^{2\epsilon}}{1+u}\left(uZ_2^{(1)} + (u+2)Z_1^{(1)}\right)\gamma_{\psi^\dagger\psi}^{(1,0)} -\right.$$
$$- \overline{g_2}s^{-2\Delta}\left(\frac{u}{1+u}Z_2^{(1)} + \frac{u+2}{1+u}Z_1^{(1)} - Z_3^{(1)}\right)\gamma_{\psi^\dagger\psi}^{(0,1)} + \overline{g_1}^2 s^{4\epsilon}\gamma_{\psi^\dagger\psi}^{(2,0)} +$$
$$\left. + \overline{g_1g_2}s^{2\epsilon-2\Delta}\gamma_{\psi^\dagger\psi}^{(1,1)} + \overline{g_2}^2 s^{-4\Delta}\gamma_{\psi^\dagger\psi}^{(0,2)}\right]. \quad (109)$$

The two-loop graphs that contribute to the calculation of Z_2 are represented by the graphs depicted in Fig. 4. For the renormalization constants Z_4 we have the expression

$$Z_4^{(2)} = -\mathcal{L}\left[\overline{g_1}\overline{\lambda}s^{2(\epsilon-\Delta)}\gamma_{(\psi^+)^2\psi^2}^{(1,0,1)} + \overline{g_2}\overline{\lambda}s^{-4\Delta}\gamma_{\psi^\dagger\psi^2}^{(0,1,1)} + \overline{\lambda}^2 s^{-4\Delta}\gamma_{\psi^\dagger\psi^2}^{(0,0,2)} +\right.$$
$$+ \overline{g_1}s^{2\epsilon}\gamma_{\psi^\dagger\psi^2}^{(1,0,0)}(-3Z_1^{(1)}) + \overline{g_2}s^{-2\Delta}\gamma_{\psi^\dagger\psi^2}^{(0,1,0)}(Z_3^{(1)} - 3Z_1^{(1)}) +$$
$$\left. + \overline{\lambda}s^{-2\Delta}\gamma_{\psi^\dagger\psi^2}^{(0,0,1)}(2Z_4^{(1)} - Z_2^{(1)})\right]. \quad (110)$$
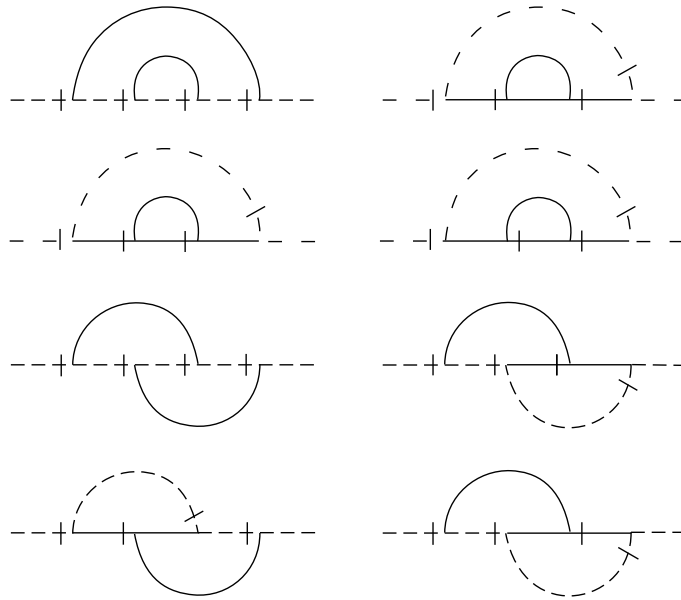
Fig. 4. Two-loop graphs for the perturbation expansion of $\Gamma_{\psi^\dagger\psi}$
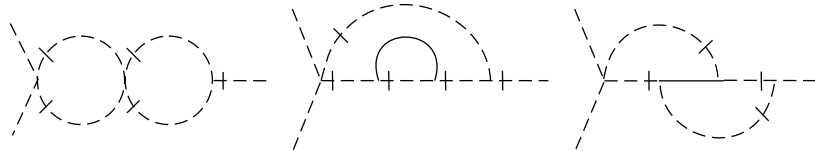

Fig. 5. Two-loop graphs for the perturbation expansion of $\Gamma_{\psi^\dagger\psi^2}$

The two-loop graphs that contribute to the calculation of Z_4 are represented by the graphs depicted in Fig. 5. From these expressions the renormalization constants Z_2 and Z_4 can be calculated in the form

$$
Z_2 = 1 - \frac{\overline{g_1}}{8u(1+u)\epsilon} + \frac{\overline{g_2}}{8u(1+u)\Delta} + \frac{A_{11}\overline{g_1}^2}{\epsilon^2} + \frac{A_{22}\overline{g_2}^2}{\Delta^2} + \frac{A_{12}\overline{g_1 g_2}}{\epsilon\Delta} +
$$
$$
+ \frac{B_{11}\overline{g_1}^2}{\epsilon} + \frac{B_{22}\overline{g_2}^2}{\Delta} + \frac{B_{12}\overline{g_1 g_2}}{\epsilon - \Delta}, \quad (111)
$$

$$
Z_4 = 1 - \frac{\overline{\lambda}}{2\Delta} - \frac{1}{16u(1+u)}\frac{\overline{g_1\lambda}}{(\epsilon-\Delta)\Delta} + \frac{1}{32u(1+u)}\frac{\overline{g_2\lambda}}{\Delta^2} +
$$
$$
+ \frac{\overline{\lambda}^2}{4\Delta^2} - \left(\frac{\overline{g_1\lambda}}{\epsilon-\Delta} - \frac{\overline{g_2\lambda}}{\Delta}\right)C(u,\xi). \quad (112)
$$

The lengthy expressions for the coefficient functions $A_{ij}(\xi, u)$, $B_{ij}(\xi, u)$ and $C(u, \xi)$ can be found in Appendix A.

In a similar way, we obtain renormalization constants Z_1 and Z_3 [36] from condition of the UV finiteness for the 1PI Green functions $\Gamma_{\tilde{v}v}|_{\omega=0}$ and $\Gamma_{\tilde{v}\tilde{v}}|_{\omega=0}$. The perturbation series for $\Gamma_{\tilde{v}v}$ can be written as

$$\Gamma_{\tilde{v}v}|_{\omega=0} = \nu_0 p^2 P_{ij}^p \left[-1 + \sum_{\substack{n_1, n_2 \geqslant 0, \\ n_1+n_2 \geqslant 1}}^{n_1+n_2=2} \alpha_{10}^{n_1} \alpha_{20}^{n_2} \gamma_{\tilde{v}v}^{(n_1, n_2)}(d) \right], \qquad (113)$$

and for $\Gamma_{\tilde{v}\tilde{v}}$ as

$$\Gamma_{\tilde{v}\tilde{v}}|_{\omega=0} = P_{ij}^p \left[g_{10}\nu_0^3 p^{2-2\Delta-2\epsilon} + \right.$$
$$\left. + g_{20}\nu_0^3 p^2 \left\{ -1 + \sum_{\substack{n_1 \geqslant 0, n_2 \geqslant -1, \\ n_1+n_2 \geqslant 1}}^{n_1+n_2=2} \alpha_{10}^{n_1} \alpha_{20}^{n_2} \gamma_{\tilde{v}\tilde{v}}^{(n_1, n_2)}(d) \right\} \right]. \qquad (114)$$

From the definition of the projection operator P_{ij}^p it is easy to see, that after contracting indices i and j we are left with the constant $d-1$. Hence, rewriting perturbation series for $\Gamma_{\tilde{v}v}$ and $\Gamma_{\tilde{v}\tilde{v}}$ in the renormalized variables (98) and contracting indices i and j, we get

$$\frac{\Gamma_{\tilde{v}v}|_{\omega=0}}{\nu p^2 (d-1)} = -Z_1 + Z_1 \left[\sum_{\substack{n_1, n_2 \geqslant 0, \\ n_1+n_2 \geqslant 1}}^{n_1+n_2=2} \alpha_1^{n_1} \alpha_2^{n_2} \gamma_{\tilde{v}v}^{(n_1, n_2)}(d) \right], \qquad (115)$$

$$\frac{\Gamma_{\tilde{v}\tilde{v}}|_{\omega=0}}{(d-1)g_2\nu^3\mu^{-2\Delta}p^2} = \frac{g_1}{g_2} s^{2\epsilon+2\Delta} +$$
$$+ Z_3 + Z_3 \sum_{\substack{n_1 \geqslant 0, n_2 \geqslant -1, \\ n_1+n_2 \geqslant 1}}^{n_1+n_2=2} \alpha_1^{n_1} \alpha_2^{n_2} \gamma_{\tilde{v}\tilde{v}}^{(n_1, n_2)}(d). \qquad (116)$$

By the same algorithm as described above in detail, for the calculation of Z_2 and Z_4 explicit expressions for the renormalization constants Z_1 and Z_3 are obtained. The results for them in the $\overline{\text{MS}}$ scheme can be found in [36].

2.3. IR Stable Fixed Points and Scaling Regimes. The coefficient functions of the RG differential operator for the Green functions

$$D_{\text{RG}} = \mu \frac{\partial}{\partial \mu} \bigg|_0 = \mu \frac{\partial}{\partial \mu} + \sum_{g_i} \beta_i \frac{\partial}{\partial g_i} - \gamma_1 \nu \frac{\partial}{\partial \nu}, \qquad (117)$$

where the bare parameters are denoted with the subscript «0», are defined as

$$\gamma_1 = \mu \frac{\partial \ln Z_1}{\partial \mu}\bigg|_0, \quad \beta_i = \mu \frac{\partial g_i}{\partial \mu}\bigg|_0, \tag{118}$$

with the charges $g_i = \{g_1, g_2, u, \lambda\}$. From this definition and the renormalization relations (98) it follows that

$$\beta_{g_1} = g_1(-2\epsilon + 3\gamma_1), \qquad \beta_{g_2} = g_2(2\Delta + 3\gamma_1 - \gamma_3),$$
$$\beta_\lambda = \lambda(2\Delta - \gamma_4 + \gamma_2), \quad \beta_u = u(\gamma_1 - \gamma_2), \tag{119}$$

where the anomalous dimensions γ_α ($\alpha = 2, 3, 4$) are defined as

$$\gamma_\alpha = \mu \frac{\partial \ln Z_\alpha}{\partial \mu}\bigg|_0. \tag{120}$$

We are interested in the IR asymptotics of small momentum $\mathbf{p}$ and frequencies ω of the renormalized functions or, equivalently, large relative distances and time differences in the $(t, \mathbf{x})$ representation. Such a behaviour is governed by the IR-stable fixed points $g^* = (g_1^*, g_2^*, u^*, \lambda^*)$, which are determined as zeroes of the β functions $\beta(g^*) = 0$. The fixed point g^* is IR stable, if real parts of all eigenvalues of the matrix $\omega_{ij} \equiv \partial \beta_i / \partial g_j|_{g=g^*}$ are strictly positive. From the knowledge of renormalization constants Z_2 and Z_4 (111), (112) and definitions (118), (120), it is possible to calculate anomalous dimensions γ_2 and γ_4

$$\gamma_2 = \frac{\overline{g_1} + \overline{g_2}}{4u(1+u)} - 4B_{11}\overline{g_1}^2 + 4B_{22}\overline{g_2}^2 - 2B_{12}\overline{g_1 g_2}, \tag{121}$$

$$\gamma_4 = -\overline{\lambda} + \overline{\lambda}(\overline{g_1} + \overline{g_2})C(u, \xi). \tag{122}$$

A straightforward calculation shows that higher-order poles cancel each other, so that the anomalous dimensions γ_2 and γ_4 are finite. For completeness, we quote also anomalous dimensions γ_1 and γ_3 [36] to the same order

$$\gamma_1 = \frac{\overline{g_1} + \overline{g_2}}{16} + \frac{(4\xi + 3)}{512(2 + \xi)}\overline{g_1}^2 + \frac{5\xi + 3}{512}\overline{g_1 g_2} - \frac{R}{256}(\overline{g_1} + \overline{g_2})^2, \tag{123}$$

$$\gamma_3 = \frac{(\overline{g_1} + \overline{g_2})^2}{16\overline{g_2}} - \frac{\xi(13 + 19\xi)}{1024(2 + \xi)}\frac{\overline{g_1}^3}{\overline{g_2}} + \frac{34\xi + 19 + 6\xi^2}{512(2 + \xi)}\overline{g_1}^2 - \frac{3\overline{g_1}^2}{512} +$$
$$+ \frac{13 + 31\xi}{1024}\overline{g_1 g_2} + \frac{1 - R}{256}\frac{(\overline{g_1} + \overline{g_2})^3}{\overline{g_2}}, \tag{124}$$

where the value $R = -0.168$ is a result from numerical integration. Zeroes of the beta functions (2.3) determine possible IR behaviour of the model. There are

four IR-stable fixed points and one IR-unstable fixed point. In this section, we present them with their regions of stability.

(i) The trivial (Gaussian) fixed point:

$$\overline{g_1}^* = \overline{g_2}^* = \overline{\lambda}^* = 0, \tag{125}$$

with no restrictions on the inverse Prandtl number u. The Gaussian fixed point is stable, when

$$\epsilon < 0, \quad \Delta > 0, \tag{126}$$

and physically corresponds to the case, when the mean-field solution is valid and fluctuation effects are negligible.

(ii) The short-range (thermal) fixed point:

$$\overline{g_1}^* = 0, \quad \overline{g_2}^* = -16\Delta + 8(1 + 2R)\Delta^2,$$
$$u^* = \frac{\sqrt{17} - 1}{2} - 1.12146\Delta, \tag{127}$$
$$\overline{\lambda}^* = -\Delta + \frac{\Delta^2}{2}(\xi - 2.64375),$$

at which local correlations of the random force dominate over the long-range correlations. This fixed point has the following basin of attraction:

$$\Delta - \frac{2R - 1}{2}\Delta^2 < 0, \quad 2\epsilon + 3\Delta - \frac{3\Delta^2}{2} < 0, \tag{128}$$
$$\Delta + \frac{1}{2}\Delta^2 < 0, \quad \Delta + 0.4529\Delta\epsilon < 0 \tag{129}$$

and corresponds to anomalous decay faster than that due to density fluctuations only, but slower than the mean-field decay.

(iii) The kinetic [37] fixed point with finite rate coefficient:

$$\overline{g_1}^* = \frac{32}{9}\frac{\epsilon(2\epsilon + 3\Delta)}{\epsilon + \Delta} + g_{12}^*(\xi)\epsilon^2,$$
$$\overline{g_2}^* = \frac{32}{9}\frac{\epsilon^2}{\Delta + \epsilon} + g_{22}^*(\xi)\epsilon^2,$$
$$u^* = \frac{\sqrt{17} - 1}{2} + u_1^*(\xi)\epsilon, \tag{130}$$
$$\overline{\lambda}^* = -\frac{2}{3}(\epsilon + 3\Delta) + \frac{1}{9\pi}(3\Delta + \epsilon)(Q\epsilon - \Delta),$$

here $Q = 1.64375$. The fixed point (130) is stable, when inequalities

$$\Omega_\pm > 0, \quad \epsilon > 0, \quad -\frac{2}{3}\epsilon < \Delta < -\frac{1}{3}\epsilon \tag{131}$$

are fulfilled, where

$$\Omega_\pm = \Delta + \frac{4}{3}\epsilon \pm \frac{\sqrt{9\Delta^2 - 12\epsilon\Delta - 8\epsilon^2}}{3} +$$
$$+ \frac{2}{9}\left(-(3 + 2R)\epsilon^2 - 3\epsilon\Delta \pm \frac{4\epsilon(\epsilon + 3\Delta)R - 6\epsilon^2 - 12\epsilon\Delta - 9\Delta^2}{\sqrt{9\Delta^2 - 12\Delta\epsilon - 8\epsilon^2}}\epsilon\right). \quad (132)$$

The decay rate controlled by this fixed point of the average number density is faster than the decay rate induced by dominant local force correlations, but still is slower than the mean-field decay rate.

(iv) The kinetic fixed point with vanishing rate coefficient:

$$\overline{g_1}^* = \frac{32}{9}\frac{\epsilon(2\epsilon + 3\Delta)}{\epsilon + \Delta} + g_{12}^*(\xi)\epsilon^2,$$
$$\overline{g_2}^* = \frac{32}{9}\frac{\epsilon^2}{\Delta + \epsilon} + g_{22}^*(\xi)\epsilon^2, \quad (133)$$
$$u^* = \frac{\sqrt{17} - 1}{2} + u_1^*(\xi)\epsilon, \quad \overline{\lambda}^* = 0.$$

This fixed point is stable, when the long-range correlations of the random force are dominant

$$\Omega_\pm > 0, \qquad \epsilon > 0, \qquad \Delta > -\frac{1}{3}\epsilon, \quad (134)$$

and corresponds to reaction kinetics with the normal (mean-field like) decay rate.

(v) Driftless fixed point given by:

$$\overline{g_1}^* = \overline{g_2}^* = 0, \quad u^* \text{ not fixed}, \quad \overline{\lambda}^* = -2\Delta, \quad (135)$$

with the following eigenvalues:

$$\Omega_1 = -2\epsilon, \quad \Omega_2 = -\Omega_4 = 2\Delta, \quad \Omega_3 = 0. \quad (136)$$

An analysis of the structure of the fixed points and the basins of attraction leads to the following physical picture of the effect of the random stirring on the reaction kinetics. Anomalous behaviour always emerges below two dimensions, when the *local* correlations are dominant in the spectrum of the random forcing (the short-range fixed point (ii)). However, the random stirring gives rise to an effective reaction rate faster than the density-fluctuation induced reaction rate even in this case. The anomaly is present (but with still faster decay, see the next Section) also, when the long-range part of the forcing spectrum is effective, but the power-like falloff of the correlations is fast (this regime is governed by the kinetic fixed point (iii)). Note that this is different from the case in which the di-

vergenceless random velocity field is time-independent, in which case there is no fixed point with $\lambda^* \neq 0$ [12]. At slower spatial falloff of correlations, however, the anomalous reaction kinetics is replaced by a mean-field-like behaviour (this corresponds to the kinetic fixed point (iv)). In particular, in dimensions $d > 1$ this is the situation for the value $\epsilon = 2$ which corresponds to the Kolmogorov spectrum of the velocity field in fully developed turbulence. Thus, long-range correlated forcing gives rise to a random velocity field, which tends to suppress the effect of density fluctuations on the reaction kinetics below two dimensions.
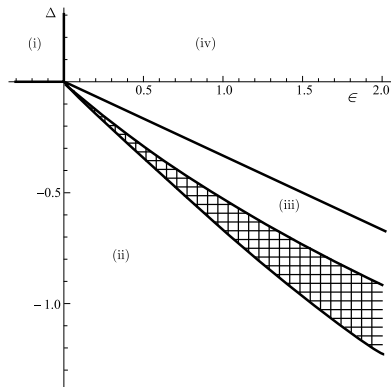

Fig. 6. Regions of stability

For better illustration, regions of stability for fixed points (i)–(iv) are depicted in Fig. 6. We see that in contrast to the one-loop approximation [15], overlap (dashed region) between regions of stability of fixed points (ii) and (iii) is observed. It is a common situation in the perturbative RG approach that higher order terms lead to either gap or overlap between neighbouring stability regions. The physical realization of the large-scale behaviour then depends on the initial state of the system.

2.4. Long-Time Asymptotics of Number Density. Since the renormalization and calculation of the fixed points of the RG are carried out at two-loop level, we are able to find the first two terms of the ϵ, Δ expansion of the average number density, which corresponds to solving the stationarity equations at the one-loop level. The simplest way to find the average number density is to calculate it from the stationarity condition of the functional Legendre transform [38] (which is often called the effective action) of the generating functional obtained by replacing the unrenormalized action by the renormalized one in the weight functional. This is a convenient way to avoid any summing procedures used [7] to take into account the higher-order terms in the initial number density n_0. We are interested in the solution for the number density, therefore we put the expectation values of the fields $\mathbf{v}$ and $\tilde{\mathbf{v}}$ equal to zero at the outset (but retain, of course, the propagator and the correlation function). Therefore, at the second-order approximation the effective renormalized action for this model is

$$\Gamma_R = S_1 + \frac{1}{4} \; \text{〔graph〕} + \frac{1}{8} \; \text{〔graph〕} + \text{〔graph〕} + \ldots, \quad (137)$$

where S_1 is the action (82) (within our convention $S_2 = 0$ in the effective action) and graphs are shown together with their symmetry coefficients. The slashed

wavy line corresponds to the field $\psi^\dagger$ and the single wavy line to the field ψ. The stationarity equations for the variational functional

$$\frac{\delta\Gamma_R}{\delta\psi^\dagger} = \frac{\delta\Gamma_R}{\delta\psi} = 0 \tag{138}$$

give rise to the equations

$$\partial_t\psi = u\nu Z_2\nabla^2\psi - 2\lambda u\nu\mu^{-2\Delta}Z_4\left(1+\psi^\dagger\right)\psi^2 +$$

$$+ 4u^2\nu^2\lambda^2\mu^{-4\Delta}\int\limits_0^\infty dt'\int dy\,(\Delta^{\psi\psi^\dagger})^2(t-t',\mathbf{x}-\mathbf{y})\psi^2(t',\mathbf{y}) +$$

$$+ 4u^2\nu^2\lambda^2\mu^{-4\Delta}\psi^\dagger(t,\mathbf{x})\int\limits_0^\infty dt'\int d\mathbf{y}\,(\Delta^{\psi\psi^\dagger})^2(t-t',\mathbf{x}-\mathbf{y})\psi^2(t',\mathbf{y}) +$$

$$+ \frac{\partial}{\partial x_i}\int\limits_0^\infty dt'\int dy\,\Delta_{ij}^{vv}(t-t',\mathbf{x}-\mathbf{y})\frac{\partial}{\partial x_j}\Delta^{\psi\psi^\dagger}(t-t',\mathbf{x}-\mathbf{y})\psi(t',\mathbf{y}) + \dots,$$

$$\tag{139}$$

$$-\partial_t\psi^\dagger = u\nu Z_2\nabla^2\psi^\dagger - 2\lambda u\nu\mu^{-2\Delta}Z_4\left[2\psi^\dagger + \left(\psi^\dagger\right)^2\right]\psi +$$

$$+ 8u^2\nu^2\lambda^2\mu^{-4\Delta}\int\limits_0^\infty dt'\int d\mathbf{y}(\Delta^{\psi\psi^\dagger})^2(t'-t,\mathbf{y}-\mathbf{x})\psi^\dagger(t',\mathbf{y})\psi(t,\mathbf{x}) +$$

$$+ 4u^2\nu^2\lambda^2\mu^{-4\Delta}\int\limits_0^\infty dt'\int d\mathbf{y}(\Delta^{\psi\psi^\dagger})^2(t'-t,\mathbf{y}-\mathbf{x})\times\left[\psi^\dagger(t',\mathbf{y})\right]^2\psi(t,\mathbf{x}) +$$

$$+ \int\limits_0^\infty dt'\int dy\,\Delta_{ji}^{vv}(t'-t,\mathbf{y}-\mathbf{x})\frac{\partial}{\partial x_i}\Delta^{\psi\psi^\dagger}(t'-t,\mathbf{y}-\mathbf{x})\frac{\partial}{\partial y_j}\psi^\dagger(t',\mathbf{y}) + \dots$$

$$\tag{140}$$

In (139) and (140), in the integral terms it is sufficient to put all renormalization constants equal to unity. Substituting the solution $\psi^\dagger = 0$ of (140) into (139) we arrive at the fluctuation-amended rate equation in the form

$$\partial_t\psi = u\nu Z_2\nabla^2\psi - 2\lambda u\nu\mu^{-2\Delta}Z_4\psi^2 +$$

$$+ 4u^2\nu^2\mu^{-4\Delta}\lambda^2\int\limits_0^\infty dt'\int dy\,(\Delta^{\psi\psi^\dagger})^2(t-t',\mathbf{x}-\mathbf{y})\psi^2(t',\mathbf{y}) + \tag{141}$$

$$+ \frac{\partial}{\partial x_i}\int\limits_0^\infty dt'\int dy\,\Delta_{ij}^{vv}(t-t',\mathbf{x}-\mathbf{y})\frac{\partial}{\partial x_j}\Delta^{\psi\psi^\dagger}(t-t',\mathbf{x}-\mathbf{y})\psi(t',\mathbf{y}) + \dots \tag{142}$$

This is a nonlinear partial integro-differential equation, whose explicit solution is not known. It is readily seen that for a homogeneous solution the term resulting from the third graph in (137) vanishes, and hence the influence of the velocity field on the homogeneous annihilation process would be only through the renormalization of the coefficients λ and D. However, in the case of a nonuniform density field ψ, the effect of velocity fluctuations is explicit in (141). Such a solution can be most probably found only numerically.

To arrive at an analytic solution, we restrict ourselves to the homogeneous density $n(t) = \langle \psi(t) \rangle$, which can be identified with the expression (62). In this case, the last term in (141) vanishes together with the Laplace operator term and the remaining coordinate integral may be calculated explicitly. The propagator is the diffusion kernel of the renormalized model (we consider first the system in the general space dimension d)

$$\Delta^{\psi\psi^{\dagger}}(t - t', \mathbf{x}) = \frac{\theta(t - t')}{[4\pi u\nu(t - t')]^{d/2}} \exp\left[-\frac{x^2}{4u\nu(t - t')}\right]. \qquad (143)$$

As noted above, for calculation of the one-loop contribution it is sufficient to put the renormalization constant $Z_2 = 1$ in the propagator $\Delta^{\psi\psi^{\dagger}}$. Therefore, evaluation of the Gaussian coordinate integral in (141) yields

$$\int d\mathbf{y}\, (\Delta^{\psi\psi^{\dagger}})^2(t - t', \mathbf{x} - \mathbf{y}) = \frac{\theta(t - t')}{[8\pi u\nu(t - t')]^{d/2}}, \qquad (144)$$

and we arrive at the ordinary integro-differential equation

$$\frac{dn(t)}{dt} = -2\lambda u\nu\mu^{-2\Delta}Z_4 n^2(t) + 4\lambda^2 u^2\nu^2\mu^{-4\Delta} \int\limits_0^t dt'\, \frac{n^2(t')}{[8\pi u\nu(t - t')]^{d/2}}. \qquad (145)$$

Spatial fluctuations in the particle density show in the integral term and affect rather heavily even the homogeneous solution. In particular, the integral in (145) diverges at the upper limit in space dimensions $d \geqslant 2$. This is a consequence of the UV divergences in the model above the critical dimension $d_c = 2$ and near the critical dimension is remedied by the UV renormalization of the model. To see this, subtract and add the term $n^2(t)$ in the integrand to obtain

$$\frac{dn(t)}{dt} = -2\lambda u\nu\mu^{-2\Delta}Z_4 n^2(t) + 4\lambda^2 u^2\nu^2\mu^{-4\Delta}n^2(t) \int\limits_0^t \frac{dt'}{[8\pi u\nu(t - t')]^{d/2}} +$$

$$+ 4\lambda^2 u^2\nu^2\mu^{-4\Delta} \int\limits_0^t dt'\, \frac{n^2(t') - n^2(t)}{[8\pi u\nu(t - t')]^{d/2}}. \qquad (146)$$

The last integral here is now convergent at least near two dimensions, provided the solution $n(t)$ is a continuous function. This is definitely the case for the iterative solution constructed below. The divergence in the first integral in (146) may be explicitly calculated below two dimensions and is canceled — in the leading order in the parameter $\Delta = (d-2)/2$ — by the one-loop term of the renormalization constant Z_4 (112). Expanding the right-hand side of (146) in the parameter $\Delta = (d-2)/2$ to the next-to-leading order, we arrive at the equation

$$\frac{dn(t)}{dt} = -2\lambda u\nu\mu^{-2\Delta}n^2(t) + 2\lambda u\nu\mu^{-2\Delta}n^2(t)\left\{\frac{\lambda}{4\pi}\left[\gamma + \ln\left(2u\nu\mu^2 t\right)\right]\right\} +$$

$$+ \frac{\lambda^2 u\nu\mu^{-2\Delta}}{2\pi}\int\limits_0^t dt'\,\frac{n^2(t') - n^2(t)}{t - t'} \quad (147)$$

without divergences near two dimensions. Here, the factor $\mu^{-2\Delta}$ has been retained intact in order not to spoil the consistency of scaling dimensions in different terms of the equation. In (147), $\gamma = 0.57721$ is Euler's constant and we have considered the coupling constant λ and the parameter $\Delta = (d-2)/2$ to be small parameters of the same order taking into account the magnitudes of the parameters in the basins of attraction of the fixed points of the RG. The leading-order approximation for $n(t)$ is given by the first term on the right-hand side of (147) and it is readily seen that, after substitution of this expression, the integral term in (147) is of the order of λ^3 and thus negligible in the present next-to-leading-order calculation. In this approximation, Eq. (147) yields

$$n(t) = \frac{n_0}{1 + 2\lambda u\nu t\left\{1 + \dfrac{\lambda}{4\pi}\left[1 - \gamma - \ln\left(2u\nu\mu^2 t\right)\right]\right\}\mu^{-2\Delta}n_0}, \quad (148)$$

where n_0 is the initial number density.

Since the fields $\Phi = \{v, \tilde{v}, \psi, \psi^\dagger\}$ are not renormalized, the renormalized connected Green functions W_R differ from the unrenormalized $W = \langle\Phi\cdots\Phi\rangle$ [19] only by the choice of parameters and thus, one may write

$$W_R(g, \nu, \mu, \ldots) = W(g_0, \nu_0, \ldots), \quad (149)$$

where $g_0 = \{g_{10}, g_{20}, u_0, \lambda_0\}$ is the full set of the bare parameters, and dots denote all variables unaffected by the renormalization procedure. The independence of renormalization mass parameter μ is expressed by the equation $\mu\partial_\mu W_R = 0$. Using this equation, the RG equation for the mean particle

number $n(t)$ is readily obtained:

$$\left(\mu \frac{\partial}{\partial \mu} + \sum_g \beta_g \frac{\partial}{\partial g} - \gamma_1 \nu \frac{\partial}{\partial \nu} \right) n(t, \mu, \nu, n_0, g) = 0. \tag{150}$$

We are interested in long-time behaviour of the system ($t \to \infty$), therefore we trade the renormalization mass for the time variable. Canonical scale invariance yields relations [34]

$$\left(\mu \frac{\partial}{\partial \mu} - 2\nu \frac{\partial}{\partial \nu} + dn_0 \frac{\partial}{\partial n_0} - d \right) n(t, \mu, \nu, n_0, g) = 0, \tag{151}$$

$$\left(-t \frac{\partial}{\partial t} + \nu \frac{\partial}{\partial \nu} \right) n(t, \mu, \nu, n_0, g) = 0, \tag{152}$$

where the first equation expresses scale invariance with respect to wave number and the second equation with respect to time. Eliminating partial derivatives with respect to the renormalization mass μ and viscosity ν, we obtain the Callan–Symanzik equation for the mean particle number

$$\left[(2 - \gamma_1)t \frac{\partial}{\partial t} + \sum_g \beta_g \frac{\partial}{\partial g} - dn_0 \frac{\partial}{\partial n_0} + d \right] n(t, \mu, \nu, n_0, g) = 0. \tag{153}$$

To separate information given by the RG, consider the dimensionless normalized mean particle number

$$\frac{n}{n_0} = \Phi \left(\nu \mu^2 t, \lambda u \frac{n_0}{\mu^d}, g \right). \tag{154}$$

For the asymptotic analysis, it is convenient to express the particle density in the combination used here. Solution of (153) by the method of characteristics yields

$$\Phi \left(\nu \mu^2 t, \lambda u \frac{n_0}{\mu^d}, g \right) = \Phi \left(\nu \mu^2 \tau, \overline{\lambda u} \frac{\overline{n}_0}{\mu^d}, \overline{g} \right), \tag{155}$$

where τ is the time scale. In Eq. (155), $\overline{g}$ and $\overline{n}_0$ are the first integrals of the system of differential equations

$$t \frac{d}{dt} \overline{g} = -\frac{\beta_g(\overline{g})}{2 - \gamma_1(\overline{g})}, \quad t \frac{d}{dt} \overline{n}_0 = d \frac{\overline{n}_0}{2 - \gamma_1(\overline{g})}. \tag{156}$$

Here $\overline{g} = \{\overline{g}_1, \overline{g}_2, \overline{u}, \overline{\lambda}\}$ with initial conditions $\overline{g}|_{t=\tau} = g$ and $\overline{n}_0|_{t=\tau} = n_0$. In particular,

$$\overline{\lambda u n}_0 = \lambda u \, n_0 \left(\frac{t}{\tau} \right) \exp \left[\int_\tau^t \frac{\gamma_4 ds}{(2 - \gamma_1)s} \right]. \tag{157}$$

The asymptotic expression of the integral on the right-hand side of (157) in the vicinity of the IR-stable fixed point g^* is of the form

$$
\int_\tau^t \frac{\gamma_4 ds}{(2-\gamma_1)s} \underset{t\to\infty}{\sim} \frac{\gamma_4^*}{2-\gamma_1^*} \ln\left(\frac{t}{\tau}\right) + \frac{2}{2-\gamma_1^*} \int_\tau^\infty \frac{(\gamma_4-\gamma_4^*)ds}{(2-\gamma_1)s} =
$$
$$
= \frac{\gamma_4^*}{2-\gamma_1^*} \ln\left(\frac{t}{\tau}\right) + \tilde{c}_4(\tau), \quad (158)
$$

corrections to which vanish in the limit $t \to \infty$. In (158) and henceforth, the notation $\gamma_1^* = \gamma_1\left(g^*\right)$ has been used. From the point of view of the long-time asymptotic behaviour, the next-to-leading term in (158) is an inessential constant. In the vicinity of the fixed point

$$
\overline{\lambda u}\,\frac{\overline{n}_0}{\mu^d} \sim \lambda u \frac{n_0}{\mu^d} \left(\frac{t}{\tau}\right)^{1+\frac{\gamma_4^*}{2-\gamma_1^*}} \widetilde{C}_n \equiv \lambda u \frac{n_0}{\mu^d} \left(\frac{t}{\tau}\right)^\alpha \widetilde{C}_n \equiv \overline{y}\,\widetilde{C}_n, \quad (159)
$$

where a shorthand notation $\overline{y}$ has been introduced for the long-time scaling of the normalized number density as well as the dimensional normalization constant

$$
\widetilde{C}_n = \mathrm{e}^{\tilde{c}_4(\tau)}
$$

and the decay exponent

$$
\alpha = \frac{1+\gamma_4^*}{2-\gamma_1^*}. \quad (160)
$$

The asymptotic behaviour of the normalized particle density is described by the scaling function $f(x,y)$

$$
\Phi\left(\nu\mu^2 t, \lambda u \frac{n_0}{\mu^d}, g\right) \sim \Phi\left(\nu\mu^2\tau, \widetilde{C}_n\overline{y}, g^*\right) \equiv f\left(\nu\mu^2\tau, \widetilde{C}_n\overline{y}\right). \quad (161)
$$

The free parameters in the variables of the scaling function $f(x,y)$ correspond to the choice of units of these variables, whereas the objective information is contained in the form of the scaling function [19, 34]. Here, it is convenient to use the explicit solution (148) to obtain the ε, Δ expansion for the inverse $h(x,y) = 1/f(x,y)$ of the scaling function. We obtain the generic expression

$$
h(x,y) = \frac{1}{f(x,y)} = 1 + 2xy\left\{1 + \frac{\lambda^*}{4\pi}\left[1-\gamma-\ln\left(2u^*x\right)\right]\right\}, \quad (162)
$$

in which the substitution of the various fixed-point values λ^* (at the leading order $\lambda^* \approx 2\pi\overline{\lambda}^*$) and u^* in the leading approximation yields the corresponding ε, Δ expansions.

Below, we list the scaling functions $h(x, y)$ and the dynamic exponents α at the stable fixed points in the next-to-leading-order approximation.

(i) At the trivial (Gaussian) fixed point (125) the mean-field behaviour takes place with

$$h(x, y) = 1 + 2xy, \quad \alpha = 1. \tag{163}$$

(ii) The thermal (short-range) fixed point (127) leads to scaling function and decay exponent

$$h(x, y) = 1 + 2xy \left\{ 1 - \frac{\Delta}{2} \left[1 - \gamma - \ln \left(\sqrt{17} - 1 \right) x \right] \right\},$$

$$\alpha = 1 + \frac{\Delta}{2} + \frac{\Delta^2}{2}. \tag{164}$$

Here, the last coefficient is actually a result of numerical calculation, which in the standard accuracy of Mathematica is equal to 0.5. We have not been able to sort out this result analytically, but think that most probably the coefficient of the Δ^2 term in the decay exponent α in (164) really is $1/2$.

(iii) The kinetic fixed point with an anomalous reaction rate (130) corresponds to

$$h(x, y) = 1 + 2xy \left\{ 1 - \frac{\epsilon + 3\Delta}{3} \left[1 - \gamma - \ln \left(\sqrt{17} - 1 \right) x \right] \right\},$$

$$\alpha = 1 + \frac{3\Delta + \epsilon}{3 - \epsilon}, \tag{165}$$

with an exact value of the decay exponent.

(iv) At the kinetic fixed point with mean-field-like reaction rate (133) we obtain

$$h(x, y) = 1 + 2xy, \quad \alpha = 1. \tag{166}$$

In the actual asymptotic expression corresponding to (161), the argument $y \rightarrow \widetilde{C}_n \overline{y}$ is different from that of the Gaussian fixed point.

To complete the picture, we recapitulate — with a little bit more detail — the asymptotic behaviour of the number density in the physical space dimension $d = 2$ predicted within the present approach [15] (it turns out that for these conclusions the one-loop calculation is sufficient). On the ray $\varepsilon \leqslant 0$, $\Delta = 0$ logarithmic corrections to the mean-field decay take place. The integral determining the asymptotic behaviour of the variable (157) yields in this case

$$\int_{\tau}^{t} \frac{\gamma_4 ds}{(2 - \gamma_1)s} \underset{t \to \infty}{\sim} -\frac{1}{2} \ln \ln \left(\frac{t}{\tau} \right) + \tilde{c}_4(\tau), \tag{167}$$

with corrections vanishing in the limit $t \to \infty$. Therefore, in the vicinity of the fixed point

$$\overline{\lambda u} \, \frac{\overline{n}_0}{\mu^d} \sim \lambda u \, \frac{n_0}{\mu^d} \left(\frac{t}{\tau} \right) \ln^{-1/2} \left(\frac{t}{\tau} \right) \widetilde{C}_n \equiv \overline{y} \, \widetilde{C}_n. \tag{168}$$

The scaling function h is of the simple form

$$h(x, y) = 1 + 2xy$$

and gives rise to asymptotic decay slower than in the mean-field case by a logarithmic factor:

$$n \sim \frac{\ln^{1/2} (t/\tau)}{2\nu\lambda u \widetilde{C}_n t}.$$

It is worth noting that this logarithmic slowing down is weaker than that brought about the density fluctuations only [23] and this change is produced even by the ubiquitous thermal fluctuations of the fluid, when the reaction is taking place in gaseous or liquid media.

On the open ray $\varepsilon > 0$, $\Delta = 0$, the kinetic fixed point with mean-field-like reaction rate (133) is stable and the asymptotic behaviour is given by (166) regardless of the value of the falloff exponent of the random forcing in the Navier–Stokes equation. In particular, only the amplitude factor in the asymptotic decay rate in two dimensions is affected by the developed turbulent flow with Kolmogorov scaling, which corresponds to the value $\varepsilon = 2$. This is in accord with the results obtained in the case of quenched solenoidal flow with long-range correlations [12, 13] as well as with the usual picture of having the maximal reaction rate in a well-mixed system.

3. ROLE OF RANDOM SOURCES AND SINKS
ON REACTION PROCESSES

3.1. Master Equation for Random Sources and Sinks. We will consider the annihilation reaction $A + A \to \varnothing$ in a random drift field in a more general setup than in the previous parts. For this purpose we introduce random sources and sinks of the reacting particles in order to maintain a steady state in the system. In most cases this is carried out by including an additive noise term in the Langevin equation of the stochastic process as was done, e.g., in (83), to have steady turbulent state. Since our analysis is based on the master equation, this is not quite appropriate here. Unfortunately, there is no unique way to introduce random sources in the master equation corresponding to the random noise of the mean-field (Langevin) description. We use the simplest choice, described in detail in [4], which is equivalent to adding processes $A \to X$ and $Y \to A$ to the

whole reaction scheme. Here, X and Y stand for particle baths of the sink and the source, respectively. In a homogeneous system these reactions lead to the master equation

$$\frac{dP(t,n)}{dt} = \mu_+ V \left[P(t, n-1) - P(t,n) \right] +$$
$$+ \mu_- \left[(n+1)P(t, n+1) - nP(t,n) \right] \ldots, \quad (169)$$

where $P(t,n)$ is the probability to find n particles at the time instant t in the system. The ellipsis in (169) represents terms describing the annihilation reaction, diffusion and advection in the system. In (169) μ_+ and μ_- are the reaction constants of the creation and annihilation reactions, respectively. The transition rate has been chosen proportional to the particle number n, which can be understood as consequence of independent processes $A \to X$ and this choice also preserves the empty state as an absorbing state. In the transition rate for creation process, V is the volume of the (for the time being) homogeneous system and will be important in passing to the continuum limit of the inhomogeneous system. The master equation (169) gives rise to the reaction-rate equation

$$\frac{d\langle n \rangle}{dt} = \mu_+ V - \mu_- \langle n \rangle + \ldots, \quad (170)$$

where $\langle n \rangle$ is the mean particle number.

We recall that the basic idea of the Doi approach [8] is to rewrite the set of master equations for probability distributions of a stochastic problem in the form of a single kinetic equation for a state vector incorporating all probabilistic information about the system constructed in a suitable Fock space. The kinetic equation is defined by the Liouville operator acting in the Fock space and generated by the set of master equations. Although the basic procedure has been thoroughly exposed in the literature, the introduction of random sources and sinks of particles in the master equation has specific features, which should be presented in detail. Therefore, let us briefly recall the basic quantities and relations of the Doi approach. For simplicity, consider probabilities $P(t,n)$ to find n particles at the time instant t on a fixed lattice site. Then the spatial dependence may be described by labeling the particle number by the coordinates of the lattice and introducing necessary sums and products over the lattice sites. The construction of corresponding Fock space was presented in Sec. 1, namely Eqs. (7)–(14). The set of master equations for a birth–death process may also be cast in the form of a single evolution equation for the state vector (14) without any explicit dependence on the occupation number

$$\frac{d|\Phi\rangle}{dt} = -\hat{H}(\hat{a}^+, \hat{a})|\Phi\rangle. \quad (171)$$

Master equations (169) give rise to the following terms in the Hamilton operator:

$$\hat{H}_g(\hat{a}^+, \hat{a}) = -\mu_+ V \left(\hat{a}^+ - I\right) - \mu_- \left(I - \hat{a}^+\right) \hat{a}, \tag{172}$$

where I is the identity operator. The expectation value of any function $A(n)$ of the random particle number

$$\langle A(t) \rangle = \sum_{n=0}^{\infty} A(n) P(t, n) \tag{173}$$

may be expressed in the form of the functional integral over the functions $\tilde{a}(t)$ and $a(t)$

$$\langle A(t) \rangle = \int \mathcal{D}\tilde{a} \, \mathcal{D}a \, A_N(1, a(t)) \, \mathrm{e}^{S_1}, \tag{174}$$

where $A_N(\tilde{a}, a)$ is the normal form [38] of the operator $A(\hat{a}^+ \hat{a})$ and S_1 is the dynamic action

$$S_1(\tilde{a}, a) = \int_0^{\infty} dt \, \left[-\tilde{a}(t) \partial_t a(t) + \mu_+ V \tilde{a}(t) - \mu_- \tilde{a}(t) a(t)\right] \ldots \tag{175}$$

Only the generic time-derivative term and terms brought about by the random source model are expressed here explicitly, while the ellipsis stands for terms corresponding to other reactions and initial conditions.

Let the transition rates $\mu_\pm$ be the random functions uncorrelated in time with a probability distribution given in terms of the moments $\langle \mu_\pm^n \rangle = E_{\pm, n}$. To keep the problem translation-invariant in time, we assume stationary stochastic processes determined on the whole time axis. Therefore, henceforth all time integrals in the action functional and, correspondingly, in the perturbation expansion shall be taken over the whole time axis. At this point we also generalize the treatment to the case of a spatially inhomogeneous system and introduce a lattice subscript as the spatial argument, i.e., $a(t) \to a_i(t)$. In this case, the volume V becomes the volume element attached to the lattice site. For simplicity, we replace the time integral with the integral sum $\int_{-\infty}^{\infty} dt \to \sum_\alpha \Delta t$ and assume that the transition rates at each time instant and lattice site $\mu_{\pm, \alpha, i}$ are independent random variables. Then, the average of the expectation value (174) over the distribution of random sources reduces to the calculation of the expectation value

$$\prod_{\alpha, i} \langle \exp\left(\mu_{+, \alpha, i} V \tilde{a}_{\alpha, i} \Delta t - \mu_{-, \alpha, i} \tilde{a}_{\alpha, i} a_{\alpha, i} \Delta t\right) \rangle. \tag{176}$$

For each particular time instant and lattice (we assume that the moments of $\mu_\pm$ are the same for all α and i and omit labels for brevity) this gives rise to the

usual cumulant expansion

$$\langle e^{\mu b \Delta t} \rangle = 1 + b \Delta t E_1 + \frac{1}{2} E_2 (b \Delta t)^2 + \frac{1}{6} E_3 (b \Delta t)^3 + \ldots =$$

$$= \exp \left(b \Delta t E_1 + \frac{E_2 - E_1^2}{2} (b \Delta t)^2 + \frac{E_3 - 3 E_1 E_2 + E_1^3}{6} (b \Delta t)^3 + \ldots \right). \quad (177)$$

Here, b stands for either $V \tilde{a}$ or $-\tilde{a} a$. Thus, for instance, the average over μ_+ assumes the form

$$\prod_{\alpha, i} \langle \exp(\mu_{+, \alpha, i} V \tilde{a}_{\alpha, i} \Delta t) \rangle =$$

$$= \exp \left(\sum_\alpha \sum_i \left[\Delta t E_{+1} V \tilde{a}_{\alpha, i} + \frac{1}{2} \left(E_{+2} - E_{+1}^2 \right) (V \tilde{a}_{\alpha, i} \Delta t)^2 \right] \right) \times$$

$$\times \exp \left(\sum_\alpha \sum_i \left[\frac{E_{+3} - 3 E_{+1} E_{+2} + E_{+1}^3}{6} (V \tilde{a}_{\alpha, i} \Delta t)^3 + \cdots \right] \right). \quad (178)$$

In the continuum limit, the function $\tilde{a}_{\alpha, i}$ is replaced by the field $\psi^+(t, \mathbf{x})$, whereas in the limit $V \to 0$, the expression $a_{\alpha, i}/V$ gives rise to the field $\psi(t, \mathbf{x})$. The sum over α together with Δt gives rise to the time integral and the sum over i together with the volume element leads to the spatial integral $\sum_i V \to \int d\mathbf{x}$. In the first term of the exponential in (178) we thus obtain

$$\sum_\alpha \sum_i \Delta t E_{+1} V \tilde{a}_{\alpha, i} \to E_{+1} \int dt \int d\mathbf{x} \, \psi^+(t, \mathbf{x}). \quad (179)$$

The continuum limit for the cumulants of the second and higher order is not so obvious. We assume the simplest nontrivial distribution for $\mu_\pm$, in which only the variance term has a finite limit, when $\Delta t \to 0$ and $V \to 0$, whereas the contributions of higher-order cumulants vanish, for instance,

$$(E_{+2} - E_{+1}^2) V \Delta t \to \sigma_+, \qquad \qquad \Delta t \to 0, \; V \to 0, \qquad (180)$$

$$(E_{+3} - 3 E_{+1} E_{+2} + E_{+1}^3)(V \Delta t)^2 \to 0, \quad \Delta t \to 0, \; V \to 0. \qquad (181)$$

Therefore, the contribution of the average over μ_+ to the effective dynamic action assumes the form

$$S_+ = \int dt \int d\mathbf{x} \left\{ E_{+1} \psi^+(t, \mathbf{x}) + \frac{1}{2} \sigma_+ \left[\psi^\dagger(t, \mathbf{x}) \right]^2 \right\}. \qquad (182)$$

For the average over μ_-, a similar argument yields

$$S_- = \int dt \int d\mathbf{x} \left\{ -E_{-1}\psi^\dagger(t,\mathbf{x})\psi(t,\mathbf{x}) + \frac{1}{2}\sigma_- \left[\psi^\dagger(t,\mathbf{x})\psi(t,\mathbf{x})\right]^2 \right\}. \quad (183)$$

These contributions to the effective dynamic action may, of course, be generated by suitably chosen normal distributions of $\mu_\pm$.

This way of introduction of random sources and sinks has the annoying feature that it does not conserve the number of particles in the system. For a comparison with the treatment of this problem in the Langevin approach, the random sources and sinks should be introduced in such a way that the particle number is conserved. The simplest way how to deal with this problem is to add to the random source a term proportional to the particle number, i.e., to use the «reaction constant» $\mu_+ V + \mu_{1+} n$ instead of $\mu_+ V$ in the master equation. The source terms on the right-hand side of the master equation (169) in this case assume the form

$$\frac{dP(t,n)}{dt} = \mu_+ V \left[P(t, n-1) - P(t,n) \right] +$$
$$+ \mu_{1+} \left[(n-1)P(t, n-1) - nP(t,n) \right] \dots \quad (184)$$

The new part of the master equation corresponds to a branching process [4].

The added term gives rise to the following contribution to the Hamilton operator:

$$\hat{H}_{g2}(\hat{a}^+, \hat{a}) = -\mu_{1+} \left(\hat{a}^+ - I \right) \hat{a}^+ \hat{a}. \quad (185)$$

Performing the steps described above we arrive at the contribution to the dynamic action in the following form:

$$S_{1+} = \int dt \int d\mathbf{x} \left\{ E_{1+1}\psi^\dagger \left(\psi^\dagger + 1 \right) \psi + \frac{1}{2}\sigma_{1+}\psi^{\dagger\,2} \left(\psi^\dagger + 1 \right)^2 \psi^2 \right\}. \quad (186)$$

Now it is easy to see that if we exclude the plain source (i.e., letting $E_{+1} = \sigma_+ = 0$) and choose $E_{1+1} = E_{-1}$, the empty state remains absorbing one and the «mass term» $\propto \psi^\dagger\psi$ disappears in the dynamic action and we arrive at the dynamic action of random sources and sinks

$$S_{\mathrm{gc}} = \int dt \int d\mathbf{x} \left\{ E_{1+1}\psi^{\dagger\,2}\psi + \frac{1}{2}\sigma_- \left(\psi^\dagger\psi \right)^2 + \frac{1}{2}\sigma_{1+}\psi^{\dagger\,2} \left(\psi^\dagger + 1 \right)^2 \psi^2 \right\}, \quad (187)$$

which conserves the average number of particles.

The effects of the high-order terms are drastically different in the two cases amenable for a scaling analysis with the aid of the renormalization group. The time derivative term in the dynamic action

$$S = - \int dt \int d\mathbf{x}\, \psi^\dagger(t,\mathbf{x})\, \partial_t \psi(t,\mathbf{x}) + \dots \quad (188)$$

must be dimensionless in order to have nontrivial dynamics. Therefore, the total scaling dimension of the number-density operator $\psi^\dagger(t, \mathbf{x})\psi(t, \mathbf{x})$ is equal to the dimension of space and thus is positive.

First, if the scaling dimension of the field $\psi^\dagger$ is equal to zero, $d_{\psi^\dagger} = 0$, then the dimension of the field ψ is positive (more precisely $d_\psi = d$) and the operator monomials in the second and third terms in (187) have the same scaling dimension. Since they are carrying the factor ψ^2, their scaling dimension is larger than that of $\psi^{\dagger 2}\psi$. Therefore, the second and third terms in (187) are IR irrelevant and should be discarded in the asymptotic analysis.

Second, if the scaling dimensions of both fields are positive, then in the operator monomials in the second and third terms in (187) there is at least one «excessive» field factor in comparison with the first term, which renders them irrelevant. Thus, in these cases the IR relevant dynamic action of random sources and sinks reduces to the single term

$$S'_{\mathrm{gc}} = \int dt \int d\mathbf{x}\, E_{1+1}\psi^{\dagger 2}\psi, \quad d_{\psi^\dagger} = 0 \vee d_{\psi^\dagger} > 0, \quad d_\psi > 0. \qquad (189)$$

Third, if the scaling dimension of the field ψ is zero, the scaling dimension of the field $\psi^\dagger$ is positive and terms with «excessive» powers of $\psi^\dagger$ are IR irrelevant. So the starting point for the subsequent RG analysis is the source and sink action in the form

$$S''_{\mathrm{gc}} = \int dt \int d\mathbf{x}\left\{ E_{1+1}\psi^{\dagger 2}\psi + \frac{1}{2}\left(\sigma_- + \sigma_{1+}\right)\left(\psi^\dagger\psi\right)^2 \right\}, \quad d_\psi = 0. \qquad (190)$$

3.2. Annihilation Process with Random Sources and Sinks. Let us analyze the dynamic action of the diffusion-limited annihilation reaction $A + A \rightarrow \varnothing$

$$S_1 = -\int dt \int d\mathbf{x}\left\{ \psi^\dagger \partial_t \psi - D_0 \psi^\dagger \nabla^2 \psi + \lambda_0 D_0[2\psi^\dagger + (\psi^\dagger)^2]\psi^2 \right\} +$$
$$+ n_0 \int d\mathbf{x}\, \psi^\dagger(\mathbf{x}, 0), \quad (191)$$

from the point of view of scaling behaviour sketched below.

In the first case with $d_{\psi+} = 0$, the nonlinear terms in action (191) are of equal scaling dimension. However, the source-sink part (189) is linear in the field ψ with positive scaling dimension in contrast to the quadratic ψ terms of (191). Therefore, the IR relevant interaction above two dimensions is (189) and the corresponding dynamic action is

$$S_{\mathrm{IR1}} = -\int dt \int d\mathbf{x}\left\{ \psi^\dagger \partial_t \psi - D_0 \psi^\dagger \nabla^2 \psi - E_{1+1}\psi^{\dagger 2}\psi \right\} + n_0 \int d\mathbf{x}\, \psi^\dagger(\mathbf{x}, 0).$$
$$(192)$$

This dynamic action does not bring about any graphs with closed loops of the density propagator, which implies supresion of the fluctuation effects. However, the scaling dimension of the interaction term is negative and may compensate for the positive dimensions of the irrelevant interaction terms. Therefore, the rest of the interaction terms are in fact dangerous irrelevant operators and in this case definitive conclusion about the IR relevant action cannot be reached on the basis of the analysis of the scaling dimensions.

In the second case with $d_{\psi^\dagger} > 0$ and $d_\psi > 0$, the fourth-order term in action (191) becomes irrelevant. Either of the remaining third-order terms alone does not generate loops, therefore density fluctuation effects are brought about only, when both fields have the same scaling dimension $d_{\psi^\dagger} = d_\psi = d/2$. In this case, the IR relevant dynamic action is

$$S_{\mathrm{IR2}} = -\int dt \int d\mathbf{x} \left\{ \psi^+ \partial_t \psi - D_0 \psi^+ \nabla^2 \psi + 2\lambda_0 D_0 \psi^+ \psi^2 - E_{1+1} \psi^{+2} \psi \right\} +$$
$$+ n_0 \int d\mathbf{x}\, \psi^+(\mathbf{x}, 0). \quad (193)$$

Here, the scaling dimension of both interaction terms is $(d/2) - 2$ and vanishes at the critical dimension $d_c = 4$, at which the dimensions of all the other interaction terms are positive and they are unambiguously irrelevant. Effective action (193) is the dynamic action of the Gribov process [40], also known as the Reggeon model. Effects of random drift in this case with the use of the Obukhov–Kraichnan compressible velocity field have been analyzed in [41].

In the third case with $d_\psi = 0$, the fourth-order term in action (191) becomes irrelevant as well due to the positive dimension of the field ψ^+. By same token, however, both terms of the source-sink action (190) are also irrelevant and we arrive at the IR relevant dynamic action

$$S_{\mathrm{IR3}} = -\int dt \int d\mathbf{x} \left\{ \psi^\dagger \partial_t \psi - D_0 \psi^\dagger \nabla^2 \psi + 2\lambda_0 D_0 \psi^\dagger \psi^2 \right\} +$$
$$+ n_0 \int d\mathbf{x}\, \psi^+(\mathbf{x}, 0). \quad (194)$$

An argument similar to that used for (192) shows that the scaling analysis with this choice of field dimensions does not allow one to resolve relevance of interaction terms. It should be recalled that the scaling dimensions of auxiliary quantities and the asymptotic behaviour of individual graphs are actually independent of the choice of the values of the field dimensions. Therefore, the effective action (193) with unambiguous classification of relevant and irrelevant interaction terms describes the critical scaling behaviour amenable to the RG analysis.

In summary, if the sources and sinks are chosen such that they conserve the mean number of particles in the system, the anomalous scaling behaviour in the system is that of the Gribov process.

A different situation arises, if the plain source term is included into the analysis. Then there is a possibility that the system does not tend to the absorbing empty state but to an active state with a finite concentration of particles. In this case, the starting point is the dynamic action with all the terms quoted above, i.e.,

$$
S = \int dt \int d\mathbf{x} \left\{ -\psi^\dagger \partial_t \psi + D_0 \psi^\dagger \nabla^2 \psi - \lambda_0 D_0 [2\psi^\dagger + (\psi^\dagger)^2] \psi^2 + E_{+1} \psi^\dagger + \right.
$$

$$
+ \frac{1}{2}\sigma_+ (\psi^\dagger)^2 + E_{1+1}\psi^\dagger (\psi^\dagger + 1)\,\psi + \frac{1}{2}\sigma_{1+}\psi^{+2}(\psi^\dagger + 1)^2 \psi^2 - E_{-1}\psi^\dagger \psi +
$$

$$
\left. + \frac{1}{2}\sigma_- (\psi^\dagger \psi)^2 \right\} + n_0 \int d\mathbf{x}\,\psi^\dagger(\mathbf{x}, 0). \quad (195)
$$

The stationarity equation brought about by this dynamic action for the field ψ is (the stationary value $\psi^\dagger = 0$ as usual)

$$
\partial_t \psi - D_0 \nabla^2 \psi = -2\lambda_0 D_0 \psi^2 + E_{+1} + E_{1+1}\psi - E_{-1}\psi. \quad (196)
$$

However, the action expanded around the stationary value is rather complicated. To keep expressions simple, continue to consider the case $E_{1+1} = E_{-1}$. Then the re-expanded action is

$$
S = \int dt \int d\mathbf{x} \left\{ -\psi^\dagger \partial_t \psi + D_0 \psi^\dagger \nabla^2 \psi - \sqrt{8}\sqrt{E_{+1}\lambda_0 D_0}\,\psi^\dagger \psi + \right.
$$

$$
+ \left(\frac{-E_{+1}}{2} + \frac{E_{-1}\sqrt{E_{+1}\,\lambda_0 D_0}}{\sqrt{2}\lambda_0 D_0} + \frac{E_{+1}\sigma_{1+}}{4\lambda_0 D_0} + \frac{E_{+1}\sigma_-}{4\lambda_0 D_0} + \frac{\sigma_+}{2} \right)\psi^{\dagger 2} +
$$

$$
+ \frac{E_{+1}\sigma_{1+}\psi^{\dagger 3}}{2\lambda_0 D_0} + \frac{E_{+1}\sigma_{1+}\psi^{\dagger 4}}{4\lambda_0 D_0} + \frac{\sqrt{2}\sqrt{E_{+1}\,\lambda_0 D_0}\,\sigma_{1+}\psi^{\dagger 3}\psi}{\lambda_0 D_0} +
$$

$$
+ \left(E_{-1} - \sqrt{2}\sqrt{E_{+1}\,\lambda_0 D_0} + \frac{\sqrt{E_{+1}\,\lambda_0 D_0}\,\sigma_{1+}}{\sqrt{2}\,\lambda_0 D_0} + \frac{\sqrt{E_{+1}\,\lambda_0 D_0}\,\sigma_-}{\sqrt{2}\,\lambda_0 D_0} \right)\psi^{\dagger 2}\psi +
$$

$$
+ \frac{\sqrt{E_{+1}\,\lambda_0 D_0}\,\sigma_{1+}\,\psi^{\dagger 4}\psi}{\sqrt{2}\,\lambda_0 D_0} - 2\,\lambda_0 D_0\,\psi^\dagger \psi^2 + \left(-\lambda_0 D_0 + \frac{\sigma_{1+}}{2} + \frac{\sigma_-}{2} \right)\psi^{\dagger 2}\psi^2 +
$$

$$
\left. + \sigma_{1+}\,\psi^{\dagger 3}\psi^2 + \frac{\sigma_{1+}\psi^{\dagger 4}\psi^2}{2} \right\} + n_0 \int d\mathbf{x}\,\psi^\dagger(\mathbf{x}, 0). \quad (197)
$$

In the critical limit $E_{+1} \to 0$. Since it is the expectation value of a non-negative random quantity μ_+, the variance σ_+ vanishes as well. In the vicinity of the

critical point we keep only the leading E_{+1} and σ_+ putting them equal to zero in terms, where they are subleading. This simplifies the action a little bit

$$
S = \int dt \int d\mathbf{x} \Bigg\{ -\psi^\dagger \partial_t \psi + D_0 \psi^\dagger \nabla^2 \psi - \sqrt{8}\sqrt{E_{+1}\lambda_0 D_0}\psi^\dagger\psi +
$$
$$
+ \left(\frac{E_{-1}\sqrt{E_{+1}}}{\sqrt{2\lambda_0 D_0}} + \frac{\sigma_+}{2} \right)\psi^{\dagger 2} + \frac{E_{+1}\sigma_{1+}\psi^{\dagger 3}}{2\lambda_0 D_0} + \frac{E_{+1}\sigma_{1+}\psi^{\dagger 4}}{4\lambda_0 D_0} + E_{-1}\psi^{\dagger 2}\psi +
$$
$$
+ \frac{\sqrt{2}\sqrt{E_{+1}\lambda_0 D_0}\,\sigma_{1+}\psi^{\dagger 3}\psi}{\lambda_0 D_0} + \frac{\sqrt{E_{+1}\lambda_0 D_0}\,\sigma_{1+}\psi^{\dagger 4}\psi}{\sqrt{2}\lambda_0 D_0} - 2\lambda_0 D_0 \psi^\dagger\psi^2 +
$$
$$
+ \left(-\lambda_0 D_0 + \frac{\sigma_{1+}}{2} + \frac{\sigma_-}{2} \right)\psi^{\dagger 2}\psi^2 + \sigma_{1+}\psi^{\dagger 3}\psi^2 + \frac{\sigma_{1+}\psi^{\dagger 4}\psi^2}{2} \Bigg\} +
$$
$$
+ n_0 \int d\mathbf{x}\,\psi^\dagger(\mathbf{x},0). \quad (198)
$$

Dimensional analysis of the canonical dimensions then yields the following cases. In the nonlinear parts without the critical parameters E_{+1} and σ_+, the previous arguments hold, but in terms having powers of these parameters as coefficients, the positive scaling dimensions of them must be taken into account. The free-field part of the action (198) suggests that the canonical dimension of E_{+1} is four. In fact, the canonical dimension of σ_+ remains a free parameter.

Proceeding in the same manner as above, we arrive at the following effective actions for the IR scaling limit. In the first case with $d_{\psi^\dagger} = 0$, the third and fourth powers of $\psi^\dagger$ and independent of ψ or first order in ψ are irrelevant (due to the coefficients proportional to E_{+1} or its square root) compared with terms $\propto \psi^{\dagger 2}$ in action (198). Nonlinear in ψ terms are irrelevant against the linear terms due to the positive dimension of ψ. Therefore, the IR effective action in this case is

$$
S = \int dt \int d\mathbf{x} \Bigg\{ -\psi^\dagger \partial_t \psi + D_0 \psi^\dagger \nabla^2 \psi - 2\sqrt{2}\sqrt{E_{+1}\lambda_0 D_0}\,\psi^\dagger\psi +
$$
$$
+ \left(\frac{E_{-1}\sqrt{E_{+1}}}{\sqrt{2\lambda_0 D_0}} + \frac{\sigma_+}{2} \right)\psi^{\dagger 2} + E_{-1}\psi^{\dagger 2}\psi \Bigg\} + n_0 \int d\mathbf{x}\,\psi^\dagger(\mathbf{x},0). \quad (199)
$$

Again, the interaction term remaining after the formal dimensional analysis does not bring about loops, although here we have a nontrivial correlation function of the field ψ. The scaling dimension of this term is negative, however, rendering the irrelevant terms dangerous and prohibiting any definitive conclusion about the relevance of individual interaction terms.

In the second case with $d_{\psi^\dagger} > 0$ and $d_\psi > 0$, higher powers than the leading corrections to the free-field action of both fields are irrelevant. This argument

leaves us with the dynamic action

$$
S = \int dt \int d\mathbf{x} \left\{ -\psi^\dagger \partial_t \psi + D_0 \psi^\dagger \nabla^2 \psi - 2\sqrt{2}\, \sqrt{E_{+1}\, \lambda_0 D_0}\, \psi^\dagger \psi + \right.
$$

$$
\left. + \left(\frac{E_{-1}\sqrt{E_{+1}}}{\sqrt{2\lambda_0 D_0}} + \frac{\sigma_+}{2} \right) \psi^{\dagger 2} + E_{-1}\, \psi^{\dagger 2}\psi - 2\lambda_0 D_0\, \psi^\dagger \psi^2 \right\} +
$$

$$
+ n_0 \int d\mathbf{x}\, \psi^\dagger(\mathbf{x}, 0). \quad (200)
$$

Contrary to the case discussed above, here the interaction term $-2\lambda_0 D_0\, \psi^\dagger \psi^2$ generates loops alone due to the presence of the correlation function of the field ψ. Therefore, two effective actions with nontrivial fluctuation contributions are possible.

a) $d_{\psi^\dagger} > d_\psi$. To keep the correlation function of the field ψ for the loops, the variance σ_+ must have a dimension less than that of $\sqrt{E_{+1}}$. This yields the effective action

$$
S = \int dt \int d\mathbf{x} \left\{ -\psi^\dagger \partial_t \psi + D_0 \psi^\dagger \nabla^2 \psi - 2\sqrt{2}\, \sqrt{E_{+1}\, \lambda_0 D_0}\, \psi^\dagger \psi + \right.
$$

$$
\left. + \frac{\sigma_+}{2}\, \psi^{\dagger 2} - 2\lambda_0 D_0\, \psi^\dagger \psi^2 \right\} + n_0 \int d\mathbf{x}\, \psi^+(\mathbf{x}, 0) \quad (201)
$$

with the critical dimension depending on the scaling dimension of σ_+ in the spirit of the description of tricritical scaling behaviour [19]. The model is logarithmic at six dimensions, however, because apart from the coefficient of the $\propto \psi^{\dagger 2}$, the action is that of critical dynamics of the φ^3 model. The upper critical dimension is determined by the scaling behaviour of σ_+ in the critical limit $\sigma_+ \to 0$, $E_{+1} \to 0$.

b) $d_{\psi^\dagger} = d_\psi = d/2$. Both third-order terms are relevant and the effective action is basically (200). In this case, the dimension of σ_+ is larger than that of $\sqrt{E_{+1}}$ and for simplicity we omit σ_+. Thus, the effective dynamic action may be written as

$$
S = \int dt \int d\mathbf{x} \left\{ -\psi^\dagger \partial_t \psi + D_0 \psi^\dagger \nabla^2 \psi - 2\sqrt{2}\, \sqrt{E_{+1}\, \lambda_0 D_0}\, \psi^\dagger \psi + \right.
$$

$$
+ \frac{E_{-1}\sqrt{E_{+1}}}{\sqrt{2\lambda_0 D_0}}\, \psi^{\dagger 2} + E_{-1}\, \psi^{\dagger 2}\psi - 2\lambda_0 D_0\, \psi^\dagger \psi^2 \left. \right\} + n_0 \int d\mathbf{x}\, \psi^\dagger(\mathbf{x}, 0). \quad (202)
$$

Note that this is a dynamic action describing the Gribov process with a random source independent of the active agent density. It is a natural assumption that the

rate of change of the density due to the random sink is proportional to a power of density. The assumption that the rate of change of the density due to the random source is proportional to a power of density is not natural. Therefore, the dynamic action (202) possibly predicts a critical behaviour of the Gribov process different from that discussed in the literature.

In the third case with $d_{\psi^\dagger} > 0$ and $d_\psi = 0$, we arrive at the effective action (201).

The analysis of scaling dimensions shows that we may actually lift most of the restrictions on the probability distribution of the transition rates of the type (180) and (181). Indeed, even if the higher order cumulants are finite, the scaling dimensions of corresponding terms in the dynamic action grow with the order of the cumulant with the exception of the case, when the transition rate is independent of the agent density.

The scaling-dimension analysis of relevant and irrelevant interaction terms presented above appears somewhat formal. In particular, the arbitrariness of the scaling dimensions of the fields is an irritating detail. Therefore, it is instructive to repeat the analysis with the use of standard power counting. Consider first the case of particle-number conserving sources and sinks. The dynamic action is then (195) at the critical point, i.e., with $E_{+1} = \sigma_+ = 0$ and $E_{1+1} = E_{-1}$:

$$S = \int dt \int d\mathbf{x} \left\{ -\psi^+ \partial_t \psi + D_0 \psi^+ \nabla^2 \psi - \lambda_0 D_0 [2\psi^+ + (\psi^+)^2] \psi^2 + \right.$$
$$\left. + E_{1+1} \psi^{+2} \psi + \frac{1}{2} \sigma_{1+} \psi^{+2} (\psi^+ + 1)^2 \psi^2 + \frac{1}{2} \sigma_- (\psi^+ \psi)^2 \right\} +$$
$$+ n_0 \int d\mathbf{x}\, \psi^+(\mathbf{x}, 0). \quad (203)$$

The divergence index of a one-irreducible graph is

$$\delta = (d+2)L - 2I, \quad (204)$$

where d is the dimension of space; L is the number of loops, and I is the number of internal lines. The usual conditions relate the number of loops, lines, vertices, external field arguments E_ψ and E_{ψ^+} as well as the number of vertices V_{ij}, in which the first subscript denotes the number of fields ψ and the second the number of fields ψ^+ in the action (203):

$$L = I + 1 - (V_{12} + V_{21} + V_{22} + V_{23} + V_{24}),$$
$$I = V_{12} + 2V_{21} + 2V_{22} + 2V_{23} + 2V_{24} - E_\psi, \quad (205)$$
$$I = 2V_{12} + V_{21} + 2V_{22} + 3V_{23} + 4V_{24} - E_{\psi^+}.$$

From these equations it follows, in particular,

$$E_\psi - E_{\psi^+} = -V_{12} + V_{21} - V_{23} - 2V_{24}. \quad (206)$$

Eliminating the number of lines I and the number of vertices V_{12}, we obtain

$$\delta = d + 2 + (d-4)V_{21} + (d-2)V_{22} + dV_{23} + (d+2)V_{24} - (d-2)E_\psi - 2E_{\psi^+}. \quad (207)$$

Multiplying relation (206) by an arbitrary coefficient a and combining with expression (207), we arrive at the following representation of the divergence index:

$$\delta = d + 2 - aV_{12} + (d-4+a)V_{21} + (d-2)V_{22} + (d-a)V_{23} +$$
$$+ (d+2-2a)V_{24} - (d-2+a)E_\psi - (2-a)E_{\psi^+}. \quad (208)$$

Here, it is immediately seen that the choice of the value of the parameter a is tantamount to choosing the values of the scaling dimensions of the fields. Since the divergence index δ is independent of a, we conclude that the choice of the scaling dimension of the fields has actually nothing to do with the UV or IR behaviour of a one-irreducible graph. There is no mass parameter in the model, therefore a divergence index δ with at least one negative coefficient of a vertex number indicates potential IR divergence, while a positive δ corresponds to the usual superficial UV divergence.

For further discussion, let us denote the generic vertex as $v_{nm}\psi^n\psi^{+m}$ and refer to any particular interaction term by its coefficient function v_{nm}. From expression (207), it would appear that vertices v_{23} and v_{24} always make a positive contribution to δ regardless of the space dimension, therefore they are definitely IR irrelevant. The vertex v_{22} is irrelevant above two dimensions, whereas the vertex v_{21} gives rise to IR divergences below four dimensions. Since there is no known regular way to cope with these directly, we have to rely on the usual connection between the IR and UV divergences in the logarithmic theory and extrapolate its results below the critical dimension with the aid of the ε expansion. Thus, if the vertex v_{21} is present, the best we can do is to carry out the RG analysis around the critical dimension $d_c = 4$, at which the vertex v_{22} is also irrelevant, and we are left with the Gribov process. From the point of view of the previous scaling dimension analysis, representation (207) corresponds to effective action (193) at the logarithmic dimension $d = 4$ with the field dimensions $d_\psi = d - 2$, $d_{\psi^+} = 2$ and dimensionless coupling constant of the vertex v_{12}.

On the other hand, with the use of relation (206) we may eliminate any one vertex number, except V_{22}, which then leads to changes in the coefficients of the remaining vertex numbers (apart from V_{22}) and to a different classification of the relevance of a given vertex, although the index of any graph does not feel the change. For instance, choosing $a = 4 - d$ in (208) to eliminate V_{21} we arrive at

$$\delta = d + 2 + (d-4)V_{12} + (d-2)V_{22} + (2d-4)V_{23} +$$
$$+ (3d-6)V_{24} - 2E_\psi - (d-2)E_{\psi^+}. \quad (209)$$

Here, v_{22}, v_{23}, and v_{24} are all irrelevant at $d > 2$, whereas v_{12} is relevant at $d < 4$. Thus, if $V_{12} > 0$, the only critical regime amenable to an RG analysis is that of the Gribov process. It is evident from the preceding discussion that eliminating a vertex number V_{ij} from the expression for the divergence index is tantamount to putting the scaling dimension of that vertex equal to zero and thus fixing the scaling dimensions of the fields correspondingly. The aim of the power-counting analysis is to discard all graphs of a perturbation expansion of a Green function with the scaling dimension (i.e., the divergence index) larger than that of the leading-order expression. The divergence index is independent of the choice of the scaling dimension of the fields. In fact, the only tunable parameter it depends on is the space dimension. Putting the scaling dimension of a vertex equal to zero means that we are looking for the critical behaviour of the model at a space dimension at which there are no restrictions on the number of these vertices in the relevant and marginal graphs.

Since we are analyzing the effect of random sources and sinks on the pair annihilation process, we should keep the v_{21} vertex in the effective IR model at any rate and therefore, for the purposes of the physical model, put the scaling dimension of this vertex equal to zero. This means that $d_\psi = 2$ and $d_{\psi^+} = d - 2$, and that the critical behaviour of the model is that of the Gribov process, which belongs to a universality class significantly different from that of the pure annihilation reaction model.

From the point of view of classification of terms in power counting, the two effective actions (192) and (194) of the previous scaling-dimensions analysis correspond to elimination of E_{ψ^+} and E_ψ, respectively, from the expression for the divergence index. This yields

$$\delta = d + 2 - 2V_{12} + (d-2)V_{21} + (d-2)V_{22} + (d-2)V_{23} + \\ + (d-2)V_{24} - dE_\psi, \quad (210)$$

$$= d + 2 + (d-2)V_{12} - 2V_{21} + (d-2)V_{22} + (2d-2)V_{23} + \\ + (3d-2)V_{24} - dE_{\psi^+}. \quad (211)$$

Relation (210) suggests that all other vertices than v_{12} are irrelevant above two dimensions leading to effective action (192). However, all these are «dangerous» irrelevant operators in the sense that the negative dimension of the vertex v_{12} may render the dimension of a graph with irrelevant operators negative anyway. Since all dangerous operators have equal dimensions, they cannot be classified by the degree of «dangerousness». Relation (211) leads to a similar phenomenon with respect vertices other than v_{21}, which has negative dimension there. However, here irrelevant operators have different dimensions and this might serve as a basis for classification of some operators more dangerous than others.

Let us recall that all expressions (207)–(211) for the divergence index are equal and reflect different choices of the ambiguous field dimensions. We see that

in a multicoupling case it is rather difficult to arrive at the consistent conclusion about the relevance of different vertices solely on the basis of the analysis of scaling dimensions of fields and vertex operators. In particular, to put either of the field dimensions equal to zero at the outset appears to be quite misleading. Assuming both field dimensions nonvanishing, put them equal to each other to obtain

$$
\delta = d + 2 + \left(\frac{d}{2} - 2 \right) V_{12} + \left(\frac{d}{2} - 2 \right) V_{21} + (d - 2)V_{22} +
$$
$$
+ \left(\frac{3d}{2} - 2 \right) V_{23} + (2d - 2)V_{24} - \frac{d}{2} E_\psi - \frac{d}{2} E_{\psi^+}. \quad (212)
$$

Here, indeed, all coefficients of vertex numbers depend of the space dimension explicitly and immediately lead to the conclusion that the critical scaling behaviour tractable within the RG is that of the Gribov process.

Consider then the action (197) for the active state. The divergence index does not give the whole truth in this case due to the presence of the «temperature» parameter E_{+1}, but it reflects the contribution of the wave-vector and frequency integral anyway. Many new types of vertices appear, but the set of conditions imposed on their numbers remains the same. In this case, it is convenient to analyze the IR and UV behaviour separately. The reason is that the temperature parameter is not involved in the power counting of UV divergences in any other way that new vertices have appeared. Thus, the power counting goes in the same fashion as above. Denoting new vertices by tilde, we obtain

$$
L = I + 1 - (\tilde{V}_{02} + \tilde{V}_{03} + \tilde{V}_{04} + V_{12} + \tilde{V}_{13} + \tilde{V}_{14} +
$$
$$
+ V_{21} + V_{22} + V_{23} + V_{24}), \quad (213)
$$
$$
I = V_{12} + \tilde{V}_{13} + \tilde{V}_{14} + 2V_{21} + 2V_{22} + 2V_{23} + 2V_{24} - E_\psi, \quad (214)
$$
$$
I = 2\tilde{V}_{02} + 3\tilde{V}_{03} + 4\tilde{V}_{04} + 2V_{12} + 3\tilde{V}_{13} + 4\tilde{V}_{14} + V_{21} +
$$
$$
+ 2V_{22} + 3V_{23} + 4V_{24} - E_{\psi^+}. \quad (215)
$$

Consequently,

$$
E_\psi - E_{\psi^+} = -2\tilde{V}_{02} - 3\tilde{V}_{03} - 4\tilde{V}_{04} - V_{12} - 2\tilde{V}_{13} - 3\tilde{V}_{14} + V_{21} - V_{23} - 2V_{24}. \quad (216)
$$

From (213) and (214), it follows that

$$
\delta = (d+2)L - 2I = d + 2 - (d+2)\tilde{V}_{02} - (d+2)\tilde{V}_{03} - (d+2)\tilde{V}_{04} - 2V_{12} - 2\tilde{V}_{13} -
$$
$$
- 2\tilde{V}_{14} + (d-2)V_{21} + (d-2)V_{22} + (d-2)V_{23} + (d-2)V_{24} - d\,E_\psi. \quad (217)
$$

Adding the relation (216) multiplied by the coefficient a to expression (217) we arrive at the representation

$$\delta = d + 2 + (2a - d - 2)\tilde{V}_{02} + (3a - d - 2)\tilde{V}_{03} + (4a - d - 2)\tilde{V}_{04} +$$
$$+ (a - 2)V_{12} + (2a - 2)\tilde{V}_{13} + (3a - 2)\tilde{V}_{14} + (d - 2 - a)V_{21} + (d - 2)V_{22} +$$
$$+ (d - 2 + a)V_{23} + (d - 2 + 2a)V_{24} - a\,E_\psi - (d - a)\,E_{\psi^+}. \quad (218)$$

Here, by the choice of the value of the parameter a we could try to find a representation convenient for the classification of the vertices. However, in this case the vertex with the negative dimension $\psi^+\psi^+$ may be absorbed in the pair correlation function of the field ψ included in the elements of the graphical representation. In calculation of the UV index the field dimensions are then fixed by the condition that the coupling constant of the term $\propto \psi^+\psi^+$ is dimensionless. There is no ambiguity in the expression for the index and denoting the number of correlation functions in a one-irreducible graph $\tilde{I}$ we arrive at the relations

$$L = I + \tilde{I} + 1 - \left(\tilde{V}_{03} + \tilde{V}_{04} + V_{12} + \tilde{V}_{13} + \tilde{V}_{14} + V_{21} + V_{22} + V_{23} + V_{24}\right),$$
$$I + 2\tilde{I} = V_{12} + \tilde{V}_{13} + \tilde{V}_{14} + 2V_{21} + 2V_{22} + 2V_{23} + 2V_{24} - E_\psi, \quad (219)$$
$$I = 3\tilde{V}_{03} + 4\tilde{V}_{04} + 2V_{12} + 3\tilde{V}_{13} + 4\tilde{V}_{14} + V_{21} + 2V_{22} + 3V_{23} + 4V_{24} - E_{\psi^+}.$$

From here, it follows that

$$\delta = (d + 2)L - 2I - 4\tilde{I} = d + 2 + \left(\frac{d}{2} + 1\right)\tilde{V}_{03} + (d + 2)\tilde{V}_{04} +$$
$$+ \left(\frac{d}{2} - 1\right)V_{12} + d\tilde{V}_{13} + \left(\frac{3d}{2} + 1\right)\tilde{V}_{14} + \left(\frac{d}{2} - 3\right)V_{21} + (d - 2)V_{22} +$$
$$+ \left(\frac{3d}{2} - 1\right)V_{23} + 2dV_{24} - \left(\frac{d}{2} - 1\right)E_\psi - \left(\frac{d}{2} + 1\right)E_{\psi^+}. \quad (220)$$

The same relation follows from (218), when the parameter a is chosen such that the coefficient in front of $\tilde{V}_{02}$ vanishes. From (220) we see that when the vertex v_{21} is marginal, all the rest are irrelevant and we arrive at the situation described by the action (201).

This, however, is not the whole story, because in the case of IR behaviour we are interested in the limit of vanishing temperature parameter. Since coupling constants of new vertices are functions of E_{+1}, they affect the limit of small E_{+1} directly. The dependence of a graph on E_{+1} in the critical limit ($\omega_i \to 0$, $\mathbf{k}_i \to 0$, $E_{+1} \to 0$, $\omega_i = O\left(\sqrt{E_{+1}}\right)$, $\mathbf{k}_i^2 = O\left(\sqrt{E_{+1}}\right)$) is readily estimated by a suitable scaling of variables in the loop integrals, if these integrals have negative dimensions, in which case the upper limit in integrals of a renormalized

graph may be sent to infinity. The result of the scaling of integration variables is a power of the parameter E_{+1} multiplied by a function of reduced frequencies and wave-vectors $\omega_i/\sqrt{E_{+1}}$, $\mathbf{k}_i E_{+1}^{-1/4}$. The wave-number integrals defining this function are UV and IR finite for all values of the reduced frequencies and wave-vectors and thus possess a finite limit, when the latter vanish. Therefore, any IR-singular behaviour is signalled by a negative overall power of the parameter E_{+1}, which takes into account both the scaling of integration variables (this gives the divergence index δ) and the additional powers of E_{+1} at the vertex factors of the graph. Thus, we might use the following «IR divergence index» brought about by the action (197) (here, σ_+ is assumed to be subleading for simplicity):

$$\delta_{\mathrm{IR}} = \delta + 2\tilde{V}_{02} + 4\tilde{V}_{03} + 4\tilde{V}_{04} + 2\tilde{V}_{13} + 2\tilde{V}_{14} = d + 2 + \left(\frac{d}{2} + 2\right)\tilde{V}_{03} + (d+2)\tilde{V}_{04} +$$

$$+ \left(\frac{d}{2} - 2\right)V_{12} + d\tilde{V}_{13} + \left(\frac{3d}{2}\right)\tilde{V}_{14} + \left(\frac{d}{2} - 2\right)V_{21} + (d-2)V_{22} +$$

$$+ \left(\frac{3d}{2} - 2\right)V_{23} + (2d-2)V_{24} - \frac{d}{2}E_\psi - \frac{d}{2}E_{\psi^+} \quad (221)$$

to guide in the classification of the interaction terms. The same form may be obtained by the generic expression (218) with the choice of a such that the coefficient in front of $\tilde{V}_{02}$ in δ_{IR} vanishes. In order to keep the IR behaviour tractable, all coefficients of the vertex numbers in (221) must be non-negative. The lowest space dimension conforming to this requirement is the upper critical dimension of the model. We see from (221) that $d_c = 4$ for model (197) and the effective action is indeed (202).

CONCLUSIONS

In conclusion, we have analyzed the effect of density and velocity fluctuations on the reaction kinetics of the single-species decay $A + A \rightarrow \varnothing$ universality class in the framework of field-theoretic renormalization group and calculated the scaling function and the decay exponent of the mean particle density for the four asymptotic patterns predicted by the RG.

We have calculated the relevant renormalization constants at two-loop level and found the decay exponent of the mean particle density at this order of the ϵ, Δ expansion for four IR stable fixed points of the RG, whose regions of stability cover the whole parametric space in the vicinity of the origin in the ϵ, Δ plane. The decay exponent assumes the mean-field value in the basins of attraction of the trivial fixed point (125) and of the kinetic fixed point (133) with dominant fluctuations of the random force of the Navier–Stokes equation. At the kinetic fixed point with finite rate coefficient (130), the decay value of the decay exponent

is determined exactly by the fixed-point equations. At the thermal (short-range) fixed point (127), the decay exponent possesses a nontrivial ϵ, Δ expansion. We have calculated three first terms of this expansion.

Using a variational approach, we have inferred a renormalized fluctuation-amended rate equation with the account of one-loop corrections. This nonlinear integro-differential equation has been solved iteratively in the framework of the ϵ, Δ expansion and the scaling function for the mean particle density has been calculated for the four IR stable regimes. The scaling function assumes the mean-field form (exactly) in the basins of attraction of the trivial fixed point and the kinetic fixed point with dominant fluctuations of the random force. At the kinetic fixed point with finite rate coefficient and at the thermal fixed point the scaling function possesses a nontrivial ϵ, Δ expansion, which we have calculated at the linear order. Fluctuations of the random advection field affect heavily the long-time asymptotic behaviour of the system: the kinetic fixed points are brought about by the velocity fluctuations as well as the nontrivial series expansion of the decay exponent at the thermal fixed point (without velocity fluctuations, the decay exponent is fixed to the one-loop value, because there are no high-order corrections to the rate constant in this case). Predictions of the renormalization-group analyses for the reaction $A + A \rightarrow \varnothing$ in quenched random fields have been corroborated by numerical simulations [10, 13]. In the case of dynamically generated random drift, this seems to be a much more demanding task, but would surely be highly desirable, since the experimental data for reaction processes is quite scarce.

We have investigated possible effects of random sources and sinks on the pair annihilation reaction $A + A \rightarrow \varnothing$. Contrary to the frequently used approach, in which the sources and sinks are introduced into the Langevin equation, we have included them directly to the master equation, where their physical sense is clear. We have considered linear in particle number creation and annihilation reactions with random rate coefficients to model the sources and sinks. On the basis of the analysis of canonical scaling dimensions we have constructed effective actions, which are the starting point for an RG analysis of the critical behaviour of the systems under consideration. In all cases, the effect of random sources and sinks to the large-scale, long-time behaviour of the Green functions is significant and changes the universality class of the model. Instead of the universality class of the pair annihilation reaction $A + A \rightarrow \varnothing$, the asymptotic behaviour of the model with random sources and sinks belongs to the universality class of the Gribov process in the critical case and to a modified Gribov process in the critical limit of the noncritical model. In the former case, it is demonstrated once again that the description of a stochastic process with the use of the Langevin equation is significantly different from the description in terms of a master equation. The random noise term in the Langevin equation corresponds rather to the account of effects of genuine random sources and sinks than to a description of the effect of

microscopic degrees of freedom on the mesoscopic process. Here, the universality classes of the same reaction process are completely different in the case of the master equation without sources and sinks in comparison to the case of Langevin equation for the same process.

In the noncritical case with a random source independent of the agent density, the Gribov process is modified to account for effects in critical behaviour, when sources and sinks asymptotically vanish. The analysis of the dependence of scaling functions on the parameters of the probability distribution of sinks and sources in infrared limit is called for. This reminds the situation, which takes place in the theory of phase transitions, where statistical correlations of the order parameter depend on a «mass» (deviation of temperature from the critical value) and the dependence of the scaling functions on the «mass» is investigated.

Appendix A
EXPLICIT FORM OF THE RENORMALIZATION CONSTANTS

$$A_{11} = -\frac{(1+\xi)u^2 + (3\xi+2)u + 6\xi + 1}{512u(1+u)^3\xi},$$

$$A_{12} = -\frac{(1+\xi)u^2 + (1+3\xi)u + 6\xi - 4}{256u(1+u)^3(1-\xi)},$$

$$A_{22} = -\frac{u+5}{512u(1+u)^3},$$

$$B_{11} = B_1(u) + B_2(u,\xi), \quad B_{12} = 4[B_1(u) + B_3(u,\xi)],$$

$$B_{22} = -B_1(u) - B_2(u,-1),$$

where the functions $B_1, B_2,$ and B_3 are given as

$$B_1(u) = \frac{1}{1024u^4(1+u)^3(u-1)}\left[-12u^3(1+u)\ln\frac{2}{1+u}+\right.$$

$$+32u^4(1+u)^2\ln 2 + 2u^3(1+u)^2(u+10)\ln\frac{4}{3} - 32u^3(1+u)^3\,\text{arctgh}\,\frac{1}{2}+$$

$$+4(2u^6 + 6u^5 + 7u^4 + 10u^3 - 4u - 1)\ln\frac{1+2u}{(1+u)^2}-$$

$$- 4u^3(1+u)[4u^2 + 20u + 9]\ln\frac{1+2u}{2+2u} + 16u^3(1+u)^3\,\text{arctgh}\,\frac{u}{u+1}+$$

$$+ 8u^3(1+u)^2(u-1)(\gamma + \psi(3/2))u^2(23u^4 + 38u^3 + 17u^2 + 22u + 4)\Big] -$$

$$- \frac{1}{128\pi u(1+u)^2(u-1)}\int\limits_{1}^{\infty} dq \int\limits_{-1}^{1} dz\, F(q,z,u),$$

$$B_2(u,\xi) = \xi\frac{(2+4\xi)u^2 + (10\xi + 8)u + 38 + 22\xi}{1024u(1+u)^3(2+\xi)},$$

$$B_3(u,\xi) = -\frac{(8\xi + 4)u^2 + (14\xi + 10)u + 22 - 10\xi}{1024u(1+u)^3}$$

with function F given by the expression

$$F(x,z,u) = (1-z^2)^{1/2}\frac{M(x,z,u)}{N(x,z,u)},$$

$$M(x,z,u) = (x^6 + 1)[z^3(24u^3 + 24u^2 + 72u + 72)-$$

$$- z(8u^3 + 12u^2 + 8u + 60)] + (x^5 + x)[-z^4(40u^3 + 88u^2 + 120u + 264)+$$

$$+ z^2(-4u^3 + 16u^2 + 108u + 168) + 4u^3 + 14u^2 + 28u + 18]+$$

$$+ (x^4 + x^2)[z^5(16u^3 + 96u^2 + 48u + 288) + z^3(12u^4+$$

$$+ 64u^2 + 128u^2 + 96u + 180) - z(4u^4 + 26u^3 + 92u^2 + 174u + 312)]+$$

$$+ x^3[-z^6(32u^2 + 96) - z^4(8u^4 + 64u^3 + 240u^2 + 144u + 600)+$$

$$+ z^2(-8u^4 + 4u^3 + 84u^2 + 108u + 452) + 2u^4 + 6u^3 + 26u^2 + 58u + 36],$$

$$N(x,z,u) = (1 + x^2 - 2xz)(1 + x^2 - xz)((1+u)x^2 + 2 - 2xz)\times$$

$$\times (1 + u + 2x^2 - 2xz),$$

$$C(u,\xi) = -\frac{1}{8u\pi}\int\limits_{-1}^{1} dz(1-z^2)^{1/2}G(z,u)+$$

$$+ \frac{1}{8u(1+u)}\left(\ln\frac{1+u}{2u} + 1 + \frac{2+u}{u}\ln\frac{u+2}{2u+2} + \xi\right), \quad \text{(A.1)}$$

where

$$G(z,u) = \frac{4}{(1-u)^2 + 4uz^2} \left\{ \frac{u-1}{2} \ln \frac{2u}{1+u} - \frac{2(1+u)z}{\sqrt{1-z^2}} \times \right.$$

$$\times \left[\frac{\pi}{2} - \arctan \sqrt{\frac{1+z}{1-z}} \right] + \frac{u(u+3)z}{\sqrt{2u(1+u) - u^2 z^2}} \times$$

$$\left. \times \left[\pi - \arctan \frac{zu+u+1}{\sqrt{2u(1+u) - u^2 z^2}} - \arctan \frac{(2+z)u}{\sqrt{2u(1+u) - u^2 z^2}} \right] \right\}. \quad (A.2)$$

Appendix B
FIXED POINTS

$$u_1^*(\xi) = \frac{8R}{3\sqrt{17}} - \frac{8192}{3\sqrt{17}} B_1(u_0^*) - \frac{1}{432\sqrt{17}(1+\xi)^2(2+\xi)} \times$$

$$\times \left((21384 - 648\sqrt{17})\xi^4 + (52512 - 2592\sqrt{17})\xi^3 + (22192 - 2736\sqrt{17})\xi^2 + \right.$$

$$\left. + (72\sqrt{17} - 29064)\xi + 720\sqrt{17} - 18768 \right), \quad (B.1)$$

$$g_{12}^*(\xi) = \frac{64(R(2+3\xi) - 1)}{27(1+\xi)} - \frac{16(2+3\xi)}{243(1+\xi)^4(2+\xi)} \times$$

$$\times \left[45\xi^4 + 213\xi^3 + 349\xi^2 + 231\xi + 50 \right], \quad (B.2)$$

$$g_{22}^*(\xi) = \frac{64(1+R)}{27(1+\xi)} - \frac{16(2+3\xi)}{243(1+\xi)^4(2+\xi)} \left[57\xi^4 + 171\xi^3 + 185\xi^2 + 93\xi + 22 \right].$$
$$(B.3)$$

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