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Filtering of Dispersed Wavetrains*

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Summary

The filtering problem with dispersed signals is examined and an equation relating the duration of the filtered signal to the bandwidth of the filter and the dispersiveness of the dispersing system is established. This equation is applied in two domains: one, the optimization of the multiple filter technique, a method used for a time–frequency analysis of a signal; and two, the choice of parameters in a 'time variable filter', a technique used to isolate from a signal a wavetrain related to a dispersion curve. We give an application of this 'time variable filter' technique to isolate the fundamental mode of Rayleigh waves from a record of an earthquake 650 km deep.

1. Filtering of a dispersed signal

(a) *Dispersion curve analysis*

A dispersed wavetrain $g(t)$ may be written as a sum of sinusoidal waves that propagate in one dimension with a speed related to their frequency. We suppose phase is zero when $x = 0$, so

$$g(t) = \int_{0}^{\infty} G(\omega) \cos\big(K(\omega)\,x - \omega t\big)\, d\omega$$

where ω is an angular frequency, K a wavenumber and x the epicentral distance. $g(t)$ may also be written as the real part of a complex function $f(t)$

$$f(t) = \int_{-\infty}^{+\infty} F(\omega) \exp\big[i\big(K(\omega)\,x - \omega t\big)\big]\, d\omega \tag{1}$$

where $F(\omega) = 0$ for $\omega < 0$ and $F(\omega) = G(\omega)$ for $\omega \geqslant 0$. The imaginary part of $f(t)$ gives a wavetrain that differs in phase by $\pi/2$. We can develop $K(\omega)\,x$ in Taylor series about ω_0 to analyse the form of the dispersion curve:

$$K(\omega)\,x = \sum_{n=0}^{\infty} a_n(\omega - \omega_0)^n \tag{2}$$

where

$$a_n = \frac{x}{n!}\,[K^{(n)}(\omega)]_{\omega_0}.$$

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The parameter a_0 depends only on the phase velocity $c(\omega_0) = \omega_0/K(\omega_0)$ for the angular frequency ω_0

$$a_0 = \omega_0\, x/c(\omega_0). \tag{3}$$

For an arbitrary angular frequency ω, the phase $K(\omega)\,x-\omega t$ is stationary if the time t is equal to

$$t(\omega) = \frac{dKx}{d\omega}(\omega).$$

Thus when $n \geqslant 1$ the parameters a_n are related to the derivatives of $t(\omega)$ with respect to ω at ω_0:

$$a_n = \frac{1}{n!}\left[\frac{d^{n-1}\,t(\omega)}{d\omega^{n-1}}\right]_{\omega_0}. \tag{4}$$

For the first three orders we can also write:

$$a_1 = x/u$$

$$a_2 = \frac{x}{2}\left[\frac{d(1/u)}{d\omega}\right]_{\omega_0}$$

$$a_3 = \frac{x}{6}\left[\frac{d^2(1/u)}{d\omega^2}\right]_{\omega_0}$$

where u is the group velocity. A third-order development is sufficient to express many dispersion curves in seismology, however we continue here our study with the general development.

(b) *Width of a filtered dispersed signal*

Let us examine the effect of filtering near an angular frequency ω_0. Our filter is a gaussian $\exp[-\alpha(\omega-\omega_0)^2]$ with a half width on the $1/e$ level $\Delta\omega = \alpha^{-\frac{1}{2}}$. Passing through the filter, the signal $f(t)$ becomes:

$$f^*_{\omega_0}(t) = \int_{-\infty}^{+\infty} F(\omega)\exp[-\alpha(\omega-\omega_0)^2]\exp[i(K(\omega)\,x-\omega t)]\,d\omega. \tag{5}$$

Our purpose is to find a relation between the width of the function $f^*_{\omega_0}(t)$ and α. We define this width Δ_t by the square root of the second moment of the function with respect to its centre t_0:

$$\Delta_t^2 = \int_{-\infty}^{+\infty} |f^*_{\omega_0}(t)|^2\,(t-t_0)^2\,dt \Big/ \int_{-\infty}^{+\infty} |f^*_{\omega_0}(t)|^2 dt. \tag{6}$$

We suppose the amplitude of the Fourier transform of the dispersed signal remains constant over the filter bandwidth and $\exp(-\alpha\omega_0^2)$ is small enough to neglect the contribution of the branch $]-\infty, 0[$ to the integral (5). So, it can be written:

$$f^*_{\omega_0}(t) \simeq F(\omega_0)\int_{-\infty}^{+\infty} \exp[-\alpha(\omega-\omega_0)^2]\exp[i(K(\omega)\,x-\omega t)]\,d\omega. \tag{7}$$

We put

$$A(\omega) = F(\omega_0)\exp[-\alpha(\omega-\omega_0)^2]$$

and

$$\phi(\omega) = \sum_{n=0}^{\infty} a_n(\omega-\omega_0)^n.$$

Applying Fourier transform to equation (7) we obtain:

$$A(\omega)\exp[i\phi(\omega)] \simeq \frac{1}{2\pi}\int_{-\infty}^{+\infty} f_{\omega_0}^*(t)\exp(i\omega t)\,dt.$$

Let us differentiate each member of this equation with respect to ω:

$$\left[\frac{dA}{d\omega} + iA\frac{d\phi}{d\omega}\right]\exp[i\phi(\omega)] \simeq \frac{1}{2\pi}\int_{-\infty}^{+\infty} itf_{\omega_0}^*(t)\exp(i\omega t)\,dt.$$

We suppose this differentiation introduces no problem when t becomes infinite. Applying inverse Fourier transform we obtain:

$$itf_{\omega_0}^*(t) \simeq \int_{-\infty}^{+\infty}\left[\frac{dA}{d\omega} + iA\frac{d\phi}{d\omega}\right]\exp[i(\phi(\omega)-\omega t)]\,d\omega$$

and then

$$i(t-t_0)f_{\omega_0}^*(t) \simeq \int_{-\infty}^{+\infty}\left[\frac{dA}{d\omega} + iA\left(\frac{d\phi}{d\omega} - t_0\right)\right]\exp[i(\phi(\omega)-\omega t)]\,d\omega.$$

Applying Parseval's relation, we obtain a good approximation for Δ_t^2:

$$\Delta_t^2 \simeq \int_{-\infty}^{+\infty}\left[\left(\frac{dA}{d\omega}\right)^2 + A^2\left(\frac{d\phi}{d\omega} - t_0\right)^2\right]d\omega \Big/ \int_{-\infty}^{+\infty} A^2\,d\omega.$$

If we put $z = \omega-\omega_0$ we have:

$$A = \exp(-\alpha z^2)F(\omega_0);\ \frac{dA}{d\omega} = -2\alpha z\exp(-\alpha z^2)F(\omega_0)$$

$$\phi(\omega) = \sum_{n=0}^{\infty} a_n z^n;\ \frac{d\phi}{d\omega} = \sum_{n=1}^{\infty} na_n z^{n-1}.$$

As $a_1 = t_0$, it follows that

$$\frac{d\phi}{d\omega} - t_0 = \sum_{n=2}^{\infty} na_n z^{n-1}.$$

And we obtain for the quantity

$$Z = \left(\frac{dA}{d\omega}\right)^2 + A^2\left(\frac{d\phi}{d\omega} - t_0\right)^2:$$

$$Z = F^2(\omega_0)\left[4\alpha^2 z^2 + \left(\sum_{n=2}^{\infty} na_n z^{n-1}\right)^2\right]\exp(-2\alpha z^2).$$

By changing the order of the terms in the sum we obtain:

$$Z = F^2(\omega_0)\exp(-2\alpha z^2)\left[4(\alpha^2+a_2^{\,2})z^2+\sum_{n=3}^{\infty} z^n \sum_{\substack{p+q=n+2\\ p,q\geqslant 2}} pqa_p a_q\right].$$

so that

$$Z = F^2(\omega_0)\exp(-2\alpha z^2)\sum_{n=2}^{\infty} b_n z^n,$$

where

$$\left| \begin{array}{l} b_2 = 4(\alpha^2+a_2^{\,2}) \\ b_n = \sum\limits_{\substack{p+q=n+2\\ p,q\geqslant 2}} pqa_p a_q \quad \text{for} \quad n>2 \end{array} \right. \tag{8}$$

a_n being defined in relations (3) and (4). Thus we obtain:

$$\Delta_t^{\,2} \simeq \sum_{n=2}^{\infty} b_n I_n/I_0 \quad \text{with} \quad I_n = \int_{-\infty}^{+\infty} \exp(-2\alpha z^2) z^n\, dz.$$

For odd n, $I_n = 0$ and using the properties of the gamma function

$$\Gamma(u) = \int_0^{\infty} \exp(-x)\, x^{u-1}\, dx$$

(Reif 1965) we have:

$$\Delta_t^{\,2} \simeq \sum_{p=1}^{\infty} b_{2p}(2\alpha)^{-p}\,\Gamma(p+\tfrac{1}{2})/\Gamma(\tfrac{1}{2}) \tag{9}$$

where the b_{2p} are defined in relations (8). In the following sections we apply this result to the analysis of two dispersion curves.

(c) *Monotonous dispersion case*

If $a_n = 0$ when $n \geqslant 3$, the phase term becomes:

$$\phi_t(\omega) = Kx-\omega t = a_0+a_1(\omega-\omega_0)+a_2(\omega-\omega_0)^2-\omega t. \tag{10}$$

To obtain a shorter expression for $\phi_t(\omega)$ we choose $c(\omega_0) = u(\omega_0)$ (equality of the group and phase velocity). By relation (3) and (4), this choice yields $\omega_0 a_1 = a_0$. Since a_0 and a_1 does not appear in the evaluation of Δ_t, this choice is not a restrictive one for our study.

$$\phi_t(\omega) = a_2(\omega-\omega_0)^2 + \omega(t_0-t).$$

With such a dispersion relation we can study the filtered signal evaluating exactly the integral (7) (Inston, Marshall & Blamey 1971). From relations (8) and (9), the width of the filtered signal is given by:

$$\Delta_t^{\,2} = \alpha + a_2^{\,2}/\alpha. \tag{11}$$

From a physical point of view it is easy to understand this dependence of $\Delta_t^{\,2}$ with respect to α. For a narrow filter, the phase term $\phi_t(\omega)$ presents slow variations on the range of definition of the filter and we can take $f^*_{\omega_0}(t)$ as the impulse response of the filter. The width Δ_t of this impulse response is inversely proportional to the filter bandwidth and so $\Delta_t^{\,2}$ is proportional to α. It is the first term of relation (11).

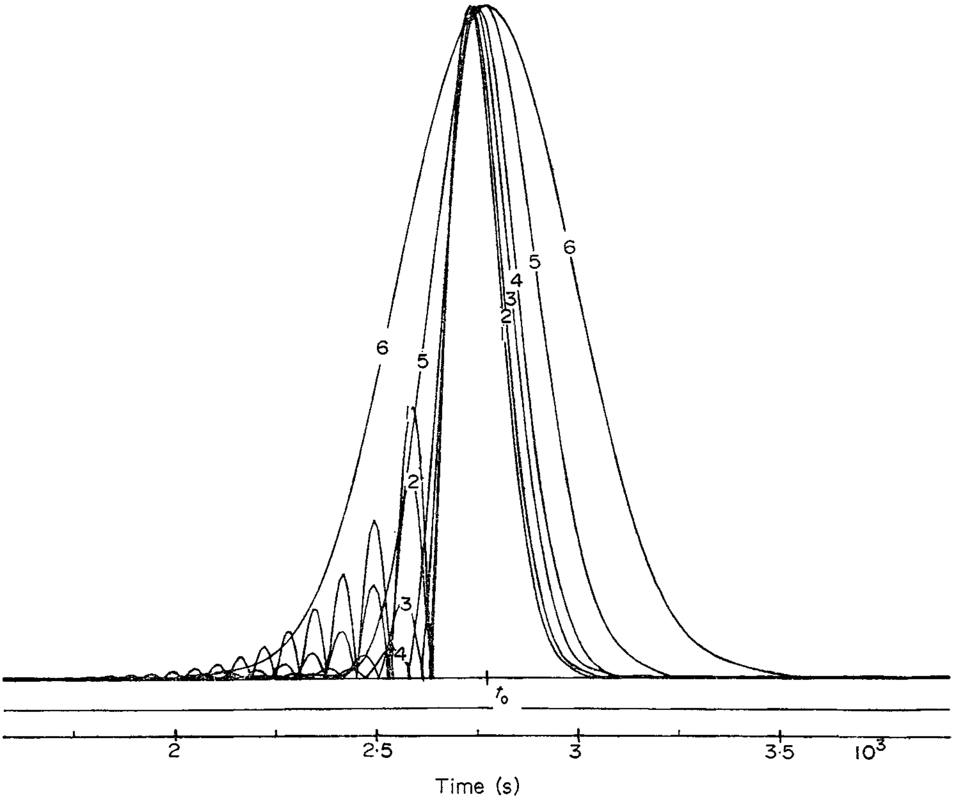

FIG. 1. Modulus of dispersed signal related to relation (14) ($a_3 = -5{\cdot}21\times10^5\ s^3$) and filtered near the angular frequency $\omega_0(2\pi/\omega_0 = 200$ s) with different bandwidths. 30db bandwidth: 1, $9{\cdot}6\times10^{-3}$ Hz; 2, $8{\cdot}0\times10^{-3}$ Hz; 3, $6{\cdot}0\times10^{-3}$ Hz; 4, $5{\cdot}0\times10^{-3}$ Hz; 5, $3{\cdot}3\times10^{-3}$ Hz; 6, 2.0×10^{-3} Hz.

When we pass the signal through a wide filter, we can assume $f^*_{\omega_0}(t)$ is the sum of wave groups of dominant angular frequency ω and arrival time $t(\omega)$, arranged side by side. The relation between $t(\omega)$ and ω is given by the dispersion relation:

$$t(\omega) = t_0+2a_2(\omega-\omega_0).$$

So the filtered signal width Δ_t becomes proportional to the product of the filter bandwidth Δ_ω and a_2. Since $\Delta_\omega = \alpha^{-\frac{1}{2}}$ we have the second term of relation (11), with the exception of the proportionality factor.

The filtered signal width Δ_t presents a minimum when $\alpha = a_2$. If we call $2\Delta_\omega$ the angular frequency interval between points where the amplitude is $1/e$ of the peak amplitude the smallest value of Δ_t occurs when

$$\Delta_\omega = a_2^{-\frac{1}{2}}. \tag{12}$$

This is the relation given by Inston, Marshall & Blamey (1971). The minimum value of Δ_t is

$$\Delta_{t\min} = \sqrt{2}\, a_2^{\frac{1}{2}}. \tag{13}$$

(d) *Extremum group velocity case*

For an extremum of group velocity we take $a_2 = 0$ (first derivative of the group velocity is zero for the central angular frequency) and $a_n = 0$ if $n > 4$.

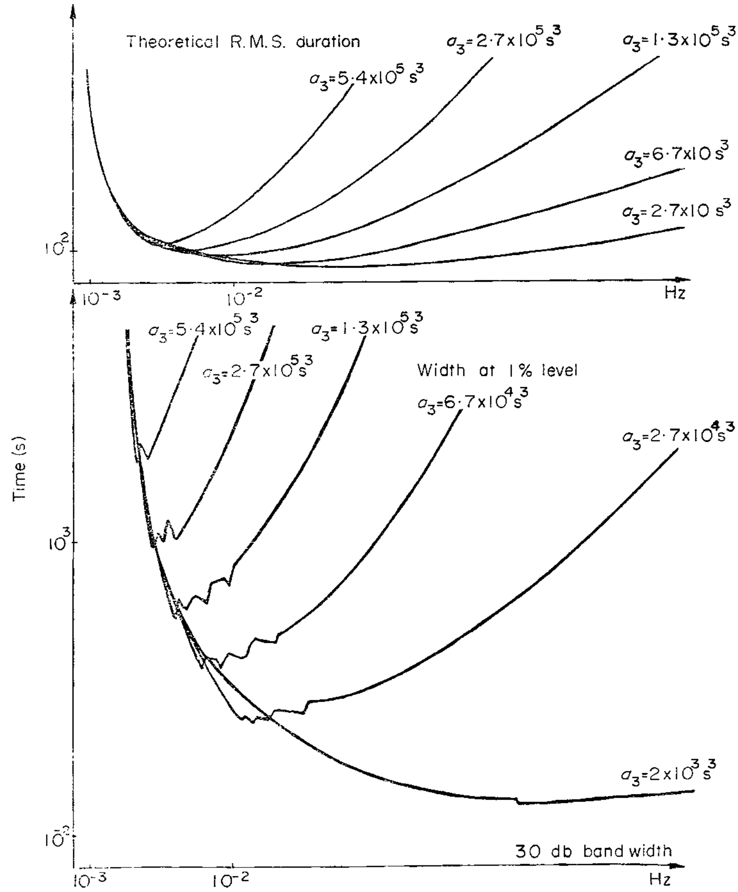

FIG. 2. Root mean square duration from relation (15) and width numerically computed on the 1 per cent level of the modulus of a dispersed signal related to the dispersion relation (14), versus the 30 db bandwidth of the filter.

If we make the same unrestrictive choice as in the Δ_t study discussed before $(c(\omega_0) = u(\omega_0))$, we obtain a shorter expression for $\phi_t(\omega)$:

$$\phi_t(\omega) = a_3(\omega-\omega_0)^3 + \omega(t_0-t). \tag{14}$$

In the case of a white spectrum this dispersion equation gives us an Airy integral.

Filtering a dispersed signal $f(t)$ related to the dispersion relation (14) we obtain a function $f^*_{\omega_0}(t)$. By relations (8) and (9) the width of $f^*_{\omega_0}(t)$ may be written:

$$\Delta_t^2 = \alpha + \tfrac{27}{16} a_3^2 \alpha^{-2}. \tag{15}$$

The same physical point of view as before shows that the first term is the impulse response of the filter and the second term depends on the dispersion of the signal.

When

$$\alpha = \tfrac{3}{2} a_3^{\frac{2}{3}} \tag{16}$$

the width Δ_t presents a minimum:

$$\Delta_{t\,\mathrm{min}} = \tfrac{9}{4} a_3^{\frac{1}{3}}. \tag{17}$$

In order to know better the function $f^*_{\omega_0}(t)$, dispersed according to equation (14), it has been numerically computed. The modulus of the function $f^*_{\omega_0}(t)$ is plotted in Fig. 1 for some bandwidths of the gaussian filter. When the bandwidth increases, $f^*_{\omega_0}(t)$ progressively passes from a gaussian to a function with secondary lobes, the main lobe becomes thinner as secondary lobes grow, and for a very wide bandwidth $f^*_{\omega_0}(t)$ tends to the Airy function.

In Fig. 2 we have taken the width of the functions $|f^*_{\omega_0}(t)|^2$ at the 1 per cent level for some values of a_3 versus the filter bandwidth. The family of curves is smoothed outside the vicinity of the minimum. Discontinuities of the curve occurred when a secondary lobe reaches the 1 per cent level.

2. Multiple filter technique

(a) *General considerations*

Every analysis of a seismic signal in terms of normal modes implies the study of dispersion curves of the recorded wavetrain. Group velocity curves may be drawn on a time–frequency diagram on which the density of energy in the recording signal is plotted. Two methods, the moving window analysis and the multiple filter technique, allow to set up such diagrams. For a complete description of these methods see Dziewonski, Bloch & Landisman (1969), see also Landisman, Dziewonski & Sato (1969). We re-examine the multiple filter technique in which the work is done in the frequency domain.

The recorded seismic signal at a station is a time function $f(t)$. We call $F(\omega)$ its Fourier transform. When searching for the density in the signal versus time for an angular frequency ω_0, the function $f(t)$ is filtered in the vicinity of this angular frequency. A gaussian is chosen for the filter function in the way of concentrating the energy (Papoulis 1962). The filtered spectrum is:

$$F^*_{\omega_0}(\omega) = F(\omega) \exp\left[-\alpha(\omega-\omega_0)^2\right] W_D(\omega-\omega_0) \Bigg/ \int_{-\infty}^{+\infty} \exp\left[-\alpha(\omega-\omega_0)^2\right] d\omega. \tag{18}$$

The filter bandwidth $\alpha^{-\frac{1}{2}}$ depends on the angular frequency ω_0 and $W_D\ (\omega-\omega_0)$ is a boxcar with unit amplitude and limits $\omega_0 - D$, $\omega_0 + D$. The application of this function becomes necessary in numerical computations. The width $2D$ of this function is chosen in such a manner that the gaussian is truncated at a 30 db level. Keeping only positive angular frequencies near ω_0 we construct a complex filtered signal $f^*_{\omega_0}(t)$ with a real and imaginary part differing in phase by $\pi/2$.

$$f^*_{\omega_0}\,(t) = 2\int_0^{\infty} F^*_{\omega_0}\,(\omega) \exp\left(-i\omega t\right) d\omega. \tag{19}$$

These operations are made for many angular frequencies ω_0, and so $|f^*_{\omega_0}(t)|^2$ is the energy density in the signal versus time t and angular frequency ω_0. Note that if the time variation of $|f_{\omega_0}(t)|^2$ well represents a variation of energy density in the signal for an angular frequency ω_0, it is not true for an angular frequency variation of $|f_{\omega_0}(t)|^2$ at a given time t because the filter bandwidth used depends on ω_0.

There is a theoretical limit of the resolution power of such a method, and further, time and frequency resolutions are not independent. If the signal is not dispersed, time resolution continuously increases with filter bandwidth: the impulse response of a gaussian filter with half width at the $1/e$ level equal to $\Delta\omega$ is a gaussian with a root mean square duration $1/\Delta\omega$. It is the classical result of the uncertainty principle relating time and frequency resolution. A choice of a temporal resolution proportional to the period implies filter bandwidths proportional to the frequency. This is the choice made by Dziewonski *et al.* (1968).

(b) *Multiple filtering of dispersed signals*

Our purpose now is to study the choice of filter bandwidth to maximize time resolution in dispersed signal analysis. The preceding developments show that an optimum time resolution is obtained for filters according to the dispersion curve of the signal. So it is necessary first to make a classical analysis of the wavetrain to obtain an idea of the dispersion curve, unless theoretical considerations lead to an approximate dispersion curve. If the dispersion curve is described by a polynomial of order $(n-1)$, the root mean square duration of the signal after passing through a gaussian filter $\exp[-\alpha(\omega-\omega_0)^2]$ is from relation (9):

$$\Delta_t^2 = \alpha + a_2^2\alpha^{-1} + \sum_{p=2}^{n} b_{2p}(2\alpha)^{-p}\,\Gamma(p+\tfrac{1}{2})/\Gamma(\tfrac{1}{2}).$$

When Δ_t is at its minimum, α is a solution of the following equation:

$$1 - \sum_{p=1}^{n} A_p\alpha^{-(p+1)} = 0 \quad \text{with} \quad A_1 = a_2^2 \quad \text{and} \quad A_p = p2^{-p}b_{2p}\Gamma(p+\tfrac{1}{2})/\Gamma(\tfrac{1}{2}). \quad (20)$$

The root of interest may be found by numerical methods.

We can hope for improvements in the time-frequency diagram if the value of α found by resolving the equation (20) greatly differs from the value $\alpha_0 = K\omega^{-2}$ used in classical multiple filtering. Even with the best parameter K, a good resolution is not possible if the slope and curvature of the group velocity curve presents strong variations in the spectral range over which analysis is undertaken. In Fig. 3 and Fig. 4 the results of classical and optimum multiple filtering applied to synthetic signals are presented. These wavetrains are dispersed according to the dispersion relation (14). The value of K has been chosen to have the best mean resolution. For this filtering we have used the fast Fourier transform algorithm (Cooley & Tukey 1965). Computing time is reduced by limiting the number of points in the integral (19) to the power of two immediately greater than the number of points that are inside the range of definition of $F_{\omega_0}(\omega)$, an interpolation of the filtered functions $f_{\omega_0}^*(t)$ is then necessary. Optimum filters have been computed by resolving equation (20) in which we have put $n = 3$. On both figures we can see the K constant method cannot give a good resolution on the whole range of periods.

3. Time variable filtering

(a) *General considerations*

A variable filtering permits us to extract from a complex signal a wavetrain according to a given dispersion curve. Such filtering is necessary when the phase of the extracted wavetrain has to be analysed and seems to be an interesting means for the study of the higher modes free oscillation of the Earth.

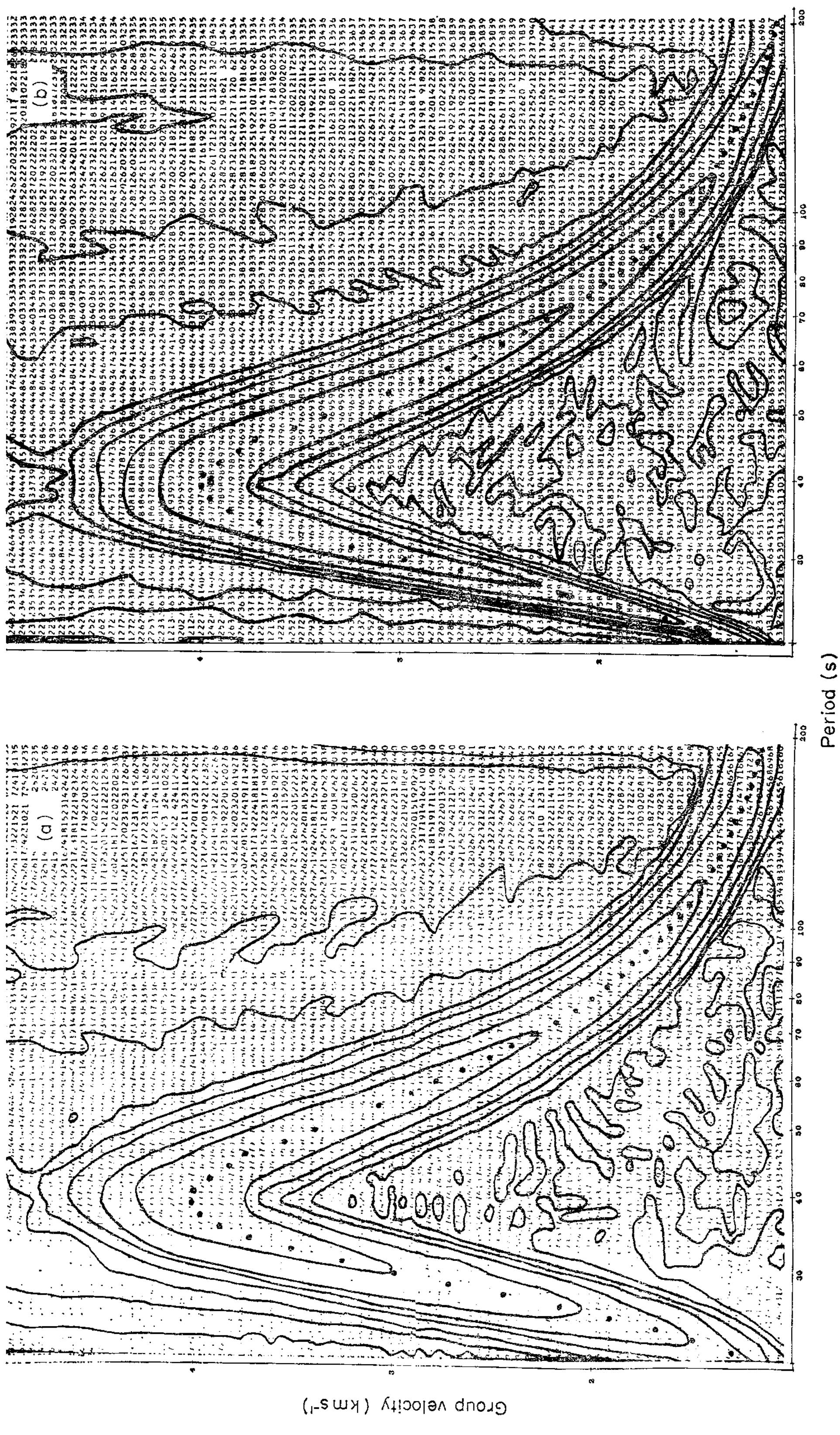

FIG. 3. Results of multiple filtering with constant K method (here, $K = 614$) in (a), and optimized multiple filtering in (b). The used signal is dispersed according to relation (14) where $a_3 = 6{\cdot}75\times10^6\ s^3$, dots are the values of the theoretical group velocity. The energy contours spacing is 10 db.

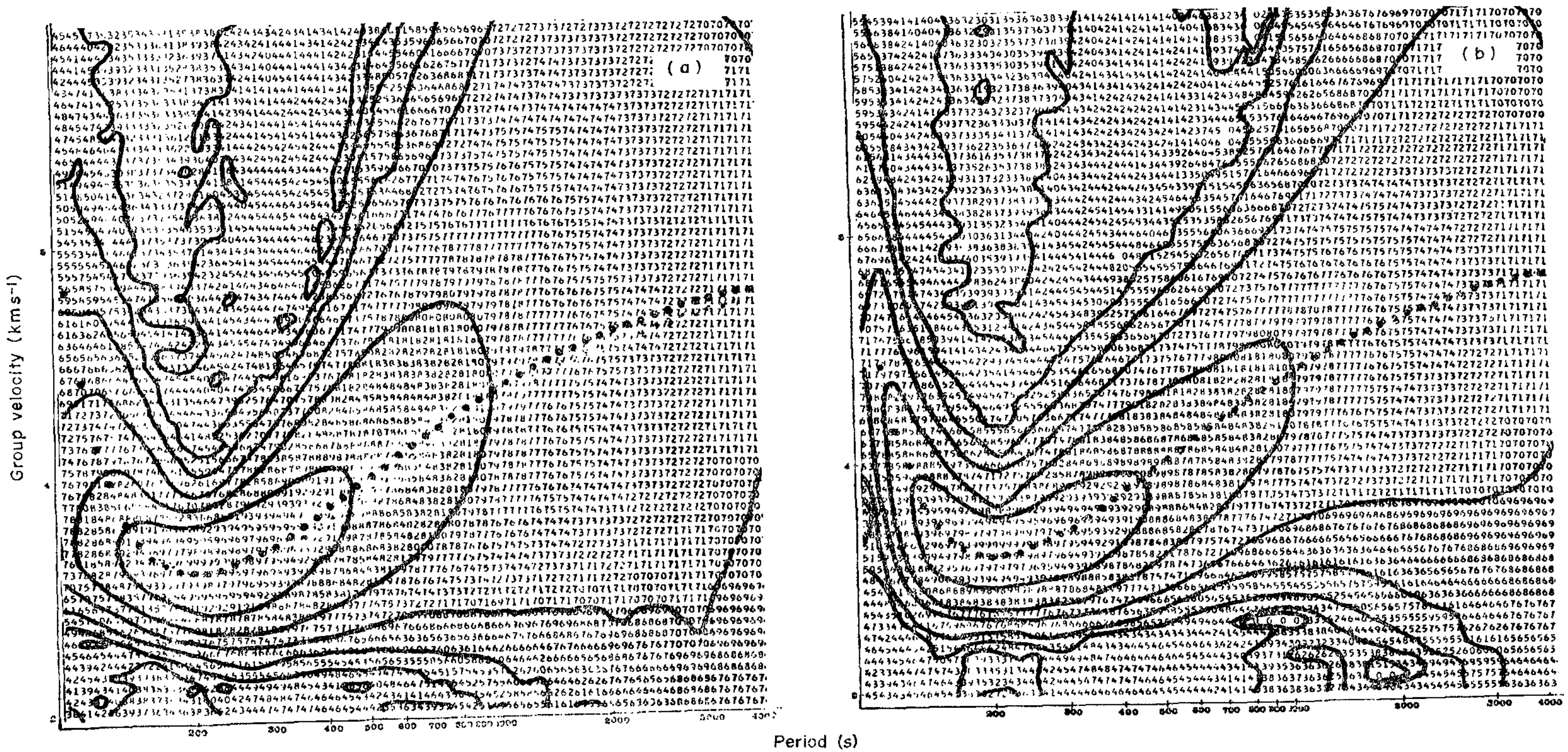

FIG. 4. Results of multiple filtering with constant K method (here, $K = 55$) in (a) and optimized multiple filtering in (b). The used signal is dispersed according to the relation (14) ($a_3 = -5{\cdot}21\times10^5\ s^3$), dots are the values of the theoretical group velocity. For periods greater than 600 s, the 30 db bandwidth of the filters have been taken equal to twice the analysed frequency when the optimum bandwidth of the filter overtakes this value. The energy contours spacing is 10 db.

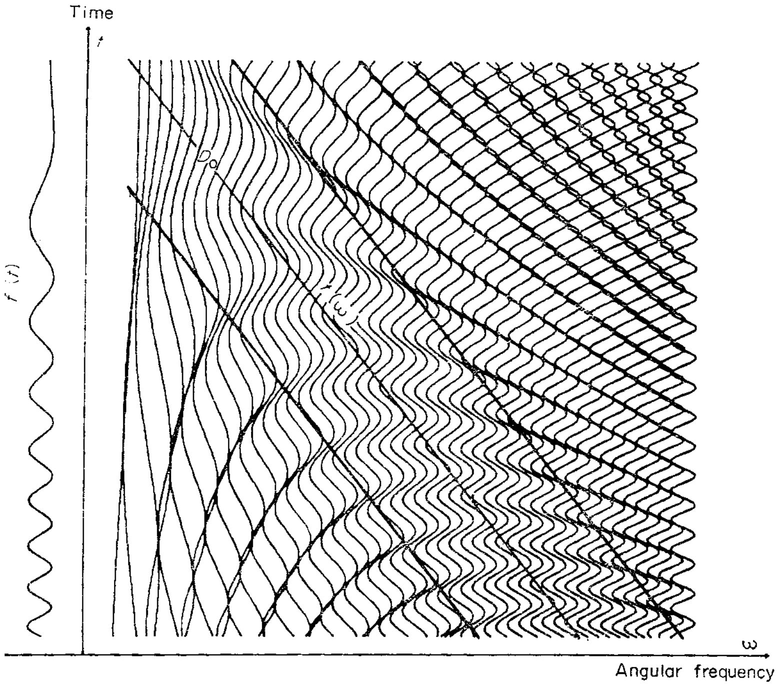

FIG. 5. Illustration of integral (22).

Our purpose is to extract a wavetrain related to a given group velocity curve $u(\omega)$ from a signal $f(t)$. This dispersion data may be written as the arrival times of groups of angular frequency ω

$$t(\omega) = \frac{x}{u(\omega)} \tag{21}$$

where x is the epicentral distance and $u(\omega)$ the group velocity. Again we call $F(\omega)$ the Fourier transform of $f(t)$. Since $f(t)$ is real we can write:

$$f(t) = 2\int_0^\infty |F(\omega)| \cos[\omega t + \phi(\omega)]\, d\omega. \tag{22}$$

For discrete values of the angular frequency, this Fourier integral may be illustrated by a scheme in which some ' Fourier components ' $|F(\omega)| \cos[\omega t + \phi(\omega)]$ are drawn; the sum of these Fourier components giving $f(t)$; this has been shown in Fig. 5. Arrival times of groups of angular frequency ω, $t(\omega)$, are plotted on the diagram. In the vicinity of this curve $t(\omega)$, there is a domain D_0 in which Fourier components constructively interfere. Outside of the domain D_0, we can neglect the contribution of Fourier components to the wavetrain we want to isolate. So we can make a re-synthesis of the wavetrain relating to the dispersion curve $u(\omega)$ by summing the Fourier components truncated at the upper and lower limits of the domain D_0. The wavetrain extracted is:

$$f^*(t) = \int_0^\infty A(\omega, t)|F(\omega)| \cos[\omega t + \phi(\omega)]\, d\omega \tag{23}$$

where $A(\omega, t)$ is a 'filter' in two dimensions that is zero outside of the domain D_0.

A time performing algorithm consists of summing numerically the Fourier components of the signal windowed by the operator

$$B(t) = 0 \quad \text{for} \quad t < t(\omega)-L(\omega); \; t > t(\omega)+L(\omega)$$

$$B(t) = 1-\left[\frac{t-t(\omega)}{L(\omega)}\right]^4 \quad \text{for} \quad t(\omega)-L(\omega) < t < t(\omega)+L(\omega)$$

where $t(\omega)$ is the arrival time of the group of angular frequency and $L(\omega)$ is the height of the domain D_0 for the angular frequency ω. Note that pulsation width $2\Delta_\omega$ of the domain D_0 for an instant t gives the precision with which the extracted wavetrain is reconstructed. In the algorithm presented, Δ_ω depends on $L(\omega)$ by the relation giving $t(\omega)$.

(b) *Choice of the optimum wavelet length*

Our problem is to choose the length L that permits to isolate as well as possible the wavetrain related to a dispersion curve $u(\omega)$ without introducing too large perturbations: short-length wavelets improve the resolution power and long-length wavelets improve the precision with which the wavetrain is reconstructed. Landisman *et al.* (1969) take for L:

$$L = T\left(\alpha+\beta\frac{\partial u}{\partial T}\right) \tag{24}$$

where T is a period, u a group velocity and α, β empirical parameters.

We have re-examined this question for a dispersed signal related to equation (10) in the case of a white spectrum. In our time variable filtering, filters are not gaussian, however a numerical computation on synthetic signals will show that the theoretical result obtained with gaussian filter in Section 1(b) may be applied. Evaluating the integral (7) it can be shown that differences between the analysed wavetrain and the reconstructed one depend only on the product $L\times a_2^{-\frac{1}{2}}$ where a_2 is defined by equation (4). So for small differences we have to choose

$$L = E\left|\; x\,\frac{d(1/u)}{d\omega}\right|^{\frac{1}{2}}$$

where E is a proportionality factor that depends on the precision wanted. Note that this temporal length is the optimum length on which the energy of the signal can be focused after filtering (see relation (13)).

In the case of an extremum of the group velocity, this equation cannot be applied. From equation (17), Fourier components of the signal constructively interfere on a length that becomes proportional to $a_3^{\frac{1}{3}}$. If the group velocity curve presents a plateau or an inflexion point, we have to use a fourth-order development at least. So in the general case we have to choose wavelet lengths proportional to the minimum values of expression (9)

$$L = E\Delta_{t\,\min} \tag{25}$$

where the dimensionless proportionality factor E depends on the precision with which the wavetrain is reconstructed. This parameter is chosen by a numerical study on synthetic wavetrains. We have plotted in Fig. 6 the residual energy $\int_0^\infty |f(t)-f^*(t)|^2\,dt$ in per cent of the analysed signal versus the error parameter E. The wavetrains analysed are related to monotonous dispersion curves (phase term given by equation (10)) and to an 'Airy phase' (phase term given by (14)).

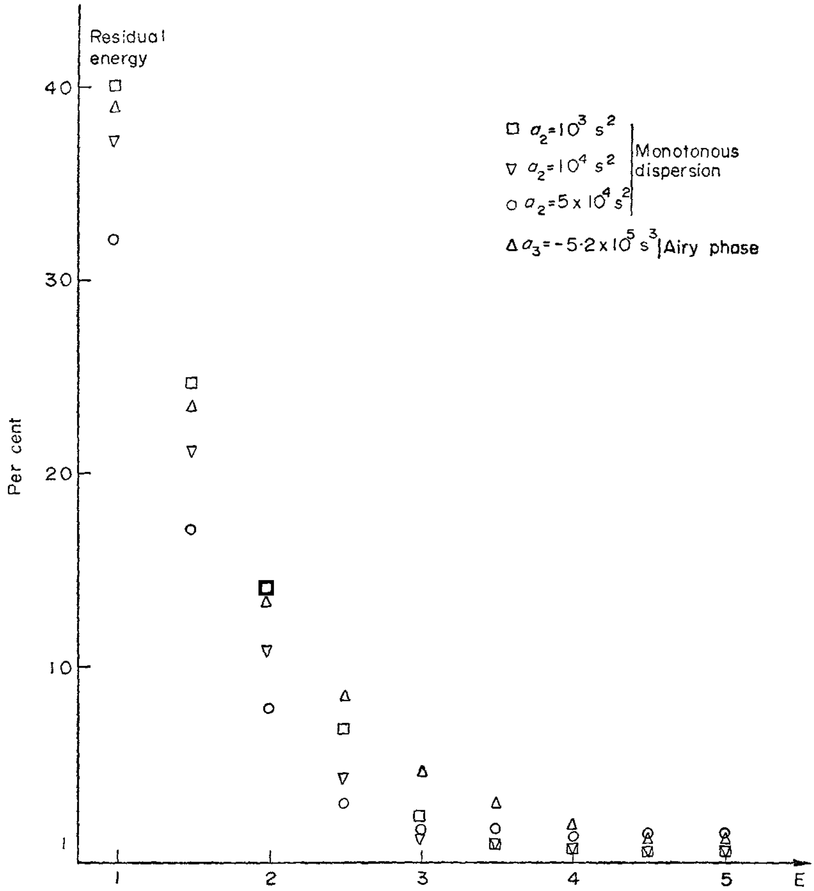

FIG. 6. Residual energy $\int_0^\infty |f^*(t)-f(t)|^2\, dt$ in per cents of the energy of the signal $\int_0^\infty |f(t)|^2\, dt$ with regard to the error parameter E.

(c) *Variable filter technique applied to a Colombia Earthquake.*

We have applied this technique in the way of extracting the fundamental mode of spheroidal vibration from a record of 16h 39m 10s duration of a deep focus earthquake of magnitude 7·1 that occurred in Colombia on 1970 July 31 at 1·5 °S and 72·6 °E at 650 km depth. The seismogram has been recorded in St Marie at 48·20° N and 7·15° W with a modified North American gravimeter. Depth and magnitude of this earthquake are favourable to the excitation of many higher modes. In extracting the fundamental Rayleigh wave we have used a routine of which the principle is discussed in Sections 2(a) and 2(b).

The temporal length $L(\omega)$ has been computed by introducing the root α of the equation (20) in the expression (9) of Δ_t. The group velocity curves used present neither inflexion point nor plateau, so they have been analysed only to third order (terms of greater order than a_3 have not been computed). To extract the 11 wave-trains that are present on the record, we have used three experimental dispersion curves (Direct path R_1, inverse path R_2 and great circle by means of the auto-correlation of the record). The group velocity curves have been studied by means of multiple filters in the range of periods 100–300 s, and the curves have been completed to 1000 s with mean free vibrations data (Gaulon 1971). Sets are multiplied by the

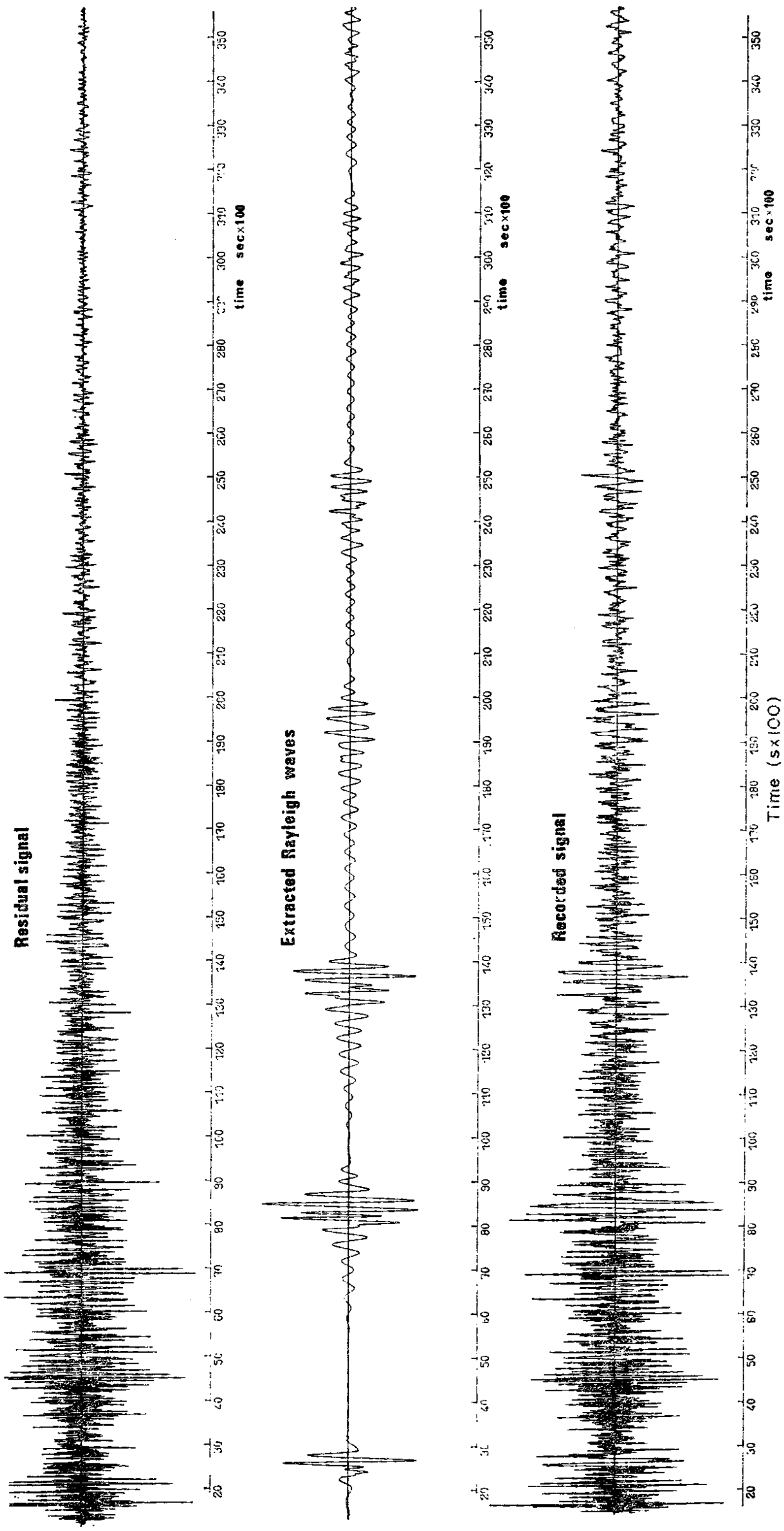

FIG. 7. Vertical seismogram of the Colombia earthquake of 1970 July 31, recorded in St Marie (France) and multiplied by the function $1-[2(t-t_0)/(t_1-t_2)]^4$, (bottom), extracted Rayleigh wave (middle) and residual signal.

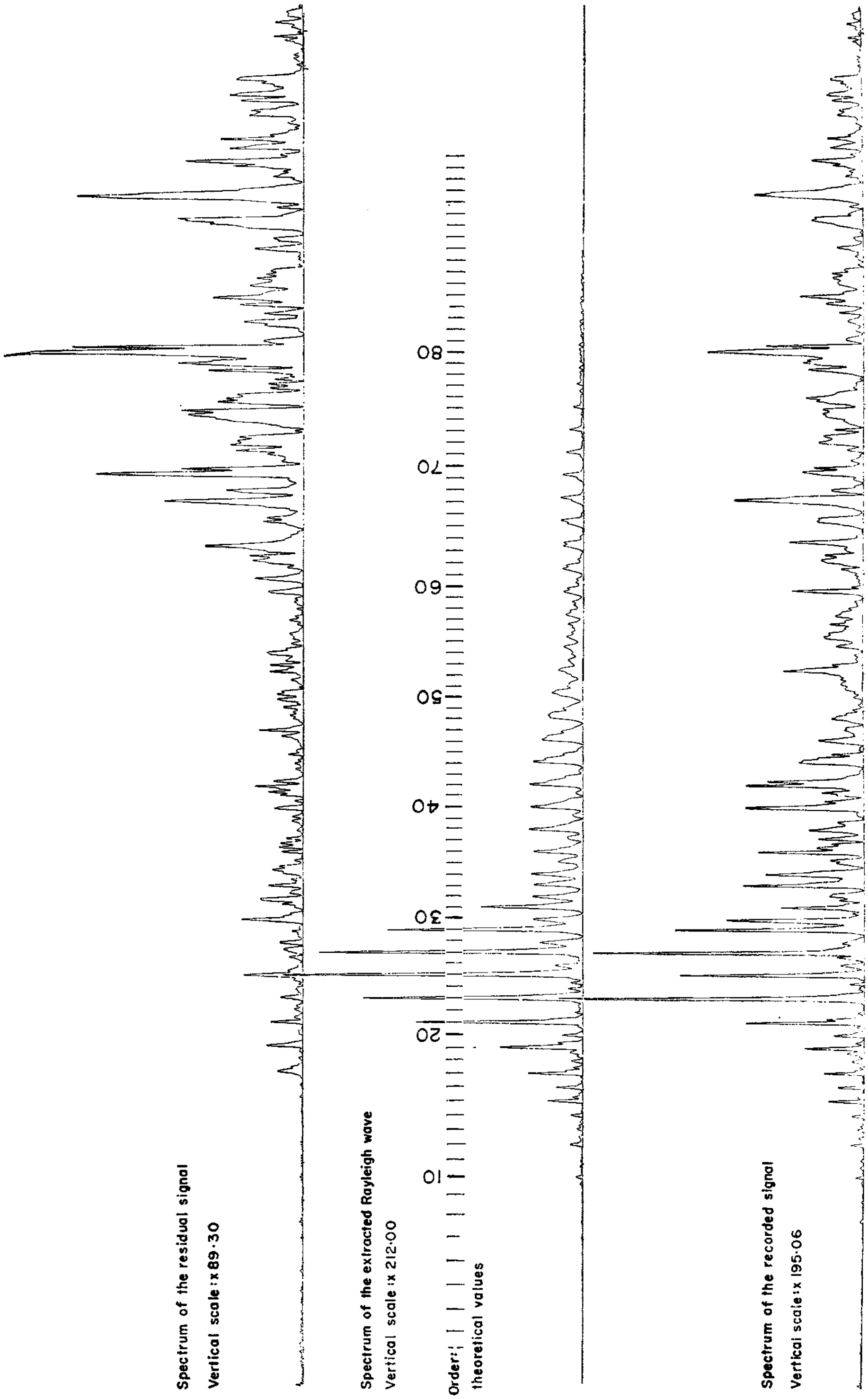

FIG. 8. Spectrum of the three functions presented on Fig. 7.

function $1-[2(t-t_0)/(t_2-t_1)]^4$ where t_1 and t_2 are the times of the beginning and the end of the record and $t_0 = (t_1+t_2)/2$. In Fig. 7 we have plotted the extracted fundamental Rayleigh wave and the difference between windowed sets and the fundamental. Spectrums of these three signals are shown in Fig. 8. Amplitude variations of the spectrum of the extracted Rayleigh wave are more regular. On the spectrum of the residual signal the level of the fundamental peak is very low and more higher modes of spheroidal oscillations may be observed than in the spectrum of the recorded signal.

Conclusion

Searching for optimum filters for dispersed wavetrain analysis permits us to improve our knowledge of multiple filter technique and clearly shows the importance of epicentral distance in the choice of filters. However we do not expect great improvements in time–frequency resolution for the analysis of dispersed surface waves, because experimental dispersion curves used in seismology are smoother than those used for our examples on synthetics wavetrains.

On the other hand, we hope for great improvements in variable filtering techniques. We can extract with a given precision the wavetrain related to any dispersion curve from the recorded signal when this wavetrain is far enough from others in the time–frequency domain.

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