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*Czechoslovak Mathematical Journal*, Vol. 44 (1994), No. 3, 513–520

Persistent URL: <http://dml.cz/dmlcz/128474>

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## FILIPPOV'S OPERATION AND SOME ATTRIBUTES

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(Received August 12, 1992)

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be given. The *Filippov* of  $f$  is defined as follows:

$$\mathcal{F}[f](x) = \bigcap_{\varepsilon > 0} \bigcap_{Z: \mu(Z)=0} \overline{\text{conv}} f(B_\varepsilon(x) \setminus Z),$$

where  $\mu$  denotes Lebesgue measure,  $\overline{\text{conv}} A$  represents the closure of the convex hull of the set  $A$  and  $B_\varepsilon(x)$  represents the open ball of radius  $\varepsilon$  about the point  $x$ . The Filippov is used in defining a generalized solution of the ordinary differential equation  $x' = f(x)$ , particularly in the case in which  $f$  is discontinuous. Information concerning the Filippov can be found in many references, including [1] through [10]. (Actually, Filippov's operation and the notion of a Filippov solution were defined for nonautonomous differential equations. However, for the present paper, consideration of the nonautonomous case essentially has only the effect of introducing an unnecessary parameter  $t$  into our results.) In this paper, we treat  $\mathcal{F}$  as a function, mapping real-valued functions into set-valued functions, and investigate the properties of  $\mathcal{F}$ . Such results add to our understanding of this operation. We note that there is an alternate definition of the Filippov (for  $f \in L^\infty$ ), equivalent [5] to the previous one, that we will frequently use:

$$\mathcal{F}[f](x) = \{y: \lim_{\varepsilon \rightarrow 0} \text{ess inf}_{B_\varepsilon(x)} f \leq y \leq \lim_{\varepsilon \rightarrow 0} \text{ess sup}_{B_\varepsilon(x)} f\}.$$

We first consider choosing an appropriate domain for  $\mathcal{F}$ . Certainly, there are a number of possibilities, but we require that the domain be restricted to  $f$ 's which are useful for differential equations in the following sense. It can be shown that the functions in  $L^\infty$  are precisely the ones which satisfy the classical local existence theorem for Filippov solutions in the case of  $x' = f(x)$ , namely Theorem 4 in [5]. Hence, we choose  $L^\infty$  as the domain for  $\mathcal{F}$ , using  $\|\cdot\|$  to denote the usual norm on  $L^\infty$ .

We now discuss the selection of a codomain for the function  $\mathcal{F}$ . We recall two standard definitions (see [1]). We shall say  $F: \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$  (= power set of  $\mathbb{R}$ ) is *bounded* if and only if  $\sup_{x \in \mathbb{R}} \{\sup\{|y|: y \in F(x)\}\} < \infty$ . Also  $F$  is said to be *upper semicontinuous* if and only if for each  $x \in \mathbb{R}$  and for each open set  $N$  containing  $F(x)$  there exists an open set  $M$  containing  $x$  such that  $F(M) \subseteq N$ . We choose for the codomain the set  $\mathcal{B} \equiv \{F: \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R}) \mid F \text{ is upper semicontinuous, } F \text{ is closed-interval valued and } F \text{ is bounded}\}$ .  $\mathcal{B}$  can be made into a metric space by defining

$$D(F, G) = \sup_{x \in \mathbb{R}} \{h(F(x), G(x))\},$$

where  $F, G \in \mathcal{B}$  and  $h$  represents the Hausdorff distance between the two sets  $F(x)$  and  $G(x)$ . It follows easily that  $\mathcal{F}[L^\infty] \subseteq \mathcal{B}$  using well-known facts such as for  $f \in L^\infty$ ,  $\mathcal{F}[f]: \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$  is upper semicontinuous. We note that  $D(F, G)$  is alternatively given by

$$D(F, G) = \sup_{x \in \mathbb{R}} \{\max\{|\min F(x) - \min G(x)|, |\max F(x) - \max G(x)|\}\},$$

where, for example,  $\max F(x) \equiv \max\{y: y \in F(x)\}$ .

We now consider questions involving the range of  $\mathcal{F}$ . In the results which follow, we shall make use of the following definition. Let  $F: \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ . Then the *Filippov* of  $F$  is defined by

$$\mathcal{F}[F](x) = \bigcap_{\varepsilon > 0} \bigcap_{Z: \mu(Z)=0} \overline{\text{conv}} \bigcup_{y \in B_\varepsilon(x) \setminus Z} F(y).$$

(Note that the purpose of this is to extend the Filippov so that it can be applied to set-valued functions. We emphasize that in Corollary 1 and Theorems 2, 6, 7 and 9 that the domain of  $\mathcal{F}$ , as mentioned earlier, is  $L^\infty$ .) Results from Jarnik's paper [8] allow us to completely characterize the range of  $\mathcal{F}$ .

**Theorem 1.** *Let  $F: \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ . Then, there exists  $f \in L^\infty$  such that  $\mathcal{F}[f] = F$  if and only if  $F$  satisfies the following conditions:*

- 1)  $F$  is upper semicontinuous,
- 2) There exists  $M > 0$  such that  $F(x) \subseteq [-M, M]$  for all  $x \in \mathbb{R}$ , and
- 3)  $\mathcal{F}[F] = F$ .

*Proof.* Suppose there exists  $f \in L^\infty$  such that  $\mathcal{F}[f] = F$ . As noted above,  $\mathcal{F}[L^\infty] \subseteq \mathcal{B}$ , thus  $F$  satisfies 1) and 2), while  $F$  satisfies property 3) by (7) in [8]. Now assuming that  $F$  satisfies 1), 2) and 3), the existence of  $f \in L^\infty$  such that  $\mathcal{F}[f] = F$  follows from the main result in [8] using a simple, but tedious, change of scale, which we omit for brevity. □

**Corollary 1.**  $\mathcal{F}$  is not onto  $\mathcal{B}$ .

**Proof.** Define  $F \in \mathcal{B}$  by

$$F(x) = \begin{cases} \{0\} & \text{for } x \neq 0, \\ [0, 1] & \text{for } x = 0. \end{cases}$$

Clearly,  $\mathcal{F}[F] \equiv \{0\}$ , so  $\mathcal{F}[F] \neq F$ . Thus, by property 3) of Theorem 1  $F$  is not the Filippov of an  $L^\infty$  function.  $\square$

For our next result concerning the range of  $\mathcal{F}$ , we shall need the following.

**Lemma 1.** For all  $F, G \in \mathcal{B}$ , we have  $D(\mathcal{F}[F], \mathcal{F}[G]) \leq D(F, G)$  and hence  $\mathcal{F}$ , thought of as mapping  $\mathcal{B} \rightarrow \mathcal{B}$ , is continuous.

**Proof.** It can easily be shown that  $\mathcal{F}[\mathcal{B}] \subseteq \mathcal{B}$ . Define  $a(x) = \lim_{\varepsilon \rightarrow 0} \operatorname{ess\,inf}_{B_\varepsilon(x)} F$ ,  $b(x) = \lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{B_\varepsilon(x)} F$ ,  $c(x) = \lim_{\varepsilon \rightarrow 0} \operatorname{ess\,inf}_{B_\varepsilon(x)} G$  and  $d(x) = \lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{B_\varepsilon(x)} G$ . Then,  $D(\mathcal{F}[F], \mathcal{F}[G]) = \sup_{\mathbf{R}} \{\max\{|a - c|, |b - d|\}\}$ . Similarly, define  $e(x) = \min F(x)$ ,  $f(x) = \max F(x)$ ,  $g(x) = \min G(x)$  and  $h(x) = \max G(x)$ . Then,  $D(F, G) = \sup_{\mathbf{R}} \{\max\{|e - g|, |f - h|\}\}$ . In the proof of the following claim, we shall use the fact that for  $A \subseteq \mathbf{R}$ ,

$$|\operatorname{ess\,sup}_A f - \operatorname{ess\,sup}_A h| \leq \operatorname{ess\,sup}_A |f - h|.$$

This is easily verified, so we omit the proof.

*Claim 1:* For all  $x \in \mathbf{R}$ ,  $|b(x) - d(x)| \leq \sup_{\mathbf{R}} |f - h|$ .

**Proof of Claim 1**

$$\begin{aligned} |b(x) - d(x)| &= \left| \lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{B_\varepsilon(x)} F - \lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{B_\varepsilon(x)} G \right| \\ &= \left| \lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{B_\varepsilon(x)} f - \lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{B_\varepsilon(x)} h \right| \\ &\leq \lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{B_\varepsilon(x)} |f - h| \\ &\leq \operatorname{ess\,sup}_{\mathbf{R}} |f - h| \leq \sup_{\mathbf{R}} |f - h|. \end{aligned}$$

*Claim 2:* For all  $x \in \mathbf{R}$ ,  $|a(x) - c(x)| \leq \sup_{\mathbf{R}} |e - g|$ .

We omit the proof of Claim 2 since it is similar to that of Claim 1. We now have, applying Claims 1 and 2, for all  $x \in \mathbb{R}$ ,

$$\begin{aligned} \max\{|a(x) - c(x)|, |b(x) - d(x)|\} &\leq \max\{\sup_{\mathbb{R}}|e - g|, \sup_{\mathbb{R}}|f - h|\} \\ &= \sup_{\mathbb{R}} \max\{|e - g|, |f - h|\} \end{aligned}$$

and thus

$$\sup_{\mathbb{R}} \max\{|a - c|, |b - d|\} \leq \sup_{\mathbb{R}} \max\{|e - g|, |f - h|\},$$

i.e.,  $D(\mathcal{F}[F], \mathcal{F}[G]) \leq D(F, G)$ . □

**Theorem 2.** *The range of  $\mathcal{F} : L^\infty \rightarrow \mathcal{B}$  is closed and unbounded in  $(\mathcal{B}, D)$ .*

*Proof.* Let  $\{F_n\}_{n=1}^\infty \subseteq \mathcal{F}[L^\infty]$  and  $F_n \rightarrow F$  in  $(\mathcal{B}, D)$ . Lemma 1 implies that  $\mathcal{F}[F_n] \rightarrow \mathcal{F}[F]$  in  $(\mathcal{B}, D)$ . By Theorem 1, for each  $n \in \mathbb{N}$ ,  $\mathcal{F}[F_n] = F_n$ , hence  $F_n \rightarrow \mathcal{F}[F]$ . Since limits are unique in  $(\mathcal{B}, D)$ , we have  $F = \mathcal{F}[F]$ . Also, since  $F \in \mathcal{B}$ , we have that i)  $F$  is upper semicontinuous and ii) there exists  $M > 0$  such that  $F(x) \subseteq [-M, M]$  for all  $x \in \mathbb{R}$ . Thus, applying Theorem 1 to  $F$ , we have  $F \in \mathcal{F}[L^\infty]$  and hence  $\mathcal{F}$  has closed range. Now, for each  $n \in \mathbb{N}$ , define  $f_n \in L^\infty$  by  $f_n(x) \equiv n$ , and also define  $f_\infty \in L^\infty$  by  $f_\infty(x) \equiv 0$ . It follows that  $\sup_{n \in \mathbb{N}} D(\mathcal{F}[f_n], \mathcal{F}[f_\infty]) = \infty$ , and hence the range is unbounded. □

We now consider the question of whether or not  $\mathcal{F}$  is one-to-one. In [2], the following appeared.

**Theorem 3.** *Let  $f, g \in \mathcal{C} \equiv \{h \in L^\infty : \text{there exists a set } A_h \text{ of full measure such that } h|_{A_h} \text{ is continuous}\}$ . If  $\mathcal{F}[f] = \mathcal{F}[g]$ , then  $f = g$  (in  $L^\infty$ ).*

To complete the one-to-one question, we shall need the following lemmas, the first of which is proven, for example, in [8].

**Lemma 2.** *Let  $A \subseteq \mathbb{R}$  be Lebesgue measurable with  $\mu A > 0$ . Then, there exist Lebesgue measurable sets  $D$  and  $E$  such that  $D \cap E = \emptyset$ ,  $D \cup E = A$  and for all  $\varepsilon > 0$ , for all  $x \in A$  with  $\mu(B_\varepsilon(x) \cap A) > 0$ , we have both  $\mu(D \cap B_\varepsilon(x)) > 0$  and  $\mu(E \cap B_\varepsilon(x)) > 0$ . ( $D$  and  $E$  are known as “metrically dense” subsets of  $A$ ).*

**Lemma 3.**  *$f \in \mathcal{C}$  if and only if  $\mathcal{F}[f]$  is a singleton a.e.*

The proof is given in [2]. We are now able to prove the following, which, in a sense, tells us that the set  $\mathcal{C}$  in Theorem 3 is the largest subset of  $L^\infty$  on which  $\mathcal{F}$  is one-to-one.

**Theorem 4.** Let  $f \in L^\infty \setminus \mathcal{C}$ . Then, there exists some  $g \in L^\infty$  such that  $\mathcal{F}[f] = \mathcal{F}[g]$  but  $f \neq g$  (in  $L^\infty$ ).

*Proof.* Define  $\bar{f}, \underline{f}: \mathbb{R} \rightarrow \mathbb{R}$  by

$$\bar{f}(x) = \lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{B_\varepsilon(x)} f \quad \text{and} \quad \underline{f}(x) = \lim_{\varepsilon \rightarrow 0} \operatorname{ess\,inf}_{B_\varepsilon(x)} f.$$

Since  $f \notin \mathcal{C}$ , by Lemma 3 there exists a set  $D \subseteq \mathbb{R}$  of positive measure such that for all  $x \in D$ ,  $\underline{f}(x) < \bar{f}(x)$ . It follows from Lusin's Theorem that there exists a set  $F$  with the following properties:

- 1)  $F \subseteq D$ ,
- 2)  $F$  is closed,
- 3)  $\mu(F) > 0$ , and
- 4)  $\bar{f}$  and  $\underline{f}$  are both continuous relative to  $F$ .

Let  $A$  and  $B$  be disjoint metrically dense subsets of  $F$  such that  $A \cup B = F$ . (Such sets exist by Lemma 2.) Define  $k_1, k_2: \mathbb{R} \rightarrow \mathbb{R}$  by

$$k_1(x) = \begin{cases} f(x) & \text{for } x \notin F, \\ \bar{f}(x) & \text{for } x \in A, \\ \underline{f}(x) & \text{for } x \in B, \end{cases} \quad k_2(x) = \begin{cases} f(x) & \text{for } x \notin F, \\ \underline{f}(x) & \text{for } x \in A, \\ \bar{f}(x) & \text{for } x \in B. \end{cases}$$

Clearly,  $k_1$  and  $k_2$  disagree on  $F$ , a set of positive measure. Without loss of generality, assume  $f$  and  $k_1$  disagree on a set of positive measure, and let  $g(x) = k_1(x)$  for all  $x \in \mathbb{R}$ .

*Claim:*  $\mathcal{F}[f] = \mathcal{F}[g]$ .

*Case 1:*  $x \notin F$ .

Since  $F$  is closed, there exists an open interval  $I \subseteq \mathbb{R} \setminus F$  containing  $x$ . We note that  $f(y) = g(y)$  for all  $y \in I$ , thus  $\mathcal{F}[f](x) = \mathcal{F}[g](x)$ .

*Case 2:*  $x \in F$  and  $x$  is a point of density of  $F$ .

We want to show

$$\lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{B_\varepsilon(x)} g = \lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{B_\varepsilon(x)} f.$$

Note that  $\lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{B_\varepsilon(x) \cap F} \bar{f} = \bar{f}(x)$  since  $\bar{f}$  is continuous on  $F$ , and

$$\lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{B_\varepsilon(x) \cap F^c} f \leq \lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{B_\varepsilon(x)} f = \bar{f}(x).$$

We then have

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{B_\varepsilon(x)} g &= \lim_{\varepsilon \rightarrow 0} \max\left\{ \operatorname{ess\,sup}_{B_\varepsilon(x) \cap F} g, \operatorname{ess\,sup}_{B_\varepsilon(x) \cap F^c} g \right\} \\
&\leq \lim_{\varepsilon \rightarrow 0} \max\left\{ \operatorname{ess\,sup}_{B_\varepsilon(x) \cap F} \bar{f}, \operatorname{ess\,sup}_{B_\varepsilon(x) \cap F^c} f \right\} \\
&= \max\left\{ \lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{B_\varepsilon(x) \cap F} \bar{f}, \lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{B_\varepsilon(x) \cap F^c} f \right\} \\
&= \bar{f}(x) = \lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{B_\varepsilon(x)} f.
\end{aligned}$$

For the opposite inequality, we note that for all  $\varepsilon > 0$  and for all  $Z \subseteq \mathbb{R}$  with  $\mu(Z) = 0$ , we have

$$\sup_{B_\varepsilon(x) \setminus Z} g \geq \sup_{(B_\varepsilon(x) \setminus Z) \cap A} g = \sup_{(B_\varepsilon(x) \setminus Z) \cap A} \bar{f} \geq \bar{f}(x),$$

since  $\bar{f}$  is continuous on  $A$  and  $x$  is a point of density of  $F$ . Thus, for all  $\varepsilon > 0$ ,

$$\operatorname{ess\,sup}_{B_\varepsilon(x)} g \geq \bar{f}(x)$$

and so

$$\lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{B_\varepsilon(x)} g \geq \bar{f}(x) = \lim_{\varepsilon \rightarrow 0} \operatorname{ess\,sup}_{B_\varepsilon(x)} f.$$

The fact that  $\lim_{\varepsilon \rightarrow 0} \operatorname{ess\,inf}_{B_\varepsilon(x)} g = \lim_{\varepsilon \rightarrow 0} \operatorname{ess\,inf}_{B_\varepsilon(x)} f$  is handled analogously.

We note that we need not handle Case 3, in which  $x \in F$  but  $x$  is not a point of density of  $F$  since these form a set of measure zero. The result follows since it can be shown that if  $\mathcal{F}[f]$  and  $\mathcal{F}[g]$  agree on a set of full measure, then  $\mathcal{F}[f](x) = \mathcal{F}[g](x)$  for all  $x \in \mathbb{R}$ .  $\square$

We now investigate whether or not  $\mathcal{F}$  is continuous. It not only turns out to be continuous, but is “Lipschitz,” in the following sense.

**Theorem 5.** *For all  $f, g \in L^\infty$ , we have  $D(\mathcal{F}[f], \mathcal{F}[g]) \leq \|f - g\|$ .*

The proof is similar to that of Lemma 1, so for brevity, we omit it. Examples can be given to show that the Lipschitz constant of 1 is sharp, and also that, in general, the inequality cannot be replaced with equality.

We now investigate other topological properties of  $\mathcal{F}$ .

**Theorem 6.**  *$\mathcal{F}$  is not an open map.*

PROOF. Let  $U \subseteq L^\infty$  be an open ball containing the zero function in  $L^\infty$  (call it  $f$ ). Define  $F \in \mathcal{B}$  by  $F(x) = \{0\}$  for all  $x \in \mathbb{R}$ . Note that  $\mathcal{F}[f] = F$  so  $F \in \mathcal{F}[U]$ . Let  $\varepsilon > 0$ . Define  $G \in \mathcal{B}$  by

$$G(x) = \begin{cases} \{0\} & \text{for } x \neq 0 \\ [0, \frac{\varepsilon}{2}] & \text{for } x = 0. \end{cases}$$

Clearly,  $G$  lies in the  $\varepsilon$ -ball centered at  $F$ .  $G$  cannot be in  $\mathcal{F}[U]$  since Filippovs which agree almost everywhere agree everywhere. Thus,  $F$  is not an interior point of  $\mathcal{F}[U]$ , so  $\mathcal{F}[U]$  is not open.  $\square$

**Theorem 7.**  $\mathcal{F}$  is not a closed map. (Here, closed map is intended to mean images of closed sets are closed.)

PROOF. Let  $A$  and  $\mathbb{R} \setminus A$  be metrically dense in  $\mathbb{R}$  and let  $A_n$  and  $B_n$ ,  $n = 1, 2, \dots$ , be metrically dense in  $[n, n+1] \setminus A$  with  $A_n \cup B_n = [n, n+1] \setminus A$  and  $A_n \cap B_n = \emptyset$ . For each  $n \in \mathbb{N}$ , define  $f_n \in L^\infty$  by

$$f_n(x) = \begin{cases} (1 + \frac{1}{n})\chi_A(x) & \text{for } x \notin [n, n+1] \\ 1 + \frac{1}{n} & \text{for } x \in [n, n+1] \cap A \\ \frac{1}{2} & \text{for } x \in A_n \\ 0 & \text{for } x \in B_n \end{cases}$$

Then,  $\mathcal{F}[f_n](x) \equiv [0, 1 + \frac{1}{n}]$ . Define  $F \in \mathcal{B}$  by  $F(x) = [0, 1]$  for all  $x \in \mathbb{R}$ . Clearly,  $\mathcal{F}[f_n] \rightarrow F$  in  $(\mathcal{B}, D)$ , but  $\mathcal{F}[f_n] \neq F$  for each  $n \in \mathbb{N}$ . Let  $K \equiv \{f_n\}_{n=1}^\infty \subseteq L^\infty$ . If  $n \neq m$ ,  $\|f_n - f_m\| \geq \frac{1}{2}$ . Thus, there can be no Cauchy sequences and hence no convergent sequences in  $K$  (except, of course, those which are eventually constant), so  $K$  is closed. However,  $F \in \overline{\mathcal{F}[K]} \setminus \mathcal{F}[K]$ . Therefore,  $\mathcal{F}$  is not a closed map.  $\square$

Another property often considered concerning functions is monotonicity. The notion of order depends on the particular application. To study monotonicity in this more abstract context, we can define a partial order  $\preceq$  on the domain  $L^\infty$  by:

$$f \preceq g \text{ iff } f(x) \leq g(x) \text{ for almost all } x \in \mathbb{R}$$

where  $f, g \in L^\infty$ . We next define a partial order on the codomain  $\mathcal{B}$ , which we also denote by  $\preceq$ :

$$F \preceq G \text{ iff for all } x \in \mathbb{R}, \text{ we have both} \\ \min F(x) \leq \min G(x) \text{ and } \max F(x) \leq \max G(x)$$



where  $F, G \in \mathcal{B}$ .

**Theorem 8.** *Let  $f, g \in L^\infty$ . If  $f \preceq g$ , then  $\mathcal{F}[f] \preceq \mathcal{F}[g]$ .*

The proof is trivial so we omit it. Although the condition  $f \preceq g$  is sufficient for  $\mathcal{F}[f] \preceq \mathcal{F}[g]$ , it is not necessary. Such an example is provided by  $k_1$  and  $k_2$  in the proof of Theorem 4.

**Theorem 9.**  *$\mathcal{F}$  is not strictly monotone.*

*Proof.* Let  $A$  and  $\mathbb{R} \setminus A$  be metrically dense in  $\mathbb{R}$  and let  $E$  and  $A \setminus E$  be metrically dense in  $A$ . Define  $f \in L^\infty$  by

$$f(x) = \begin{cases} 0 & \text{for } x \notin A, \\ \frac{1}{2} & \text{for } x \in E, \\ 1 & \text{for } x \in A \setminus E. \end{cases}$$

Clearly,  $f$  is strictly less than  $g \equiv \chi_A$ . We note that  $\mathcal{F}[g] \equiv [0, 1]$  by the metric density of  $A$  and  $A^c$ . Let  $x \in \mathbb{R}$  and  $\varepsilon > 0$ . Metric density implies that the sets  $(\mathbb{R} \setminus A) \cap B_\varepsilon(x)$  and  $(A \setminus E) \cap B_\varepsilon(x)$  each have positive measure. Thus,  $\mathcal{F}[f] \equiv [0, 1]$ . Therefore,  $\mathcal{F}[f] = \mathcal{F}[g]$ .  $\square$

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