Wharton

Financial Institutions Center

Financial Fragility, Liquidity and Asset Prices

by
Franklin Allen
Douglas Gale

01-37-B

The Wharton Financial Institutions Center

The Wharton Financial Institutions Center provides a multi-disciplinary research approach to the problems and opportunities facing the financial services industry in its search for competitive excellence. The Center's research focuses on the issues related to managing risk at the firm level as well as ways to improve productivity and performance.

The Center fosters the development of a community of faculty, visiting scholars and Ph.D. candidates whose research interests complement and support the mission of the Center. The Center works closely with industry executives and practitioners to ensure that its research is informed by the operating realities and competitive demands facing industry participants as they pursue competitive excellence.

Copies of the working papers summarized here are available from the Center. If you would like to learn more about the Center or become a member of our research community, please let us know of your interest.

Franklin Allen
Co-Director

Richard J. Herring
Co-Director

Financial Fragility, Liquidity and Asset Prices [*]

Franklin Allen
Department of Finance
Wharton School
University of Pennsylvania
Philadephia, PA 19104
allenf@wharton.upenn.edu

Douglas Gale
Department of Economics
New York University
269 Mercer Street
New York, NY 10003
douglas.gale@nyu.edu

January 23, 2003

Abstract

A financial system is fragile if a small shock has a large effect. Sunspot equilibria, where the endogenous variables depend on extrinsic uncertainty, provide an extreme illustration. However, fundamental equilibria, where outcomes depend only on intrinsic uncertainty, can also be fragile. We study the relationship between sunspot equilibria and fundamental equilibria in a model of financial crises. The amount of liquidity is endogenously chosen and determines asset prices. The model has multiple equilibria, but only some of these are the limit of fundamental equilibria when the fundamental uncertainty becomes vanishingly small. We show that under certain conditions the only robust equilibria are those in which extrinsic uncertainty gives rise to asset price volatility and financial crises.

JEL Classification: D5, D8, G2

1 Excess sensitivity and sunspots

There are numerous historical examples of *financial fragility*, where shocks that are small in relation to the economy as a whole have a significant impact on the financial system. For example, Kindleberger (1978, pp. 107-108) argues that the immediate cause of a financial crisis

> "... may be trivial, a bankruptcy, a suicide, a flight, a revelation, a refusal of credit to some borrower, some change of view which leads a significant actor to unload. Prices fall. Expectations are reversed. The movement picks up speed. To the extent that speculators are leveraged with borrowed money, the decline in prices leads to further calls on them for margin or cash, and to further liquidation. As prices fall further, bank loans turn sour, and one or more mercantile houses, banks, discount houses, or brokerages fail. The credit system itself appears shaky and the race for liquidity is on."

Fisher (1933, pp. 341-42) elaborated one mechanism by which a small shock might lead to a recession. He identified overindebtedness and deflation as the key factors.

> "Assuming, accordingly, that, at some point of time, a state of overindebtedness exists, this will tend to lead to liquidation, through the alarm of debtors or creditors or both. Then we may deduce the following chain of consequences... (1) *Debt liquidation* leads to *distress selling* and to (2) *Contraction of deposit currency*, as bank loans are paid off, and to a slowing down of velocity of circulation. This contraction of deposits and of their velocity, precipitated by distress selling, causes (3) *A fall in the level of prices*, in other words, a swelling of the dollar. Assuming [no reflation] there must be (4) *A still greater fall in the net worths of business*, precipitating bankruptcies...." [emphasis in original]

Schnabel and Shin (2002), in documenting the financial crisis of 1763, argue that the bankruptcy of the de Neufville brothers' banking house forced them to sell their stocks of commodities. In the short run, the liquidity of the market is limited and commodity sales result in lower prices. These price declines in turn put other intermediaries under strain and they were forced to sell. A collapse in commodity prices and a financial crisis followed. Schnabel and Shin draw a parallel between the 1763 crisis and the Long Term Capital Management (LTCM) episode in 1998. They suggest that without the rescue coordinated by the Federal Reserve Bank of New York, there might have been a collapse in asset prices and a widespread financial crisis similar to that in 1763.

Allen and Gale (1998) describe a model in which financial crises are caused by exogenous asset-return shocks. When asset returns are very low, banks are unable to meet their commitments and are forced to default and liquidate assets. Forced assets sales by a large number of banks will depress asset prices and exacerbate the crisis, as Fisher described.

By a *crisis* we mean a profound drop in the value of asset prices which affects the solvency of a large number of banks and their ability to meet their commitments to their depositors. If the price movement is large enough some banks are forced into liquidation, but there may also be crises in which banks avoid default, although their balance sheet is under extreme pressure. In the current paper we investigate *endogenous* crises, where small or negligible shocks

set off self-reinforcing and self-amplifying price changes. We use a simple version of the general equilibrium model introduced in Allen and Gale (2003), henceforth AG. The key element of the model is the role of liquidity in determining asset prices. Banks initially choose the amount of liquidity to hold in their portfolios. Subsequent small shocks cause a collapse in asset prices and a financial crisis because the amount of liquidity available is fixed by their initial portfolio choices.

We model liquidity preference in the standard way by assuming that consumers have stochastic time preferences. There are three dates, $t = 0, 1, 2$, and all consumption occurs at dates 1 and 2. There are two types of consumers, *early consumers*, who only value consumption at date 1 and *late consumers*, who only value consumption at date 2. Consumers are identical at date 0 and learn their true type, "early" or "late", at the beginning of date 1.

There are two types of investments in this economy, short-term investments, which yield a return after one period and long-term investments, which take two periods to mature. There is a trade-off between liquidity and returns: long-term investments have a higher yield but take longer to mature.

Banks are modeled as risk-sharing institutions that pool consumers' endowments and invest them in a portfolio of long- and short-term investments. In exchange for their endowments, banks give consumers a deposit contract that allows a consumer the option of receiving either a fixed amount of consumption at date 1 or a fixed amount of consumption at date 2. This provides individual consumers with insurance against liquidity shocks by intertemporally smoothing the returns paid to depositors. There is free entry into the banking sector: in a competitive equilibrium banks must maximize the expected utility of depositors in order to attract customers.

As a benchmark, we first consider an economy in which there is no aggregate uncertainty. More precisely, we assume that both asset returns and the total number of early consumers in the economy are non-stochastic. Despite the absence of aggregate uncertainty, individual banks can be uncertain about the number of early consumers among their depositors and, hence, about their demand for liquidity at date 1. The asset market plays a crucial role in providing liquidity at date 1. Banks that have an above average demand for liquidity can sell long-term assets to banks that have a lower than average demand. Because the number of early consumers is constant across the entire economy, the aggregate demand for and supply of liquidity are in balance and the asset market serves to reallocate liquidity among banks as necessary.

The asset market is important because it integrates the financial system and allows banks to share liquidity. However, it is also a source of financial fragility. To see this, we have to understand the relationship between asset prices and the bank's decision to sell assets. If the asset price is high, the bank can meet its liquidity needs by selling a small proportion of its long-term assets. As the asset price falls, the bank has to sell a larger proportion of its long-term assets. In extreme cases, the asset price is so low that the bank cannot meet its commitments and defaults. At that point it is forced to liquidate all its long-term assets. This "backward bending" supply curve for long-term assets explains why the asset market can clear at more than one price. The asset price

and the quantity of long-term assets supplied move in opposite directions, so that the *value* of assets supplied does not change very much.

We distinguish equilibria in several ways. In the first place, we distinguish *fundamental equilibria*, in which the endogenous variables are functions of the exogenous primitives or fundamentals of the model (endowments, preferences, technologies) from *sunspot equilibria*, in which endogenous variables may be influenced by extraneous variables that have no direct impact on fundamentals. In a fundamental equilibrium, a crisis is driven by exogenous shocks to fundamentals, such as asset returns or liquidity demands. In the absence of aggregate real shocks, asset prices are non-stochastic and a crisis cannot arise. In a sunspot equilibrium, by contrast, asset prices fluctuate in the absence of aggregate real shocks and crises appear to occur spontaneously.

So far, we have suggested there might be multiple equilibria, only some of which are characterized by crises. We want to go further and suggest that some equilibria are more *robust* than others. To test for robustness we perturb the benchmark economy by introducing a small amount of aggregate uncertainty. Specifically, we assume there is a small amount of uncertainty about the total number of early consumers in the economy. However small the amount of aggregate uncertainty, the equilibria of the perturbed economy always exhibit crises with positive probability. Furthermore, the probability of a crisis is bounded away from zero as the aggregate uncertainty becomes vanishingly small. Thus, in a robust equilibrium of the limit economy there must be extrinsic uncertainty. The fundamental equilibrium of the limit economy is not robust.[1]

These results help us understand the relationship between two traditional views of financial crises. One is that they are spontaneous events, unrelated to changes in the real economy. Historically, banking panics were attributed to "mob psychology" or "mass hysteria" (see, e.g., Kindleberger (1978)). The modern theory explains banking panics as equilibrium coordination failures (Bryant (1980), Diamond and Dybvig (1983)). An alternative view is that financial crises are a natural outgrowth of the business cycle (Gorton (1988), Calomiris and Gorton (1991), Calomiris and Mason (2000), Allen and Gale (1998, 2000a-d)). The formal difference between these two views is whether a crisis is generated by intrinsic or extrinsic uncertainty. *Intrinsic uncertainty* is caused by stochastic fluctuations in the primitives or fundamentals of the economy. Examples would be exogenous shocks that effect liquidity preferences or asset returns. *Extrinsic uncertainty*, often referred to as sunspots, by definition has no effect on the fundamentals of the economy. [2,3]

---

[1]The limit economy has two types of equilibria with extrinsic uncertainty. In a trivial sunspot equilibrium, prices are random but the allocation is essentially the same as in the fundamental equilibrium and no banks default. In a non-trivial sunspot equilibrium, the equilibrium allocation is random as well as the prices.

[2]Strictly speaking, much of the banking literature exploits multiple equilibria without addressing the issue of sunspots. We adopt the sunspots framework here because it encompasses the standard notion of equilibrium and allows us to address the issue of equilibrium selection.

[3]The theoretical analysis of sunspot equilibria began with the seminal work of Cass and Shell (1982) and Azariadis (1981), which gave rise to two streams of literature. The Cass-Shell paper is most closely related to work in a Walrasian, general-equilibrium framework;

Our model combines the most attractive features of both traditional approaches. Like the sunspot approach, it produces large effects from small shocks. Like the real business cycle approach, it makes a firm prediction about the conditions under which crises will occur.[4] However, it is important to distinguish the present model of systemic or economy-wide *crises* from models of individual bank runs or *panics*. Panics are the result of coordination failure (Bryant (1980), Diamond and Dybvig (1983)). If late consumers expect a run, it is optimal for them to join it; if late consumers do not expect a run, it is optimal for them to wait until date 2 to withdraw. Whether a run occurs depends entirely on the depositors' expectations, not on the value of the bank's assets. Furthermore, a run on a single bank can occur independently of what is happening to other banks.

In the crisis model, by contrast, default only occurs if it is unavoidable, that is, the value of the bank's portfolio is too low to allow the bank to meet its commitments.[5] Furthermore, default occurs as part of a general crisis. Banks fail because asset prices are too low and asset prices are low because banks are selling assets when liquidity is scarce. From the point of view of a single, price-taking bank, default results from an exogenous shock to asset prices. From the point of view of the banking sector as a whole, the "shock" is endogenous. But note that only the simultaneous failure of a non-negligible number of banks can affect asset prices.

Our work is related to the wider literature on general equilibrium with incomplete markets (GEI). As is well known, sunspots do not matter when markets are complete. (For a precise statement, see Shell and Goenka, (1997)). The incompleteness in our model reveals itself in two ways. First, sunspots are assumed to be non-contractible, that is, the deposit contract is not explicitly contingent on the sunspot variable. In this respect we are simply following the incomplete contracts literature (see, for example, Hart (1995)). Secondly, there are no markets for Arrow securities contingent on the sunspot variable, so financial institutions cannot insure themselves against asset price fluctuations associated with the sunspot variable. This is the standard assumption of the GEI literature (see, for example, Geanakoplos et al. (1990) or Magill and Quinzii (1996)).

There is a small but growing literature related to financial fragility. Financial multipliers were introduced by Bernanke and Gertler (1989). In the model of Kiyotaki and Moore (1997), the impact of illiquidity at one link in the credit chain has a large impact further down the chain. Chari and Kehoe (2000) show

---

the Azariadis paper is most closely related to the macroeconomic dynamics literature. For a useful survey of applications in macroeconomics, see Farmer (1999); for an example of the current literature in the general equilibrium framework see Gottardi and Kajii (1995, 1999).

[4]Models of sunspot phenomena typically have many equilibrium, including as a special case the fundamental equilibria in which extrinsic uncertainty has no effect on endogenous variables. Thus, financial fragility remains a possibility but not a necessity.

[5]This is a refinement of the equilibrium concept. We assume that late consumers withdraw at the last date whenever it is incentive compatible for them to do so. Bank runs occur only when it is impossible for the bank to meet its obligations in an incentive-compatible way. Such runs are called essential in AG to distinguish them from the coordination failures in Diamond and Dybvig (1983).

that herding behavior can cause a small information shock to have a large effect on capital flows. Lagunoff and Schreft (2001) show how overlapping claims on firms can cause small shocks to lead to widespread bankruptcy. Bernardo and Welch (2002) develop a model of runs on financial markets and asset price collapses based on the anticipation of liquidity needs.

The rest of the paper is organized as follows. Section 2 contains the basic assumptions of the model. Section 3 describes the optimal contracts offered by banks and the rules governing default and liquidation. Section 4 defines equilibrium. A few special cases of the model are considered in Section 5. Section 6 contains a full analysis of the equilibria of the economy. We consider the limit economy with no aggregate uncertainty and then perturb the economy by introducing aggregate uncertainty. We show that, in any equilibrium of the economy with aggregate uncertainty, crises occur with positive probability. We also show that the limit of a sequence of equilibria corresponding to a sequence of perturbed economies is an equilibrium in the limit economy and we characterize the limit equilibria. Section 7 contains a discussion. Proofs are gathered in Section 8.

2 Assets and preferences

The model we use is a special case of AG.

Dates. There are three dates $t = 0, 1, 2$ and a single good at each date. The good can be used for consumption or investment.

Assets. There are two assets, a short-term asset (the *short* asset) and a long-term asset (the *long* asset).

- The short asset is represented by a storage technology. Investment in the short asset can take place at date 1 or date 2. One unit of the good invested at date t yields one unit at date $t + 1$, for $t = 0, 1$.

- The long asset takes two periods to mature and is more productive than the short asset. Investment in the long asset can only take place at date 0. One unit invested at date 0 produces $r > 1$ units at date 2.

Consumers. There is a continuum of ex ante identical consumers, whose measure is normalized to unity. Each consumer has an endowment $(1, 0, 0)$ consisting of one unit of the good at date 0 and nothing at subsequent dates. There are two (ex post) types of consumers at date 1, *early consumers*, who only value consumption at date 1, and *late consumers*, who only value consumption at date 2. If η denotes the probability of being an early consumer and c_t denotes consumption at date $t = 1, 2$, the consumer's ex ante utility is

$$u(c_1, c_2; \eta) = \eta U(c_1) + (1 - \eta)U(c_2).$$

The period utility function $U : \mathbf{R}_+ \to \mathbf{R}$ is twice continuously differentiable and satisfies the usual neoclassical properties, $U'(c) > 0, U''(c) < 0$, and $\lim_{c \searrow 0} U'(c) = \infty$.

Uncertainty. There are three sources of intrinsic uncertainty in the model. First, each individual consumer faces idiosyncratic uncertainty about her preference type (early or late consumer). Secondly, each bank faces idiosyncratic uncertainty about the number of early consumers among the bank's depositors. For example, different banks could be located in regions subject to independent liquidity shocks. Thirdly, there is aggregate uncertainty about the fraction of early consumers in the economy. Aggregate uncertainty is represented by a state of nature θ, a non-degenerate random variable with a finite support and a density function $f(\theta)$. The bank's idiosyncratic shock is represented by a random variable α, with finite support and a density function $g(\alpha)$. The probability of being an early consumer at a bank in state (α, θ) is denoted by $\eta(\alpha, \theta)$, where

$$\eta(\alpha, \theta) \equiv \alpha + \varepsilon\theta$$

and $\varepsilon \geq 0$ is a constant. We adopt the usual "law of large numbers" convention and assume that the fraction of early consumers at a bank in state (α, θ) is identically equal to the probability $\eta(\alpha, \theta)$. The economy-wide average of α is assumed to be constant and equal to the mean $\bar{\alpha} = \sum \alpha g(\alpha)$. Thus, there is aggregate intrinsic uncertainty only if $\varepsilon > 0$.

Information. All uncertainty is resolved at date 1. The true value of θ is publicly observed,[6] the true value of α for each bank is publicly observed, and each consumer learns his type, i.e., whether he is an early consumer or a late consumer.

Asset markets. There are no asset markets for hedging against aggregate uncertainty at date 0, for example, there are no Arrow securities contingent on the state of nature θ. At date 1, there is a market in which future (date-2) consumption can be exchanged for present (date-1) consumption. If $p(\theta)$ denotes the price of future consumption in terms of present consumption at date 1, then one unit of the long asset is worth $p(\theta)r$ at date 1 in state θ.

Markets are incomplete at date 0 but complete at date 1. We assume that market participation is incomplete: financial institutions such as banks can participate in the asset market at date 1 but individual consumers cannot.[7]

3 Banking

Banks are financial institutions that provide investment and liquidity services to consumers. They do this by pooling the consumers' resources, investing them

---

[6]It is not strictly necessary to assume that θ is observed. In equilibrium, all that agents need to know is the equilibrium price $p(\theta)$, which may or may not reveal θ.

[7]As Cone (1983) and Jacklin (1986) showed, if consumers have access to the capital market, it is impossible for banks to offer risk sharing that is superior to the market.

in a portfolio of short- and long-term assets, and offering consumers future consumption streams with a better combination of asset returns and liquidity than individual consumers could achieve by themselves. Banks also have access to the interbank capital market, from which consumers are excluded.

Banks compete by offering deposit contracts to consumers in exchange for their endowments and consumers respond by choosing the most attractive of the contracts offered. Free entry ensures that banks earn zero profits in equilibrium. The deposit contracts offered in equilibrium must maximize consumers' welfare subject to the zero-profit constraint. Otherwise, a bank could enter and make a positive profit by offering a more attractive contract.

Anything a consumer can do, the bank can do. So there is no loss of generality in assuming that consumers deposit their entire endowment in a bank at date 0.[8] The bank invests y units per capita in the short asset and $1 - y$ units per capita in the long asset and offers each consumer a deposit contract, which allows the consumer to withdraw either d_1 units at date 1 or d_2 units at date 2. Without loss of generality, we set $d_2 = \infty$. This ensures that consumers receive the residue of the bank's assets at date 2. Then the deposit contract is characterized by the middle-period payment $d_1 \equiv d$.

If $p(\theta)$ denotes the price of future consumption at date 1 in state θ, then the value of the bank's assets at date 1 is $y + p(\theta)r(1 - y)$.

A consumer's type is private information. An early consumer cannot misrepresent his type because he needs to consume at date 1; but a late consumer can claim to be an early consumer, withdraw d at date 1, store it until date 2 and then consume it. The deposit contract is *incentive compatible* if and only if the residual payment to late consumers at date 2 is at least d. Since the late consumers are residual claimants at date 2, it is possible to give them at least d units of consumption if and only if

$$\eta(\alpha, \theta)d + (1 - \eta(\alpha, \theta))p(\theta)d \leq y + p(\theta)r(1 - y). \tag{1}$$

---

[8]This is not simply an application of the Modigliani-Miller theorem. The consumer may do strictly better by putting all his "eggs" in the bank's "basket". Suppose that the deposit contract allows the individual to hold m units in the safe asset and deposit $1 - m$ units in the bank. The bank invests y in the short asset and $1 - m - y$ in the long asset. If the bank does not default at date 1, the early consumers receive $d + m$ and the late consumers receive

$$\frac{p(\theta)r(1 - m - y) + y - \eta(\theta)d}{(1 - \eta(\theta))p(\theta)} + m.$$

If the bank defaults, early and late consumers receive

$$p(\theta)r(1 - m - y) + y + m.$$

Suppose that $m > 0$ and consider a reduction in m and an increase in y and d of the same amount. It is clear that the early consumers' consumption is unchanged. So is the late consumers' consumption if the bank defaults. The change in the late consumers' consumption when the bank does not default is

$$\frac{\Delta y - \eta(\theta)\Delta d}{(1 - \eta(\theta))p(\theta)} + \Delta m = \frac{-\Delta m}{p(\theta)} + \Delta m \geq 0$$

because $p(\theta) \leq 1$ and $\Delta m < 0$. Thus, it is optimal for the bank (and for the consumer) to choose $m = 0$.

8

The left hand side is a lower bound for the present value of consumption when early consumers are given d and late consumers are given at least d. The right hand side is the value of the portfolio. Thus, condition (1) is necessary and sufficient for the deposit contract d to satisfy incentive compatibility and the budget constraint simultaneously. We often refer to the inequality in (1) as the *incentive constraint* for short.

In what follows, we assume that bank runs occur only if they are unavoidable. In other words, late consumers will withdraw at date 2 as long as the bank can satisfy the incentive constraint. If (1) is violated, then all consumers will want to withdraw at date 1. In the event of bankruptcy, the bank is *required* to liquidate its assets in an attempt to provide the promised amount d to the consumers who withdraw at date 1. Whatever withdrawal decisions consumers make, the consumers who withdraw at date 2 will receive less than the consumers who withdraw at date 1. Hence, in equilibrium, all consumers must withdraw at date 1. Then each consumer receives the liquidated value of the portfolio $y + p(\theta)r(1 - y)$.

Let $x_t(d, y, \alpha, \theta)$ denote the consumption at date t if the bank chooses (d, y) and the bank is in state (α, θ) at date 1. Let $x = (x_1, x_2)$, where

$$x_1(d, y, \alpha, \theta) = \begin{cases} d & \text{if (1) is satisfied} \\ y + p(\theta)r(1 - y) & \text{otherwise,} \end{cases}$$

$$x_2(d, y, \alpha, \theta) = \begin{cases} \frac{y + p(\theta)r(1 - y) - \eta d}{(1 - \eta)p(\theta)} & \text{if (1) is satisfied} \\ y + p(\theta)r(1 - y) & \text{otherwise,} \end{cases}$$

and $\eta = \eta(\alpha, \theta)$. Using this notation, the bank's decision problem can be written as

$$\begin{aligned} \max \quad & E\left[u\left(x(d, y, \alpha, \theta), \eta(\alpha, \theta)\right)\right] \\ \text{s.t.} \quad & 0 \le d, 0 \le y \le 1. \end{aligned} \tag{DP1}$$

An ordered pair (d, y) is *optimal* for the given price function $p(\cdot)$ if it solves (DP1).

4   Equilibrium

The bank's decision problem is "non-convex". To ensure the existence of equilibrium, we take advantage of the convexifying effect of large numbers and allow for the possibility that ex ante identical banks will choose different deposit contracts d and portfolios y. Each consumer is assumed to deal with a single bank and each bank offers a single contract. In equilibrium, consumers will be indifferent between banks offering different contracts. Consumers allocate themselves to different banks in proportions consistent with equilibrium.

To describe an equilibrium, we need some additional notation. A partition of consumers at date 0 is defined by an integer $m < \infty$ and an array $\rho = (\rho_1, ..., \rho_m)$

9

of numbers $\rho_i \geq 0$ such that $\sum_{i=1}^m \rho_i = 1$. Consumers are divided into m groups and each group i contains a measure ρ_i of consumers. We impose an arbitrary bound m on the number of groups to rule out pathological cases.[9] The banks associated with group i offer a deposit contract d_i and a portfolio y_i, both expressed in per capita terms. An *allocation* consists of a partition (m, ρ) and an array $(d, y) = \{(d_i, y_i)\}_{i=1}^m$ such that $d_i \geq 0$ and $0 \leq y_i \leq 1$ for $i = 1, ..., m$.

To define the market clearing conditions we need some additional notation. Let $\bar{x}(d_i, y_i, \alpha, \theta) = (\bar{x}_1(d_i, y_i, \alpha, \theta), \bar{x}_2(d_i, y_i, \alpha, \theta))$ denote the bank's demand for goods. If the bank is solvent then a fraction $\eta(\alpha, \theta)$ get paid $x_1(d_i, y_i, \alpha, \theta)$ at date 1 and a fraction $(1 - \eta(d_i, y_i, \alpha, \theta))$ get paid $x_2(d_i, y_i, \alpha, \theta))$ at date 2. Then the bank's total demand for goods is given by

$$\bar{x}(d_i, y_i, \alpha, \theta) = (\eta(\alpha, \theta)x_1(d_i, y_i, \alpha, \theta), (1 - \eta(d_i, y_i, \alpha, \theta))x_2(d_i, y_i, \alpha, \theta)).$$

If the bank is bankrupt, on the other hand, then everyone gets paid $x_1(d_i, y_i, \alpha, \theta) = x_2(d_i, y_i, \alpha, \theta)$ at date 1 and the bank's total demand for goods is

$$\bar{x}(d_i, y_i, \alpha, \theta) = (x_1(d_i, y_i, \alpha, \theta), 0).$$

An allocation (m, ρ, d, y) is *attainable* if it satisfies the market-clearing conditions

$$\sum_i \rho_i E\left[\bar{x}_1(d_i, y_i, \alpha, \theta)\right] \leq \sum_i \rho_i y_i, \tag{2}$$

and

$$\sum_i \rho_i \left\{E\left[\bar{x}_1(d_i, y_i, \alpha, \theta) + \bar{x}_2(d_i, y_i, \alpha, \theta)\right]\right\}$$
$$= \sum_i \rho_i \left\{y_i + r(1 - y_i)\right\},$$

for any state θ. In the market-clearing conditions, we take expectations with respect to α because the cross-sectional distribution of idiosyncratic shocks is assumed to be the same as the probability distribution. The first inequality says that the total demand for consumption at date 1 is less than or equal to the supply of the short asset. The inequality may be strict, because an excess supply of liquidity can be re-invested in the short asset and consumed at date 2. The second condition says that total consumption at date 2 is equal to the return from the investment in the long asset *plus* the amount invested in the short asset at date 1, which is the difference between the left and right hand sides of (2).

In equilibrium, it must be the case that $p(\theta) \leq 1$. Otherwise banks could make an arbitrage profit at date 1 by selling goods forward and investing the proceeds in the short asset. If $p(\theta) < 1$ then no one is willing to invest in the short asset at date 1 and (2) must hold as an equation. A price function $p(\cdot)$ is

---

[9]In general, two groups are sufficient for existence of equilibrium.

admissible (for the given allocation) if it satisfies the following complementary slackness condition:

> For any state θ, $p(\theta) \leq 1$ and $p(\theta) < 1$ implies that (2) holds as an equation.

Now we are ready to define an equilibrium.

An *equilibrium* consists of an attainable allocation (m, ρ, d, y) and an admissible price function $p(\cdot)$ such that, for every group $i = 1, ..., m$, (d_i, y_i) is optimal given the price function $p(\cdot)$.

5 A first look at equilibrium

5.1 Autarkic equilibria

In order to illustrate the model and its properties, we begin with some special cases. Consider first the case in which there is no aggregate uncertainty ($\varepsilon = 0$) and banks receive no idiosyncratic shocks (α is a constant). In this case banks have no need to rely on the asset market to provide liquidity. Autarky is optimal. To see this, consider the problem faced by a central planner. In the absence of aggregate uncertainty, optimal risk sharing requires the consumption allocation to be non-stochastic. An early consumer receives c_1 and a late consumer receives c_2, where c_1 and c_2 are constants. The total demand for consumption is αc_1 at date 1 and $(1 - \alpha)c_2$ at date 2. The planner provides consumption at each date by holding the asset with the highest return. Thus, consumption at date 1 is provided by holding the short asset and consumption at date 2 is provided by holding the long asset. So the planner chooses y to satisfy $\alpha c_1 = y$ and $(1 - \alpha)c_2 = r(1 - y)$. In the first best (ignoring the incentive constraint), the allocation (c_1, c_2, y) is chosen to solve the following problem:

$$
\begin{aligned}
\max \quad & \alpha U(c_1) + (1 - \alpha)U(c_2) \\
\text{s.t.} \quad & 0 \leq y \leq 1 \\
& \alpha c_1 = y \\
& (1 - \alpha)c_2 = r(1 - y).
\end{aligned}
$$

This problem uniquely determines (c_1, c_2, y). Note that the optimal consumption allocation must satisfy the first-order condition

$$
U'(c_1) = rU'(c_2),
$$

which implies that $c_1 < c_2$, so the incentive constraint is strictly satisfied. This means that the incentive-efficient and Pareto-efficient allocations are identical.

Now consider what a bank can achieve. On the one hand, the bank cannot do better than the first best. On other hand, it can achieve the first best by choosing the same portfolio y and setting $d = c_1$. To show that this is an equilibrium, we only need to find a price that makes the portfolio choice optimal at date 0 and clears the market at date 1. The first order condition for (DP1)

for the choice of y given that the incentive constraint is satisfied, there is no default and $\varepsilon = 0$, is

$$E\left[U'(c_2)\left(\frac{1 - p(\theta)r}{p(\theta)}\right)\right] = 0. \tag{3}$$

Since there is no aggregate uncertainty, it is natural to assume that there is a non-stochastic price $p(\theta) = \bar{p}$ at date 1. It then follows from (3) that

$$\bar{p} = \frac{1}{r}.$$

At this price, one unit invested in either asset at date 0 yields one unit at date 1. The banks will hold only the long asset between date 1 and date 2 because at this price the yield on the short asset, one, is dominated by the return on the long asset, r. Thus, $(d, y, \bar{p})$ defines an equilibrium.

The equilibrium is special in several ways. First, there is no aggregate uncertainty, intrinsic or extrinsic. Secondly, the allocation is Pareto-efficient. Finally, the market plays no role, apart from determining the equilibrium price. Banks can achieve the first best while remaining autarkic.

Autarky is not actually necessary for equilibrium: since banks are indifferent between the two assets at date 0, they may choose to hold different portfolios initially and then use the asset market at date 1 to reshuffle their liquidity holdings. Thus, an attainable allocation (ρ, m, d, y) is an equilibrium allocation for the price $\bar{p}$ if $d_i = c_1$ for all i and the market-clearing condition

$$\sum_i \rho_i \alpha d_i = \alpha c_1 = \sum_i \rho_i y_i$$

is satisfied. So there is a continuum of equilibria differing in the portfolios chosen but identical in terms of consumption and consumer welfare.

One further point we can make using this special case is to show how random prices can arise in equilibrium. Consider the symmetric equilibrium in which all banks choose the same portfolio $y_i = \alpha c_1$ and suppose that in place of the constant price $\bar{p}$ we choose a price function $p(\theta)$. In order to maintain equilibrium, we need to satisfy two market-clearing conditions. First, we have to make banks willing to hold the portfolio y between date 0 and date 1 and, secondly, we have to make the banks willing to hold (only) the long asset between date 1 and date 2. It follows from (3) that the first condition is satisfied if

$$E\left[\frac{1}{p(\theta)}\right] = r. \tag{4}$$

The second condition is satisfied if $p(\theta) \le 1$ for each θ. There are clearly many price functions that will satisfy these conditions. The uncertainty about prices that can arise in equilibrium is an example of extrinsic uncertainty or sunspot activity. This is a trivial example because the sunspot activity has no effect on the equilibrium allocation. We refer to this as a trivial sunspot equilibrium in the sequel.

To make this quite concrete, consider the following example. Let $U(c_t) = \log(c_t)$ be the period utility function, let $\alpha = 0.8$ be the fraction of early consumers and let

$$\theta = \begin{cases} 0 & \text{w. pr. } 0.65 \\ 1 & \text{w. pr. } 0.35. \end{cases}$$

As a benchmark, consider the autarkic equilibrium corresponding to the economy with no aggregate uncertainty, $\varepsilon = 0$. There is a constant price

$$p(\theta) = \bar{p} = \frac{1}{r}.$$

The value of the bank's portfolio at date 1 is $y + \bar{p}r(1 - y) = 1$ in each state. Since the bank's objective function is Cobb-Douglas, it will spend a fraction α of its budget on early consumers and a fraction $1 - \alpha$ on late consumers. Thus, $\alpha c_1 = \alpha$ and $(1 - \alpha)\bar{p}c_2 = (1 - \alpha)$, or

$$(c_1, c_2) = (1, r).$$

The short term asset is used to provide for the consumption needs of the early consumers and the long term asset is used to provide for the needs of the late consumers. If every bank chooses the same portfolio y, the market-clearing conditions require

$$y = \alpha c_1 = \alpha.$$

The numerical values for this equilibrium, which we term the fundamental equilibrium, are summarized below.

$$\begin{aligned} E[U] &= 0.081, \\ y &= 0.800, \\ (c_1, c_2) &= (1.000, 1.500), \\ \bar{p} &= 0.667 \end{aligned}$$

There is a second equilibrium whose equilibrium parameters are summarized below.

$$\begin{aligned} E[U] &= 0.081, \\ y &= 0.800, \\ (c_1(\theta), c_2(\theta)) &= (1.000, 1.500), \theta = 0, 1, \\ \begin{bmatrix} p(0) \\ p(1) \end{bmatrix} &= \begin{bmatrix} 0.943 \\ 0.432 \end{bmatrix}. \end{aligned}$$

There is no exogenous aggregate uncertainty, because $\varepsilon = 0$, and yet the asset prices are very different in the two states. We term this equilibrium a trivial sunspot equilibrium. There are many other trivial sunspot equilibrium in this example.

13

Note that price fluctuations in trivial sunspot equilibria have no impact on the equilibrium allocation because banks are autarkic. A portfolio containing a positive proportion of the long asset will have a high value in state 0, where the price of long assets at date 1 is high, $p(0) = 0.943$, and a low value in state 1, where the price of the long asset is low, $p(0) = 0.432$; however, the maturity structure of the portfolio is perfectly matched to the demand for consumption, so changes in portfolio value are matched by changes in the present value of consumption.

In general, the construction of a trivial sunspot equilibrium requires that all banks hold the same amount of the short asset $y_i = \alpha d_i$. Although there is a continuum of trivial sunspot equilibrium, differing in the equilibrium price function, they have identical allocations, including the choice of portfolio.

5.2 Aggregate intrinsic uncertainty

The second case we want to look at is one in which there is a small amount of aggregate (intrinsic) uncertainty, that is, $\varepsilon > 0$. For the moment we maintain the assumption that α is a constant so banks face no idiosyncratic uncertainty. Somewhat surprisingly, it turns out that introducing a small amount of aggregate uncertainty imposes different qualitative features on the equilibrium. In particular, there is no counterpart of the "natural" fundamental equilibrium of the limit economy in which prices are constant.

In any equilibrium with $\varepsilon > 0$, there must be a positive probability of default. To see this, suppose that, contrary to what we want to show, there does exist an equilibrium in which the banks never default. Then the market clearing condition at date 1 implies

$$(\alpha + \varepsilon\theta) \sum_i \rho_i d_i \leq \sum_i \rho_i y_i,$$

for every value of θ. If the inequality is strict there is an excess supply of liquidity and $p(\theta) = 1$. The right hand side represents the supply of liquidity, which is inelastic and independent of θ. The left hand side represents the demand for liquidity, which is inelastic and linearly dependent on θ. Then if $\varepsilon > 0$ and the distribution of θ is not degenerate, the inequality must be strict at least with positive probability. This means that $p(\theta) = 1$ with positive probability. Clearly, we cannot have $p(\theta) = 1$ with probability one because this would violate condition (3). In fact, $p(\theta) = 1$ with probability one implies that the return on the long asset between date 0 and date 1 is $p(\theta)r > 1$, so the short asset is dominated, $y_i = 0$, and the market clearing condition cannot be satisfied at date 1. In order to satisfy condition (3) with a positive supply of liquidity at date 1 we must have $p(\theta) < 1/r$ with positive probability. To sum up, either default occurs with positive probability in equilibrium or there is non-negligible price asset-price volatility.

We can illustrate these features of the equilibrium with the example introduced earlier, except that now we set $\varepsilon = 0.01$. In this case there is no

14

equilibrium that is close to the fundamental equilibrium. However, there is an equilibrium close to the non-trivial sunspot equilibrium.

As we argued above, there must be default in equilibrium. But if default occurs with positive probability, the banks cannot all choose the same portfolio y and the same deposit contract d. If they do, all banks will go bankrupt in the bad state, there will be no one left to buy the liquidated assets of the defaulting banks, and the price will fall to zero. Anticipating this, a bank could do better by remaining solvent in this state and buying up all the assets.

There must be at least two groups of banks, let us call them safe (denoted by a superscript s) and risky (denoted by the superscript r), who choose different portfolios and offer different deposit contracts. Let $\rho_1 = 0.996$ be the measure of safe banks and $\rho_2 = 0.004$ the measure of risk banks. The other parameters for the equilibrium are listed below.

$$E[U^s] = E[U^r] = 0.078,$$

$$\left[\begin{array}{c} y^s \\ y^r \end{array}\right] = \left[\begin{array}{c} 0.810 \\ 0.016 \end{array}\right],$$

$$\left[\begin{array}{c} (c_1^s(0), c_2^s(0)) \\ (c_1^s(1), c_2^s(1)) \\ (c_1^r(0), c_2^r(0)) \\ (c_1^r(1), c_2^r(1)) \end{array}\right] = \left[\begin{array}{c} (0.998, 1.488) \\ (0.998, 1.520) \\ (1.405, 1.488) \\ (0.651, 0.651) \end{array}\right],$$

$$\left[\begin{array}{c} p(0) \\ p(1) \end{array}\right] = \left[\begin{array}{c} 0.940 \\ 0.430 \end{array}\right].$$

Notice that both safe and risky banks achieve the same expected utility for their investors, but the safe banks choose a much higher investment in the short asset whereas the risky banks invest most of their deposits in the long asset. When $\theta = 0$, the asset price is high and risky banks can meet the demand for liquidity by selling a portion of their long assets in the market at date 1. The safe banks supply the necessary liquidity in exchange for long assets. When $\theta = 1$, the asset price collapses, making risky banks unable to meet their commitments and forcing them to default and liquidate their assets. The price of the long asset is determined by the amount of "cash in the market", that is, $p(1)$ equals the ratio of the excess liquidity of the safe banks to the long assets of the risky banks.

Thus, introducing a small amount of uncertainty changes the properties of equilibrium. Notice that these properties hold for every value of $\varepsilon > 0$, however small. What this suggests is that some of the equilibria of the limit economy with $\varepsilon = 0$ are not robust, in the sense that a small amount of aggregate uncertainty ($\varepsilon > 0$) leads to a big change in equilibrium properties. To see which equilibria are robust, we need to characterize the equilibrium set of the limit economy with $\varepsilon = 0$ (Section 6.1) and then characterize the limits of equilibria of the perturbed economies with $\varepsilon > 0$ (Section 6.5). But first we consider the effect of idiosyncratic risk on banks.

5.3 Non-autarkic equilibria

In the preceding examples, markets could be used in equilibrium, but they played no essential role in achieving the first-best outcome. To illustrate a less trivial role for markets, consider the case where there is again no aggregate uncertainty ($\varepsilon = 0$) but banks face idiosyncratic risk (α is a non-degenerate random variable). Once again, because there is no aggregate uncertainty, it is natural to consider an equilibrium in which the price of future consumption $\bar{p}$ is non-stochastic. As before, equilibrium requires $\bar{p} = 1/r$. If a bank chooses (d, y) at date 0 the incentive constraint at date 1 is

$$\alpha d + \bar{p}(1 - \alpha)d \leq y + \bar{p}r(1 - y) = 1.$$

Since the value of α is uncertain, the bank may well choose (d, y) so that there is a positive probability of default at date 1. The avoidance of default requires the bank to lower d or increase y and either change will be costly in terms of the consumers' expected utility. For example, if the proportion of early consumers is low ($\alpha = \alpha_L$), with high probability and high ($\alpha = \alpha_H$), with low probability, then the choice of d will be close to what it would be in an equilibrium with a non-stochastic $\alpha = \alpha_L$. This value of d satisfies $\alpha_L d + \bar{p}(1 - \alpha_L)d \leq 1$ but it may be that $\alpha_H d + \bar{p}(1 - \alpha_H)d > 1$. If the probability of α_H is not too great, it is optimal for the bank to default in that state, rather than distort its decisions in the more likely state $\alpha = \alpha_L$. Default in this case is a means of introducing greater flexibility into the risk sharing contract.

While uncertainty about liquidity preference, as measured by α, is a plausible reason for default, it is not our main interest here. So, in what follows, we assume the parameters of the model are such that it is never optimal for the bank to choose default in the equilibrium with the non-stochastic price $\bar{p}$ and in the absence of aggregate uncertainty. Then the bank's choice of (d, y) will always satisfy the incentive constraint and the decision problem can, without loss of generality, be written as follows:

$$\begin{aligned}
\max \quad & E\left[\alpha U(d) + (1 - \alpha)U\left(r\frac{1 - \alpha d}{1 - \alpha}\right)\right] \\
\text{s.t.} \quad & \alpha d + \bar{p}(1 - \alpha)d \leq 1.
\end{aligned}$$

In the absence of default, the early consumers receive d and the late consumers receive the remainder. Since the value of the portfolio is $y + \bar{p}r(1 - y) = 1$ at date 1 the late consumers receive $1 - \alpha d$ in present value terms, which is worth $r(1 - \alpha d)/(1 - \alpha)$ per capita at date 2. This explains the objective function of the bank. We can add the incentive constraint without loss of generality because we have assumed that it is not optimal to allow default. The value of d that solves this problem must satisfy the first order condition

$$E\left[\alpha U'(d) - \alpha r U'\left(r\frac{1 - \alpha d}{1 - \alpha}\right)\right] \geq 0$$

with equality if the incentive constraint is not binding. Once again, the deposit contract is uniquely determined by this condition, but the choice of portfolio

16

is not. Since banks are indifferent between the two assets at date 0, they can hold any amount of the short asset as long as the market clearing condition is satisfied at date 1. Whatever amount of the short asset they hold, they need to use markets to provide liquidity for their depositors. Because the bank's demand for liquidity αd is random, there is no way the bank can remain autarkic and achieve optimal risk sharing.

Precisely because all banks rely in a non-trivial way on the market for liquidity, a change in price will have real effects on the consumption allocation available to the bank's depositors. If sunspot equilibria occur, they have real effects.

This section has provided a first look at equilibrium. We have shown that the introduction of a small amount of aggregate uncertainty can have a large effect on the set of equilibria. The next section provides a full characterization of the set of equilibria and how it varies with ε.

6 Analysis

6.1 Equilibria in the limit

In this section we first characterize the equilibria of the limit economy in which $\varepsilon = 0$. There is no aggregate intrinsic uncertainty in the model, but there may still be aggregate extrinsic uncertainty (sunspots). We first classify equilibria in the limit economy according to the impact of extrinsic uncertainty. An equilibrium (m, ρ, d, y, p) in the limit economy is a *fundamental equilibrium* (F) if $x(d_i, y_i, \alpha, \theta)$ is constant for each i and α and if $p(\theta)$ is constant. In that case, the sunspot variable θ has no influence on the equilibrium values. An equilibrium (m, ρ, d, y, p) of the limit economy is a *trivial sunspot equilibrium* (T) if $x(d_i, y_i, \alpha, \theta)$ is constant for each i and α and $p(\theta)$ is not constant. In this case, the sunspot variable θ has no effect on the allocation of consumption but it does affect the equilibrium price $p(\theta)$. An equilibrium (m, ρ, d, y, p) which is neither an F nor a T is called a *non-trivial sunspot equilibrium* (N), that is, an N is an equilibrium in which the sunspot variable has some non-trivial impact on the allocation of consumption.

We saw examples of these different kinds of equilibria in Section 5. In the economy with no aggregate uncertainty ($\varepsilon = 0$) and no idiosyncratic uncertainty for banks (α constant), the F has a non-stochastic price $\bar{p}$ and a non-stochastic consumption allocation (c_1, c_2). For the same economy, we also saw that there are many T, in which the real allocation is the same as in the F but the equilibrium price $p(\theta)$ varies with θ. By contrast, in the economy with no aggregate uncertainty ($\varepsilon = 0$) and idiosyncratic uncertainty for banks (α is not a constant), the banks must use the market to obtain liquidity so fluctuations in asset prices do have real effects. In other words, any sunspot equilibrium is an N.

We can also classify equilibria according to the variety of choices made by different groups of banks. An equilibrium (m, ρ, d, y, p) is *pure* if each group of

banks makes the same choice:

$$(d_i, y_i) = (d_j, y_j), \forall i, j = 1, ..., m.$$

An equilibrium (m, ρ, d, y, p) is *semi-pure* if the consumption allocations are the same for each group of banks:

$$x(d_i, y_i, \alpha, \theta) = x(d_j, y_j, \alpha, \theta), \forall i, j = 1, ..., m.$$

Otherwise, (m, ρ, d, y, p) is a *mixed* equilibrium where different types of bank provide different allocations (but the same ex ante expected utility). In our first look, we saw examples of each of these three types. For example, in the case of the economy with no aggregate uncertainty ($\varepsilon = 0$) and no idiosyncratic uncertainty for banks (α constant), there is a unique pure F, namely the one in which $\alpha d_i = y_i$ for all i. However, for the same economy there is also a large number of semi-pure equilibria F in which the consumption allocation is the same for all banks but the portfolios are different. In a T, however, price fluctuations require that each bank be autarkic ($\alpha d_i = y_i$) so any T is pure. The role of mixed equilibria is to ensure existence when default introduces non-convexities.

6.2 Equilibrium with non-stochastic α

In the special case with α constant the following theorem partitions the equilibrium set into two cases with distinctive properties.

Theorem 1 *Suppose that α is constant and let (ρ, m, x, y, p) be an equilibrium of the limit economy, in which $\varepsilon = 0$. There are two possibilities:*

> (*i*) (ρ, m, x, y, p) *is a semi-pure, fundamental equilibrium*
> *in which the probability of default is zero;*
> (*ii*) (ρ, m, x, y, p) *is a pure, trivial sunspot equilibrium*
> *in which the probability of default is zero.*

Proof. See Section 8. ∎

By definition, an equilibrium must be either an F, T, or N. What Theorem 1 shows is that an N does not occur and each of the remaining cases is associated with distinctive properties in terms of symmetry and probability of default.

Because the incentive constraint does not bind in either the F or T, both achieve the first-best or Pareto-efficient allocation. No equilibrium can do better. Any bank can guarantee this level of utility by choosing $\alpha d_i = y_i$, where d_i is the deposit contract chosen in the F. For this choice of (d_i, y_i) prices have no effect on the bank's budget constraint and the depositors will receive the first-best consumption. In an N, by contrast, agents receive noisy consumption allocations. Because they are risk averse, the noise in their consumption allocations is inefficient. Since we have seen that the bank can guarantee more to the depositors, this kind of equilibrium cannot exist.

6.3 Equilibrium with stochastic α

Banks may choose (d_i, y_i) so that they are forced to default in some states because of uncertainty about the idiosyncratic shock α. In order to distinguish crises caused by aggregate extrinsic uncertainty from defaults caused by idiosyncratic shocks, we will assume that the parameters are such that default is never optimal in the F. In that case, the bank's optimal choice of (d_i, y_i) must satisfy the incentive constraint, so the bank's optimal decision problem can be written as follows. At the equilibrium price $\bar{p} = 1/r$, the value of the bank's assets at date 1 is $y_i + \bar{p}r(1 - y_i) = 1$, independently of the choice of y_i. In the absence of default, the budget constraint implies that the consumption at date 2 is given by $r(1 - \alpha d_i)/(1 - \alpha)$. The incentive constraint requires that $r(1 - \alpha d_i)/(1 - \alpha) \geq d_i$. Thus, the decision problem can be written as:

$$\max \quad E\left[\alpha U(d_i) + (1 - \alpha)U\left(r(1 - \alpha d_i)/(1 - \alpha)\right)\right]$$
$$\text{st} \quad r(1 - \alpha d_i)/(1 - \alpha) \geq d_i, \forall \alpha.$$

This is a convex programming problem and has a unique solution for d_i. As noted, y_i is indeterminate, but the equilibrium allocation must satisfy the market-clearing condition

$$\sum_i \rho_i \bar{\alpha} d_i = \sum_i \rho_i y_i$$

where $\bar{\alpha} = E[\alpha]$. Note that there is a single *pure* F $(\rho, m, d, y, \bar{p})$, in which $y_i = E[\alpha d_i]$ for every $i = 1, ..., m$.

In the case of idiosyncratic shocks, the following theorem partitions the equilibrium set into two cases, F and N.

Theorem 2 *Let (ρ, m, d, y, p) be an equilibrium of the limit economy, in which $\varepsilon = 0$. There are two possibilities:*

> *(i) (ρ, m, d, y, p) is a semi-pure, fundamental equilibrium in which the probability of default is zero;*
> *(ii) (ρ, m, d, y, p) is a non-trivial sunspot equilibrium which is pure if the probability of default is zero.*

Proof. See Section 8. ∎

The fundamental equilibrium is semi-pure for the usual reasons and there is no default by assumption. Unlike the case with no idiosyncratic shocks (non-stochastic α), there can be no trivial sunspot equilibrium. To see this, suppose that some bank group i with measure ρ_i chooses (d_i, y_i) and that default occurs with positive probability. Then consumption for both early and late consumers is equal to $y_i + p(\theta)r(1 - y_i)$, which is independent of $p(\theta)$ only if $y_i = 1$. In that case there is no point in choosing $d > 1$ and so no need for default. If there is no probability of default, then the consumption of the late consumers is

$$\frac{y_i + p(\theta)r(1 - y_i) - \alpha d_i}{(1 - \alpha)p(\theta)}.$$

This is independent of $p(\theta)$ only if $y_i = \alpha d_i$, which cannot hold unless α is a constant. Thus, there can be no T.

The only remaining possibility is an N. If the probability of default is zero, then the bank's decision problem is a convex programming problem and the usual methods suffice to show uniqueness of the optimum choice of (d, y) under the maintained assumptions.

As long as there is no possibility of default, an equilibrium (m, ρ, d, x, p) must be either an F or a T. If there is a positive probability of default, the equilibrium (m, ρ, d, x, p) must be an N. Any N is characterized by a finite number of prices. To see this, let $B \subset \{1, ..., m\}$ denote the groups that default at some state θ. If $p(\theta) < 1$ then complementary slackness and the market-clearing condition (2) imply that

$$\sum_{i \in B} \rho_i w_i(\theta) + \sum_{i \notin B} \alpha \rho_i d_i = \sum_i \rho_i y_i.$$

Substituting for $w_i(\theta) = y_i + p(\theta)r(1 - y_i)$ gives

$$\sum_{i \in B} \rho_i \left(y_i + p(\theta)r(1 - y_i) \right) + \sum_{i \notin B} \alpha \rho_i d_i = \sum_i \rho_i y_i,$$

or

$$p(\theta) = \frac{\sum_i \rho_i y_i - \sum_{i \in B} \rho_i y_i - \sum_{i \notin B} \alpha \rho_i d_i}{\sum_{i \in B} \rho_i r(1 - y_i)},$$

where the denominator must be positive. If not, then $\sum_{i \in B} \rho_i r(1 - y_i) = 0$, which implies that $y_i = 1$ for every $i \in B$. The budget constraint implies that $x_{i1}(\theta) \leq 1$, in which case there can be no default, contradicting the definition of B. So the denominator must be positive. The expression on the right hand side can only take on a finite number of values because there is a finite number of sets B. Thus, there can be at most a finite number of different prices observed in equilibrium.

We have proved the following corollary.

Corollary 3 *Let (ρ, m, d, y, p) be an equilibrium. There are three possibilities:*

In an F, $p(\cdot)$ has a single value: $p(\theta) = 1/r$;
In a T, $p(\cdot)$ can have a finite or infinite number of values;
In an N, $p(\cdot)$ has a finite number of values.

The properties of equilibria in the limit economy are summarized in the following table.

Symmetry	Type	Default	# of $p(\cdot)$
Semi-pure	F	No	One
Pure	T	No	Finite or infinite
Mixed or pure	N	Possible	Finite

6.4 Limit equilibria

An equilibrium of the limit economy with no aggregate intrinsic uncertainty is *robust* if the equilibrium does not change very much when the economy is perturbed by introducing a small amount of intrinsic uncertainty. In other words, an equilibrium *in the limit* is robust if it is the *limit* as $\varepsilon \searrow 0$ of sequences of equilibria of the corresponding perturbed economies.

We have already seen that introducing a small amount of aggregate uncertainty $\varepsilon > 0$ changes the properties of equilibria. However, we have to be very careful in interpreting these results because the changes can be quite subtle. A few examples will illustrate the possibilities. The first is an example of a robust T, that is, a trivial sunspot equilibrium that is the limit of equilibria with vanishing aggregate uncertainty. The second example illustrates a non-robust T, that is, a trivial sunspot equilibrium that is *not* the limit of equilibria with vanishing aggregate uncertainty. The third example illustrates a robust N, that is, a non-trivial sunspot equilibrium that is the limit of equilibria with vanishing aggregate uncertainty. The fundamental equilibrium is *never* robust, of course.

Example 1. Consider the example studied in Section 5. We saw that when $\varepsilon = 0$ there is a fundamental equilibrium with no price variation and trivial sunspot equilibria with price variation (see Table 1 for a summary). When $\varepsilon > 0$ there were two groups of banks, safe and risky, adopting different strategies at date 0 and that the risky banks defaulted with positive probability at date 1. There are two parameters that determine the impact of default in equilibrium. One is the proportion of banks choosing the risky strategy. The other is the probability of the bad state $\theta = 1$ in which the risky banks default. As we let $\varepsilon \searrow 0$, price volatility is bounded away from zero so there must be non-negligible price volatility in the limit. Further, it is optimal for a bank to choose the risky strategy and default with positive probability for each $\varepsilon > 0$ and the same must be true in the limit. However, the proportion of risky banks converges to zero as $\varepsilon \searrow 0$, so in the limit there is no default. In the terminology introduced earlier, the limit of the fundamental equilibrium with a vanishingly small amount of aggregate uncertainty is a trivial sunspot equilibrium. This is in fact the equilibrium discussed in Section 5, but with the addition of the risky banks with $\rho_2 = 0$. The equilibrium parameters are described in Table 1, Panel 1B.

Example 2. Introducing a small amount of aggregate uncertainty destabilizes the equilibrium in which prices are constant $\bar{p} = 1/r$. It can also destabilize equilibria in which prices are not constant. Consider the following variation on Example 1, in which $U(c_t) = Ln(c_t)$, $r = 1.5$, $\alpha = 0.5$, and

$$\theta = \begin{cases} 0 & w.\ pr.\ 0.7 \\ 1 & w.\ pr.\ 0.3 \end{cases}$$

generate the equilibria shown in Table 2. The safe banks hold enough liquidity to meet depositors' demand when $\theta = 1$. This means they hold surplus liquidity when $\theta = 0$ and so the market-clearing price is $p(0) = 1$. This is the only price

at which they are willing to hold surplus liquidity. It is not optimal to adopt the risky default strategy in this equilibrium, which proves that it cannot be the limit of a sequence of equilibria with $\rho_2 > 0$ and a positive probability of default. This is a failure of lower hemi-continuity.

Example 3 Introducing a small amount of aggregate uncertainty in Example 1 destabilizes the equilibrium in which prices are constant $\bar{p} = 1/r$, but it does not destabilize the first-best allocation. This is because α is constant, so that a bank can achieve the first best without using markets and without being affected by price volatility. Hence, when $\varepsilon > 0$ is small, the bank can do almost as well. Thus, even if there is price volatility and a positive probability of default, the welfare loss is small and vanishes as $\varepsilon \searrow 0$. In order to get a big welfare impact from a small amount of aggregate uncertainty, we need idiosyncratic risk for banks.

Consider the following variation on Example 1, in which $U(c_t) = Ln(c_t)$, $r = 1.5$,

$$\alpha = \begin{cases} 0.75 & w. \; pr. \; 0.5 \\ 0.85 & w. \; pr. \; 0.5 \end{cases}$$

and

$$\theta = \begin{cases} 0 & w. \; pr. \; 0.65 \\ 1 & w. \; pr. \; 0.35 \end{cases}$$

The equilibria that occur in this example are shown in Table 3. It can be seen that the fundamental equilibrium in Panel $3.1A$ is unchanged from Panel $1.1A$. However the sunspot equilibria in Panels $1.1B$ and $3.1B$ are not the same. Trade occurs at the low price and this adversely affects the welfare of the safe banks as well as the risky banks. In Panel $3.1B$, d^s is lowered to 0.995 from 1 in Panel $1.1B$ to reduce the effects of trading at this low price. Similarly, y^s is increased to 0.810 from 0.800. Expected utility is lowered from 0.081 in Panel $3.1A$ to 0.080 in Panel $3.1B$. As with the autarkic equilibria introducing a small amount of aggregate uncertainty eliminates equilibria with a non-stochastic price. Only the sunspot equilibrium is robust. Panel 3.2 shows the equilibrium with intrinsic uncertainty $\varepsilon = 0.010$ close to the sunspot equilibrium. In this case $\rho_1 = 0.996$ and $\rho_2 = 0.004$ so that there is entry by both types of bank.

6.5   Limit theorems

The definition of equilibrium presented in Section 4, prices $p(\theta)$ and consumption $x(\cdot, \theta)$ are functions of θ. When $\varepsilon = 0$, a change in θ has no effect on preferences, so any dependence of $p(\theta)$ or $x(\cdot, \theta)$ on θ represents extrinsic uncertainty. When $\varepsilon > 0$, on the other hand, a change in θ has an effect on preference, so so any dependence of $p(\theta)$ or $x(\cdot, \theta)$ on θ represents intrinsic uncertainty. Thus, an equilibrium for an economy with aggregate uncertainty, $\varepsilon > 0$, is by definition "fundamental" in the sense that endogenous variables depend only on real exogenous shocks. To avoid confusion with the fundamental equilibrium in the

22

limit economy with $\varepsilon = 0$, we use the letter A to denote *equilibria with aggregate uncertainty.*

The next theorem characterizes the properties of equilibria in perturbed economies.

Theorem 4 *Let (m, ρ, d, y, p) denote an A. Then either there is a positive probability of default or there is non-trivial price volatility, i.e., $p(\theta) = 1$ with positive probability and $p(\theta) < 1/r$ with positive probability.*

Proof. See Section 8. ∎

We note several other properties of equilibrium with $\varepsilon > 0$. First, there cannot be a state in which all banks are in default, for this would imply $p(\theta) = 0$ which is inconsistent with equilibrium. Hence, any equilibrium with default must be mixed. Secondly, as we have seen, if there is no default there must be a positive probability that $p(\theta) = 1$. Finally, in the absence of default, for any value of $\varepsilon > 0$, the volatility of asset prices, as measured by the variance of $p(\cdot)$, is bounded away from zero. Both assets are held at date 0 in equilibrium and this requires that the high returns to the long asset, associated with $p(\theta) = 1$, must be balanced by low returns associated with $p(\theta) < 1/r$. These properties are all preserved in the limit as $\varepsilon \to 0$.

The next result shows that the limit of a sequence of equilibria is an equilibrium in the limit.

Theorem 5 *Consider a sequence of perturbed economies corresponding to $\varepsilon = 1/q$, where q is a positive integer, and let $(m^q, \rho^q, d^q, y^q, p^q)$ be the corresponding As. For some convergent subsequence $q \in Q$ let*

$$(m^0, \rho^0, d^0, y^0, p^0) = \lim_{q \in Q}(m^q, \rho^q, d^q, y^q, p^q).$$

If $d_i^0 > 0$, and $0 < y_i^0 < 1$ for $i = 1, ..., m^0$ then $(m^0, \rho^0, x^0, y^0, p^0)$ is an equilibrium of the limit economy.

Proof. See Section 8. ∎

Theorem 5 shows that, under certain conditions, the limit of a sequence of equilibria as $\varepsilon \to 0$ is an equilibrium of the limit economy where $\varepsilon = 0$. We are also interested in the opposite question, namely, which equilibria of the limit economy in which $\varepsilon = 0$ are limits of equilibria from the perturbed economy in which $\varepsilon > 0$? This requirement of lower hemi-continuity is a test of *robustness*: if a small perturbation of the limit economy causes an equilibrium to disappear, we argue that the equilibrium is not robust. Since there are many equilibria of the limit economy, it is of interest to see whether any of these equilibria can be eliminated by being shown to be non-robust.

We have shown in Theorem 4 that equilibria of the perturbed economy are characterized by default or non-trivial asset price uncertainty. Furthermore, because the random variable θ has a finite support, the probability of these events is bounded away from zero, uniformly in ε. The fundamental equilibrium of the limit economy with $\varepsilon = 0$ has none of these properties. However, this

does not by itself prove that the fundamental equilibrium is not robust. It could be the limit of a sequence of equilibria of the perturbed economy if the fraction of banks that defaults in equilibrium converges to zero as $q \to \infty$. However, if the fundamental equilibrium were the limit referred to in Theorem 5, then it must be the case that (a) default is optimal in the limit and (b) there is no price volatility in the limit. These two properties can be shown to be inconsistent.

Corollary 6 *If $(m^0, \rho^0, x^0, y^0, p^0)$ is the equilibrium of the limit economy mentioned in Theorem 5, then $(m^0, \rho^0, x^0, y^0, p^0)$ is not an F of the limit economy, i.e., it must be either an N or a T.*

Proof. Suppose that, contrary to what we wish to prove, $(m^0, \rho^0, x^0, y^0, p^0)$ is the fundamental equilibrium. Then $p^0(\theta) = 1/r < 1$ with probability one and, hence, $p^q(\theta) < 1$ for all θ and all q sufficiently large. From Theorem 4 we know that this implies some group i defaults with positive probability for all sufficiently large q. Thus, in the limit, default must be optimal for group i and $\rho_i^q \to \rho_i^0 = 0$. However, we assumed in Section 6.3 that default is not optimal for stochastic α and it is clear that it is not optimal with non-stochastic α because

$$U(1) < \max_{\alpha c_1 + (1-\alpha)c_2/r = 1} \left\{ \alpha U(c_1) + (1 - \alpha)U(c_2) \right\}.$$

This contradiction proves that $(m^0, \rho^0, x^0, y^0, p^0)$ is not an F, i.e., the F is not robust. ∎

This shows that the limit of a sequence of equilibria of the perturbed economy must be either a T or N, but not an F. So this approach of regarding sunspot equilibria as a limiting case of A actually eliminates the F in the limit economy or, in other words, selects the sunspot equilibria as the only robust equilibria.

<div align="center">

Corresponding types of equilibria
— Place Figure 1 here —

</div>

7   Discussion

In this paper, we have investigated the relationship between intrinsic and extrinsic uncertainty in a model of financial crises. Our general approach is to regard extrinsic uncertainty as a limiting case of intrinsic uncertainty. In our model, small shocks to the demand for liquidity are always associated with large fluctuations in asset prices. These price fluctuations cause financial crises to occur with positive probability. This is the sense in which there is financial fragility. In the limit, as the liquidity shocks become vanishingly small, the model converges to one with extrinsic uncertainty. The limit economy has three kinds of equilibria,

- fundamental equilibria, in which there is neither aggregate uncertainty nor a positive probability of crisis;

- trivial sunspot equilibria, in which prices fluctuate but the real allocation is the same as in the fundamental equilibrium;

- and non-trivial sunspot equilibria, in which prices fluctuate and financial crises can occur with positive probability.

Introducing small shocks into the limit economy destabilizes the first type of equilibrium, leaving the second and third as possible limits of equilibria of the perturbed economy. If α is a constant, the limiting equilibrium as $\varepsilon \to 0$ is a trivial sunspot equilibrium. If α is random, the limiting equilibrium as $\varepsilon \to 0$ is a non-trivial sunspot equilibrium. We argue that only the sunspot equilibria are robust, in the sense that a small perturbation of the model causes a small change in these equilibria. This selection criterion provides an argument for the relevance of extrinsic uncertainty and the necessity of financial crises.

Although crises in the limit economy arise from extrinsic uncertainty, the causation is quite different from the bank run story of Diamond and Dybvig (1983). In the Diamond-Dybvig story, bank runs are spontaneous events that depend on the decisions of late consumers to withdraw early. Given that almost all agents withdraw at date 1, early withdrawal is a best response for every agent; but if late consumers were to withdraw at date 2, then late withdrawal is a best response for every late consumer. So there are two "equilibria" of the coordination game played by agents at date 1, one with a bank run and one without. This kind of coordination failure plays no role in the present model. In fact, coordination failure is explicitly ruled out: a bank run occurs only if the bank cannot simultaneously satisfy its budget constraint and its incentive constraint. (For a fuller discussion of these issues, see Allen and Gale (2003)). When bankruptcy does occur, it is the result of low asset prices. Asset prices are endogenous, of course, and there is a "self-fulfilling" element in the relationship between asset prices and crises. Banks are forced to default and liquidate assets because asset prices are low and asset prices are low as a result of mass bankruptcy and the association liquidation of bank assets.

One interesting difference between the present story and Diamond and Dybvig (1983) is that here a financial crisis is a *systemic event*. A crisis occurs only if the number of defaulting banks is large enough to affect the equilibrium asset price. In the Diamond-Dybvig model, by contrast, bank runs are an idiosyncratic phenomenon. Whether a run occurs at a particular bank depends on the decisions taken by the bank's depositors. It is only by coincidence that runs are experienced by several banks at the same time.

At the heart of our theory is a pecuniary externality: when one group of banks defaults and liquidates its assets, it forces down the price of assets and this may cause another group of banks to default. This pecuniary externality may be interpreted as a form of contagion.

Allen and Gale (2000a) describes a model of contagion in a multi-region economy. Bankruptcy is assumed to be costly: long-term projects can be liquidated prematurely but a fraction of the returns are lost. This deadweight loss from liquidation creates a spillover effect in the adjacent regions where the

claims on the bankrupt banks are held. If the spillover effect is large enough, the banks in the adjacent regions will also be forced into default and liquidation. Each successive wave of bankruptcies increases the loss of value and strengthens the impact of the spillover effect on the next region. Under certain conditions, a shock to one small region can propagate throughout the economy. By contrast, in the present model, a bank's assets are always marked to market. Given the equilibrium asset price p, bankruptcy does not change the value of the bank's portfolio. However, if a group of banks defaults, the resulting change in the price p may cause other banks to default, which will cause further changes in p, and so on. The "contagion" in both models is instantaneous.

Several features of the model are special and deserve further consideration.

Pecuniary externalities "matter" in our model because markets are incomplete: if banks could trade Arrow securities contingent on the states θ, they would be able to insure themselves against changes in asset values (Allen and Gale (2003)). No trade in Arrow securities would take place in equilibrium, but the existence of the markets for Arrow securities would have an effect. The equilibrium allocation would be incentive-efficient, sunspots would have no real impact, and there would be no possibility of crises.

It is important that small shocks lead to large fluctuations in asset prices (and large pecuniary externalities). We have seen, in the case of trivial sunspot equilibria, that small price fluctuations have no real effect. What makes the pecuniary externality large in this example is inelasticity of the supply and demand for liquidity. Inelasticity arises from two assumptions. First, the supply of liquidity at date 1 is fixed by the decisions made at date 0. Secondly, the assumption of Diamond-Dybvig preferences implies that demand for consumption at date 1 is interest-inelastic. This raises a question about the robustness of the results when more general preferences are allowed.

One justification for the Diamond-Dybvig preferences is that they provide a cheap way of capturing, within the standard, Walrasian, auction-market framework, some realistic features of alternative market clearing mechanisms. In an auction market, prices and quantities adjust simultaneously in a *tatonnement* process until a full equilibrium is achieved. An alternative mechanism is one in which quantities are chosen before prices are allowed to adjust. An example is the use of market orders. If depositors were required to make a withdrawal decision before the asset price was determined in the interbank market, the same inelasticity of demand would be observed even if depositors had preferences that allowed for intertemporal substitution. There may be other institutional structures that have the qualitative features of our example. An investigation of these issues goes far beyond the scope of the present paper, but it is undoubtedly one of the most important topics for future research.

The model of banks that we have used is special, but the same general arguments apply to other types of intermediary. As long as intermediaries use non-contingent contracts and markets are incomplete, small liquidity shocks will result in extreme price volatility and intermediaries will be subject to default.

In analyzing this model of liquidity and crises, we have ignored the possibility of intervention by the central bank or government. A full understanding of

the laisser-faire case should be seen as a prelude to the analysis of optimal intervention. In the same way, the analysis of a "real" model is a prelude to the introduction of fiat money into the model. These are both important topics for future research.

8 Proofs

8.1 Proof of Theorem 1

By definition, an equilibrium (ρ, m, d, y, p) must be either an F, T, or N. The theorem is proved by considering each case in turn.

Case (i). If (ρ, m, d, y, p) is an F then by definition the price $p(\theta)$ is constant and, for each group i, the consumption allocation $x(d_i, y_i, \alpha, \theta)$ is constant. In particular, $x_1(d_i, y_i, \alpha, \theta) = d_i$ with probability one so there is no default in equilibrium.

Let $\bar{p}$ denote the constant price and $c_i = (c_{i1}, c_{i2})$ the consumption allocation chosen by banks in group i. The decision problem of a bank in group i is

$$
\begin{aligned}
\max \quad & [\alpha U(c_{i1}) + (1-\alpha)U(c_{i2})] \\
\text{s.t.} \quad & c_{i1} \leq c_{i2}, 0 \leq y_i \leq 1 \\
& \alpha c_{i1} + (1-\alpha)\bar{p}c_{i2} \leq y_i + \bar{p}r(1-y_i).
\end{aligned}
$$

Clearly, y_i will be chosen to maximize $y_i + \bar{p}r(1-y_i)$. Then the strict concavity of $U(\cdot)$ implies that c_i is uniquely determined and independent of i. Thus, the equilibrium is semi-pure: $c_i = c_j$ for any i and j.

Case (ii). Suppose that (ρ, m, d, y, p) is a T. Then by definition $p(\theta)$ is not constant and, for each group i, the consumption allocation $x(d_i, y_i, \alpha, \theta)$ is constant. In particular, $x_1(d_i, y_i, \alpha, \theta) = d_i$ with probability one so there is no default in equilibrium.

Let c_i denote the consumption allocation chosen by banks in group i. The budget constraint at date 1 reduces to

$$
\alpha c_{i1} - y_i = -p(\theta)((1-\alpha)\alpha c_{i2} - r(1-y_i)), \text{ a.s.}
$$

Since $p(\theta)$ is not constant, this equation can be satisfied only if

$$
\alpha c_{i1} - y_i = (1-\alpha)c_{i2} - r(1-y_i) = 0.
$$

Then the choice of (c_i, y_i) must solve the problem

$$
\begin{aligned}
\max \quad & E\left[\alpha U(c_{i1}) + (1-\alpha)U(c_{i2})\right] \\
\text{s.t.} \quad & c_{i1} \leq c_{i2}, 0 \leq y_i \leq 1 \\
& \alpha c_{i1} = y_i, (1-\alpha)c_{i2} = r(1-y_i).
\end{aligned}
$$

The strict concavity of $U(\cdot)$ implies that this problem uniquely determines the value of c_i and hence y_i, independently of i. Consequently, the equilibrium is pure.

27

Case (iii). Suppose that (ρ, m, d, y, p) is an N. The allocation of consumption for group i is $x(d_i, y_i, \alpha, \theta)$, and the expected utility of each group is the same

$$E\left[u(x(d_i, y_i, \alpha, \theta), \alpha\right] = E\left[u(x(d_j, y_j, \alpha, \theta), \alpha\right], \forall i, j.$$

The mean allocation $\sum_i \rho_i x(d_i, y_i, \alpha, \theta)$ satisfies the market clearing conditions for every θ and hence consumption bundle $E\left[\sum_i \rho_i x(d_i, y_i, \alpha, \theta)\right]$ is feasible for the planner. Since agents are strictly risk averse,

$$u\left(E\left[\sum_i \rho_i x(d_i, y_i, \alpha, \theta)\right], \alpha\right) > \sum_i \rho_i E\left[u(x(d_i, y_i, \alpha, \theta), \alpha)\right].$$

This contradicts the equilibrium conditions, since the individual bank could choose

$$y_0 = \alpha d_0 = \alpha E\left[\sum_i \rho_i x(d_i, y_i, \alpha, \theta)\right]$$

and achieve a higher utility.

8.2 Proof of Theorem 2

Again we let (ρ, m, d, y, p) be a fixed but arbitrary equilibrium and consider each of three cases in turn.

Case (i). If (ρ, m, d, y, p) is an F, $p(\theta)$ is constant and, for each group i and each α, the consumption allocation $x(d_i, y_i, \alpha, \theta)$ is constant. In particular, $x_1(d_i, y_i, \alpha, \theta) = d_i$ with probability one so there is no default in equilibrium.

Let $\bar{p}$ denote the constant price and $c_i(\alpha) = (c_{i1}(\alpha), c_{i2}(\alpha))$ the consumption allocation chosen by banks in group i. The decision problem of a bank in group i is

$$\begin{aligned}
\max \quad & E\left[\alpha U(c_{i1}(\alpha)) + (1-\alpha)U(c_{i2}(\alpha))\right] \\
\text{s.t.} \quad & c_{i1}(\alpha) \le c_{i2}(\alpha), 0 \le y_i \le 1 \\
& \alpha c_{i1}(\alpha) + (1-\alpha)\bar{p}c_{i2}(\alpha) \le y_i + \bar{p}r(1-y_i).
\end{aligned}$$

Clearly, y_i will be chosen to maximize $y_i + \bar{p}r(1-y_i)$. Then the strict concavity of $U(\cdot)$ implies that $c_i(\alpha)$ is uniquely determined and independent of i (but not of α). Thus, the equilibrium is semi-pure: $c_i = c_j$ for any i and j.

Case (ii). Suppose that (ρ, m, d, y, p) is a T. Then by definition $p(\theta)$ is not constant and, for each group i and α, the consumption allocation $x(d_i, y_i, \alpha, \theta)$ is constant. In particular, $x_1(d_i, y_i, \alpha, \theta) = d_i$ with probability one so there is no default in equilibrium.

Let $c_i(\alpha)$ denote the consumption allocation chosen by banks in group i. The budget constraint at date 1 reduces to

$$\alpha c_{i1}(\alpha) - y_i = -p(\theta)((1-\alpha)\alpha c_{i2}(\alpha) - r(1-y_i)), \text{ a.s.}$$

Since $p(\theta)$ is not constant, this equation can be satisfied only if

$$\alpha c_{i1}(\alpha) - y_i = (1-\alpha)c_{i2}(\alpha) - r(1-y_i) = 0.$$

Since α is not constant this can only be true if $c_{i1}(\alpha) = c_{i2}(\alpha) = 0$, a contradiction. Thus, there cannot be a T when α is not constant.

Case (iii). The only remaining possibility is that (ρ, m, d, y, p) is an N. If there is no default in this equilibrium, then each bank in group i solves the problem

$$\max \quad E\left[\alpha U(d_i) + (1-\alpha)U\left(\frac{y_i + p(\theta)r(1-y_i) - \alpha d_i}{(1-\alpha)p(\theta)}\right)\right]$$
$$\text{st} \quad \frac{y_i + p(\theta)r(1-y_i)}{(1-\alpha)p(\theta)} \geq d_i.$$

This is a convex programming problem and it is easy to show that the strict concavity of $U(\cdot)$ uniquely determines (d_i, y_i). Thus an N without default is pure.

8.3   Proof of Theorem 4

Suppose that the probability of default in (ρ, m, d, y, p) is zero. Then for each group i, $x_1(d_i, y_i, \alpha, \theta) = d_i$ and the market-clearing condition (2) implies

$$\sum_i E\left[\rho_i \eta(\alpha, \theta)\right] d_i \leq \sum_i \rho_i y_i. \tag{5}$$

There are two cases to consider. In the first case, $\sum_i \rho_i y_i = 0$. Then $d_i = 0$ for every i and the utility achieved in equilibrium is

$$E\left[\eta(\alpha, \theta)U(0) + (1 - \eta(\alpha, \theta))U\left(\frac{r}{1 - \eta(\alpha, \theta)}\right)\right].$$

By holding a small amount $\delta > 0$ of the short asset, positive consumption could be guaranteed at the first date. Optimality requires that

$$E\left[\eta(\alpha, \theta)U\left(\frac{\delta}{\eta(\alpha, \theta)}\right)\right] + E\left[(1 - \eta(\alpha, \theta))U\left(\frac{r(1-\delta)}{1 - \eta(\alpha, \theta)}\right)\right]$$
$$\leq \quad E\left[\eta(\alpha, \theta)U(0) + (1 - \eta(\alpha, \theta))U\left(\frac{r}{1 - \eta(\alpha, \theta)}\right)\right].$$

for any $\delta > 0$. In the limit as $\delta \to 0$,

$$E\left[U'(0)\right] - E\left[U'\left(\frac{r}{1 - \eta(\alpha, \theta)}\right)\right] \leq 0,$$

which contradicts the assumption that $U'(0) = \infty$.

In the second case, $\sum_i \rho_i y_i > 0$. Then the market-clearing condition (5) and the fact that $\eta(\alpha, \theta) = \alpha + \varepsilon\theta$ together imply that

$$\sum_i \rho_i \eta(\alpha, \theta)d_i < \sum_i \rho_i y_i$$

29

with positive probability. The complementary slackness condition implies that $p(\theta) = 1$ with positive probability and the short asset will be dominated by the long asset at date 0 unless $p(\theta) < 1/r$ with positive probability. Thus, the price volatility is non-trivial if the probability of default is zero.

8.4   Proof of Theorem 5

Continuity and the convergence of $\{(\rho^q, m, d^q, y^q, p^q)\}$ immediately implies the following properties of the limit point $(\rho^0, m, d^0, y^0, p^0)$:

(i) $\sum_i \rho_i^0 = 1$ and $\rho_i^0 \geq 0$ for every i so (ρ^0, m) is a partition.

(ii) For every i, $(d_i^0, y_i^0) \in \mathbf{R}_+ \times [0, 1]$. The market-clearing conditions

$$\sum_i \rho_i^0 E\left[\alpha \bar{x}_1(d_i^0, y_i^0, \alpha, \theta)|\theta\right] \leq \sum_i \rho_i^0 y_i^0,$$

and

$$\sum_i \rho_i^0 E\left\{\alpha \bar{x}_1(d_i^0, y_i^0, \alpha, \theta) + (1-\alpha)\bar{x}_2(d_i^0, y_i^0, \alpha, \theta)|\theta\right\}$$
$$= \sum_i \rho_i^0 \left\{y_i^0 + r(1 - y_i^0)\right\}$$

are satisfied in the limit and the complementary slackness condition holds. Thus, (d^0, y^0) is an attainable allocation.

It remains to show that (d_i^0, y_i^0) is optimal for each i. Let $W^q(d_i, y_i, \alpha, \theta)$ denote the utility associated with the pair (d_i, y_i) in the perturbed economy corresponding to $\varepsilon = 1/q$ when the price function is p^q and let $W^0(d_i, y_i, \alpha, \theta)$ denote the utility associated with the pair (d_i, y_i) in the limit economy corresponding to $\varepsilon = 0$, where the price function is p^0. The function $W_i^0(\cdot)$ is discontinuous at the bankruptcy point defined implicitly by the condition

$$\left(\alpha + (1-\alpha)p^0(\theta)\right) d_i = y_i + p^0(\theta)r(1 - y_i). \tag{6}$$

If (6) occurs with probability zero in the limit, then it is easy to see from the assumed convergence properties that

$$W^q(d_i, y_i, \alpha, \theta) \to W^0(d_i, y_i, \alpha, \theta), \text{ a.s.}$$

and hence

$$\lim_{q \to \infty} E\left[W^q(d_i, y_i, \alpha, \theta)\right] = E\left[W^0(d_i, y_i, \alpha, \theta)\right].$$

Let (d_i^0, y_i^0) denote the pair corresponding to the limiting consumption allocation x_i^0 and let $\{(d_i^q, y_i^q)\}$ denote the sequence of equilibrium choices converging to (d_i^0, y_i^0). There may exist a set of states (α, θ) with positive measure such that (d_i^q, y_i^q) implies default in state (α, θ) for arbitrarily large q but that (d_i^0, y_i^0) does not imply default in state (α, θ). Then at least we can say that

$$\liminf_q W^q(d_i^q, y_i^q, \alpha, \theta) \leq W^0(d_i^0, y_i^0, \alpha, \theta), \text{ a.s.}$$

and this implies that

$$\liminf_q E\left[W^q(d_i^q, y_i^q, \alpha, \theta)\right] \leq E\left[W^0(d_i^0, y_i^0, \alpha, \theta)\right].$$

Now suppose, contrary to what we want to prove, that (d_i^0, y_i^0) is not optimal. Then there exists a pair (d_i, y_i) such that $E\left[W^0(d_i, y_i, \alpha, \theta)\right] > E\left[W^0(d_i^0, y_i^0, \alpha, \theta)\right]$. If $d_i = 0$ then (6) holds with probability zero and it is clear that for some sufficiently large value of q, $E\left[W^0(d_i, y_i, \alpha, \theta)\right] > E\left[W^q(d_i^q, y_i^q, \alpha, \theta)\right]$, contradicting the equilibrium conditions. If $d_i > 0$, then either the critical condition (6) holds with probability zero or we can find a slightly lower value $d' < d$ that does satisfy the critical condition. To see this, note first that the critical condition uniquely determines the value of $p^0(\theta)$ as long as

$$(1 - \alpha)d \neq r(1 - y)$$

which is true for almost every value of d. Secondly, if the value of $p^0(\theta)$ for which the critical condition is satisfied is an atom, we can always find a slightly smaller value $d_i' < d_i$ such that the value of $p^0(\theta)$ for which the critical condition is satisfied is not an atom. Furthermore, reducing d slightly will at most reduce the payoff by a small amount, so for $d_i' < d_i$ and close enough to d_i we still have $E\left[W^0(d_i', y_i, \alpha, \theta)\right] > E\left[W^0(d_i^0, y_i^0, \alpha, \theta)\right]$. Then this leads to a contradiction in the usual way.

References

Allen, F. and D. Gale (1998). "Optimal Financial Crises," *Journal of Finance* **53,** 1245-1284.

Allen, F. and D. Gale (2000a). "Financial Contagion," *Journal of Political Economy* **108,** 1-33.

Allen, F. and D. Gale (2000b). "Optimal Currency Crises," *Carnegie Rochester Series on Public Policy* **53**, 177-230.

Allen, F. and D. Gale (2000c). "Bubbles and Crises," *The Economic Journal* **110**, 236-256

Allen, F. and D. Gale (2003). "Financial Intermediaries and Markets," New York University: www.nyu.edu/econ/user/galed/fim4rev.pdf.

Alonso, I. (1996). "On Avoiding Bank Runs," *Journal of Monetary Economics* **37**, 73-87.

Azariadis, C. (1981). "Self-Fulfilling Prophecies," *Journal of Economic Theory* **25**, 380-96.

Bernanke, B. and M. Gertler (1989) "Agency Costs, Net Worth, and Business Fluctuations," *American Economic Review* **79**, 14-31.

Bernardo, A and I. Welch (2002). "Financial Market Runs," working paper, Yale School of Management.

Bhattacharya, S. and D. Gale (1987). "Preference Shocks, Liquidity and Central Bank Policy," in William Barnett and Kenneth Singleton, eds., *New Approaches to Monetary Economics,* New York, NY: Cambridge University Press.

Bryant, J. (1980). "A Model of Reserves, Bank Runs, and Deposit Insurance," *Journal of Banking and Finance* **4,** 335-344.

Calomiris, C. and G. Gorton (1991). "The Origins of Banking Panics, Models, Facts, and Bank Regulation," in Glenn Hubbard, ed., *Financial Markets and Financial Crises* Chicago, IL: University of Chicago Press.

Calomiris, C. and J. Mason (2000). "Causes of U.S. Bank Distress During the Depression," NBER Working Paper W7919.

Cass, D. and K. Shell (1983). "Do Sunspots Matter?" *Journal of Political Economy* **91**, 193-227.

Champ, Bruce, Bruce Smith, and Stephen Williamson, 1996. "Currency Elasticity and Banking Panics: Theory and Evidence," *Canadian Journal of Economics* **29**, 828-864.

Chang, R. and A. Velasco (1998). "Financial Crises in Emerging Markets: A Canonical Model," working paper, New York University.

Chang, R. and A. Velasco (2000). "Financial Fragility and the Exchange Rate Regime," *Journal of Economic Theory* **92**, 1-34.

Chari, V. (1989). "Banking Without Deposit Insurance or Bank Panics: Lessons From a Model of the U.S. National Banking System," *Federal Reserve Bank of Minneapolis Quarterly Review* **13** (Summer), 3-19.

Chari, V. and R. Jagannathan (1988). "Banking Panics, Information, and Rational Expectations Equilibrium," *Journal of Finance* **43**, 749-60.

Cone, K. (1983). *Regulation of Depository Institutions.* Stanford University Ph.D. dissertation.

Diamond, D. and P. Dybvig (1983). "Bank Runs, Deposit Insurance, and Liquidity," *Journal of Political Economy* **91**, 401-419.

Farmer, R. (1999). *The Macroeconomics of Self-Fulfilling Prophecies.* Cambridge and London: MIT Press.

Fisher, I. (1933). "The Debt-Deflation Theory of Great Depressions," *Econometrica* **1**, 337-357.

Geanakoplos, J. (1990). "An Introduction to General Equilibrium with Incomplete Asset Markets," *Journal of Mathematical Economics* **19**, 1-38.

Gorton, G. (1988). "Banking Panics and Business Cycles," *Oxford Economic Papers* **40**, 751-781.

Gottardi, P. and A. Kajii (1995). "Generic Existence of Sunspot Equilibria: The Real Asset Case," University of Pennsylvania, CARESS WP 95/12.

Gottardi, P. and A. Kajii (1999). "The Structure of Sunspot Equilibria: The Role of Multiplicity," *Review of Economic Studies* **66**, 713-732.

Hart, O. (1995). *Firms, Contracts and Financial Structure*, Oxford: Oxford University Press.

Hellwig, M. (1994). "Liquidity Provision, Banking, and the Allocation of Interest Rate Risk," *European Economic Review* **38**, 1363-1389.

Jacklin, C. (1986). "Demand Deposits, Trading Restrictions, and Risk Sharing," in E. Prescott and N. Wallace, eds., *Contractual Arrangements for Intertemporal Trade,* Minneapolis, MN: University of Minnesota Press.

Jacklin, C., and S. Bhattacharya (1988). "Distinguishing Panics and Information-based Bank Runs: Welfare and Policy Implications," *Journal of Political Economy* **96**, 568-592.

Kindleberger, C. (1978). *Manias, Panics, and Crashes: A History of Financial Crises,* New York, NY: Basic Books.

Lagunoff, R. and S. Schreft (2001). "A Model of Financial Fragility," *Journal of Economic Theory* **99**, 220-264.

Magill, M. and M. Quinzii (1996). *Theory of Incomplete Markets, Volume I,* Cambridge and London: MIT Press.

Mitchell, W. (1941). *Business Cycles and Their Causes,* Berkeley, CA: University of California Press.

Peck, J. and K. Shell (1999). "Bank Portfolio Restrictions and Equilibrium Bank Runs," working paper, Cornell University.

Postlewaite, A. and X. Vives (1987). "Bank Runs as an Equilibrium Phenomenon," *Journal of Political Economy* **95**, 485-491.

Rochet, J.-C. and J. Tirole, (1996). "Interbank Lending and Systemic Risk," *Journal of Money, Credit, and Banking* 28, 733-762.

Schnabel, I. and H. S. Shin (2002). "Foreshadowing LTCM: The Crisis of 1763," working paper, London School of Economics.

Shell, K. and A. Goenka (1997). "When Sunspots Don't Matter," *Economic Theory* **9**, 169-178.

Wallace, N. (1988). "Another Attempt to Explain an Illiquid Banking System: The Diamond and Dybvig Model with Sequential Service Taken Seriously," *Federal Reserve Bank of Minneapolis Quarterly Review* **12** (Fall), 3-16.

Wallace, N. (1990). "A Banking Model in which Partial Suspension is Best," *Federal Reserve Bank of Minneapolis Quarterly Review* **14** (Fall), 11-23.

Table 1
Equilibria for Example 1

#	ε	$\begin{bmatrix} E[U^s] \\ E[U^r] \end{bmatrix}$	$\begin{bmatrix} y^s \\ y^r \end{bmatrix}$	$\begin{matrix} (c_1^s(0), c_2^s(0)) \\ (c_1^s(1), c_2^s(1)) \\ (c_1^r(0), c_2^r(0)) \\ (c_1^r(1), c_2^r(1)) \end{matrix}$		$\begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$
1.1A	0	0.081	0.800	(1.000, 1.500)		0.667
1.1B	0	$\begin{bmatrix} 0.081 \\ 0.081 \end{bmatrix}$	$\begin{bmatrix} 0.800 \\ 0 \end{bmatrix}$	(1.000, 1.500) (1.000, 1.500) (1.414, 1.500) (0.648, 0.648)		$\begin{bmatrix} 0.943 \\ 0.432 \end{bmatrix}$
1.2	0.010	$\begin{bmatrix} 0.078 \\ 0.078 \end{bmatrix}$	$\begin{bmatrix} 0.810 \\ 0.016 \end{bmatrix}$	(0.998, 1.488) (0.998, 1.520) (1.405, 1.488) (0.651, 0.651)		$\begin{bmatrix} 0.940 \\ 0.430 \end{bmatrix}$

Table 2
Equilibria for Example 2

#	ε	$\begin{bmatrix} E[U^s] \\ E[U^r] \end{bmatrix}$	$\begin{bmatrix} y^s \\ y^r \end{bmatrix}$	$\begin{matrix} (c_1^s(0), c_2^s(0)) \\ (c_1^s(1), c_2^s(1)) \\ (c_1^r(0), c_2^r(0)) \\ (c_1^r(1), c_2^r(1)) \end{matrix}$		$\begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$
2.1A	0	0.203	0.500	(1.000, 1.500)		0.667
2.1B	0	$\begin{bmatrix} 0.203 \\ 0.111 \end{bmatrix}$	$\begin{bmatrix} 0.500 \\ 0 \end{bmatrix}$	(1.000, 1.500) (1.000, 1.500) (1.500, 1.500) (0.563, 0.563)		$\begin{bmatrix} 1.000 \\ 0.375 \end{bmatrix}$
2.2	0.010	$\begin{bmatrix} 0.199 \\ 0.110 \end{bmatrix}$	$\begin{bmatrix} 0.508 \\ 0 \end{bmatrix}$	(0.996, 1.497) (0.996, 1.507) (1.500, 1.500) (0.561, 0.561)		$\begin{bmatrix} 1 \\ 0.374 \end{bmatrix}$

Table 3
Equilibria for Example 3

#	ε	$\begin{bmatrix} E[U^s] \\ E[U^r] \end{bmatrix}$	$\begin{bmatrix} y^s \\ y^r \end{bmatrix}$	$\begin{bmatrix} (c_1^s(0), c_2^{sL}(0), c_2^{sH}(0)) \\ (c_1^s(1), c_2^{sL}(1), c_2^{sH}(1)) \\ (c_1^r(0), c_2^{rL}(0), c_2^{rH}(0)) \\ (c_1^r(1), c_2^{rL}(1), c_2^{rH}(1)) \end{bmatrix}$		$\begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$
$3.1A$	0	0.081	0.800	$(1.000, 1.500)$		0.667
$3.1B$	0	$\begin{bmatrix} 0.080 \\ 0.080 \end{bmatrix}$	$\begin{bmatrix} 0.798 \\ 0 \end{bmatrix}$	$\begin{array}{l} (0.998, 1.648, 1.283) \\ (0.998, 1.648, 1.283) \\ (1.364, 1.502, 1.503) \\ (0.682, 0.682, 0.682) \end{array}$		$\begin{bmatrix} 0.910 \\ 0.455 \end{bmatrix}$
3.2	0.010	$\begin{bmatrix} 0.077 \\ 0.077 \end{bmatrix}$	$\begin{bmatrix} 0.809 \\ 0 \end{bmatrix}$	$\begin{array}{l} (0.995, 1.414, 1.624) \\ (0.995, 1.700, 1.366) \\ (1.360, 1.502, 1.503) \\ (0.678, 0.678, 0.678) \end{array}$		$\begin{bmatrix} 0.940 \\ 0.430 \end{bmatrix}$

Figure 1A
Equilibrium correspondence
Stochastic α

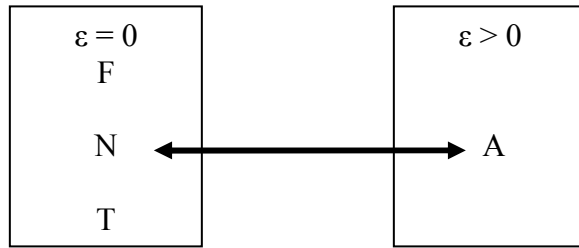| ε = 0		ε > 0
F		
N ◄───┼──►│ A
T		

Figure 1B
Equilibrium correspondence
Non-stochastic α

ε = 0		ε > 0
F		
N		A
T ◄───┼──►│