Finding a Maximum Matching in a Sparse Random Graph in O(n) Expected Time

Prasad Chebolu

Joint work with Alan Frieze and Páll Melsted

Overview

- Preliminaries and previous work

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- The Karp-Sipser algorithm

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- Our algorithm

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- Analysis

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Maximum cardinality matching is a fundamental problem in combinatorial optimization.

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- Micali and Vazirani [1980] - $O(m\sqrt{n})$

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$G_{n,p}$ is a graph on n vertices where each edge appears independently with probability p.

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- Bast et al. [2005] - $O(n \log n)$ for sparse graphs, i.e. $p = \frac{c}{n}$

Our Work

Theorem

The maximum cardinality matching can be found in $O(n)$ expected time in $G_{n,m}$ where $m = c_1 n$.

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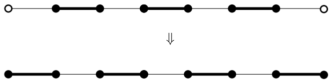
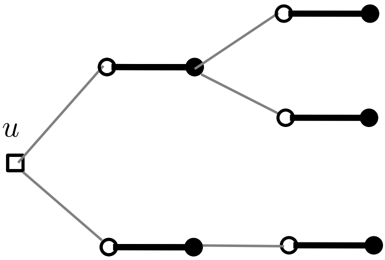


Augmenting Trees

Augmenting trees are rooted at an unmatched vertex, all leaves are connected with a matching edge and all paths from leaves to the root alternate between matching and nonmatching edges.



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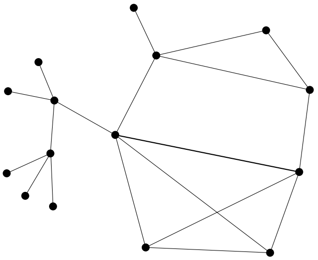

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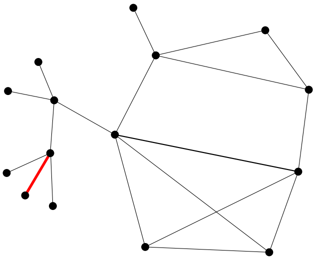

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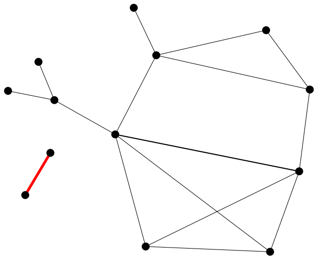

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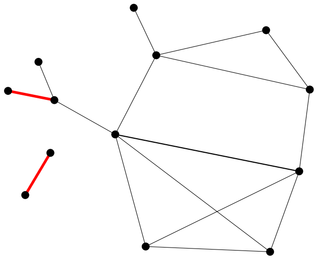

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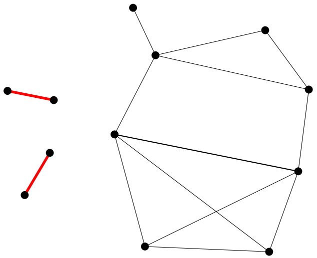

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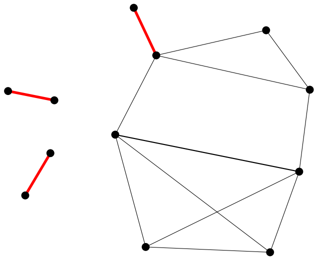

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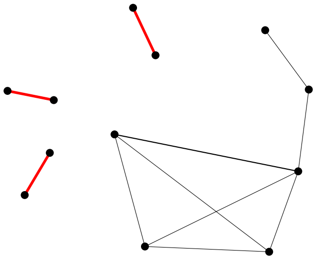

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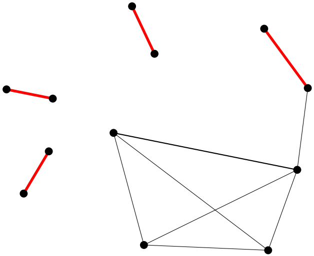

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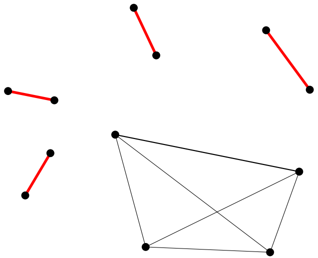

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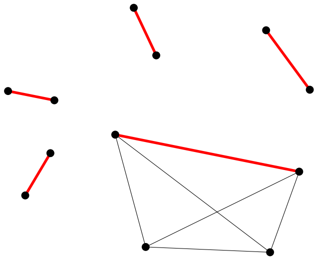

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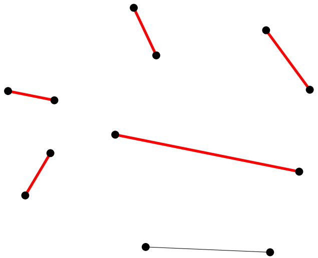

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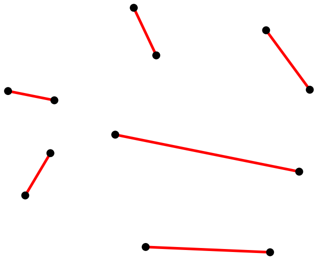

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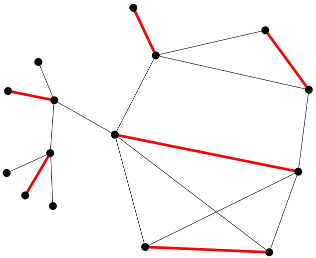

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We start in phase 1 and go into phase 2 when there are no vertices of degree 1 for the first time.

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The Karp-Sipser algorithm makes no mistakes in phase 1.

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- If $c \leq e$ then **whp** at the end of Phase 1, G has only $o(n)$ vertices
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Frieze and Pittel (2004):
If $c > e$ then **whp** the graph at the end of Phase 2 has a perfect matching modulo isolated odd cycles.

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- Run KS to find initial matching. Let G be the graph at the end of Phase 2
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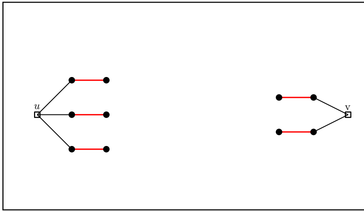

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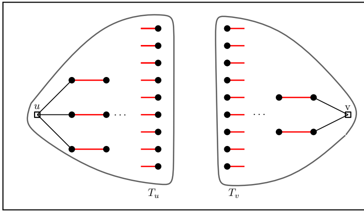

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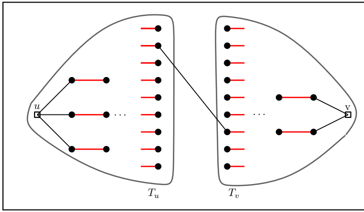

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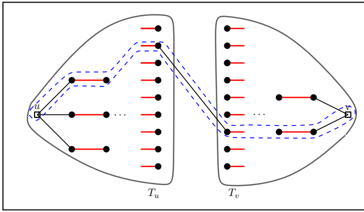

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When vertices are removed from the graph they either have degree ≥ 2, 1 or 0. We refer to them as regular, pendant and isolated.

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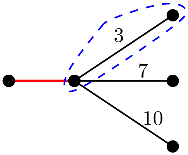
If KS removes a regular vertex, the first edge incident to it becomes its regular witness edge.
It ensures this vertex had degree at least 2



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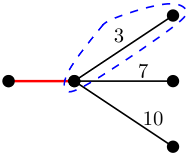

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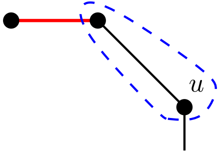

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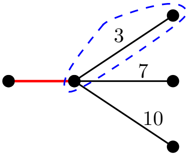

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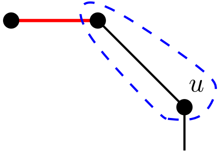

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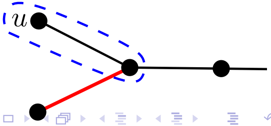

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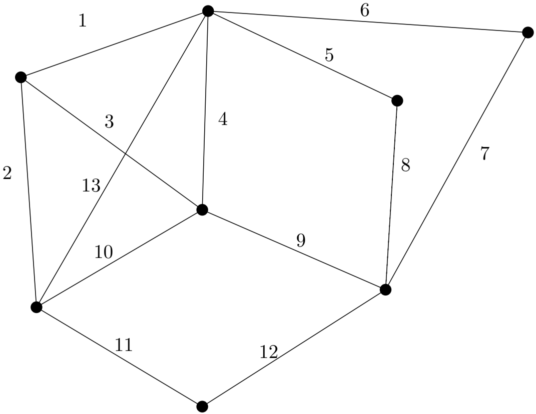

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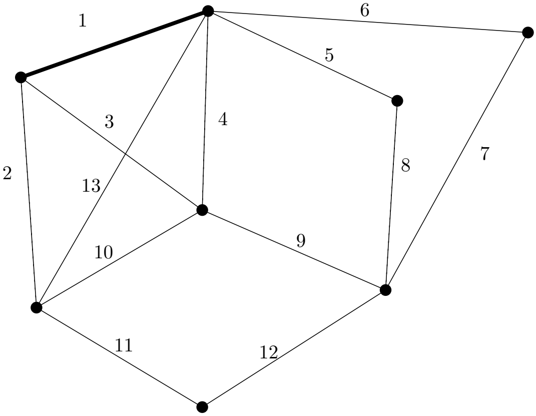

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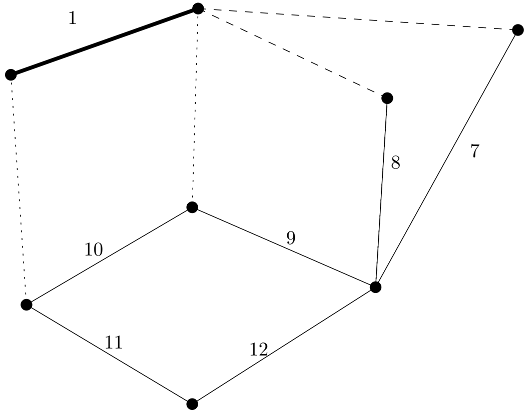

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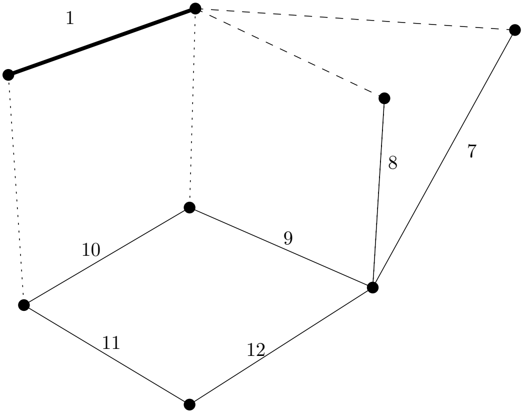

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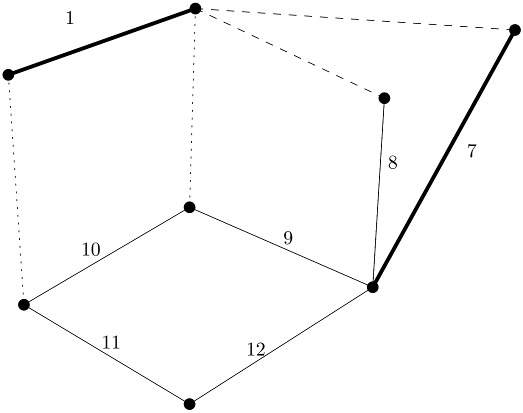

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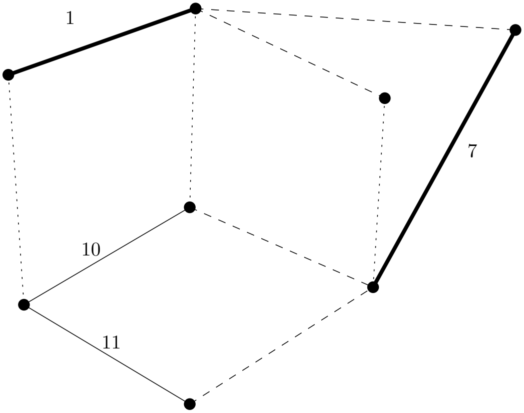

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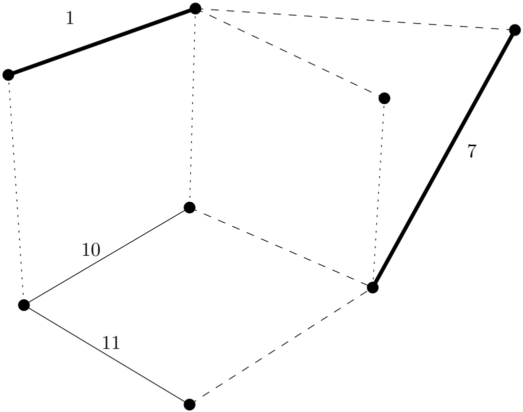

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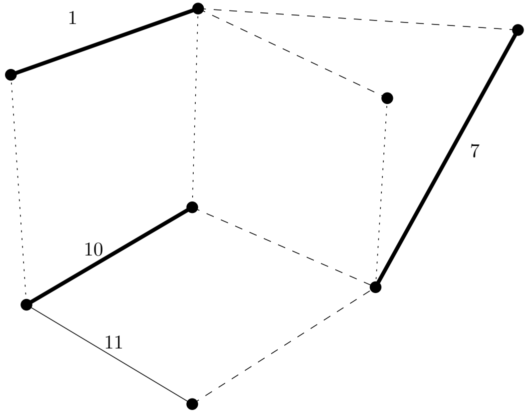

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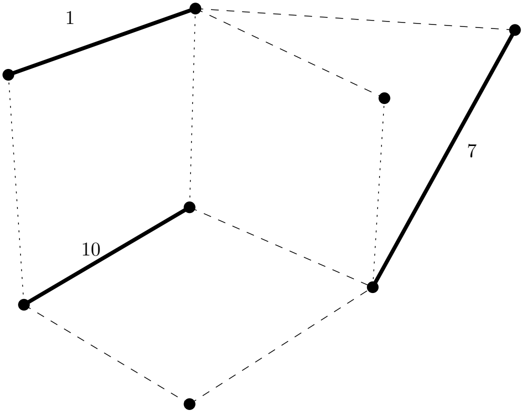

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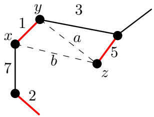

M	M	W	?	M	?	W	?	⋯
1	2	3	4	5	6	7	8	

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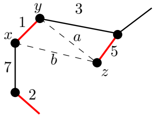

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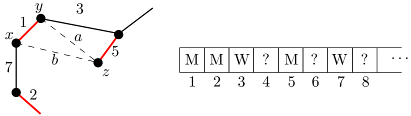

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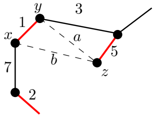

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Whether an edge can go into a box or not depends only on the matching and witness edges.

This allows us to sample the random graph, conditioned on the output of KS

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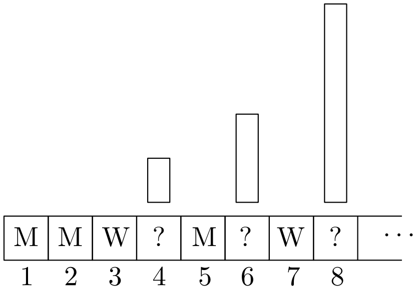

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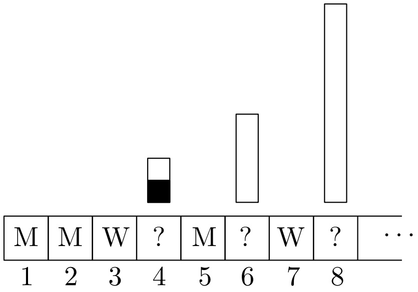

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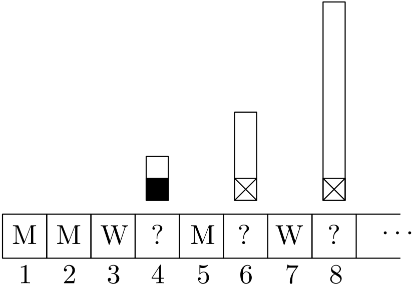

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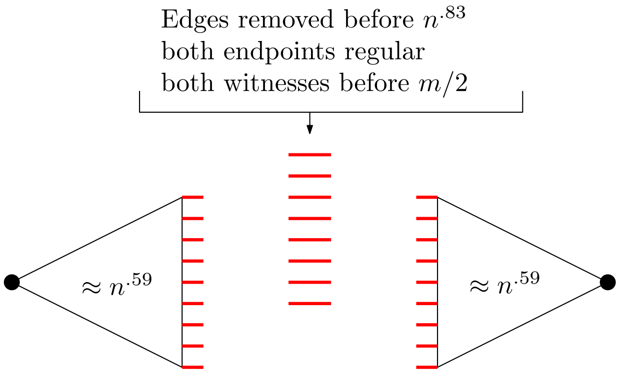
  - How much randomness do we have?

Start with large augmenting trees



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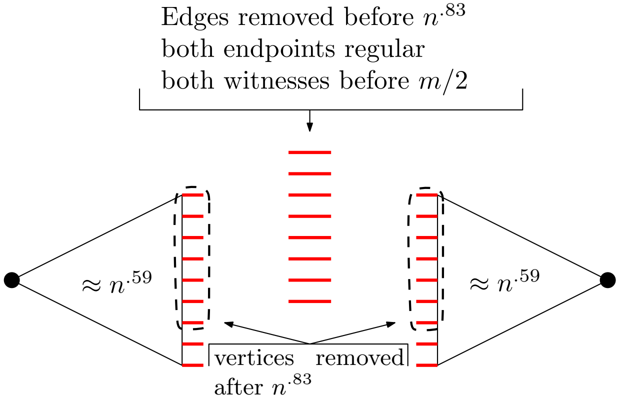
Good edges can help us connect across



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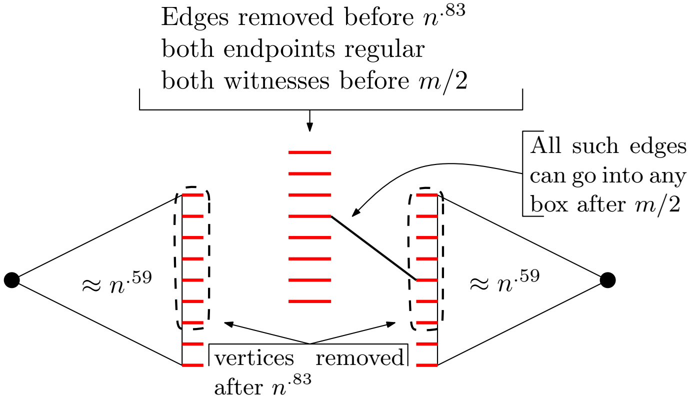

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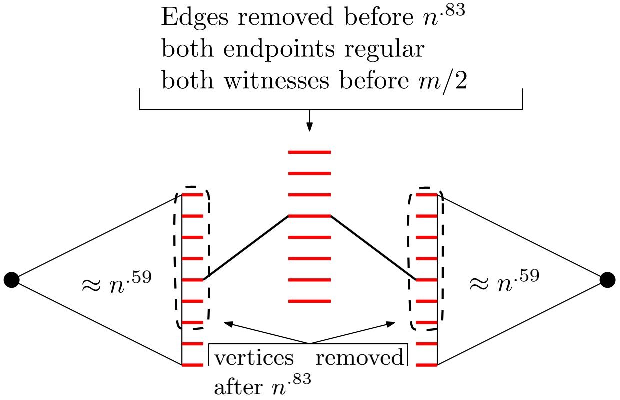
Such pairs of edges ensure the algorithm finds an augmenting path after two rounds



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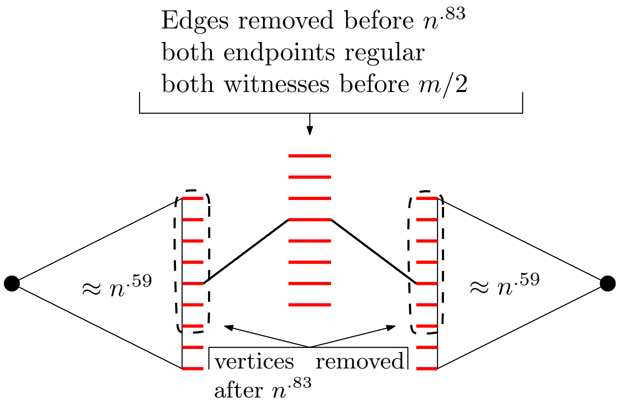

We have skipped a few technical details here.

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Still, this can be shown to take no more than $o(n)$ time.

Thank you