FINDING A MAXIMUM CLIQUE

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Abstract

An algorithm for finding a maximum clique in an arbitrary graph is described. The algorithm has a worst-case time bound of $k(1.286)^n$ for some constant k, where n is the number of vertices in the graph. Within a fixed time, the algorithm can analyze a graph with 2 3/4 as many vertices as the largest graph which the obvious algorithm (examining all subsets of vertices) can analyze.

Keywords

Algorithm, Clique, Graph, Independent Set

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Introduction

Let $G = (V, E)$ be a graph with $|\hat{V}| = n$ vertices. Consider the problem of discovering a clique (a set of vertices which determine a complete subgraph) of maximum size in G. Cook [1] has shown that if the clique problem has an algorithm with a time bound polynomial in n, then any algorithmically solvable problem has an algorithm with a time bound polynomial in the size of the problem data. Thus any improvement over the obvious clique-finding algorithm is an interesting forward step.

Suppose we examine every subset $S \subseteq V$ to see if S determines a clique, and then we choose the largest clique found. This is the obvious algorithm. Since V has $\mathcal{P}(V) = 2^n$ subsets, the algorithm has a time bound of $O(n2^n)$. However, the algorithm may be improved.

A Fast Algorithm for Finding a Maximum Clique

If we do not examine all subsets of V but only a sufficiently large number of them, we may get a faster method for determining a maximum clique. The basic idea is partion V into two sets, S and V - S. Let G_S and G_{V-S} be the subgraphs of G determined by these two vertex sets. Then any clique in G determines a clique in G_S and a clique in G_{V-S}. Further, any clique C in G_S

may be combined with any clique in $G_{A(C)-S}$ to give a clique in G, if A(C) is the set of vertices adjacent to one or more vertices in C. By finding each clique in G_S, solving a corresponding clique problem in G_{V-S}, and combining all these solutions, we may find a maximum clique in G.

A lemma will state this result more precisely. If $S \subseteq V$, let G_S be the subgraph of G with vertex set S. Let A(S) be the set of vertices adjacent to one or more vertices in S. Finally, let $||G||$ be the size of a maximum clique in G.

<u>Lemma 1</u>: Let G = (V , E) be a graph. Let $S \subseteq V$. Then:

$$\text{(I)} \qquad ||G|| = \max_{\substack{C \text{ a clique} \\ \text{in } G_S}} \left\{ |C| + ||G_{A(C)-S}|| \right\}$$

<u>Proof</u>: If X is a clique in G, $X \cap S$ is a clique in G_S, and $X \cap (V-S)$ is a clique in $G_{A(X \cap S)-S}$. Expression (I) is then immediate.

In fact, the maximum in (I) need not be taken over <u>all</u> cliques in G_S but only over a subset of them. Let $S \subseteq G$ and let X,Y be cliques in G_S. Suppose that $||G_{Y \cup (A(Y)-S)}|| \leq ||G_{X \cup (A(X)-S)}||$. Then X is said to <u>dominate</u> Y. A set of cliques $\mathcal{C} \subseteq \mathcal{P}(S)$ is said to be <u>dominant</u> in G_S if every clique in G_S is dominated by at least one clique in $\mathcal{C}$. Dominance is a transitive relation. A clique X may be shown to dominate a clique Y by giving a simple method of transforming any clique in $G_{Y \cup (A(Y)-S)}$ into a clique of equal size in $G_{X \cup (A(X)-S)}$. For instance, suppose that if C is a clique in $G_{Y \cup (A(Y)-S)}$, then

$(C \cap (A(X) - S)) \cup X$ is always a clique as large as C. Then X dominates Y.

Lemma 2: Let $S \subseteq V$. Let $\mathcal{C}$ be a dominant set of cliques in G_S. Then:

$$\text{(II)} \qquad \| G \| = \max_{C \in \mathcal{C}} \{ |C| + \| G_{A(C)-S} \| \}$$

Proof: Let Y be a clique in G_S. Then some clique $X \in \mathcal{C}$ dominates Y in G_S. If Z is a clique in $G_{Y \cup (A(Y)-S)}$, there is a clique at least as big as Z in $G_{X \cup (A(X)-S)}$. Thus the maximum in (I) need only be taken over the dominant set of cliques $\mathcal{C}$.

Thus to find a maximum clique in G, we carefully choose a subset S of vertices, and we solve one smaller clique problem for each clique in a dominant set of cliques for G_S. The procedure is applied recursively to solve the subproblems. The set S depends on the nature of G; thus the algorithm has several cases. (In one case, the clique problem is solved directly.) Exposition of the cases is tedious; we shall skip details in a few places.

The entire algorithm has a time bound $t(n) = kb^n$ for some constant b and k. We shall calculate b separately for each case; the maximum of these values will give a bound for the complete algorithm.

The Possible Subproblems

The function $t_i(n)$ is a time bound for the algorithm if case (i) always applies.

(1) If G contains a vertex v of degree n-1 or n-2, let $S = \{v\} \cup (V-A(v))$. Clique $\{v\}$ dominates all cliques in G_S. Thus $||G|| = 1 + ||G_{V-S}||$ and only one subproblem must be solved. If this case applies, $t_1(n) = t_1(n-1) + p(n)$ for some polynomial $p(n)$.

(2) Suppose G contains only vertices of degree n-3. Then $\bar{G}$, the complement graph of G, consists exclusively of cycles. We may easily find a maximum set of independent (pairwise non-adjacent) vertices in $\bar{G}$. Such a set is a maximum clique in G. If this case applies, $t_2(n) = p(n)$ for some polynomial $p(n)$.

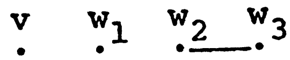
(3) If G contains a vertex v of degree n-3 and a non-adjacent vertex of degree n-4 or less, let

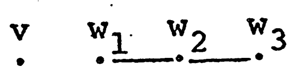
$$S = \{v\} \cup (V - A(\{v\})) = \{v_1, w_1, w_2\} .$$

If $(w_1, w_2) \notin E$, there is one subproblem of size n-3. If $(w_1, w_2) \in E$, there are two subproblems, one of size n-3 and one of size $|A(\{w_1, w_2\}) - S| \le n-5$. In the worst case $t_3(n) = t_3(n-3) + t(n-5) + p(n)$ for some polynomial $p(n)$, and $t_3(n) = (1.17)^n$, ignoring constants and polynomial terms.

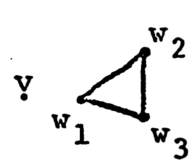
(4) If G contains a vertex v of degree $n-4$, let $S = \{v\} \cup (V - A(v)) = \{v, w_1, w_2, w_3\}$. Let $A_i = A(\{w_i\}) - S$, for $i = 1,2,3$. The subproblems depend on the subgraph G_S and the A_i.

(4a) $G_S = \ldots.$ $||G|| = 1 + ||G_{V-S}|$. There is one subproblem of size $n-4$. $t_{4a}(n) = t_{4a}(n-4) + p(n)$ for some polynomial $p(n)$.

(4b) $G_S =$  . If $|A_2 \cap A_3| = n-5$, there is one subproblem of size $n-5$. If $|A_2 \cap A_3| \leq n-6$, there are two subproblems, one of size $n-4$ and one of size $|A_2 \cap A_3|$. In this case $t_{4b}(n) = t_{4b}(n-4) + t_{4b}(n-6) + p(n)$ for some polynomial $p(n)$, and $t_{4b}(n-4) = (1.15)^n$, ignoring constants and polynomial terms.

(4c) $G_S =$  . If $|A_1 \cap A_2| \leq |A_2 \cap A_3| = n-6$, there are two subproblems, one of size $n-4$ and one of size $n-6$. In this case $t_{4c}(n) = t_{4c}(n-4) + t_{4c}(n-6) + p(n)$ and $t_{4c}(n) = (1.15)^n$.

If $|A_1 \cap A_2| \leq |A_2 \cap A_3| \leq n-7$, there are three subproblems. In this case $t_{4c}(n-4) + 2t_{4c}(n-7) + p(n)$ and $t_{4c}(n) = (1.22)^n$.

(4d) $G_S =$  There are several cases, depending upon $|A_1 \cap A_2 \cap A_3|$.

If $|A_1 \cap A_2 \cap A_3| \geq n-7$, there are two subproblems, one of size $n-4$ and one of size $n-7$. $t_{4d}(n) = t_{4d}(n-4) + t_{4d}(n-7) + p(n)$ for some polynomial $p(n)$, and $t(n) = (1.14)^n$.

If $|A_1 \cap A_2 \cap A_3| = n-8$, there are at most three sub-problems, of sizes $n-4$, $n-6$, and $n-8$.

$$t_{4d}(n) = t_{4d}(n-4) + t_{4d}(n-6) + t_{4d}(n-8) + p(n),$$

and $t_{4d}(n) = (1.215)^n$.

If $|A_1 \cap A_2 \cap A_3| = n-9$, there are at most three sub-problems, of sizes $n-4$, $n-6$, and $n-9$. This case is better than the one just above.

If $|A_1 \cap A_2 \cap A_3| \geq n-10$, there may be five subproblems, one of size $n-4$, three of size $n-8$, and one of size $n-10$. In this case $t_{4d}(n) = t_{4d}(n-4) + 3t_{4d}(n-8) + t_{4d}(n-10) + p(n)$ for some polynomial $p(n)$, and $t_{4d}(n) = (1.26)^n$.

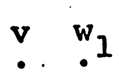
(5) If G contains a vertex v of degree $n-6$, let $S = \{v\}$. There are two subproblems, one of size $n-1$ and one of size $n-6$.

$$t_5(n) = t_5(n-1) + t_5(n-6) + p(n) \qquad t_5(n) = (1.286)^n$$

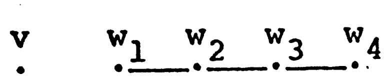
(6) If G contains a vertex of degree $n-5$, let

$$S = \{v\} \cup (V - A(v)) = \{v, w_1, w_2, w_3, w_4\}.$$

Let $A_i = A(w_i) - S$, for $i = 1,2,3,4$. The subproblems depend upon the subgraph G_S and the A_i.

(6a) $G_S =$ 

$G_{\{w_2, w_3, w_4\}}$. Any possible set of subproblems is better than some set of subproblems which arises in case (4).

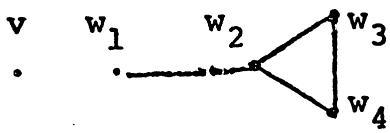
(6b) $G_S =$ 

If $|A_2 \cap A_3| = n-7$, there are two subproblems, of sizes $n-5$ and $n-7$. If $|A_2 \cap A_3| \leq n-8$ but $|A_1 \cap A_2| = n-7$, there may be three subproblems, one of size

$n-5$ and two of size $n-7$. In this case

$t_{6b}(n) = t_{6b}(n-5) + 2t_{6b}(n-7) + p(n)$ and $t_{6b}(n) = (1.21)^n$.

If $|A_1 \cap A_2|, |A_2 \cap A_3|, |A_3 \cap A_4| \leq n-8$, there may be four subproblems. In this case, $t_{6b}(n) = t_{6b}(n-5) + 3t_{6b}(n-8) + p(n)$, and $t_{6b}(n) = (1.22)^n$.

(6c) G_s = 

If $|A_1 \cap A_2| = n-8$, there may be at most four subproblems. A recursive bound on $t(n)$ in all cases is:

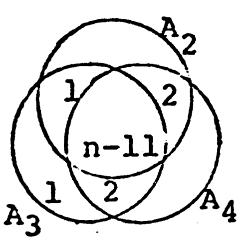
$t_{6c}(n) = t_{6c}(n-5) + t_{6c}(n-7) + t_{6c}(n-8) + t_{6c}(n-10) + p(n)$, and

$t_{6c}(n) = (1.21)^n$.

If $|A_2 \cap A_3 \cap A_4| \geq n-9$, there are at most three subproblems, and the bound above works in all cases.

If $|A_2 \cap A_3 \cap A_4| = n-10$, there are at most four subproblems, and the bound above works in all cases.

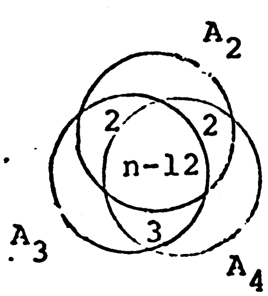
Suppose $|A_2 \cap A_3 \cap A_4| = n-11$. In the worst case there are five subproblems. A Venn diagram illustrates the situation.



$$t_{6c}(n) = t_{6c}(n-5) + 3t_{6c}(n-9) + t_{6c}(n-11) + p(n)$$

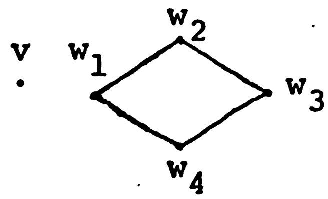
$$t_{6c}(n) = (1.22)^n$$

Suppose $|A_2 \cap A_3 \cap A_4| = n-12$. In the worst case there are six subproblems. A Venn diagram illustrates the situation.



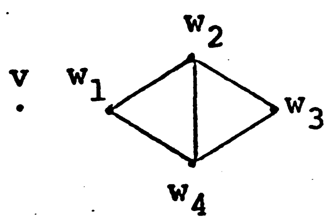
$$t_{6c}(n) = t_{6c}(n-5) + 2t_{6c}(n-9) + 2t_{6c}(n-10) + t_{6c}(n-12) + p(r$$

$$t_{6c}(n) = (1.24)^n$$

(6d) G_S = 

If $|A_1 \cap A_4| = n-7$, there are at most three subproblems. $t_{6d}(n-5) + 2t_{6d}(n-7) + p(n)$, and $t_{6d}(n) = (1.20)^n$.

If $|A_1 \cap A_2| \leq |A_2 \cap A_3| \leq |A_3 \cap A_4| \leq |A_4 \cap A_1| \leq n-8$, there may be five subproblems. $t_{6d}(n) = t_{6d}(n-5) + 4t_{6d}(n-8) + p(n)$, and $t_{6d}(n) = (1.25)^n$.

(6e) G_S = 

If $|A_2 \cap A_4| = n-8$, there are at most four subproblems, of sizes $n-5$, $n-8$, $n-10$, $n-10$.

If $|A_1 \cap A_2| = n-8$, there are at most four subproblems, of sizes $n-5$, $n-8$, $n-8$, $n-10$.

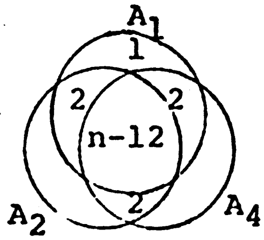
If $|A_1 \cap A_2 \cap A_4| = n-9$, there are at most two subproblems.

If $|A_1 \cap A_2 \cap A_4| = n-10$, there are at most five subproblems.
$t_{6e}(n) = t_{6e}(n-5) + 2t_{6e}(n-9) + t_{6e}(n-10) + t_{6e}(n-11) + p(n)$.
$t_{6e}(n) = (1.22)^n$

If $|A_1 \cap A_2 \cap A_4| = |A_2 \cap A_3 \cap A_4| = n-11$, there are at most seven subproblems. $t_{6e}(n) = t_{6e}(n-5) + 4t_{6e}(n-9) + 2t_{6e}(n-11) + p(n)$.
$t_{6e}(n) = (1.26)^n$.

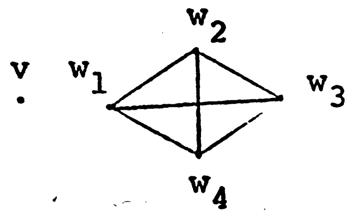
If $|A_1 \cap A_2 \cap A_4| = n-11$, $|A_2 \cap A_3 \cap A_4| \leq n-12$, there are at most seven subproblems. The bound above applies in this case.

If $|A_1 \cap A_2 \cap A_4| \leq |A_2 \cap A_3 \cap A_4| \leq n-12$, there are at most eight subproblems. A Venn diagram illustrates the situation, which is symmetric for w_1, w_2, w_4, and for w_2, w_3, w_4.



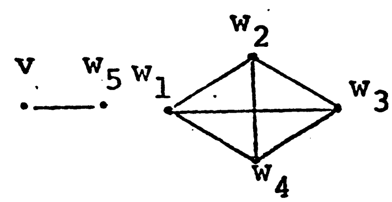
$$t_{6e}(n) = t_{6e}(n-5) + 5t_{6e}(n-10) + 2t_{6e}(n-12) + p(n).$$

$$t_{6e}(n) = (1.26)^n$$

(6f) $G_S =$ 

The situation is now really complicated.

Cases (6f)-(6k) handle the possibilities. Suppose some vertex w_5 is non-adjacent to w_1, w_2, w_3, and w_4. Then let $S = \{v, w_1, w_2, w_3, w_4, w_5\}$.

$G_S =$ 

We now use the fact that since case (5) does not apply, all vertices are of degree n-5. Thus clique $\{v, w_5\}$ dominates all cliques in G_S except those containing three or more vertices.

If $|A_1 \cap A_2 \cap A_3 \cap A_4| \geq n-13$, there can be at most five subproblems. Case (4d) has a worse bound than this case.

If $|A_1 \cap A_2 \cap A_3 \cap A_4| \leq n-14$, there can be six subproblems. In the worst case, $t_{6f}(n) = t_{6f}(n-6) + 4t_{6f}(n-12) + t_{6f}(n-14) + p($ Several cases are worse than this one.

(6g) No vertex except v is non-adjacent to w_1, w_2, w_3, and w_4, and $|A_1 \cap A_2| = n-8$. If $|A_1 \cap A_3 \cap A_4| \geq n-11$, there are at most four subproblems, and $t_{6g}(n) = t_{6g}(n-5) + t_{6g}(n-8) + t_{6g}(n-9) + t'_{6g}(n-11) + p(n)$. Case(6d) above is worse.

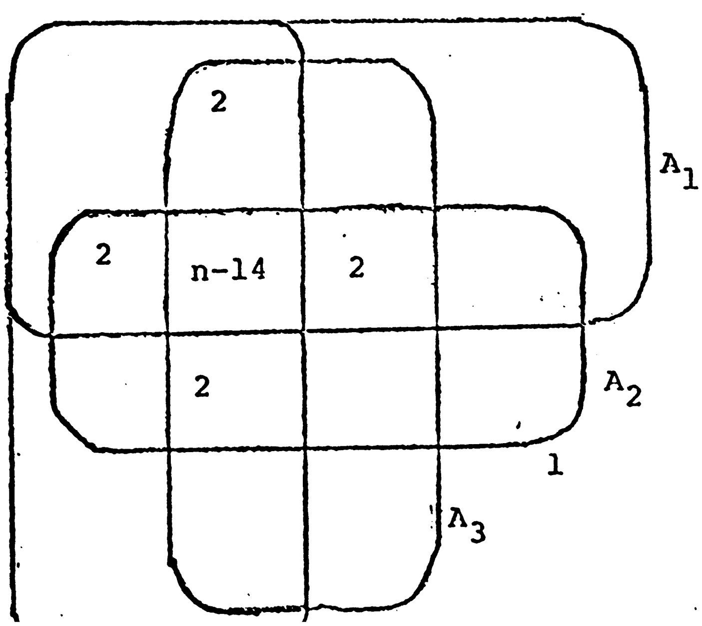
If $|A_1 \cap A_3 \cap A_4| \leq n-12$, there may be six subproblems, and $t_{6g}(n) = t_{6g}(n-5) + t_{6g}(n-9) + 3t_{6g}(n-10) + t_{6g}(n-12) + p(n)$. Case (6e) above is worse.

(6h) $|A_1 \cap A_2| = n-9$. In this case cliques $\{w_1, w_3\}$ and $\{w_2, w_3\}$ are dominated by $\{w_1, w_2, w_3\}$. Similarly $\{w_1, w_4\}$ and $\{w_2, w_4\}$ are dominated by $\{w_1, w_2, w_4\}$. There are at most eight subproblems, and $t_{6h}(n) = t_{6h}(n-5) + 2t_{6h}(n-9) + 4t_{6h}(n-11) + t_{6h}(n-13) + p(n)$. $t_{6h}(n) = (1.25)^n$. All cases with fewer than eight subproblems are better than case (6e).

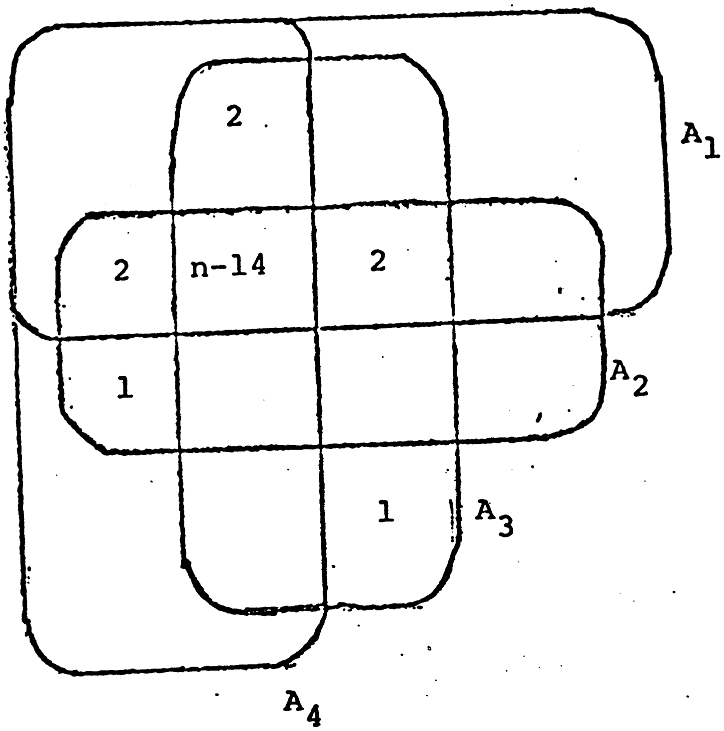
(6i) We may now assume that $|A_i \cap A_j| \leq n-10$ for all $i \neq j$. Suppose $|A_1 \cap A_2 \cap A_3| \geq n-11$. Then there are at most nine subproblems. $t_{6i}(n) = t_{6i}(n-5) + 4t_{6i}(n-10) + 3t_{6i}(n-12) + t_{6i}(n-14) + p(n)$. $t_{6i}(n) = (1.26)^n$.

If $|A_1 \cap A_2 \cap A_3 \cap A_4| \geq n-13$, then there are at most eight subproblems. $t_{6i}(n) = t_{6i}(n-5) + 6t_{6i}(n-10) + t_{6i}(n-12) + p(n)$. $t_{6i}(n) = (1.26)^n$.

(6j) $|A_1 \cap A_2 \cap A_3 \cap A_4| = n-14$. Consider the Venn diagram below.



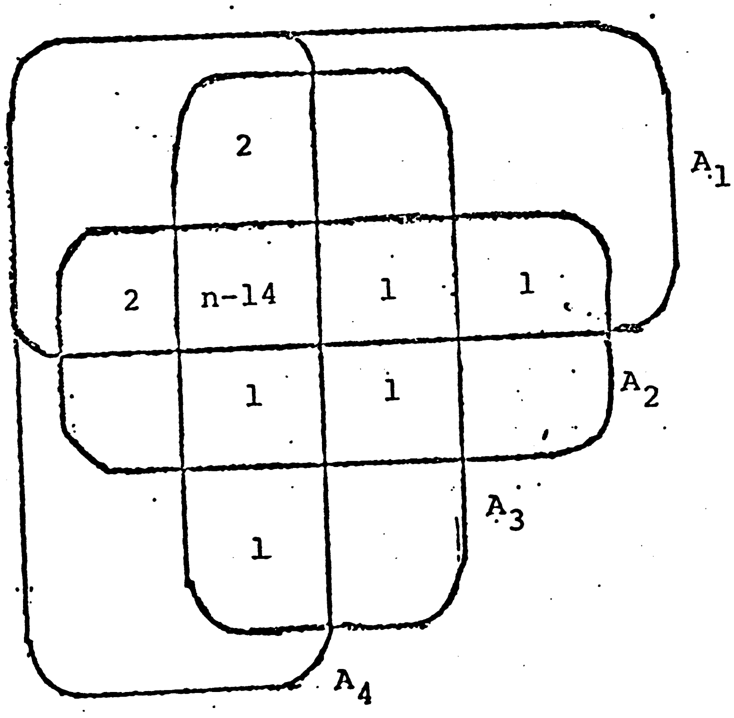
This situation is impossible, since every vertex in V-S is adjacent to w_1, w_2, w_3, or w_4. Thus at least one 3-clique in S in non-dominant, and at least one 2-clique as well. If only one of the 3-cliques is non-dominant, the worst situation is:



$$t_{6j}(n) = t_{6j}(n-5) - 4t_{6j}(n-10) + 3t_{6j}(n-12) + t_{6j}(n-14) + p(n).$$

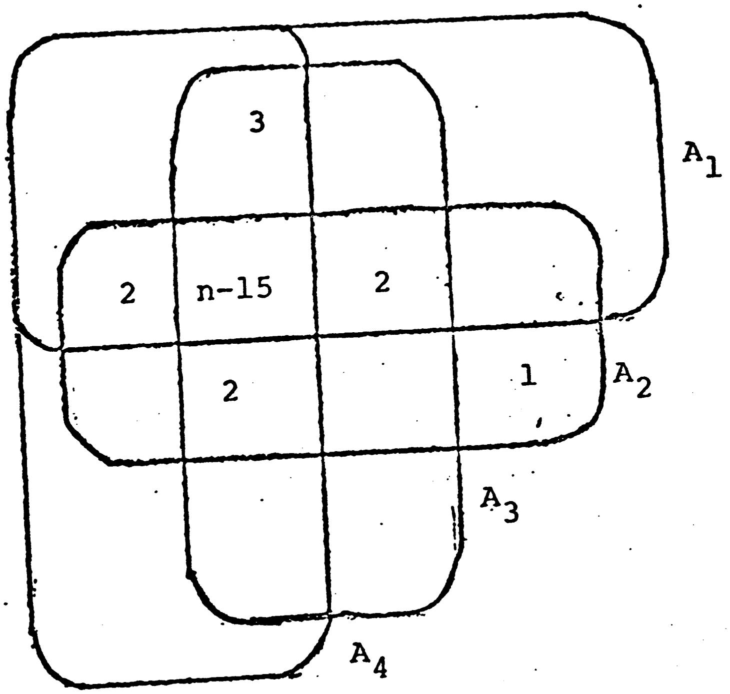
$$t_{6j}(n) = (1.25)^n$$

If two of the 3-cliques are non-dominanat, the worst situation is:



$$t_{6j}(n) = t_{6j}(n-5) + 3t_{6j}(n-10) + 2t_{6j}(n-12) + t_{6j}(n-14) + p(n),$$ which gives a better bound than above.

(6k) $|A_1\ A_2\ A_3\ A_4| \leq n-15$. The worst case is:



$$t_{6k}(n) = t_{6k}(n-5) + 3t_{6k}(n-10) + 3t_{6k}(n-11) + t_{6k}(n-12) + 3t_{6k}(n-13) + t_{6k}(n-15) + p(n).$$

$$t_{6k}(n) = (1.28)^n$$

These are the only possible cases we need to consider. Whatever the form of G, the clique problem must be reducible in one of the ways described above. By applying the reductions recursively, we may find a maximum clique in G. The cases may look complicated, but the algorithm can be implemented as a straightforward backtracking procedure; deciding between cases does not require too deep a decision tree, or too much extra work. (The polynomial $p(n) \leq kn^2$ in all cases.)

A Time Bound

Let $t(n)$ be the time required to find a maximal clique in a graph with n vertices using the algorithm outlined above. $\max_i t_i(n)$ gives an upper bound for $t(n)$. The maximum occurs in case (5). Thus $t(n) \leq (1.286)^n$, ignoring polynomial terms. This bound is asymptotically correct; if we multiply by a constant the bound is correct for all n. Thus for some k, $t(n) \leq k(1.286)^n$. Since $\log_2(1.286) = .364$, $t(n) \leq k2^{.364n}$. Within a fixed time, the recursive algorithm can handle a graph with about 2 3/4 as many vertices as the obvious algorithm can handle.

Conclusions

A recursive algorithm for finding a maximal clique in a graph has been described. The algorithm has a worst-case time bound of $k(1.286)^n$ for some constant k, if n is the number of vertices in the graph. This algorithm is a substantial improvement over the obvious algorithm. It is not clear whether the algorithm can be improved much more, or whether there is a non-exponential time algorithm for finding a maximal clique.

REFERENCES

[1] Cook, S., "The Complexity of Theorem-Proving Procedures," ACM Conference on Theory of Computation (May, 1971), pp. 151 - 158.