Filter for Positive Stochastic Nonlinear Switching Systems With Phase-Type Semi-Markov Parameters and Application

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***Abstract*—In this article, the issue of positive $\mathcal{L}_1$ filter design is investigated for positive nonlinear stochastic switching systems subject to the phase-type semi-Markov jump process. Many complicated factors, such as semi-Markov jump parameters, positivity, T–S fuzzy strategy, and external disturbance, are taken into consideration. Practical systems under positivity constraint conditions and unpredictable structural changes are characterized by positive semi-Markov jump systems (S-MJSs). First, by the key properties of the supplementary variable and the plant transformation technique, phase-type S-MJSs are transformed into Markov jump systems (MJSs), which means that, to an extent, these two kinds of stochastic switching systems are mutually represented. Second, with the help of the normalized membership function, the associated nonlinear MJSs are transformed into the local linear MJSs with specific T–S fuzzy rules. Third, by choosing the linear copositive Lyapunov function (LCLF), stochastic stability (SSY) criteria are given for the corresponding system with $\mathcal{L}_1$ performance. Some solvability conditions for positive $\mathcal{L}_1$ filter are constructed under a linear programming framework. Finally, an epidemiological model illustrates the effectiveness of the theoretical findings.**

***Index Terms*—Filter design, linear programming, T–S fuzzy.**

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I. Introduction

NONLINEARITY always finds its utilization to characterize many practical physical systems and industrial processes. Then, it becomes difficult to tackle nonlinearity issues and many traditional control strategies for analyzing linear systems are no longer used for nonlinear systems. In order to handle complex nonlinear systems, the T–S fuzzy model provides a potent modeling framework to characterize complex nonlinear systems [1], [2]. With the help of the T–S fuzzy approach, nonlinear systems can be converted into local linear subsystems. Based on this approach, many results for linear systems can be applied to nonlinear systems. During the past years, the T–S fuzzy model has been widely investigated, and research topics on the T–S fuzzy model include stability and stabilization, $\mathcal{H}_\infty$ control, finite-time control, adaptive control, input saturation, filter design, and input quantization [3]–[14].

It should be pointed out that many experts have begun to study a special kind of T–S fuzzy systems, i.e., positive T–S fuzzy systems [15], due to their potential applications in absolute temperature of physics, population position of biology, price of economics, etc. For more details, we refer readers to [16]–[27] and the references therein. Compared with traditional T–S fuzzy systems, the special characteristic of positive T–S fuzzy systems relies on the positivity of their state signals, output signals, and input signals, and it is more challenging to study positive T–S fuzzy systems. This positive requirement always generates new ideas and results. Recently, there are numerous results available in the literature (see [28]–[35]), in which stability analysis and control synthesis are the main research topics.

It is noted that Markov jump systems (MJSs), as an important modeling approach, have attracted substantial attention in industrial applications, including biomedicine, aerospace, and other aspects (see [34], [36]–[42]). However, for general MJSs, there exists one obvious limitation, that is, the sojourn

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time (ST) of the Markov process obeys exponential distribution, in which the distribution function is memoryless. In fact, it is difficult to satisfy this severe constraint for the ST distribution subject to memoryless characteristics in practice. In such a case, the corresponding stochastic switching systems with ST obeying nonexponential distribution can be characterized as semi-MJSs (S-MJSs). Nevertheless, the ST-dependent property brings many difficulties for S-MJSs. Consequently, large amounts of topics toward S-MJSs have been brought (see [43]–[55]).

On the other hand, several effective strategies for estimation and filter have been proposed to estimate $z(t)$, in which the $\mathcal{H}_\infty$ filter is a popular approach to investigate the external noise without exactly known statistics. During the filter design process of positive systems [25], sufficient conditions formulated in the linear matrix inequality framework are proposed, in which the $\mathcal{L}_2$-norm (or ℓ_2-norm) is adopted to generate $\mathcal{H}_\infty$ performance index with the alternative quadratic Lyapunov function. However, considering the non-negative property, it is natural to apply the $\mathcal{L}_1$-norm (or ℓ_1-norm) to measure the input and output variables. Thus, it is better to choose the LCLF. Compared with the traditional quadratic Lyapunov function, LCLF makes full use of the features of positive systems, and its derivative calculation is convenient. Under a linear programming framework, the corresponding judgment basis can save operational time.

Although some excellent results for positive stochastic switching systems have sprung up, there still exist obvious disadvantages. The ST subject to exponential distribution in [34], [35], and [34], [38]–[42] has become one of the most important constraint conditions, whereas many dynamical systems do not always satisfy the rigorous requirement. Specifically, the considered systems [34], [38]–[42] are positive linear MJSs without nonlinearity. Next, the traditional positive $\mathcal{L}_1$ filter design [40] only provides an estimation of system states, which further implies that no information about the transient performance can be given under its framework. Additionally, [15], [16], [22], [23], and [25]–[30] describe the deterministic switching case while the dynamical systems subject to the stochastic semi-Markov process (SMP) in this article can be more suitable to describe practical systems affected by inevitable random factors. Moreover, many factors, such as SMP, positivity, external disturbance, and nonlinearity, play an important part in describing practical complex stochastic switching systems. Therefore, a critical issue about positive $\mathcal{L}_1$ filter for positive nonlinear S-MJSs is whether there exists a positive $\mathcal{L}_1$ filter to estimate the state signals for positive nonlinear S-MJSs. However, up to now, there are no theoretical results, which motivate our study. Specifically, there are two innovations to be addressed during the positive $\mathcal{L}_1$ filter design.

Q1: Different from special MJSs [34], [34], [35], [38]–[42] subject to exponential distribution, how to propose the stochastic stability (SSY) for S-MJSs with nonexponential distribution?

Q2: Compared with the traditional positive $\mathcal{L}_1$ filter design [40], how to design a novel positive $\mathcal{L}_1$ filter to achieve the better performance?

In this article, we will address the positive $\mathcal{L}_1$ filter for a class of positive nonlinear S-MJSs subject to phase-type SMP. Compared with the existing works, the contributions of this article are summarized as follows.

1) In contrast with [34], [35], and [34], [38]–[42], one unrealistic assumption, i.e., the ST in stochastic switching systems follows an exponential distribution, is removed in this article by applying the S-MJSs model. Considering the equivalent relationship between the phase-type SMP and the Markov process, phase-type S-MJSs are transformed into MJSs.
2) By fuzzy blending, the associated nonlinear MJSs are converted into the local linear MJSs. Furthermore, with the help of the LCLF, sufficient conditions are proposed to realize SSY with $\mathcal{L}_1$ performance.
3) Compared with [40], the positive $\mathcal{L}_1$ fuzzy upper-bounding filter and lower-bounding filter for positive nonlinear S-MJSs are constructed to obtain a better performance under a linear programming optimization framework.

This article is organized as follows. In Section II, the system description, and some necessary definitions and lemmas are presented. Section III shows SSY with $\mathcal{L}_1$ analysis. In Section IV, a positive filter design is investigated in the form of linear programming. An epidemiological model is provided to illustrate the effectiveness of the theoretical findings in Section V. Concluding remarks are given in Section VI.

Notations: $\mathcal{A} \geq\geq (>> 0)$ means that all entries of matrix $\mathcal{A}$ are non-negative (positive); 1-norm of $||x||_1$ stands for $||x||_1 = \sum_{k=1}^{n} |x_k|$, where x_k is the kth element of $x \in \mathcal{R}^n$; and $\mathbf{1}_n$ denotes all-ones vector in $\mathcal{R}^n$. For given $\eta(t) : \mathcal{R} \to \mathcal{R}^m$, the $\mathcal{L}_1$-norm is defined by $||\eta(t)||_{\mathcal{L}_1} = \int_0^\infty ||\eta(t)||_1 dt$. $\mathcal{L}_1[0, +\infty)$ is the space of absolute integrable vector-valued functions on $[0, +\infty)$, i.e., we say $\eta(t) : [0, +\infty) \to \mathcal{R}^m$ is in $\mathcal{L}_1[0, +\infty)$ if $\int_0^\infty ||\eta(t)||_1 dt < \infty$. $\Im$ is the weak infinitesimal operator. $\mathcal{E}\{\cdot\}$ stands for the mathematical expectation. iff means if and only if.

II. Preliminaries

Consider a class of stochastic systems as

$$\begin{aligned} \dot{x}(t) &= \hat{g}\big(x(t), \eta(t), \hat{\omega}_t\big) \\ y(t) &= \hat{h}\big(x(t), \eta(t), \hat{\omega}_t\big) \end{aligned} \tag{1}$$

where $\hat{g}(x(t), \eta(t), \hat{\omega}_t)$ and $\hat{h}(x(t), \eta(t), \hat{\omega}_t)$ mean the smooth nonlinearities; $x(t) \in \mathcal{R}^n$ and $y(t) \in \mathcal{R}^p$ stand for the state and the measured output; and $\eta(t) \in \mathcal{R}^m$ is the disturbance input and belongs to $\mathcal{L}_1[0, +\infty)$. Let $\{\hat{\omega}_t, t \geq 0\}$ be a stochastic process (SP) in $\{1, 2, \ldots, m+1\}$, where $1, 2, \ldots, m$ are transient and $m+1$ is absorbing. The infinitesimal generator (IG) is $\mathcal{W} = \begin{bmatrix} \mathcal{Q} & \mathcal{Q}^0 \\ 0_{1\times m} & 0 \end{bmatrix}$, where $\mathcal{Q} = (\mathcal{Q}_{\mu\nu})_{m\times m}$, $\mathcal{Q}_{\mu\mu} < 0$, and $\mathcal{Q}_{\mu\nu} \geq 0$, for $\nu \neq \mu$, and $\mathcal{Q}^{-1}$ exists. The non-negative column vector $\mathcal{Q}^0$ satisfies $\mathcal{Q}e + \mathcal{Q}^0 = 0$, where the vector e has all entries equal to one. The initial distribution vector (δ, a_{m+1}) satisfies $\delta e + a_{m+1} = 1$, where $\delta = (\delta_1, \delta_2, \ldots, \delta_m)$.

Lemma 1 [37]: The probability distribution (PD) $\mathcal{H}(t)$ dependent on the initial distribution vector (δ, a_{m+1}) is given as $\mathcal{H}(t) = 1 - \delta\exp(\mathcal{Q}t)e$, for $t \geq 0$, when the time arrives at the absorbing state $m + 1$.

Definition 1 [43]: The PD $\mathcal{H}(t)$ on $[0, +\infty)$ is called a continuous distribution of phase-type iff it is the distribution of the time related to a finite Markov process with an absorbing state and all the other transient states. The pair $(\delta, \mathcal{Q})$ is defined as a representation of the PD $\mathcal{H}(t)$.

Definition 2 [43]: Consider a finite set Ξ. The SP $\hat{\omega}_t$ is called phase-type SMP, if the followings hold.

1) The sample paths of the SP $\hat{\omega}_t$ are right-continuous functions with left-hand limits in probability one.
2) Denote the sth jump point of the SP $\hat{\omega}_t$ by ϕ_s, where $0 = \phi_0 < \phi_1 < \phi_2 < \cdots < \phi_s < \cdots$, and ϕ_s $(s = 0, 1, 2, \ldots,)$ are Markov of the SP $\hat{\omega}_t$.
3) $\mathcal{H}_{\mu\nu}(t) = \mathcal{P}r(\phi_{s+1} - \phi_s \leq t|\hat{\omega}_{\phi_s} = \mu, \hat{\omega}_{\phi_{s+1}} = \nu) = \mathcal{H}_\mu(t)$, $\mu, \nu \in \Xi$, $t \geq 0$ are independent of ν and s.
4) $\mathcal{H}_\mu(t)$, $\mu \in \Xi$ is a phase-type distribution.

Remark 1: The time between transitions is subject to phase-type distribution. It is noted that the phase-type distribution is a generalization of exponential distribution and retains the analytical characteristics of an exponential distribution. Moreover, the family of phase-type distribution has a dense property in all the families of distributions on $[0, +\infty)$. Therefore, by choosing a phase-type distribution, the original distribution can be approximated in a compact domain to arbitrary accuracy.

Let $(\delta^{(\mu)}, \mathcal{Q}^{(\mu)})$, $\mu \in \Xi$ stand for the $m^{(\mu)}$ order representation of $\mathcal{H}_\mu(t)$ and $\Xi^{(\mu)}$ be the set of all transient states, where

$$\delta^{(\mu)} = \left(\delta_1^{(\mu)}, \delta_2^{(\mu)}, \ldots, \delta_{m^{(\mu)}}^{(\mu)}\right)$$
$$\mathcal{Q}^{(\mu)} = \left(\mathcal{Q}_{\nu k}^{(\mu)}, \nu, k \in \Xi^{(\mu)}\right).$$

Let

$$\pi_{\mu\nu} = \mathcal{P}r\left(\hat{\omega}_s = \mu|\hat{\omega}_{s+1} = \nu\right), \mu, \nu \in \Xi$$
$$\mathcal{P} = \left(\pi_{\mu\nu}\right), \mu, \nu \in \Xi$$
$$(\delta, \mathcal{Q}) = \left(\delta^{(\mu)}, \mathcal{Q}^{(\mu)}\right), \mu \in \Xi.$$

Based on the above analysis, the PD of $\mathcal{H}_\mu(t)$ is determined by $\{\mathcal{P}, (\delta, \mathcal{Q})\}$. For every s $(s = 0, 1, \ldots,)$, $\phi_s \leq t \leq \phi_{s+1}$, define

$$\mathcal{J}(t) = \text{ the phase of } \mathcal{H}_{\hat{\omega}_t}(\cdot) \text{ at time } t - \phi_s. \tag{2}$$

For any $\mu \in \Xi$, define

$$\mathcal{Q}_\mu^{\mu,0} = -\sum_{k=1}^{m^{(\mu)}} \mathcal{Q}_{\nu k}^{(\mu)}, \quad \nu = 1, 2, \ldots, m_\mu$$
$$\mathcal{G} = \left\{\left(\mu, k^{(\mu)}\right)|\mu \in \Xi, k^{(\mu)} = 1, 2, \ldots, m_\mu\right\}. \tag{3}$$

Lemma 2 [43]: $\mathcal{O}(t) = (\hat{\omega}_t, \mathcal{J}(t))$ is a Markov chain in state space $\mathcal{G}$. The IG of $\mathcal{O}_t$ given by $\mathcal{W} = (w_{ij})$, $i, j \in \mathcal{G}$ is determined by the pair of $(\hat{\omega}_t, \mathcal{J}(t))$ given by $\{\mathcal{P}, (\delta, \mathcal{Q})\}$ as

$$w_{(\mu,k^{(\mu)})(\mu,k^{(\mu)})} = \mathcal{Q}_{k^{(\mu)}k^{(\mu)}}^{(\mu)}, \left(\mu, k^{(\mu)}\right) \in \mathcal{G}$$
$$w_{(\mu,k^{(\mu)})(\mu,\bar{k}^{(\mu)})} = \mathcal{Q}_{k^{(\mu)}\bar{k}^{(\mu)}}^{(\mu)}, k^{(\mu)} \neq \bar{k}^{(\mu)}, \left(\mu, k^{(\mu)}\right) \in \mathcal{G} \text{ and } \left(\mu, \bar{k}^{(\mu)}\right) \in \mathcal{G}$$
$$w_{(\mu,k^{(\mu)})(\nu,k^{(\nu)})} = \pi_{\mu\nu}\mathcal{Q}_{k^{(\mu)}}^{(\mu,0)}\delta_{k^{(\nu)}}^{(\nu)}, \mu \neq \nu, \left(\mu, k^{(\mu)}\right) \in \mathcal{G} \text{ and } \left(\nu, k^{(\nu)}\right) \in \mathcal{G}.$$

According to (3), $\mathcal{G}$ has $\Omega = \sum_{\mu\in\Xi} m_\mu$ elements, which means that the state space of $\mathcal{O}(t)$ has Ω elements. Define the number of $(\mu, k^{(\mu)})$ by $\sum_{\tau=1}^{\mu-1} m^{(\tau)} + k$, $(1 \leq k \leq m^{(\mu)})$ and this transformation by $\Upsilon(\cdot)$. Hence, one has

$$\Upsilon(\mu, k) = \sum_{\tau=1}^{\mu-1} m^{(\tau)} + k, \mu \in \Xi, 1 \leq k \leq m^{(\mu)}.$$

Furthermore, define

$$\omega_t = \Upsilon(\mathcal{O}(t))$$
$$\varrho_{\Upsilon(\mu,k)\Upsilon(\mu',k')} = w_{\Upsilon(\mu,k)\Upsilon(\mu',k')}.$$

Therefore, ω_t is an associated Markov process of $\hat{\omega}_t$ in $\wp = \{1, 2, \ldots, \Omega\}$ and the IG is $\wp = (\varrho_{\alpha\beta})$, $1 \leq \alpha, \beta \leq \Omega$, so that

$$\begin{aligned} &Pr\{\omega_{t+\Delta} = \beta|\omega_t = \alpha\} \\ &= Pr\{\Upsilon(\mathcal{O}(t+\Delta)) = \beta|\Upsilon(\mathcal{O}(t)) = \alpha\} \\ &= \begin{cases} \varrho_{\alpha\beta}\Delta + o(\Delta), & \alpha \neq \beta \\ 1 + \varrho_{\alpha\alpha}\Delta + o(\Delta), & \alpha = \beta \end{cases} \end{aligned}$$

where $\varrho_{\alpha\beta} \geq 0$ stands for the transition rate from α to β for $\alpha \neq \beta$, and $\sum_{\beta=1,\beta\neq\alpha}^{\Omega} \varrho_{\alpha\beta} = -\varrho_{\alpha\alpha}$.

According to Lemma 2, every finite phase-type SMP can be transformed into a Markov chain. Then, consider the following system which is equivalent to (1):

$$\begin{aligned} \dot{x}(t) &= g(x(t), \eta(t), \omega_t) \\ y(t) &= h(x(t), \eta(t), \omega_t) \end{aligned} \tag{4}$$

where $g(x(t), \eta(t), \omega_t)$ and $h(x(t), \eta(t), \omega_t)$ mean the smooth nonlinearities; and $x(t) \in \mathcal{R}^n$, $\eta(t) \in \mathcal{R}^m$, and $y(t) \in \mathcal{R}^p$ stand for the state, the disturbance input, and the measured output.

Next, consider the T–S fuzzy method to approximate system (4). Then, we have the θth rule as follows.

Plant Rule θ: IF $\phi_1(t)$ is $\mathcal{Q}_1^\theta$, $\phi_2(t)$ is $\mathcal{Q}_2^\theta$, and $\cdots$ and $\phi_\iota(t)$ is $\mathcal{Q}_l^\theta$, THEN

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}_\theta(\omega_t)x(t) + \mathcal{C}_\theta(\omega_t)\eta(t) \\ y(t) &= \mathcal{D}_\theta(\omega_t)x(t) + \mathcal{E}_\theta(\omega_t)\eta(t) \end{aligned} \tag{5}$$

where $\mathcal{Q}_{\vartheta_1}^\theta$ $(\theta = 1, 2, \ldots, \varpi, \vartheta_1 = 1, 2, \ldots, \iota)$ denotes the fuzzy sets with linear membership functions representing a fuzzy subspace in which the implication $\mathcal{R}$ can be applied for reasoning [1]. $\phi_1(t)$, $\phi_2(t)$, $\ldots$, and $\phi_\iota(t)$ denote the premise variables that depend on the system states. When $\omega_t = \alpha \in \wp$, $\mathcal{A}_\theta(\omega_t)$, $\mathcal{C}_\theta(\omega_t)$, $\mathcal{D}_\theta(\omega_t)$, and $\mathcal{E}_\theta(\omega_t)$ are, respectively, denoted as $\mathcal{A}_{\theta\alpha}$, $\mathcal{C}_{\theta\alpha}$, $\mathcal{D}_{\theta\alpha}$, and $\mathcal{E}_{\theta\alpha}$. Thus, one has

$$\begin{aligned} \dot{x}(t) &= \sum_{\theta=1}^{\varpi} p_\theta(\phi(t))(\mathcal{A}_{\theta\alpha}x(t) + \mathcal{C}_{\theta\alpha}\eta(t)) \\ y(t) &= \sum_{\theta=1}^{\varpi} p_\theta(\phi(t))(\mathcal{D}_{\theta\alpha}x(t) + \mathcal{E}_{\theta\alpha}\eta(t)) \end{aligned} \tag{6}$$

where $\phi(t) = \begin{bmatrix}\phi_1(t) & \phi_2(t) & \cdots & \phi_l(t)\end{bmatrix}^T$, and $p_\theta(\phi(t))$ means the membership function as

$$p_\theta(\phi(t)) = \frac{\prod_{\vartheta_1=1}^{\iota} \mathcal{Q}_{\vartheta_1}^\theta\left(\phi_{\vartheta_1}(t)\right)}{\sum_{\theta=1}^{\varpi}\prod_{\vartheta_1=1}^{\iota} \mathcal{Q}_{\vartheta_1}^\theta\left(\phi_{\vartheta_1}(t)\right)} \tag{7}$$

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and $\mathcal{Q}^{\theta}_{\vartheta_1}(\phi_{\vartheta_1}(t)) \in [0, 1]$ represents the grade of the membership of $\phi_{\vartheta_1}(t)$ in $\mathcal{Q}^{\theta}_{\vartheta_1}$. In fact, since $\mathcal{Q}^{\theta}_{\vartheta_1}(\phi_{\vartheta_1}(t)) \geq 0$, it is clear that

$$\sum_{\theta=1}^{\varpi} p_\theta(\phi(t)) = 1, p_\theta(\phi(t)) \geq 0. \tag{8}$$

Lemma 3 [18], [19], [34]: System (6) is positive if $\mathcal{A}_{\theta\alpha}$ is the Metzler matrix, $\mathcal{C}_{\theta\alpha} \geq\geq 0$, $\mathcal{D}_{\theta\alpha} \geq\geq 0$, and $\mathcal{E}_{\theta\alpha} \geq\geq 0$.

Definition 3 [34]: For the initial conditions $x_0 \geq\geq 0$ and ω_0, system (6) with ($\eta(t) = 0$) is said to be stochastically stable if $\mathcal{E}\{\int_0^\infty ||x(s)||_1 ds|(x_0, \omega_0)\} < \infty$ holds.

Definition 4 [34]: For $\gamma > 0$, system (6) is said to be stochastically stable with $\mathcal{L}_1$ performance index, if system (6) with zero input $\eta(t)$ is said to be stochastically stable, and $\mathcal{E}\{\int_0^\infty ||y(s)||_1 ds\} \leq \gamma\mathcal{E}\{\int_0^\infty ||\eta(s)||_1 ds\}$ holds, for $x_0 = 0$ and $\eta(t) \neq 0$.

III. SSY With $\mathcal{L}_1$ Analysis

Based on the linear stochastic Lyapunov function, sufficient conditions for SSY of the resulting closed-loop system (RCLAS) (6) with $\eta(t) = 0$ will be provided in Theorem 1.

Theorem 1: If there exists $\sigma_\alpha \in \mathcal{R}^n_+$ $\forall\alpha \in \wp$, such that

$$\mathcal{A}^T_{\theta\alpha}\sigma_\alpha + \sum_{\beta=1}^{\Omega} \varrho_{\alpha\beta}\sigma_\beta << 0 \tag{9}$$

then system (6) ($\eta(t) = 0$) realizes SSY.

Proof: Construct the LCLF

$$\mathcal{S}(x(t), \omega_t) = x^T(t)\sigma_{\omega_t}. \tag{10}$$

Next, one has the weak infinitesimal operator as

$$\Im\mathcal{S}(x(t), \alpha) = \sum_{\theta=1}^{\varpi} p_\theta(\phi(t))x^T(t)\left(\mathcal{A}^T_{\theta\alpha}\sigma_\alpha + \sum_{\beta=1}^{\Omega} \varrho_{\alpha\beta}\sigma_\beta\right). \tag{11}$$

Applying (9) yields

$$\Im\mathcal{S}(x(t), \alpha) = x^T(t)\mu_\alpha \leq -\mu_0\|x(t)\|_1 < 0 \tag{12}$$

where

$$\mu_0 = \min_{\alpha=1,2,\ldots,\Omega}\left\{\min_{s=1,2,\ldots,n}\{-[\mu_\alpha]_s\}\right\}$$
$$\mu_\alpha = \sum_{\theta=1}^{\varpi} p_\theta(\phi(t))\left(\mathcal{A}^T_{\theta\alpha}\sigma_\alpha + \sum_{\beta=1}^{\Omega} \varrho_{\alpha\beta}\sigma_\beta\right).$$

By the use of Dynkin's formula, it gives rise to

$$\begin{aligned}&\mathcal{E}[\mathcal{S}(x(t), \alpha)] - \mathcal{S}(x_0, \omega_0)\\ &\quad= \mathcal{E}\left[\int_0^t \Im\mathcal{S}(x(s), \omega_s)ds\right]\\ &\quad\leq -\mu_0\mathcal{E}\left[\int_0^t \|x(s)\|_1 ds|(x_0, \omega_0)\right]\end{aligned}$$

which implies

$$\begin{aligned}&\mu_0\mathcal{E}\left[\int_0^t \|x(s)\|_1 ds|(x_0, \omega_0)\right]\\ &\quad\leq \mathcal{S}(x_0, \omega_0) - \mathcal{E}[\mathcal{S}(x(t), \alpha)] \leq \mathcal{S}(x_0, \omega_0).\end{aligned}$$

Then

$$\mathcal{E}\left[\int_0^\infty \|x(t)\|_1 dt|(x_0, \omega_0)\right] \leq \frac{\mathcal{S}(x_0, \omega_0)}{\mu_0} < \infty.$$

Therefore, system (6) with ($\eta(t) = 0$) realizes SSY . ■

Remark 2: The SSY theory is the basis for studying S-MJSs. Sufficient conditions are proposed to realize SSY for the corresponding S-MJSs in Theorem 1, which can lay the foundation for the later studies about $\mathcal{L}_1$ performance analysis and filter design.

Remark 3: For Q1, compared with special MJSs [34], [34], [35], [38]–[42] subject to exponential distribution, phase-type S-MJSs are transformed into MJSs via the supplementary variable and the plant transformation technique. Then, with the help of the normalized membership function, the associated nonlinear MJSs are transformed into the local linear MJSs with specific T–S fuzzy rules. Furthermore, the corresponding SSY criteria are constructed by choosing the LCLF [see (10)–(12)].

Sufficient conditions are proposed to realize SSY with $\mathcal{L}_1$ performance for the system (6) in Theorem 2.

Theorem 2: If there exists $\sigma_\alpha \in \mathcal{R}^n_+$ $\forall\alpha \in \wp$, such that

$$\mathcal{A}^T_{\theta\alpha}\sigma_\alpha + \sum_{\beta=1}^{\Omega} \varrho_{\alpha\beta}\sigma_\beta + \mathcal{D}^T_{\theta\alpha}\mathbf{1}_p << 0 \tag{13}$$

$$\mathcal{C}^T_{\theta\alpha}\sigma_\alpha + \mathcal{E}^T_{\theta\alpha}\mathbf{1}_p - \gamma\mathbf{1}_m << 0 \tag{14}$$

then system (6) realizes SSY with $\mathcal{L}_1$ performance.

Proof: From (13), we can get (9). Therefore, system (6) ($\eta(t) = 0$) realizes SSY.

For LCLF (10), one has

$$\begin{aligned}&\Im\mathcal{S}(x(t), \alpha) + \|y(t)\|_{\mathcal{L}_1} - \gamma\|\eta(t)\|_{\mathcal{L}_1}\\ &\leq \sum_{\theta=1}^{\varpi} p_\theta(\phi(t))\Bigg[x^T(t)\left(\mathcal{A}^T_{\theta\alpha}\sigma_\alpha + \sum_{\beta=1}^{\Omega} \varrho_{\alpha\beta}(h)\sigma_\beta\right)\\ &\quad + \eta^T(t)\mathcal{C}^T_{\theta\alpha}\sigma_\alpha + \mathbf{1}^T_p\mathcal{D}_{\theta\alpha}x(t) + \mathbf{1}^T_p\mathcal{E}_{\theta\alpha}\eta(t) - \gamma\mathbf{1}^T_m\eta(t)\Bigg]\\ &= \sum_{\theta=1}^{\varpi} p_\theta(\phi(t))\Bigg[x^T(t)\Bigg(\mathcal{A}^T_{\theta\alpha}\sigma_\alpha + \sum_{\beta=1}^{\Omega} \varrho_{\alpha\beta}(h)\sigma_\beta\\ &\quad + \mathcal{D}^T_{\theta\alpha}\mathbf{1}_p\Bigg) + \eta^T(t)\big(\mathcal{C}^T_{\theta\alpha}\sigma_\alpha + \mathcal{E}^T_{\theta\alpha}\mathbf{1}_p - \gamma\mathbf{1}_m\big)\Bigg].\end{aligned} \tag{15}$$

For the zero-initial condition, it follows from (13) and (14) that:

$$\Im\mathcal{S}(x(t), \alpha) + \|y(t)\|_{\mathcal{L}_1} - \gamma\|\eta(t)\|_{\mathcal{L}_1} < 0 \tag{16}$$

which means

$$\begin{aligned}\mathcal{E}\left[\int_0^\infty ||y(s)||_1 ds\right] &= \lim_{\upsilon\to\infty} \mathcal{E}\left[\int_0^\upsilon ||y(s)||_1 ds|(x_0, \omega_0)\right]\\ &\leq \gamma\mathcal{E}\left[\int_0^\infty ||\eta(s)||_1 ds\right].\end{aligned}$$

Therefore, system (6) realizes SSY with $\mathcal{L}_1$ performance index. ■

IV. $\mathcal{L}_1$ FILTER

In this section, the positive $\mathcal{L}_1$ fuzzy upper-bounding filter and lower-bounding filter are constructed for positive nonlinear S-MJSs. Consider the following fuzzy systems:

$$\begin{aligned}\dot{x}(t) &= \sum_{\theta=1}^{\varpi} p_\theta(\phi(t))(\mathcal{A}_{\theta\alpha}x(t) + \mathcal{C}_{\theta\alpha}\eta(t))\\ y(t) &= \sum_{\theta=1}^{\varpi} p_\theta(\phi(t))(\mathcal{D}_{\theta\alpha}x(t) + \mathcal{E}_{\theta\alpha}\eta(t))\\ z(t) &= \sum_{\theta=1}^{\varpi} p_\theta(\phi(t))\mathcal{L}_{\theta\alpha}x(t) \end{aligned}\tag{17}$$

where $x(t) \in \mathcal{R}^n$, $\eta(t) \in \mathcal{R}^m$, $y(t) \in \mathcal{R}^p$, and $z(t) \in \mathcal{R}^q$ mean the state, the disturbance input, the output signal, and the estimated signal.

Remark 4: As shown previously, the premise variables $\phi(t)$ depend on the system states. For partly available system states, the premise variables $\phi(t)$ can be designed to be related to these available system states. Then, it is suitable to construct the fuzzy filter by using the premise variables $\phi(t)$ dependent on available system states.

Furthermore, a pair of positive full-order $\mathcal{L}_1$ filters is constructed as follows.

Plant Rule θ: IF $\phi_1(t)$ is $\mathcal{Q}_1^\theta$, $\phi_2(t)$ is $\mathcal{Q}_2^\theta$, and $\cdots$ and $\phi_l(t)$ is $\mathcal{Q}_l^\theta$, THEN

$$\dot{\hat{x}}(t) = \bar{\mathcal{A}}_{\theta\alpha}\hat{x}(t) + \bar{\mathcal{B}}_{\theta\alpha}y(t), \hat{z}(t) = \bar{\mathcal{C}}_{\theta\alpha}\hat{x}(t) \tag{18}$$

and

$$\dot{\check{x}}(t) = \underline{\mathcal{A}}_{\theta\alpha}\check{x}(t) + \underline{\mathcal{B}}_{\theta\alpha}y(t), \check{z}(t) = \underline{\mathcal{C}}_{\theta\alpha}\check{x}(t) \tag{19}$$

where $\hat{x}(t) \in \mathcal{R}^n$, $\check{x}(t) \in \mathcal{R}^n$, $\hat{z}(t) \in \mathcal{R}^q$, and $\check{z}(t) \in \mathcal{R}^q$. $\bar{\mathcal{A}}_{\theta\alpha}$, $\bar{\mathcal{B}}_{\theta\alpha}$, $\underline{\mathcal{A}}_{\theta\alpha}$, $\underline{\mathcal{B}}_{\theta\alpha}$, $\bar{\mathcal{C}}_{\theta\alpha}$, and $\bar{\mathcal{C}}_{\theta\alpha}$ denote the filter parameters. Then, one has the overall fuzzy upper-bounding filter and lower-bounding filter model as

$$\begin{aligned}\dot{\hat{x}}(t) &= \sum_{\theta=1}^{\varpi} p_\theta(\phi(t))\big(\bar{\mathcal{A}}_{\theta\alpha}\hat{x}(t) + \bar{\mathcal{B}}_{\theta\alpha}y(t)\big)\\ \hat{z}(t) &= \sum_{\theta=1}^{\varpi} p_\theta(\phi(t))\bar{\mathcal{C}}_{\theta\alpha}\hat{x}(t)\end{aligned}\tag{20}$$

and

$$\begin{aligned}\dot{\check{x}}(t) &= \sum_{\theta=1}^{\varpi} p_\theta(\phi(t))\big(\underline{\mathcal{A}}_{\theta\alpha}\check{x}(t) + \underline{\mathcal{B}}_{\theta\alpha}y(t)\big)\\ \check{z}(t) &= \sum_{\theta=1}^{\varpi} p_\theta(\phi(t))\underline{\mathcal{C}}_{\theta\alpha}\check{x}(t).\end{aligned}\tag{21}$$

Remark 5: In the practical industrial process, the sensors are adopted to measure the physical quantities of the system. For example, the speed of the elevator in the operating process can be measured by a high-speed rotary encoder; the voltage, current, and power in power systems can be measured by the corresponding transmitter; and the temperature, pressure, and liquid level in the chemical process can be measured by thermocouple, pressure sensor, and liquid level sensor. When the system states are measurable, the state-feedback controller is constructed to obtain dynamical performance. However, due to the limitation of the sensor and special working environment, it is difficult to measure the state signals directly. In such a case, it needs the input and output information to reconstruct the state variables of the system or estimate a linear combination of the state variables of the system. This idea is not only used in the control of the system but also is applied in the development of measurement technology to form a new direction, i.e., soft measurement technology. In order to realize this idea, the proposed observer and filter provide potential methods and techniques.

Remark 6: It is well known that the state observer and filter are always adopted to estimate the system states. When the original systems do not contain the external disturbance, if the original systems are observable, we can obtain the estimation value of system states through the state observer. Different from the state observer, the filter focuses on the optimal estimation of system states under the external disturbance. In this article, the external disturbance exists in the dynamical systems, and it is reasonable to consider the filter design. For positive systems, when the statistical characteristics of system disturbances are difficult to be determined, the disturbance can be regarded as any signal belonging to $\mathcal{L}_1[0, +\infty)$. Then, the $\mathcal{L}_1$ norm of the disturbance input to the estimation error can be seen as the performance index of the filter [56], [57]. Furthermore, the $\mathcal{L}_1$ filter is designed by making the performance index less than a given value and is widely applied in radar design [56], fault detection [29], and signal processing [57].

Remark 7: For Q2, the aforementioned paper [40] presents the $\mathcal{L}_1$ filter design for positive systems and only provides an estimation of system states, which means that there is no information about the transient performance under its framework. However, results that do not consider the transient behavior of positive systems may not be suitable in practical applications. For example, the biological population should be limited to a certain number, not too many and not too few. In such a case, it can maintain the balance and stability of the ecosystems. Here, the upper-bounding filter and lower-bounding filter are utilized to estimate the states of positive systems at all times under the transient performance.

Remark 8: Compared with some existing works [34], [35], [38]–[55], the new filter design algorithms have many advantages. First, in contrast with traditional MJSs [34], [35], [38]–[42], there exists one unrealistic assumption, i.e., the ST in MJSs follows an exponential distribution. S-MJSs with the ST obeying a nonexponential distribution relax this strict restriction, which are more suitable to describe practical systems subject to sudden change of the parameters or structures. Thus, MJSs can be regarded as a special case of S-MJSs. Second, compared with the traditional positive filter design method [40], the proposed filter named as the upper-bounding filter and lower-bounding filter makes full consideration of the transient performance in practical systems. Third, different from S-MJSs without positive constraint [43]–[55], the positive constraint is adopted to study S-MJSs, which can describe positive systems subject to complex stochastic factors.

Remark 9: Here, two positive $\mathcal{L}_1$ fuzzy filters are constructed to estimate the states of system (6). It follows from Lemma 1 that the filters (20) and (21) are positive if $\bar{\mathcal{A}}_{\theta\alpha}$ and $\underline{\mathcal{A}}_{\theta\alpha}$ are Metzler matrices, $\bar{\mathcal{B}}_{\theta\alpha} \geq\geq 0$, $\underline{\mathcal{B}}_{\theta\alpha} \geq\geq 0$, $\bar{\mathcal{C}}_{\theta\alpha} \geq\geq 0$, and $\underline{\mathcal{C}}_{\theta\alpha} \geq\geq 0$.

Define the error states $\hat{x}_e(t) = \hat{x}(t) - x(t)$, $\hat{\xi}(t) = \begin{bmatrix} x^T(t) & \hat{x}_e^T(t) \end{bmatrix}^T$, and $\hat{e}(t) = \hat{z}(t) - z(t)$. Combining (17) with (20) yields

$$\dot{\hat{\xi}}(t) = \sum_{\theta=1}^{\varpi}\sum_{\vartheta=1}^{\varpi} p_\theta(\phi(t))p_\vartheta(\phi(t))\Big(\bar{\mathcal{A}}_{\xi\theta\vartheta\alpha}\hat{\xi}(t) + \bar{\mathcal{B}}_{\xi\theta\vartheta\alpha}\eta(t)\Big)$$
$$\hat{e}(t) = \sum_{\theta=1}^{\varpi} p_\theta(\phi(t))\bar{\mathcal{C}}_{\xi\theta\alpha}\hat{\xi}(t) \tag{22}$$

where $\bar{\mathcal{A}}_{\xi\theta\vartheta\alpha} = \begin{bmatrix} \mathcal{A}_{\theta\alpha} & 0 \\ \bar{\mathcal{B}}_{\theta\alpha}\mathcal{D}_{\vartheta\alpha} + \bar{\mathcal{A}}_{\theta\alpha} - \mathcal{A}_{\theta\alpha} & \bar{\mathcal{A}}_{\theta\alpha} \end{bmatrix}$, $\bar{\mathcal{B}}_{\xi\theta\vartheta\alpha} = \begin{bmatrix} \mathcal{C}_{\theta\alpha} \\ \bar{\mathcal{B}}_{\theta\alpha}\mathcal{E}_{\vartheta\alpha} - \mathcal{C}_{\theta\alpha} \end{bmatrix}$, $\bar{\mathcal{C}}_{\xi\theta\alpha} = \begin{bmatrix} \bar{\mathcal{C}}_{\theta\alpha} - \mathcal{L}_{\theta\alpha} & \bar{\mathcal{C}}_{\theta\alpha} \end{bmatrix}$.

Then, a positive $\mathcal{L}_1$ fuzzy upper-bounding filter (20) is designed to realize positivity and SSY with $\mathcal{L}_1$ performance for system (22).

Similarly, we can define $\breve{x}_e(t) = x(t) - \breve{x}(t)$, $\breve{\xi}(t) = \begin{bmatrix} x^T(t) & \breve{x}_e^T(t) \end{bmatrix}^T$, and $\breve{e}(t) = z(t) - \breve{z}(t)$ to get

$$\dot{\breve{\xi}}(t) = \sum_{\theta=1}^{\varpi}\sum_{\vartheta=1}^{\varpi} p_\theta(\phi(t))p_\vartheta(\phi(t))\Big(\underline{\mathcal{A}}_{\xi\theta\vartheta\alpha}\breve{\xi}(t) + \underline{\mathcal{B}}_{\xi\theta\vartheta\alpha}\eta(t)\Big)$$
$$\breve{e}(t) = \sum_{\theta=1}^{\varpi} p_\theta(\phi(t))\underline{\mathcal{C}}_{\xi\theta\alpha}\breve{\xi}(t) \tag{23}$$

where $\underline{\mathcal{A}}_{\xi\theta\vartheta\alpha} = \begin{bmatrix} \mathcal{A}_{\theta\alpha} & 0 \\ \mathcal{A}_{\theta\alpha} - \underline{\mathcal{A}}_{\theta\alpha} - \underline{\mathcal{B}}_{\theta\alpha}\mathcal{D}_{\vartheta\alpha} & \underline{\mathcal{A}}_{\theta\alpha} \end{bmatrix}$, $\underline{\mathcal{B}}_{\xi\theta\vartheta\alpha} = \begin{bmatrix} \mathcal{C}_{\theta\alpha} \\ \mathcal{C}_{\theta\alpha} - \underline{\mathcal{B}}_{\theta\alpha}\mathcal{E}_{\vartheta\alpha} \end{bmatrix}$, and $\underline{\mathcal{C}}_{\xi\theta\alpha} = \begin{bmatrix} \mathcal{L}_{\theta\alpha} - \underline{\mathcal{C}}_{\theta\alpha} & \underline{\mathcal{C}}_{\theta\alpha} \end{bmatrix}$.

Then, a positive $\mathcal{L}_1$ fuzzy lower-bounding filter (21) is designed to realize positivity and SSY with $\mathcal{L}_1$ performance for system (23).

Next, it follows from Theorem 2 that we can realize $\mathcal{L}_1$ SSY for system (22) with a positive fuzzy upper-bounding filter (20) directly.

Corollary 1: If there exists $\bar{\sigma}_\alpha \in \mathcal{R}_+^{2n}$ $\forall\alpha \in \wp$, such that

$$\bar{\mathcal{A}}_{\xi\theta\vartheta\alpha}^T\bar{\sigma}_\alpha + \sum_{\beta=1}^{\Omega}\varrho_{\alpha\beta}\bar{\sigma}_\beta + \bar{\mathcal{C}}_{\xi\theta\alpha}^T\mathbf{1}_q << 0 \tag{24}$$
$$\bar{\mathcal{B}}_{\xi\theta\vartheta\alpha}^T\bar{\sigma}_\alpha - \gamma\mathbf{1}_m << 0 \tag{25}$$

then system (22) realizes SSY with $\mathcal{L}_1$ performance satisfying $\mathcal{E}\{\int_0^\infty ||\hat{e}(s)||_1 ds\} \leq \gamma\mathcal{E}\{\int_0^\infty ||\eta(s)||_1 ds\}$ when $\hat{\xi}(0) = 0$.

Substituting (22) into Corollary 1, it will result in the solution for positive $\mathcal{L}_1$ fuzzy upper-bounding filter (20) in Theorem 3.

Theorem 3: Consider

$$\mathcal{A}_{\theta\alpha} = \begin{bmatrix} \mathcal{A}_{\theta\alpha1} & \mathcal{A}_{\theta\alpha2} & \cdots & \mathcal{A}_{\theta\alpha n} \end{bmatrix}^T$$
$$\mathcal{C}_{\theta\alpha} = \begin{bmatrix} \mathcal{C}_{\theta\alpha1} & \mathcal{C}_{\theta\alpha2} & \cdots & \mathcal{C}_{\theta\alpha n} \end{bmatrix}^T$$
$$\mathcal{L}_{\theta\alpha} = \begin{bmatrix} \mathcal{L}_{\theta\alpha1} & \mathcal{L}_{\theta\alpha2} & \cdots & \mathcal{L}_{\theta\alpha q} \end{bmatrix}^T$$
$$e_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T, e_2 = \begin{bmatrix} 0 & 1 & \cdots & 0 \end{bmatrix}^T$$
$$,\ldots, e_n = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}^T \tag{26}$$

with $\mathcal{A}_{\theta\alpha s} \in \mathcal{R}^n$, $\mathcal{C}_{\theta\alpha s} \in \mathcal{R}^m$, $\mathcal{L}_{\theta\alpha t} \in \mathcal{R}^n$, $\theta = 1, 2, \ldots, \varpi$, $s = 1, 2, \ldots, n$, $t = 1, 2, \ldots, q$ $\forall\alpha \in \Pi$. If there exist $\bar{\sigma}_{1\alpha} \in \mathcal{R}_+^n$, $\bar{\sigma}_{2\alpha} \in \mathcal{R}_+^n$, $\varsigma_{\theta\alpha} \in \mathcal{R}_+^n$, $\bar{a}_{\theta\alpha s} \in \mathcal{R}^n$, $\bar{b}_{\theta\alpha s} \in \mathcal{R}_+^p$, and $\bar{c}_{\theta\alpha t} \in \mathcal{R}_+^n$ $\forall\alpha \in \wp$, such that

$$\mathcal{A}_{\theta\alpha}^T\bar{\sigma}_{1\alpha} + \sum_{s=1}^{n}\bar{a}_{\theta\alpha s} + \mathcal{D}_{\vartheta\alpha}^T\sum_{s=1}^{n}\bar{b}_{\theta\alpha s} - \mathcal{A}_{\theta\alpha}^T\bar{\sigma}_{2\alpha} + \sum_{\beta=1}^{\Omega}\varrho_{\alpha\beta}\bar{\sigma}_{1\beta} + \sum_{t=1}^{q}\bar{c}_{\theta\alpha t} - \mathcal{L}_{\theta\alpha}^T\mathbf{1}_q << 0 \tag{27}$$
$$\sum_{s=1}^{n}\bar{a}_{\theta\alpha s} + \sum_{\beta=1}^{\Omega}\varrho_{\alpha\beta}\bar{\sigma}_{2\beta} + \sum_{t=1}^{q}\bar{c}_{\theta\alpha t} << 0 \tag{28}$$
$$\sum_{s=1}^{n}\bar{a}_{\theta\alpha s} + \sum_{\beta=1}^{\Omega}\bar{\varrho}_{\alpha\beta}\bar{\sigma}_{2\beta} + \sum_{t=1}^{q}\bar{c}_{\theta\alpha t} << 0 \tag{29}$$
$$\mathcal{C}_{\theta\alpha}^T\bar{\sigma}_{1\alpha} + \mathcal{E}_{\vartheta\alpha}^T\sum_{s=1}^{n}\bar{b}_{\theta\alpha s} - \mathcal{C}_{\theta\alpha}^T\bar{\sigma}_{2\alpha} - \gamma\mathbf{1}_m << 0 \tag{30}$$
$$\mathcal{A}_{\theta\alpha s}\bar{\sigma}_{2\alpha s} - \bar{a}_{\theta\alpha s} - \mathcal{D}_{\vartheta\alpha}^T\bar{b}_{\theta\alpha s} \leq\leq 0 \tag{31}$$
$$\mathcal{C}_{\theta\alpha s}\bar{\sigma}_{2\alpha s} - \mathcal{E}_{\vartheta\alpha}^T\bar{b}_{\theta\alpha s} \leq\leq 0, \quad \mathcal{L}_{\theta\alpha t} - \bar{c}_{\theta\alpha t} \leq\leq 0 \tag{32}$$
$$\bar{a}_{\theta\alpha s} + \varsigma_{\theta\alpha s}e_s \geq\geq 0 \tag{33}$$

where $\varsigma_{\theta\alpha} = [\varsigma_{\theta\alpha1}\ \varsigma_{\theta\alpha2}\ \cdots\ \varsigma_{\theta\alpha n}]^T$ and $\bar{\sigma}_{2\alpha} = [\bar{\sigma}_{2\alpha1}\ \bar{\sigma}_{2\alpha2}\ \cdots\ \bar{\sigma}_{2\alpha n}]^T$, then system (22) realizes positivity and SSY with $\mathcal{L}_1$ performance satisfying $\mathcal{E}\{\int_0^\infty ||\hat{e}(s)||_1 ds\} \leq \gamma\mathcal{E}\{\int_0^\infty ||\eta(s)||_1 ds\}$ when $\hat{\xi}(0) = 0$. Moreover, the upper-bounding filter parameters are given as

$$\bar{\mathcal{A}}_{\theta\alpha} = \begin{bmatrix} \bar{\sigma}_{2\alpha1}^{-1}\bar{a}_{\theta\alpha1} & \bar{\sigma}_{2\alpha2}^{-1}\bar{a}_{\theta\alpha2} & \cdots & \bar{\sigma}_{2\alpha n}^{-1}\bar{a}_{\theta\alpha n} \end{bmatrix}^T$$
$$\bar{\mathcal{B}}_{\theta\alpha} = \begin{bmatrix} \bar{\sigma}_{2\alpha1}^{-1}\bar{b}_{\theta\alpha1} & \bar{\sigma}_{2\alpha2}^{-1}\bar{b}_{\theta\alpha2} & \cdots & \bar{\sigma}_{2\alpha n}^{-1}\bar{b}_{\theta\alpha n} \end{bmatrix}^T$$
$$\bar{\mathcal{C}}_{\theta\alpha} = \begin{bmatrix} \bar{c}_{\theta\alpha1} & \bar{c}_{\theta\alpha2} & \cdots & \bar{c}_{\theta\alpha q} \end{bmatrix}^T. \tag{34}$$

Proof: From (33), we obtain that $\bar{\mathcal{A}}_{\theta\alpha}$ is the Metzler matrix. From (34), it implies that $\bar{\mathcal{B}}_{\theta\alpha} \geq\geq 0$ and $\bar{\mathcal{C}}_{\theta\alpha} \geq\geq 0$, which means positivity of the filter (20).

From (31) and (32), we have

$$\bar{\mathcal{B}}_{\theta\alpha}\mathcal{D}_{\vartheta\alpha} + \bar{\mathcal{A}}_{\theta\alpha} - \mathcal{A}_{\theta\alpha} \geq\geq 0$$
$$\bar{\mathcal{B}}_{\theta\alpha}\mathcal{E}_{\vartheta\alpha} + \bar{\mathcal{C}}_{\theta\alpha} - \mathcal{C}_{\theta\alpha} \geq\geq 0, \quad \bar{\mathcal{C}}_{\theta\alpha} - \mathcal{L}_{\theta\alpha} \geq\geq 0. \tag{35}$$

Together with Metzler matrices $\mathcal{A}_{\theta\alpha}$ and $\bar{\mathcal{A}}_{\theta\alpha}$, (34) implies positivity of system (22). Then, letting $\bar{\sigma}_\alpha = \begin{bmatrix} \bar{\sigma}_{1\alpha}^T & \bar{\sigma}_{2\alpha}^T \end{bmatrix}^T$ and substituting $\bar{\mathcal{A}}_{\xi i\vartheta\alpha}$, $\bar{\mathcal{B}}_{\xi i\vartheta\alpha}$, and $\bar{\mathcal{C}}_{\xi\theta\alpha}$ into (9), (24), and (25), we can get (27)–(30). ■

Similar to Theorem 3, we can get the fuzzy lower-bounding filter (21) in Theorem 4.

Theorem 4: Consider

$$\mathcal{A}_{\theta\alpha} = \begin{bmatrix} \mathcal{A}_{\theta\alpha1} & \mathcal{A}_{\theta\alpha2} & \cdots & \mathcal{A}_{\theta\alpha n} \end{bmatrix}^T$$
$$\mathcal{C}_{\theta\alpha} = \begin{bmatrix} \mathcal{C}_{\theta\alpha1} & \mathcal{C}_{\theta\alpha2} & \cdots & \mathcal{C}_{\theta\alpha n} \end{bmatrix}^T$$
$$\mathcal{L}_{\theta\alpha} = \begin{bmatrix} \mathcal{L}_{\theta\alpha1} & \mathcal{L}_{\theta\alpha2} & \cdots & \mathcal{L}_{\theta\alpha q} \end{bmatrix}^T$$
$$e_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T, e_2 = \begin{bmatrix} 0 & 1 & \cdots & 0 \end{bmatrix}^T$$
$$,\ldots, e_n = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}^T$$

with $\mathcal{A}_{\theta\alpha s} \in \mathcal{R}^n$, $\mathcal{C}_{\theta\alpha s} \in \mathcal{R}^m$, $\mathcal{L}_{\theta\alpha t} \in \mathcal{R}^n$, $\theta = 1, 2, \ldots, \varpi$, $s = 1, 2, \ldots, n$, $t = 1, 2, \ldots, q$ $\forall\alpha \in \Pi$. If there exist $\underline{\sigma}_{1\alpha} \in \mathcal{R}^n_+$, $\underline{\sigma}_{2\alpha} \in \mathcal{R}^n_+$, $\varsigma_{\theta\alpha} \in \mathcal{R}^n_+$, $\underline{a}_{\theta\alpha s} \in \mathcal{R}^n$, $\underline{b}_{\theta\alpha s} \in \mathcal{R}^p_+$, and $\underline{c}_{\theta\alpha t} \in \mathcal{R}^n_+$ $\forall\alpha \in \wp$, such that

$$\mathcal{A}^T_{\theta\alpha}\underline{\sigma}_{1\alpha} - \sum_{s=1}^{n}\underline{a}_{\theta\alpha s} - \mathcal{D}^T_{\vartheta\alpha}\sum_{s=1}^{n}\underline{b}_{\theta\alpha s} + \mathcal{A}^T_{\theta\alpha}\underline{\sigma}_{2\alpha} + \sum_{\beta=1}^{\Omega}\varrho_{\alpha\beta}\underline{\sigma}_{1\beta} - \sum_{t=1}^{q}\underline{c}_{\theta\alpha t} + \mathcal{L}^T_{\theta\alpha}\mathbf{1}_q << 0 \quad (36)$$

$$\sum_{s=1}^{n}\underline{a}_{\theta\alpha s} + \sum_{\beta=1}^{\Omega}\varrho_{\alpha\beta}\underline{\sigma}_{2\beta} + \sum_{t=1}^{q}\underline{c}_{\theta\alpha t} << 0 \quad (37)$$

$$\mathcal{C}^T_{\theta\alpha}\underline{\sigma}_{1\alpha} - \mathcal{E}^T_{\vartheta\alpha}\sum_{s=1}^{n}\underline{b}_{\theta\alpha s} + \mathcal{C}^T_{\theta\alpha}\underline{\sigma}_{2\alpha} - \gamma\mathbf{1}_m << 0 \quad (38)$$

$$-\mathcal{A}_{\theta\alpha s}\underline{\sigma}_{2\alpha s} + \underline{a}_{\theta\alpha s} + \mathcal{D}^T_{\vartheta\alpha}\underline{b}_{\theta\alpha s} \leq\leq 0 \quad (39)$$

$$-\mathcal{C}_{\theta\alpha s}\underline{\sigma}_{2\alpha s} + \mathcal{E}^T_{\vartheta\alpha}\underline{b}_{\theta\alpha s} \leq\leq 0, \quad -\mathcal{L}_{\theta\alpha t} + \underline{c}_{\theta\alpha t} \leq\leq 0 \quad (40)$$

$$\underline{a}_{\theta\alpha s} + \varsigma_{\theta\alpha s}e_s \geq\geq 0 \quad (41)$$

where $\varsigma_{\theta\alpha} = [\varsigma_{\theta\alpha 1} \ \varsigma_{\theta\alpha 2} \ \cdots \ \varsigma_{\theta\alpha n}]^T$ and $\underline{\sigma}_{2\alpha} = [\underline{\sigma}_{2\alpha 1} \ \underline{\sigma}_{2\alpha 2} \ \cdots \ \underline{\sigma}_{2\alpha n}]^T$, then system (23) realizes positivity and SSY with $\mathcal{L}_1$ performance satisfying $\mathcal{E}\{\int_0^\infty ||\breve{e}(s)||_1 ds\} \leq \gamma\mathcal{E}\{\int_0^\infty ||\eta(s)||_1 ds\}$ when $\breve{\xi}(0) = 0$. Moreover, the lower-bounding filter parameters are given as

$$\begin{aligned}\underline{\mathcal{A}}_{\theta\alpha} &= \left[\underline{\sigma}^{-1}_{2\alpha 1}\underline{a}_{\theta\alpha 1} \quad \underline{\sigma}^{-1}_{2\alpha 2}\underline{a}_{\theta\alpha 2} \quad \cdots \quad \underline{\sigma}^{-1}_{2\alpha n}\underline{a}_{\theta\alpha n}\right]^T\\ \underline{\mathcal{B}}_{\theta\alpha} &= \left[\underline{\sigma}^{-1}_{2\alpha 1}\underline{b}_{\theta\alpha 1} \quad \underline{\sigma}^{-1}_{2\alpha 2}\underline{b}_{\theta\alpha 2} \quad \cdots \quad \underline{\sigma}^{-1}_{2\alpha n}\underline{b}_{\theta\alpha n}\right]^T\\ \underline{\mathcal{C}}_{\theta\alpha} &= \left[\underline{c}_{\theta\alpha 1} \quad \underline{c}_{\theta\alpha 2} \quad \cdots \quad \underline{c}_{\theta\alpha q}\right]^T. \end{aligned} \quad (42)$$

Remark 10: In Section III, sufficient conditions are proposed for SSY with $\mathcal{L}_1$ performance of system (6). In Section IV, the positive $\mathcal{L}_1$ fuzzy upper-bounding filter and lower-bounding filter are constructed to obtain the overall error systems (22) and (23). Then, Corollary 1 for SSY with $\mathcal{L}_1$ performance of system (22) is derived via Theorem 2. Furthermore, substituting the parameters of (22) into Corollary 1 results in the solution for $\mathcal{L}_1$ fuzzy upper-bounding filter (20) in Theorem 3. Similarly, we directly obtain the solution for positive $\mathcal{L}_1$ fuzzy lower-bounding filter (21) in Theorem 4.

Remark 11: In this article, linear programming is developed to solve the filter gains. It is noted that all the computations are offline, and so with the help of existing convex optimization softwares, it is not difficult to solve the corresponding linear programming. Moreover, the computation complexity will also increase along with increasing the size of linear programming. Therefore, it is important to select the tradeoff value between the size of linear programming and the system performance such that the optimal results for the computational burden can be obtained. In addition, the linear programming in Theorems 3 and 4 is feasible, which can be illustrated by the practical epidemiological model in Section V.

Remark 12: For the filter design of S-MJSs, it is an old topic without taking positive constraint into account [52], [53]. However, the $\mathcal{L}_1$ filter is first investigated for nonlinear phase-type S-MJSs subject to positive constraint, which becomes a new topic worth studying. Different from the traditional quadratic Lyapunov method combining with linear matrix inequality technique [52], [53], linear Lyapunov function combining with the linear programming technique is proposed to get the solution for filter parameters in this article, which can make full use of the features of positive systems and save the operational time. Moreover, the upper-bounding filter and lower-bounding filter are constructed to focus on transient performance. Due to the constraint conditions of positivity and the transient performance, it is full of challenges to investigate this kind of system.

Remark 13: SSY is first investigated for phase-type S-MJSs in [43], and then extended to sliding-mode control [48]. In this article, with the help of the supplementary variable and the plant transformation technique [43], [48], phase-type S-MJSs are transformed into MJSs. However, there are three key differences between this article and [48]. First, the considered problems are quite different, that is, this article is concerned with filter design for nonlinear phase-type S-MJSs while [48] studied the sliding-mode control for S-MJSs. Second, one highlight of our work is a positive constraint; this is not involved in [48]. Third, the investigated plant is composed of nonlinear subsystems, while only linear subsystems are considered in [48].

Remark 14: In this article, the TR is completely known, which may have some conservatism. However, in practice, it is difficult to obtain such available knowledge for TR, and the cost is probably huge. Therefore, it is of great significance to carry out the research for phase-type S-MJSs under the framework of incomplete TR. How to built SSY criteria for phase-type S-MJSs with incomplete TR is an interesting topic for future study.

V. Case Study

In this section, a practical application to the epidemiological model will be given to show the effectiveness of the theoretical findings. With the development of modern science and technology, various epidemiological viruses, such as the SARS virus, Ebola virus, and Covid-19, are endangering the health of mankind and bringing immeasurable loss to human society. Thus, it is of great significance to carry out epidemiological research for the prevention, control, and eradication of diseases.

In practical systems, there always exists the random parameters change, which will lead to a sudden change of the parameters or structures of the system. Positive S-MJSs play an important part in the analysis and synthesis of the epidemiological model. Due to the number of infectives and susceptives satisfying non-negative property and the abrupt change of living environment, the epidemiological model can be described by nonlinear positive S-MJSs.

Consider an epidemiological model from [34] as

$$\dot{x}_\chi(t) = \left(1 - x_\chi(t)\right)\sum_{\pi=1}^{n}\frac{\nu_{\chi\pi}\mathcal{R}_\pi}{\mathcal{R}_\chi}x_\pi(t) - \left(\rho_\chi + \varsigma_\chi\right)x_\chi(t) \quad (43)$$

where n is the number of groups, $\nu_{\chi\pi}$ is the rate standing for susceptibles of group χ infected by infectives of group π, ρ_χ is the rate standing for an infective individual cured, and ς_χ is the death rate of group χ. $x_\chi(t) = (\mathcal{Z}_\chi(t))/(\mathcal{R}_\chi)$, $\mathcal{Z}_\chi(t) + \mathcal{V}_\chi(t) = \mathcal{R}_\chi$, where the total number $\mathcal{R}_\chi$ is constant and $\mathcal{Z}_\chi(t)$ and $\mathcal{V}_\chi(t)$ are the numbers of infectives and susceptives, and $0 \leq x_\chi(t) \leq 1$.

For the epidemiological model, there exist some random factors, such as random circumstance variation. The phase-type SMP $\hat{\omega}_t$ is adopted to describe the SP in $\mathcal{S} = \{1, 2\}$. For the first state, the ST is subject to a negative exponential distribution with parameter λ_1. For the second state, the ST is subject to a two-order Erlang distribution. During the second state, the ST is divided into two parts and follows negatively exponentially distribution with parameters λ_2 and λ_3. We assume that $\pi_{12} = \pi_{21} = 1$. Obviously, one has

$$\mathcal{P} = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \delta^1 = \left(\delta_1^1\right) = (1)$$
$$\mathcal{Q}^1 = \left(\mathcal{Q}_{11}^1\right) = (-\lambda_1), \delta^2 = \left(\delta_1^2, \delta_2^2\right) = (1, 0)$$
$$\mathcal{Q}^2 = \begin{bmatrix} \mathcal{Q}_{11}^2 & \mathcal{Q}_{12}^2 \\ \mathcal{Q}_{21}^2 & \mathcal{Q}_{22}^2 \end{bmatrix} = \begin{bmatrix} -\lambda_2 & \lambda_2 \\ 0 & -\lambda_3 \end{bmatrix}.$$

It is easy to know that the state space of $\mathcal{O}(t) = (\hat{\omega}_t, \mathcal{J}(t))$ is $\mathcal{G} = ((1, 1), (2, 1), (2, 2))$. Furthermore, all the elements of $\mathcal{G}$ are listed as

$$\Upsilon((1, 1)) = 1, \Upsilon((2, 1)) = 2, \Upsilon((2, 2)) = 3.$$

Hence, the IG of $\Upsilon(\mathcal{O}(t))$ is

$$\begin{bmatrix} -\lambda_1 & \lambda_1 & 0 \\ 0 & -\lambda_2 & \lambda_2 \\ \lambda_3 & 0 & -\lambda_3 \end{bmatrix}. \tag{44}$$

Define $\omega_t = \Upsilon(\mathcal{O}(t))$. Then, ω_t is the associated MP of $\hat{\omega}_t$ in $\{1, 2, 3\}$ and the IG of ω_t is proposed in (44). Therefore, when $n = 2$, phase-type S-MJSs can be expressed by the associated MJSs as

$$\begin{aligned} \dot{x}_1(t) = \Bigg(&\nu_{11}(\omega_t) - (\rho_1 + \varsigma_1(\omega_t))x_1(t) + \frac{\nu_{12}(\omega_t)\mathcal{R}_2}{\mathcal{R}_1}x_2(t) \\ &-\nu_{11}(\omega_t)x_1^2(t) - \frac{\nu_{12}(\omega_t)\mathcal{R}_2}{\mathcal{R}_1}x_1(t)x_2(t) + b_1\eta(t) \\ \dot{x}_2(t) = \Bigg(&\nu_{22}(\omega_t) - (\rho_2 + \varsigma_2(\omega_t))x_2(t) + \frac{\nu_{21}(\omega_t)\mathcal{R}_1}{\mathcal{R}_2}x_1(t) \\ &-\nu_{22}(\omega_t)x_2^2(t) - \frac{\nu_{21}(\omega_t)\mathcal{R}_1}{\mathcal{R}_2}x_1(t)x_2(t) + b_2\eta(t) \\ y(t) = \ & x_1(t) + x_2(t) \end{aligned} \tag{45}$$

where $\eta(t)$ and $y(t)$ mean the disturbance signal and output signal. For $\omega_t = \alpha \in \wp$, $\nu_{11}(\omega_t)$, $\nu_{12}(\omega_t)$, $\nu_{21}(\omega_t)$, $\nu_{22}(\omega_t)$, $\varsigma_1(\omega_t)$, and $\varsigma_2(\omega_t)$ are, respectively, denoted as $\nu_{11\alpha}$, $\nu_{12\alpha}$, $\nu_{21\alpha}$, $\nu_{22\alpha}$, $\varsigma_{1\alpha}$, and $\varsigma_{2\alpha}$.

Next, the T–S fuzzy model is adopted to describe system (45).

Choose $\phi_1(t) = x_1(t)$ and $\phi_2(t) = x_2(t)$ with the following.

Rule 1: IF $\phi_1(t) = 0$ and $\phi_2(t) = 0$, THEN

$$\dot{x}(t) = \mathcal{A}_{1\alpha}x(t) + \mathcal{C}_{1\alpha}\eta(t), y(t) = \mathcal{D}_{1\alpha}x(t).$$

Rule 2: IF $\phi_1(t) = 0$ and $\phi_2(t) = 1$, THEN

$$\dot{x}(t) = \mathcal{A}_{2\alpha}x(t) + \mathcal{C}_{2\alpha}\eta(t), y(t) = \mathcal{D}_{2\alpha}x(t).$$

Rule 3: IF $\phi_1(t) = 1$ and $\phi_2(t) = 0$, THEN

$$\dot{x}(t) = \mathcal{A}_{3\alpha}x(t) + \mathcal{C}_{3\alpha}\eta(t), y(t) = \mathcal{D}_{3\alpha}x(t).$$

Rule 4: IF $\phi_1(t) = 1$ and $\phi_2(t) = 1$, THEN

$$\dot{x}(t) = \mathcal{A}_{4\alpha}x(t) + \mathcal{C}_{4\alpha}\eta(t), y(t) = \mathcal{D}_{4\alpha}x(t)$$

where

$$\mathcal{A}_{1\alpha} = \begin{bmatrix} \nu_{11\alpha} - (\rho_1 + \varsigma_{1\alpha}) & \frac{\nu_{12\alpha}W_2}{W_1} \\ \frac{\nu_{21\alpha}W_1}{W_2} & \nu_{22\alpha} - (\rho_2 + \varsigma_{2\alpha}) \end{bmatrix}$$
$$\mathcal{A}_{2\alpha} = \begin{bmatrix} \nu_{11\alpha} - (\rho_1 + \varsigma_{1\alpha}) & \frac{\nu_{12\alpha}W_2}{W_1} \\ 0 & -(\rho_2 + \varsigma_{2\alpha}) \end{bmatrix}$$
$$\mathcal{A}_{3\alpha} = \begin{bmatrix} -(\rho_1 + \varsigma_{1\alpha}) & 0 \\ \frac{\nu_{21\alpha}W_1}{W_2} & \nu_{22\alpha} - (\rho_2 + \varsigma_{2\alpha}) \end{bmatrix}$$
$$\mathcal{A}_{4\alpha} = \begin{bmatrix} -(\rho_1 + \varsigma_{1\alpha}) & 0 \\ 0 & -(\rho_2 + \varsigma_{2\alpha}) \end{bmatrix}$$
$$\mathcal{C}_{1\alpha} = \mathcal{C}_{2\alpha} = \mathcal{C}_{3\alpha} = \mathcal{C}_{4\alpha} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$
$$\mathcal{D}_{1\alpha} = \mathcal{D}_{2\alpha} = \mathcal{D}_{3\alpha} = \mathcal{D}_{4\alpha} = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

Then, we can get the normalized membership functions $h_1(\phi(t)) = (1-\phi_1(t))(1-\phi_2(t))$, $h_2(\phi(t)) = \phi_2(t)(1-\phi_1(t))$, $h_3(\phi(t)) = \phi_1(t)(1-\phi_2(t))$, $h_4(\phi(t)) = \phi_1(t)\phi_2(t)$, and

$$\begin{aligned} \dot{x}(t) &= \sum_{\theta=1}^{4} p_\theta(\phi(t))(\mathcal{A}_{\theta\alpha}x(t) + \mathcal{C}_{\theta\alpha}\eta(t)) \\ y(t) &= \sum_{\theta=1}^{4} p_\theta(\phi(t))\mathcal{D}_{\theta\alpha}x(t). \end{aligned} \tag{46}$$

In this example, we design a pair of positive $\mathcal{L}_1$ fuzzy filters to realize positivity and SSY with $\mathcal{L}_1$ performance index. For given $\nu_{111} = 0.4$, $\nu_{121} = \nu_{211} = 0.2$, $\nu_{221} = 0.3$, $\varsigma_{11} = \varsigma_{12} = 0.1$, $\rho_1 = \rho_2 = 0.9$, $\nu_{112} = 0.4$, $\nu_{122} = \nu_{212} = 0.5$, $\nu_{222} = 0.3$, $\varsigma_{21} = \varsigma_{22} = 0.2$, $\mathcal{R}_1 = 200$, $\mathcal{R}_2 = 400$, $b_1 = b_2 = 0.1$, $\gamma = 0.75$, $\lambda_1 = 0.5$, $\lambda_2 = 0.8$, and $\lambda_3 = 1.2$, solving Theorems 3 and 4 results in the upper-bounding filter parameters as

$$\bar{\mathcal{A}}_{11} = \begin{bmatrix} -302.3727 & 121.4154 \\ 122.0028 & -320.5451 \end{bmatrix}, \quad \bar{\mathcal{B}}_{11} = \begin{bmatrix} 94.1069 \\ 97.4527 \end{bmatrix}$$
$$\bar{\mathcal{A}}_{12} = \begin{bmatrix} -282.8843 & 115.4182 \\ 116.6706 & -321.4207 \end{bmatrix}, \quad \bar{\mathcal{B}}_{12} = \begin{bmatrix} 88.8369 \\ 95.9435 \end{bmatrix}$$
$$\bar{\mathcal{A}}_{21} = \begin{bmatrix} -335.5093 & 130.1400 \\ 129.5759 & -317.7431 \end{bmatrix}, \quad \bar{\mathcal{B}}_{21} = \begin{bmatrix} 103.3071 \\ 99.9830 \end{bmatrix}$$
$$\bar{\mathcal{A}}_{22} = \begin{bmatrix} -315.9290 & 128.4350 \\ 128.9693 & -333.1110 \end{bmatrix}, \quad \bar{\mathcal{B}}_{22} = \begin{bmatrix} 99.3677 \\ 102.2645 \end{bmatrix}$$
$$\bar{\mathcal{A}}_{31} = \begin{bmatrix} -333.3204 & 134.6720 \\ 134.7916 & -337.2606 \end{bmatrix}, \quad \bar{\mathcal{B}}_{31} = \begin{bmatrix} 105.3043 \\ 105.9725 \end{bmatrix}$$
$$\bar{\mathcal{A}}_{32} = \begin{bmatrix} -370.7321 & 138.8333 \\ 137.4146 & -317.6937 \end{bmatrix}, \quad \bar{\mathcal{B}}_{32} = \begin{bmatrix} 113.9058 \\ 105.4865 \end{bmatrix}$$
$$\bar{\mathcal{A}}_{41} = \begin{bmatrix} -336.1857 & 134.9032 \\ 134.8754 & -335.0191 \end{bmatrix}, \quad \bar{\mathcal{B}}_{41} = \begin{bmatrix} 105.8201 \\ 105.5870 \end{bmatrix}$$

This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.

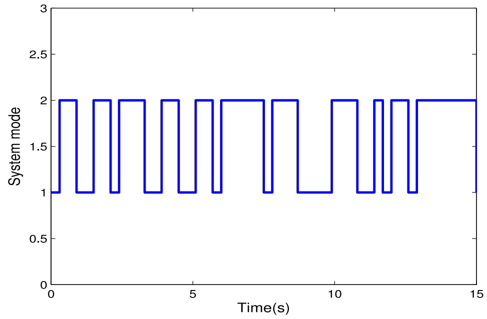

Fig. 1. System mode.

$$\bar{\mathcal{A}}_{42}=\begin{bmatrix}-357.2635 & 139.8319\\ 139.1801 & -332.1714\end{bmatrix},\quad \bar{\mathcal{B}}_{42}=\begin{bmatrix}112.3770\\ 108.7454\end{bmatrix}$$
$$\bar{\mathcal{C}}_{11}=\begin{bmatrix}245.6154 & 245.6154\end{bmatrix},\quad \bar{\mathcal{C}}_{12}=\begin{bmatrix}247.6667 & 247.6667\end{bmatrix}$$
$$\bar{\mathcal{C}}_{21}=\begin{bmatrix}226.4216 & 226.4216\end{bmatrix},\quad \bar{\mathcal{C}}_{22}=\begin{bmatrix}247.6667 & 247.6667\end{bmatrix}$$
$$\bar{\mathcal{C}}_{31}=\begin{bmatrix}226.4216 & 226.4216\end{bmatrix},\quad \bar{\mathcal{C}}_{32}=\begin{bmatrix}247.6667 & 247.6667\end{bmatrix}$$
$$\bar{\mathcal{C}}_{41}=\begin{bmatrix}245.6154 & 245.6154\end{bmatrix},\quad \bar{\mathcal{C}}_{42}=\begin{bmatrix}247.6667 & 247.6667\end{bmatrix}$$

and the lower-bounding filter parameters as

$$\underline{\mathcal{A}}_{11}=\begin{bmatrix}-310.2328 & 123.1140\\ 123.3476 & -318.1023\end{bmatrix},\quad \underline{\mathcal{B}}_{11}=\begin{bmatrix}96.0482\\ 97.3694\end{bmatrix}$$
$$\underline{\mathcal{A}}_{12}=\begin{bmatrix}-286.2784 & 118.1835\\ 119.6304 & -330.1813\end{bmatrix},\quad \underline{\mathcal{B}}_{12}=\begin{bmatrix}90.2895\\ 98.4778\end{bmatrix}$$
$$\underline{\mathcal{A}}_{21}=\begin{bmatrix}-318.2043 & 127.3531\\ 127.5774 & -325.9682\end{bmatrix},\quad \underline{\mathcal{B}}_{21}=\begin{bmatrix}99.0951\\ 100.3563\end{bmatrix}$$
$$\underline{\mathcal{A}}_{22}=\begin{bmatrix}-306.0753 & 128.2436\\ 129.5469 & -347.2548\end{bmatrix},\quad \underline{\mathcal{B}}_{22}=\begin{bmatrix}97.6072\\ 104.9350\end{bmatrix}$$
$$\underline{\mathcal{A}}_{31}=\begin{bmatrix}-340.7930 & 135.7346\\ 135.5236 & -333.5147\end{bmatrix},\quad \underline{\mathcal{B}}_{31}=\begin{bmatrix}107.0646\\ 105.8599\end{bmatrix}$$
$$\underline{\mathcal{A}}_{32}=\begin{bmatrix}-383.4960 & 137.5949\\ 135.4871 & -306.2357\end{bmatrix},\quad \underline{\mathcal{B}}_{32}=\begin{bmatrix}115.2054\\ 102.8026\end{bmatrix}$$
$$\underline{\mathcal{A}}_{41}=\begin{bmatrix}-336.8392 & 135.0682\\ 135.0292 & -335.3160\end{bmatrix},\quad \underline{\mathcal{B}}_{41}=\begin{bmatrix}105.9674\\ 105.6728\end{bmatrix}$$
$$\underline{\mathcal{A}}_{42}=\begin{bmatrix}-357.0106 & 138.6530\\ 137.8872 & -328.7307\end{bmatrix},\quad \underline{\mathcal{B}}_{42}=\begin{bmatrix}111.4045\\ 107.0840\end{bmatrix}$$
$$\underline{\mathcal{C}}_{11}=\begin{bmatrix}245.8530 & 245.8530\end{bmatrix},\quad \underline{\mathcal{C}}_{12}=\begin{bmatrix}248.1580 & 248.1580\end{bmatrix}$$
$$\underline{\mathcal{C}}_{21}=\begin{bmatrix}226.4641 & 226.4641\end{bmatrix},\quad \underline{\mathcal{C}}_{22}=\begin{bmatrix}248.1580 & 248.1580\end{bmatrix}$$
$$\underline{\mathcal{C}}_{31}=\begin{bmatrix}226.4641 & 226.4641\end{bmatrix},\quad \underline{\mathcal{C}}_{32}=\begin{bmatrix}248.1580 & 248.1580\end{bmatrix}$$
$$\underline{\mathcal{C}}_{41}=\begin{bmatrix}245.8530 & 245.8530\end{bmatrix},\quad \underline{\mathcal{C}}_{42}=\begin{bmatrix}248.1580 & 248.1580\end{bmatrix}.$$

For given $\eta(t) = e^{-t}(1-\sin(t))$, $\omega_0 = 1$, $x(0) = \begin{bmatrix}0.6 & 0.8\end{bmatrix}^T$, $\hat{x}(0) = \begin{bmatrix}0.8 & 0.9\end{bmatrix}^T$, and $\check{x}(0) = \begin{bmatrix}0.2 & 0.2\end{bmatrix}^T$, Fig. 1 describes the stochastic switching rule. Figs. 2 and 3 show the state responses $x(t)$ and the estimated states $\hat{x}(t)$, $\check{x}(t)$ of the RCLAS. Fig. 4 plots the error signals $\hat{e}(t)$ and $\check{e}(t)$. From Figs. 2 and 3, the system states are bounded by $\check{x}(t)$ and $\hat{x}(t)$. And also, one can clearly see that $\hat{x}(t)$, $x(t)$, $\check{x}(t)$, $\hat{e}(t)$,

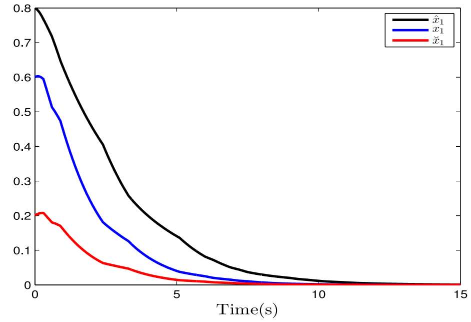

Fig. 2. Estimated state $\hat{x}_1$, system state x_1, and estimated state $\check{x}_1$.

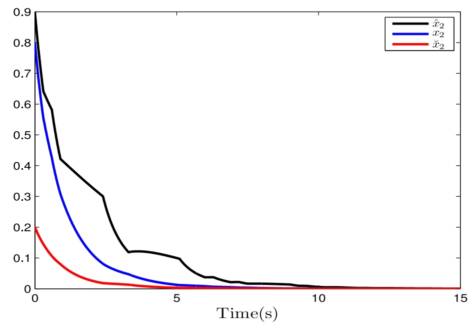

Fig. 3. Estimated state $\hat{x}_2$, system state x_2, and estimated state $\check{x}_2$.

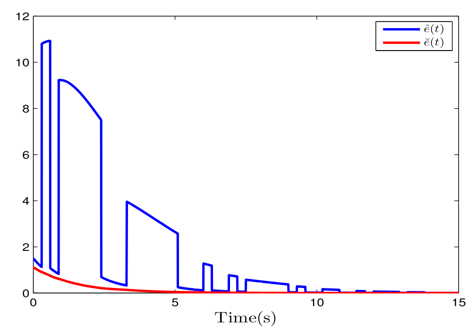

Fig. 4. Estimated errors $\hat{e}(t)$ and $\check{e}(t)$.

and $\check{e}(t)$ could arrive at the equilibrium point, which means that positivity and SSY with $\mathcal{L}_1$ performance can be realized.

Remark 15: For the filter solution, there are several open parameters that should be chosen beforehand. First, the system parameters $\nu_{11\alpha}$, $\nu_{12\alpha}$, $\nu_{21\alpha}$, $\nu_{22\alpha}$, $\varsigma_{1\alpha}$, $\varsigma_{2\alpha}$, $\mathcal{R}_1$, $\mathcal{R}_2$, b_1, and b_2 are known *a priori* according to the practical situation. Second, the transition rate parameters λ_1, λ_2, and λ_3 are selected according to the practical statistical measurement. Finally, for given $\mathcal{L}_1$-gain performance γ, we can get the filter gains $\bar{\mathcal{A}}_{\theta\alpha}$, $\bar{\mathcal{B}}_{\theta\alpha}$, $\bar{\mathcal{C}}_{\theta\alpha}$, $\underline{\mathcal{A}}_{\theta\alpha}$, $\underline{\mathcal{B}}_{\theta\alpha}$, and $\underline{\mathcal{C}}_{\theta\alpha}$ by finding feasible solutions of vectors $\bar{\sigma}_{2\alpha}$, $\bar{a}_{\theta\alpha s}$, $\bar{b}_{\theta\alpha s}$, $\bar{c}_{\theta\alpha t}$, $\underline{\sigma}_{2\alpha}$, $\underline{a}_{\theta\alpha s}$, $\underline{b}_{\theta\alpha s}$, and $\underline{c}_{\theta\alpha t}$ $\forall\alpha \in \wp$. During the whole solution process, the above parameters are given in advance without line search.

Remark 16: Compared with [34], this article is concerned with the filter design for S-MJSs while the observer design is studied for positive MJSs [34]. Second, the epidemiological model is described as positive MJSs that belong to the simple stochastic switching systems [34]. However, considering the complex stochastic living environment, the epidemiological model is described as S-MJSs to make the modeling process more accordant with practical circumstances in this article.

Remark 17: In the above simulation part, we give a detailed modeling process of the epidemiological model that illustrates the effectiveness of the proposed filter design. However, it is based on the positive filter design theory analysis without taking the real experimental test into account. Unfortunately, we have not carried out a real experimental test in the epidemiological model. How to build the real experimental test for the epidemiological model to show the feasibility of filter design is an interesting topic in future works.

VI. Conclusion

In this article, we have investigated positive $\mathcal{L}_1$ fuzzy filter design for positive nonlinear stochastic switching systems subject to phase-type semi-Markov jump parameters. By the use of LCLF, sufficient conditions for positivity and SSY with $\mathcal{L}_1$ performance index are proposed via the T–S fuzzy method. A pair of positive $\mathcal{L}_1$ filters design problem has been addressed in standard linear programming. Moreover, for reducing the occupancy of network bandwidth resources, filter design for positive S-MJSs via the event-triggered communication scheme is significant in future works.

A drawback is that the positive filter does not guarantee SSY with $\mathcal{L}_1$ performance of T–S fuzzy S-MJSs under stochastic disturbance, which may be an issue under some circumstances. How to apply the proposed method to positive filter design under stochastic disturbance is an important subject that requires further investigation.

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