ORIGINAL PAPER

FINDING A ZERO OF THE SUMS OF THREE MAXIMAL MONOTONE OPERATORS

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Abstract. *In this paper, we present a method for finding the zero of the sum of finite family of maximal monotone operators on real Hilbert spaces. In the case where the number of maximal monotone operators is three, we define a function such that its fixed points are solutions of our problem. Some illustrative examples are given at the end of this paper.*
Keywords: *maximal monotone operator; resolvent; Yoshida approximation; variational inclusion.*

1. INTRODUCTION

In the present work, we give a method to determine the set of solutions of the following variational inclusion: we seek to find x in a Hilbert space $\mathbb{H}$ such that,

$$0 \in \mathcal{A}(x) + \mathcal{B}(x) + \mathcal{C}(x), \tag{1}$$

where $\mathcal{A}, \mathcal{B}$ et $\mathcal{C}$ are maximal monotone operators in a Hilbert space $\mathbb{H}$, both subject to a Lipschitz perturbation $\mathcal{B}: \mathbb{H} \rightarrow \mathbb{H}$.

We denote by$(\mathcal{A} + \mathcal{B} + \mathcal{C})^{-1}$ (0) the set of solutions of (1). For all $\gamma > 0$, we consider the nonlinear equations system,

$$\begin{cases} J_\gamma^{\mathcal{A}}(x) = \frac{x+y+z}{3} \\ J_\gamma^{\mathcal{B}}(y) = \frac{x+y+z}{3} \\ J_\gamma^{\mathcal{C}}(z) = \frac{x+y+z}{3}. \end{cases} \tag{2}$$

The γ-resolvent and the γ-Yoshida approximation of a maximal monotone $\mathcal{A}$ are given, respectively by

$$J_\gamma^{\mathcal{A}}(x) = \left(id + \gamma\mathcal{A}(x)\right)^{-1}, \quad \mathcal{A}_\gamma(x) = \frac{1}{\gamma}\left(id - J_\gamma^{\mathcal{A}}(x)\right),$$

where id is the identity operator, the same for $\mathcal{B}$ and $\mathcal{C}$.

Our main objective in this paper is to prove that: If (1) admits a solution if and only if (2) admits a solution Through this equivalence, we will obtain an algorithm for calculating the solutions by defining a function whose fixed points are the set of solutions to the problem (1).

In the case when the set of solutions is non empty, the problem (1), $n = 2$, has been studied in [1-9], by applying the proximal point algorithm to the Yoshida approximate of

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maximal monotone operator. The proximal point algorithm for solving the problem (1), $n = 2$, is expressed as: $0 \in \mathcal{A}(u_{k+1})+\mathcal{B}(u_{k+1}) + (u_{k+1} - u_k)$; $k = 0, 1, 2, \dots$, where $\gamma > 0$ is the proximal parameter.

This paper is organized as follows: in the first section, we will recall some useful basic definitions and properties that can help as to present our results regarding the study of Problem (1). The second section presents a new method is given to determine set of solution of this problem. The technique used in this work can be applied in the future to generalize and develop the results obtained in [1, 10, 11].

2. PRELIMINARIES

Let $\mathbb{H}$ be a real Hilbert space and $\mathcal{A} : \mathbb{H} \to \mathbb{H}$ a set-valued map. We denote by dom$\mathcal{A}$ the domain of $\mathcal{A}$ that is,

$$dom\mathcal{A} := \{x \in \mathbb{H} : \mathcal{A}(x) \neq \emptyset\},$$

and the graph of $\mathcal{A}$ is given by

$$graph\mathcal{A} := \{(x, y) \in \mathbb{H} \times \mathbb{H} : x \in dom\,\mathcal{A}, y \in \mathcal{A}(x)\}.$$

Definition 2.1. The operator $\mathcal{A}$ is said to be monotone, if:

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0, \text{ for all } (x_i, y_i) \in graph\mathcal{A},\ i = 1, 2.$$

Definition 2.2. [12] $\mathcal{A}$ monotone operator $\mathcal{A}$ is called maximal, if its graph has not an extension to a graph of another monotone operator.

We can rephrase the définition of a maximal monotone operator in terms of its graph. If $\mathcal{A}$ monotone operator, then $\mathcal{A}$ is maximal if and only if

$$u \in \mathcal{A}(x) \Leftrightarrow \forall (y, v) \in graph(\mathcal{A});\ \langle u - v, x - y \rangle \geq 0.$$

Proposition 2.1. [12] $\mathcal{A}_\gamma(y) \in \mathcal{A}\left(J_\gamma^{\mathcal{A}}(y)\right), \forall y \in \mathbb{H}$.

Proposition 2.2. [12] For any $\mu > 0, \gamma > 0$, we have $J_\gamma^{\mathcal{A}}(x) = J_\mu^{\mathcal{A}}\left(\frac{\mu}{\gamma}x + \left(1 - \frac{\mu}{\gamma}\right)J_\gamma^{\mathcal{A}}(x)\right)$.

3. MAIN RESULTS

We can now state the main result of this work by giving the method for finding the set of solutions of (1).

Theorem 2.1. For all $\gamma > 0$: If $\theta(\mathrm{a}) = \mathrm{J}_\gamma^{\mathcal{C}}\left(2\mathrm{J}_\gamma^{\mathcal{A}}(\mathrm{a}) - \mathrm{a} - \gamma\mathcal{B}\left(\mathrm{J}_\gamma^{\mathcal{A}}(\mathrm{a})\right)\right) - \mathrm{J}_\gamma^{\mathcal{A}}(\mathrm{a}) + \mathrm{a}$. Then

$$\mathrm{J}_\gamma^{\mathcal{A}}(\mathrm{S}(\theta)) = (\mathcal{A} + \mathcal{B} + \mathcal{C})^{-1}(0),$$

where $\mathrm{S}(\theta)$ denotes the set of all fixed points of θ.

The proof of Theorem 2.1 is split on following Proposition.

Proposition 3.1 For all $\gamma > 0$,

$$(\mathcal{A} + \mathcal{B} + \mathcal{C})^{-1}(0) = \{J_\gamma^{\mathcal{A}}(x) \in X : (x,y,z) \text{ is a solution of } (2)\}.$$

Proof: For all $\gamma > 0$, we have

1) If $k \in (\mathcal{A} + \mathcal{B} + \mathcal{C})^{-1}(0)$ then, $0 \in \gamma\mathcal{A}(k) + \gamma\mathcal{B}(k) + \gamma\mathcal{B}(k)$

i.e, $\exists (s,t,r) \in \mathbb{H}$ $such\ that\ s \in \gamma\mathcal{A}(k), t \in \gamma\mathcal{B}(k), r \in \gamma\mathcal{C}(k)\ and\ s+t+r=0.$

We get

$$\begin{cases} s+k \in k+\gamma\mathcal{A}(k) \\ t+k \in k+\gamma\mathcal{B}(k) \\ r+k \in k+\gamma\mathcal{C}(k) \\ s+t+r=0. \end{cases}$$

This implies that

$$\begin{cases} J_\gamma^{\mathcal{A}}(s+k)=k \\ J_\gamma^{\mathcal{B}}(t+k)=k \\ J_\gamma^{\mathcal{C}}(r+k)=k \\ s+t+r=0. \end{cases}$$

By making an appropriate change of variables $x=s+k, y=t+k, z=r+k$. Then, we obtain

$$\begin{cases} J_\gamma^{\mathcal{A}}(x)=k \\ J_\gamma^{\mathcal{B}}(y)=k \\ J_\gamma^{\mathcal{C}}(z)=k \\ x+y+z=3k, \end{cases}$$

Observing that $k=\frac{x+y+z}{3}$, then

$$\begin{cases} J_\gamma^{\mathcal{A}}(x)=\dfrac{x+y+z}{3} \\ J_\gamma^{\mathcal{B}}(y)=\dfrac{x+y+z}{3} \\ J_\gamma^{\mathcal{C}}(z)=\dfrac{x+y+z}{3}. \end{cases}$$

2) If $(x,y,z)$is a solution of (2). Then,

$$\begin{cases} J_\gamma^{\mathcal{A}}(x)=\dfrac{x+y+z}{3} \\ J_\gamma^{\mathcal{B}}(y)=\dfrac{x+y+z}{3} \\ J_\gamma^{\mathcal{C}}(z)=\dfrac{x+y+z}{3}, \end{cases}$$

by making the "change of variables" $x=s+k, y=t+k, z=r+k$ and $k=\frac{x+y+z}{3}$, we obtain

$$\begin{cases} J_\gamma^{\mathcal{A}}(s+k)=k \\ J_\gamma^{\mathcal{B}}(t+k)=k \\ J_\gamma^{\mathcal{C}}(r+k)=k \\ s+t+r=0. \end{cases}$$

This implies that

$$\begin{cases} s+k \in k+\gamma\mathcal{A}(k) \\ t+k \in k+\gamma\mathcal{B}(k) \\ r+k \in k+\gamma\mathcal{C}(k) \\ s+t+r=0. \end{cases}$$

Further, we have $s \in \gamma\mathcal{A}(k),\ t \in \gamma\mathcal{B}(k),\ r \in \gamma\mathcal{C}(k)\ and\ \ s + t + r = 0$. That is

$$k = J_\gamma^A(x) \in (\mathcal{A} + \mathcal{B} + \mathcal{C})^{-1}(0).$$

It is clear that, if k a solution of

$$\begin{cases} J_\gamma^A(s + k) = k \\ J_\gamma^B(t + k) = k \\ J_\gamma^C(r + k) = k \\ s + t + r = 0. \end{cases}$$

Then, $k \in dom\mathcal{A} \cap dom\mathcal{B} \cap dom\mathcal{C}$ and k is a solution of (1).

Proof: Theorem 2.1

1) If x is a fixed point of θ, then $J_\gamma^{\mathcal{C}}\left(2J_\gamma^{\mathcal{A}}(x) - x - \gamma\mathcal{B}\left(J_\gamma^{\mathcal{A}}(x)\right)\right) = J_\gamma^{\mathcal{A}}(x)$.

We pose $x = a,$ $z = 2J_\gamma^{\mathcal{A}}(a) - a - \gamma\mathcal{B}\left(J_\gamma^{\mathcal{A}}(a)\right)$ and $y = J_\gamma^{\mathcal{A}}(a) + \gamma\mathcal{B}\left(J_\gamma^{\mathcal{A}}(a)\right)$.

Then

$$\begin{cases} J_\gamma^{\mathcal{C}}(z) = J_\gamma^{\mathcal{A}}(x) \\ J_\gamma^{\mathcal{B}}(y) = J_\gamma^{\mathcal{A}}(x) \\ x + y + z = 3J_\gamma^{\mathcal{A}}(x). \end{cases}$$

Therefore

$$\begin{cases} J_\gamma^{\mathcal{A}}(x) = \dfrac{x + y + z}{3} \\ J_\gamma^{\mathcal{B}}(y) = \dfrac{x + y + z}{3} \\ J_\gamma^{\mathcal{C}}(z) = \dfrac{x + y + z}{3}. \end{cases}$$

From, Proposition (2.1) conclus that $J_\gamma^{\mathcal{A}}(x) \in (\mathcal{A} + \mathcal{B} + \mathcal{C})^{-1}(0)$.

2) We prove that $x \in S(\theta)$. Since

$$J_\gamma^{\mathcal{A}}(x) \in (\mathcal{A} + \mathcal{B} + \mathcal{C})^{-1}(0).$$

Then,

$$\begin{cases} J_\gamma^{\mathcal{A}}(x) = \dfrac{x + y + z}{3} \\ J_\gamma^{\mathcal{B}}(y) = \dfrac{x + y + z}{3} \\ J_\gamma^{\mathcal{C}}(z) = \dfrac{x + y + z}{3}. \end{cases}$$

So,

$$\begin{cases} z = 3J_\gamma^{\mathcal{A}}(x) - x - y \\ J_\gamma^{\mathcal{B}}(y) = \dfrac{x + y + z}{3} \\ J_\gamma^{\mathcal{C}}(z) = \dfrac{x + y + z}{3}. \end{cases}$$

We obtain that

$$\begin{cases} J_\gamma^{\mathcal{B}}(y) = J_\gamma^{\mathcal{A}}(x) \\ J_\gamma^{\mathcal{C}}\left(3J_\gamma^{\mathcal{A}}(x) - x - y\right) = J_\gamma^{\mathcal{A}}(x). \end{cases}$$

This implies

$$\begin{cases} \mathrm{y} = J_\gamma^{\mathcal{A}}(x) + \mathcal{B}\left(J_\gamma^{\mathcal{A}}(x)\right) \\ \mathrm{J}_\gamma^{\mathcal{C}}\left(J_\gamma^{\mathcal{A}}(x) - \mathrm{x} - \mathrm{y}\right) = J_\gamma^{\mathcal{A}}(x). \end{cases}$$

That is

$$\mathrm{J}_\gamma^{\mathrm{C}}\left(2J_\gamma^{\mathcal{A}}(x) - \mathrm{x} - \mathrm{B}\left(J_\gamma^{\mathcal{A}}(x)\right)\right) = J_\gamma^{\mathcal{A}}(x).$$

Finally, $x \in S(\theta)$.

Example: $\mathcal{A}(x) = \{2x\}, \mathcal{B}(x) = \{3x\}, \mathcal{C}(x) = \{4x\}$, are maximal monotone operators in a Hilbert space $\mathbb{R}$ (real space) then,

$$\begin{cases} J_\gamma^{\mathcal{A}}(x) = \frac{1}{3}x \\ J_\gamma^{\mathcal{B}}(y) = \frac{1}{4}y \\ J_\gamma^{\mathcal{C}}(z) = \frac{1}{5}z. \end{cases}$$

Solve the system

$$\begin{cases} J_\gamma^{\mathcal{A}}(x) = \frac{x+y+z}{3} \\ J_\gamma^{\mathcal{B}}(y) = \frac{x+y+z}{3} \\ J_\gamma^{\mathcal{C}}(z) = \frac{x+y+z}{3}. \end{cases}$$

This gives

$$\begin{cases} \frac{x+y+z}{3} = \frac{1}{3}x \\ \frac{x+y+z}{3} = \frac{1}{4}y \\ \frac{x+y+z}{3} = \frac{1}{5}z. \end{cases}$$

Then, $(0,0,0)$ is a unique solution of (2). Finally, $0 = \mathrm{J}_\gamma^{\mathcal{A}}(0) \in (\mathcal{A} + \mathcal{B} + \mathcal{C})^{-1}(0)$.

For complicated systems we can use the following algorithm.

Algorithm: Starting from the fixed point property of the posed problem, we can define the following sequence,

$$x_{n+1} = J_\gamma^{\mathcal{C}}\left(2J_\gamma^{\mathcal{A}}(x_n) - x_n - \gamma\mathcal{B}\left(J_\gamma^{\mathcal{A}}(x_n)\right)\right) - J_\gamma^{\mathcal{A}}(x_n) + x_n.$$

If $lim_{n\to\infty} x_n = x$ then, $J_\gamma^{\mathcal{A}}(x)$ is a solution of (1).

Example: $\mathcal{A}(x) = \{2x\}, \mathcal{B}(x) = \{3x\}, \mathcal{C}(x) = \{4x\}$ are maximal monotone operators in a Hilbert space $\mathbb{R}$. Then,

$$\begin{cases} J_\gamma^{\mathcal{A}}(x) = \frac{1}{3}x \\ J_\gamma^{\mathcal{B}}(y) = \frac{1}{4}y. \\ J_\gamma^{\mathcal{C}}(z) = \frac{1}{5}z \end{cases}$$

From, $x_{n+1} = J_\gamma^{\mathcal{C}}\left(2J_\gamma^{\mathcal{A}}(x_n) - x_n - \gamma B\left(J_\gamma^{\mathcal{A}}(x_n)\right)\right) - J_\gamma^{\mathcal{A}}(x_n) + x_n$. We find,

$$x_{n+1} = \frac{1}{5}\left(\frac{2}{3}x_n - x_n - \gamma x_n\right) - \frac{1}{3}x_n + x_n.$$

Then,

$$x_{n+1} = \frac{9-5\gamma}{15}x_n.$$

Since $9 - 5\gamma < 15$. Finally $lim_{n\to\infty} x_n = 0$ then, $J_\gamma^{\mathcal{A}}(0) = 0$ is a solution of (1).

4. GEOMETRY INTERPRETATION

First, we define the function $f: \mathbb{H}^3 \to \mathbb{H}^3$ as follows:

$$f(x,y,z) = \begin{pmatrix} J_\gamma^{\mathcal{A}}(x) - \frac{x+y+z}{3} \\ J_\gamma^{\mathcal{B}}(y) - \frac{x+y+z}{3} \\ J_\gamma^{\mathcal{C}}(z) - \frac{x+y+z}{3} \end{pmatrix}.$$

We note that the points $f(x,y,z) = 0$ are the set of solutions of the system

$$\begin{cases} J_\gamma^{\mathcal{A}}(x) = \frac{x+y+z}{3} \\ J_\gamma^{\mathcal{B}}(y) = \frac{x+y+z}{3} \\ J_\gamma^{\mathcal{C}}(z) = \frac{x+y+z}{3} \end{cases}.$$

Proposition 4.1. If *(x,y,z)* is solution of $f(x,y,z) = 0$, then $J_\gamma^{\mathcal{A}}(x)$ is a solution of (1).

Proof: We suppose that*(x,y,z)* is a solution of $f(x,y,z) = 0$.

So,

$$\begin{cases} J_\gamma^{\mathcal{A}}(x) = \dfrac{x+y+z}{3} \\ J_\gamma^{\mathcal{B}}(y) = \dfrac{x+y+z}{3} \\ J_\gamma^{\mathcal{C}}(z) = \dfrac{x+y+z}{3} \end{cases}$$

by making the "change of variables" $x = s + k, y = t + k, z = r + k$ and $k = \frac{x+y+z}{3}$.

Then,

$$\begin{cases} J_\gamma^{\mathcal{A}}(s+k) = k \\ J_\gamma^{\mathcal{B}}(t+k) = k \\ J_\gamma^{\mathcal{C}}(r+k) = k \\ s+t+r = 0 \end{cases}.$$

This implies that

$$\begin{cases} s+k \in k + \gamma\mathcal{A}(k) \\ t+k \in k + \gamma\mathcal{B}(k) \\ r+k \in k + \gamma\mathcal{C}(k) \\ s+t+r = 0 \end{cases}$$

This implies that $s \in \gamma\mathcal{A}(k),\ t \in \gamma\mathcal{B}(k),\ r \in \gamma\mathcal{C}(k)\ and\ s+t+r = 0.$

Finally, $k = J_\gamma^{\mathcal{A}}(x) \in (\mathcal{A} + \mathcal{B} + \mathcal{C})^{-1}(0).$

Algorithm. If $f:\mathbb{R}^3 \to \mathbb{R}^3$, with

$$f(x,y,z) = \begin{pmatrix} J_\gamma^{\mathcal{A}}(x) - \dfrac{x+y+z}{3} \\ J_\gamma^{\mathcal{B}}(y) - \dfrac{x+y+z}{3} \\ J_\gamma^{\mathcal{C}}(z) - \dfrac{x+y+z}{3} \end{pmatrix}$$

The Jacobian matrix of f:

$$Jf(x;\ y,z) = \begin{pmatrix} \dfrac{\partial J_\gamma^{\mathcal{A}}}{\partial x}(x) - \dfrac{1}{3} & -\dfrac{1}{3} & -\dfrac{1}{3} \\ -\dfrac{1}{3} & \dfrac{\partial J_\gamma^{\mathcal{B}}}{\partial y}(y) - \dfrac{1}{3} & -\dfrac{1}{3} \\ -\dfrac{1}{3} & -\dfrac{1}{3} & \dfrac{\partial J_\gamma^{\mathcal{C}}}{\partial z}(z) - \dfrac{1}{3} \end{pmatrix}$$

we calculate the determinant of the Jacobian:

$$det\, Jf = \left(\frac{\partial J_\gamma^{\mathcal{A}}}{\partial x}(x) - \frac{1}{3}\right)\left(\frac{\partial J_\gamma^{\mathcal{B}}}{\partial y}(y) - \frac{1}{3}\right)\left(\frac{\partial J_\gamma^{\mathcal{C}}}{\partial z}(z) - \frac{1}{3}\right) - \frac{1}{9}\left(\frac{\partial J_\gamma^{\mathcal{A}}}{\partial x}(z) - \frac{1}{3}\right) - \frac{2}{27} - \frac{1}{9}\left(\frac{\partial J_\gamma^{\mathcal{C}}}{\partial z}(y) - \frac{1}{3}\right)$$

Newton-Raphson method, or Newton Method, is a powerful technique for solving equations numerically:

$$(x_{k+1}, y_{k+1}, z_{k+1}) = (x_k, y_k, z_k) - J^{-1} f(x_k, y_k, z_k), k = 0,1,2, \ldots$$

5. MATLAB PROGRAMMING

```
clear all; close all;  clc ;
x=5;
y=5;
z=5;
variable=[x;y;z]; t=[0.5;0. 5; 0.5];
delta=1; iteration=1;
while(abs(delta) > 1e-30)
```

$f=[(J_\gamma^{\mathcal{A}}\big(variable(1)\big) - \big(variable(1) + variable(2) + variable(3)\big)/3$
$\quad ;J_\gamma^{\mathcal{B}}\big(variable(2)\big) - \big(variable(1) + variable(2) + variable(3)\big)/3$
$\quad ;J_\gamma^{\mathcal{C}}(variable(3)) - (variable(1) + variable(2) + variable(3))/3$

$$Jacob=\begin{bmatrix} \left(\frac{\partial J_\gamma^{\mathcal{A}}}{\partial x}\big(variable(1)\big) - \frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right) \\ ;\left(-\frac{1}{3}\right)\left(\frac{\partial J_\gamma^{\mathcal{B}}}{\partial y}\big(variable(2)\big) - \frac{1}{3}\right)\left(-\frac{1}{3}\right) \\ ;\left(-\frac{1}{3}\right)\left(\frac{\partial J_\gamma^{\mathcal{C}}}{\partial z}\big(variable(3)\big) - \frac{1}{3}\right)\left(-\frac{1}{3}\right) \end{bmatrix}$$

```
delta=inv(jacob)*(t-f);
 variable=variable+delta;
 iteration=iteration+1;
end
format long
```

x=variable(1)
y=variable(2)
z=variable(3)

Example: $\mathcal{A}(x) = \{2x\}, \mathcal{B}(x) = \{3x\}, \mathcal{C}(x) = \{4x\}$, are maximal monotone operators in a Hilbert space $\mathbb{R}$. Then,

$$\begin{cases} J_\gamma^{\mathcal{A}}(x) = \frac{1}{3}x \\ J_\gamma^{\mathcal{B}}(y) = \frac{1}{4}y \\ J_\gamma^{\mathcal{C}}(z) = \frac{1}{5}z. \end{cases}$$

By MATLAB Programming, we find $(x,,y,z) = (0,0,0)$.

6. CONCLUSION

The equivalence between the posed problem and the equation means that in order to study problem (1) it is sufficient to study the system equations (2) and also the majority of researchers who worked on Problem (1) use approximation and our insight is that this equation will greatly help in developing algorithms in order to find solutions to the problem posed.

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