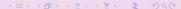
Finding all Bessel type solutions for Linear Differential Equations with Rational Function Coefficients

Introduction

Quan Yuan

March 19, 2012



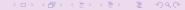
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Main Question

Introduction

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- Given a second order homogeneous differential equation $a_2y'' + a_1y' + a_0 = 0$, where a_i 's are rational functions, can we find solutions in terms of Bessel functions?
- A homogeneous equation corresponds a second order differential operator $L := a_2 \partial^2 + a_1 \partial + a_0$.



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An Analogy

Introduction

o•ooooooo introduction

- $\frac{I_{\nu}(x)\sqrt{x}}{e^x}$ converges when $x \to +\infty$. $I_{\nu}(x)$ and e^x have similar asymptotic behavior when $x \to +\infty$.
- The idea behind finding closed form solutions is to reconstruct them from the asymptotic behavior at the singular points.
- Before studying how to find Bessel type solutions, let's see how this strategy works for exponential solutions $e^{f(x)}$.



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Generalized Exponents

Introduction

oo•oooooo introduction

- To find exponential solutions $y = e^{f(x)}$, we need to know the asymptotic behavior of y at each singularity.
- Generalized exponents (up to equivalence) effectively determine asymptotic behavior up to a meromorphic function.



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Finding Exponential Solutions

Let $L \in \mathbb{C}(x)[\partial]$. Suppose $y = e^{f(x)}$ is a solution of L, where $f \in \mathbb{C}(x)$. Question: How to find f?

Poles of f

Let
$$p \in \mathbb{C} \cup \{\infty\}$$
.

```
p is a pole of f \implies p is an essential singularity of y \implies p is an irregular singularity of L.
```



Quan Yuan

Finding Exponential Solutions

Suppose L has order n and p is an irregular singularity of L (notation $p \in S_{irr}$).

- L has n generalized exponents at p, one of which gives the polar part of f at x = p.
- There are finitely many combinations of generalized exponents at all irregular singularities. Each combination give us a candidate for f.
- Try all candidate f's will give us the exponential solutions.



Slide 6/46

Introduction

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- The same process as finding $e^{f(x)}$ will give us all solutions of the form $I_{\nu}(f)$, $f \in \mathbb{C}(x)$.
- 2 We want to find all solutions of L that can be expressed in terms of Bessel functions.
- 3 As we shall see, $(1) \not\Longrightarrow (2)$.



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Introduction 000000000 introduction

Finding Bessel Type Solutions-Challenges

- **1** Let $g \in \mathbb{C}(x)$ and $f = \sqrt{g}$. Then $I_{\nu}(f)$ satisfies an equation in $\mathbb{C}(x)[\partial].$
- ② So it is not sufficient to only consider $f \in \mathbb{C}(x)$. We need to allow for f's with $f^2 \in \mathbb{C}(x)$.
- **3** As for $e^{f(x)}$ solutions, we find at each $p \in S_{irr}$:

Polar part of $f \implies half of polar part of g$ \implies half of g (half of f).

An Example

lf

$$f = 1x^{-3} + 2x^{-2} + 3x^{-1} + O(x^0),$$

then

$$g = x^{-6} + 4x^{-5} + 10x^{-4} + ?x^{-3} + O(x^{-2}).$$

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Find Bessel type Solutions-Challenges

- Let $r \in \mathbb{C}(x)$, then $\exp(\int r) I_{\nu}(\sqrt{g(x)})$ also satisfies an equation in $\mathbb{C}(x)[\partial]$.
- Let $r_0, r_1 \in \mathbb{C}(x)$, then $r_0 I_{\nu}(\sqrt{g(x)}) + r_1 (I_{\nu}(\sqrt{g(x)}))'$ satisfies an equation in $\mathbb{C}(x)[\partial]$ too.
- So to solve L "in terms of" Bessel functions, we also need to allow sums, products, differentiations, exponential integrals.
- Note: our "in terms of" is the same as that in Singer's (1985) definition. (more on that later.)



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Introduction

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Find Bessel type Solutions

Introduction

oooooooo
introduction

To summarize the three cases, when we say solve equations in terms of Bessel Functions we mean find solutions which have the form

$$e^{\int rdx}(r_0B_{\nu}(\sqrt{g})+r_1(B_{\nu}(\sqrt{g}))')$$

where $B_{\nu}(x)$ is one of the Bessel functions, and $r, r_0, r_1, g \in \mathbb{C}(x)$. (Later in the talk: completeness theorem regarding this form.)

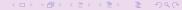


Conclusions

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Notation

- Let C_K be a number field with characteristic 0.
- Let $K = C_K(x)$ be the rational function field over C_K .
- Let $\partial = \frac{d}{dx}$.
- Then K is a differential field with derivative ∂ and $C_K := \{c \in K | \partial(c) = 0\}$ is the constant field of K.



Differential Operators

Introduction

- $L := \sum_{i=0}^{n} a_i \partial^i$ is a differential operator over K, where $a_i \in K$.
- $K[\partial]$ is the ring of all differential operators over K.
- L corresponds to a homogeneous differential equation Ly = 0.
- We say y is a solution of L, if Ly = 0.
- Denote V(L) as the vector space of solutions. (Defined inside a so-called *universal extension*).
- p is a singularity of L, if p is a root of a_n or p is a pole of a_i , $i \neq n$.



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Introduction Notation

• The two linearly independent solutions $J_{\nu}(x)$ and $Y_{\nu}(x)$ of $L_{B1} = x^2 \partial^2 + x \partial + (x^2 - \nu^2)$ are called Bessel functions of first and second kind, respectively.

- Solutions $I_{\nu}(x)$ and $K_{\nu}(x)$ of $L_{B2} = x^2 \partial^2 + x \partial (x^2 + \nu^2)$ are called the modified Bessel functions of first and second kind, respectively.
- The change of variables $x \to x\sqrt{-1}$ sends $V(L_{B1})$ to $V(L_{B2})$ and vice versa. So we can start our algorithm with $L_B := L_{B2}$. And let $B_{\nu}(x)$ refer to one of the Bessel functions.
- If $\nu \in \frac{1}{2} + \mathbb{Z}$, then L_B is reducible.



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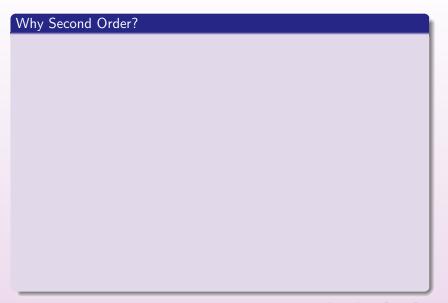
Questions

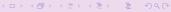
- Given an irreducible second order differential operator $L = a_2 \partial^2 + a_1 \partial + a_0$, with $a_0, a_1, a_2 \in K$. Can we solve it in terms of Bessel Functions?
- More precisely can we find solutions which have the form

$$e^{\int rdx}(r_0B_{
u}(\sqrt{g})+r_1(B_{
u}(\sqrt{g}))')$$

where $B_{\nu}(x)$ is one of the Bessel functions.







Why Second Order?

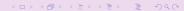
- Definition (Singer 1985): $L \in \mathbb{C}(x)[\partial]$, and if a solution y can be expressed in terms of solutions of second order equations, then y is a *eulerian solution*.
- Note: any solution of $L \in \mathbb{C}(x)[\partial]$ that can be expressed in terms of Bessel functions is a eulerian solution.



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- Singer proved that solving such L can be reduced to solving second order L's
- van Hoeij developed an algorithm that reduces to order 2.



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- Singer proved that solving such L can be reduced to solving second order L's
- van Hoeij developed an algorithm that reduces to order 2.
- such reduction to order 2 is valuable, if we can actually solve such second order equations.
- In summary, to solve n's order equation in terms of Bessel, we need an algorithm that solve 2nd order equations in terms of Bessel functions.



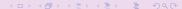
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n **Preliminaries** Local Invariant Solving Solving-details Proof of Uniqueness Conclusion 00 000 000 000000000 00000000000 000

Main Problem

Questions

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Why Bessel?
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Why Bessel?

If we can find a Bessel Solver, then we can find all ${}_{p}F_{q}$ type solutions of second order equations excepts (p, q) = (2, 1)

- \bullet $_0F_1$ and $_1F_1$ functions can be written in terms of either Whittaker functions or Bessel functions.
- Whittaker functions has already been handled. (Debeerst, van Hoeij, and Koepf)
- T. Fang and V. Kunwar are working on ${}_{2}F_{1}$ solver.



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Questions

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Why Irreducible?



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Main Problem

Questions

Why Bessel?

If we can find a Bessel Solver, then we can find all ${}_pF_q$ type solutions of second order equations excepts (p,q)=(2,1)

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- T. Fang and V. Kunwar are working on ${}_2F_1$ solver.

Why Irreducible?

If the second order operator is reducible, it has Liouvillian solutions. Kovacic's algorithm can find such solutions.



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Completeness

Introduction

Main Problem

Questions

For Bessel type solutions, is it sufficient to consider solutions with form

$$e^{\int rdx}(r_0B_{\nu}(\sqrt{g})+r_1(B_{\nu}(\sqrt{g}))')$$

where $B_{\nu}(x)$ is one of the Bessel functions, and $r, r_0, r_1, g \in K$?

To answer that, we need to answer:

- what about $B''_{\nu}, B'''_{\nu}, \ldots$?
- what about sums, products, derivatives, exponential integrals?
- 3 what about $r, r_0, r_1, g \in \overline{K}$?



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Completeness

Theorem of Completeness

Let $K = C_K(x) \subseteq \mathbb{C}(x)$. Let $L \in K[\partial]$. Let $r, f, r_0, r_1 \in \mathbb{C}(x)$ and

$$e^{\int r dx} (r_0 B_{\nu}(f) + r_1 (B_{\nu}(f))')$$

be a non-zero solution of f. Then $\exists \widetilde{r}, \widetilde{r_0}, \widetilde{r_1}, \widetilde{f}, \widetilde{\nu}$ with $\widetilde{f}^2 \in K$ such that

$$e^{\int \widetilde{r} dx} (\widetilde{r_0} B_{\widetilde{\nu}}(\widetilde{f}) + \widetilde{r_1} (B_{\widetilde{\nu}}(\widetilde{f}))')$$

is a non-zero solution of L.

Moreover, $(\nu - \frac{n}{2})^2 \in C_K$ for some $n \in \mathbb{Z}$, and $\widetilde{r}, \widetilde{r_0}, \widetilde{r_1} \in K(\nu^2)$. (If $n \in 2\mathbb{Z}$, we may assume $\nu^2 \in C_K$)



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Introduction

Transformations

There are three types of transformations that preserve order 2:

- change of variables \xrightarrow{f}_{C} : $y(x) \mapsto y(f(x))$, $f(x) \in K$. (for L_R , $f^2 \in K$)
- 2 exp-product \longrightarrow_E : $y \mapsto \exp(\int r \, dx) \cdot y$, $r \in K$
- 3 gauge transformation \longrightarrow_G : $y \mapsto r_0 y + r_1 y'$, $r_0, r_1 \in K$.

L can be solved in terms of Bessel functions when $L_B \longrightarrow_{CFG} L$. Where \longrightarrow_{CEG} is any combination of $\longrightarrow_{C}, \longrightarrow_{E}, \longrightarrow_{G}$.



Transformations

Introduction

Transformation

There are three types of transformations that preserve order 2:

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- 3 gauge transformation \longrightarrow_G : $y \mapsto r_0 y + r_1 y'$, $r_0, r_1 \in K$.

L can be solved in terms of Bessel functions when $L_B \longrightarrow_{CFG} L$. Where \longrightarrow_{CEG} is any combination of \longrightarrow_{C} , \longrightarrow_{E} , \longrightarrow_{G} .

Note

- The composition of 2 & 3 is an equivalence relation (\sim_{FG}). And there exist some algorithms to find such relations.
- If $L_1 \longrightarrow_{CFG} L_2$, then there exist an operator $M \in K[\partial]$ such that $L_1 \xrightarrow{f} C M \sim_{FG} L$.

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Main Problem

Main Problem

Given an irreducible second order differential operator $L \in K[\partial]$, can we find solutions with the form:

$$e^{\int rdx}(r_0B_{\nu}(f)+r_1(B_{\nu}(f))')$$

Where $f^2 \in K$ and $r, r_0, r_1 \in K(\nu^2)$.



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Main Problem

Given an irreducible second order differential operator $L \in K[\partial]$, can we find solutions with the form:

$$e^{\int rdx}(r_0B_{\nu}(f)+r_1(B_{\nu}(f))')$$

Where $f^2 \in K$ and $r, r_0, r_1 \in K(\nu^2)$.

Rephrase the Main Problem

Given an irreducible second linear order differential operator $L \in K[\partial]$, find f and ν with $f^2 \in K$ and $(\nu + \frac{n}{2})^2 \in C_K$ s.t there exist M and $L_R \xrightarrow{f} C M \sim_{FG} L$



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Related Work

\rightarrow_C , \longrightarrow_E

- Bronstein, M., and Lafaille, S. (ISSAC 2002) solve using only $\longrightarrow_{\mathcal{C}}$ and $\longrightarrow_{\mathcal{F}}$.



\rightarrow_C , \longrightarrow_E

- Bronstein, M., and Lafaille, S. (ISSAC 2002) solve using only $\longrightarrow_{\mathcal{C}}$ and $\longrightarrow_{\mathcal{F}}$.
- An analogy about $\longrightarrow_{\mathcal{C}}$ and $\longrightarrow_{\mathcal{F}}$: Suppose you solve polynomial equations using only $x \mapsto c \cdot x$ and $x \mapsto x + c$. then $x^6 - 24x^3 - 108x^2 - 72x + 132$ will not be solved in terms of solutions of $x^6 - 12$, even though it does have a solution in $\mathbb{Q}(\sqrt[6]{12})$. Likewise omitting \longrightarrow_G means not solving the non-trivial case!



Introduction 00000000 Transformation

Related Work

No Square Root

- Debeerst, R, van Hoeij, M, and Koepf. W. (ISSAC 2008) solve under \longrightarrow_{CEG} without dealing with square root case.
- Note for square root case, we only have half information of non-square-root case.



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Related Work

No Square Root

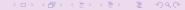
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- Note for square root case, we only have half information of non-square-root case.



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Assume the input is L, and $L_R \xrightarrow{f} C M \sim_{FC} L$:

If M were known, it would be easy to compute f from M. However, the input is not M, but an operator $L \sim_{FG} M$. So we must compute f not from M, but only from the portion of M that is invariant under \sim_{FG} . The portion is **exponent difference** $(mod \mathbb{Z}).$



Quan Yuan Bessel Type Solutions March 19, 2012 Slide 23/46 Exponent Differences

Generalized Exponents

Assume $L \in K[\partial]$ with order 2:

Define

$$t_p := \left\{ egin{array}{ll} x - p & ext{if } p
eq \infty \\ rac{1}{x} & ext{if } p = \infty \end{array}
ight.$$

- ullet there are two generalized exponents $e_1,e_2\in\mathbb{C}[t_{\scriptscriptstyle D}^{-rac{1}{2}}]$ at each point x = p.
- We can think of e_1 , e_2 as truncated Puiseux series. They determine the asymptotic behavior of solutions.
- If a solution contains $ln(t_p)$, then we say L is **logarithmic** at x = p. (only occurs when $e_1 - e_2 \in \mathbb{Z}$)
- $\Delta(L, p) := \pm (e_1 e_2)$ is the **exponent difference**.



Singularities

Exponent Differences

A singularity p of $L \in K[\partial]$ is:

- removable singularity if and only if $\Delta(L, p) \in \mathbb{Z}$ and L is not logarithmic at x = p.
- non-removable regular singularity (denoted by S_{reg}) if and only if $\Delta(L, p) \in \mathbb{C} \setminus \mathbb{Z}$ or L is logarithmic at x = p.
- irregular singularity (denoted by S_{irr}) if and only if $\Delta(L,p) \in \mathbb{C}[t_p^{-\frac{1}{2}}] \setminus \mathbb{C}.$



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Introduction

Exponent Difference

Preliminaries

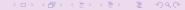
- $L_R \xrightarrow{t} C M$ then:



- $L_R \xrightarrow{f} C M$ then:
 - ① if p is a zero of f with multiplicity $m_p \in \frac{1}{2}\mathbb{Z}^+$, then p is an removable singularity or $p \in S_{reg}$, and $\Delta(M, p) = m_p \cdot 2\nu$.



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 - ② p is a pole of f with pole order $m_p \in \frac{1}{2}\mathbb{Z}^+$ such that $f = \sum_{i=-m_p}^{\infty} f_i t_p^i$, if and only if $p \in S_{irr}$ and $\Delta(M, p) = 2 \sum_{i < 0} i \cdot f_i t_p^i$



Exponent Differences

Exponent Difference

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- $\Delta(L, p)$ is invariant under \longrightarrow_F .



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- $\Delta(L, p)$ is invariant under \longrightarrow_F .
- $\bullet \longrightarrow_G$ shifts $\Delta(L, p)$ by integers.



Exponent Differences

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- $\Delta(L, p)$ is invariant under \longrightarrow_F .
- $\bullet \longrightarrow_G$ shifts $\Delta(L, p)$ by integers.
- removable singularity can disappear under \sim_{FG} .
- \sim_{FG} preserve S_{reg} and S_{irr} .



Local Information

- some (not necessarily all!) zeroes of A from S_{reg} .



Local Information

- some (not necessarily all!) zeroes of A from S_{reg} .
- the polar parts of f (from S_{irr}), then by squaring that we know the polar parts of g partially. (as a truncated Laurent series at each irregular singularity).



Introduction

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- an upper bound for the degree of A (denoted by d_A).



Local Information

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- an upper bound for the degree of A (denoted by d_A).
- Now we need to compute A.



Assume $L_B \xrightarrow{f} C M \sim_{FG} L$.

- The exponent differences of L give us whether $\nu \in \mathbb{Z}$, $\nu \in \mathbb{Q} \setminus \mathbb{Z}$, $\nu \in C_{\kappa} \setminus \mathbb{Q}$ or $\nu \notin C_{\kappa}$.



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Assume $L_B \xrightarrow{f} C M \sim_{FG} L$.

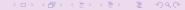
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- if $\nu \notin \mathbb{Q}$, we first compute candidates for f, and use them to compute candidates for ν .



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- The exponent differences of L give us whether $\nu \in \mathbb{Z}$, $\nu \in \mathbb{Q} \setminus \mathbb{Z}, \ \nu \in C_{\kappa} \setminus \mathbb{Q} \text{ or } \nu \notin C_{\kappa}.$
- if $\nu \notin \mathbb{Q}$, we first compute candidates for f, and use them to compute candidates for ν .
- If $\nu \in \mathbb{Q}$, then exponent differences give a list of the candidates for the denominator of ν .



Assume $L_R \xrightarrow{f} C M \sim_{FG} L$

- The exponent differences of L give us whether $\nu \in \mathbb{Z}$, $\nu \in \mathbb{O} \setminus \mathbb{Z}, \ \nu \in C_{\kappa} \setminus \mathbb{O} \text{ or } \nu \notin C_{\kappa}.$
- if $\nu \notin \mathbb{Q}$, we first compute candidates for f, and use them to compute candidates for ν .
- If $\nu \in \mathbb{Q}$, then exponent differences give a list of the candidates for the denominator of ν .
- It is sufficient to consider only $Re(\nu) \in [0, \frac{1}{2}]$, because $\nu \mapsto \nu + 1$ and $\nu \mapsto 1 - \nu$ are special case of \longrightarrow_{G}



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Introduction 00000000 An Example

An Example

$$L := \partial^2 - \frac{1}{x-1}\partial + \frac{1}{18} \frac{18 - 23x + 4x^2 - 20x^3 + 12x^4}{(x-1)^4 x^3}$$

From generalized exponent, we can obtain the following:



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Introduction

An Example

$$L:=\partial^2-\tfrac{1}{x-1}\partial+\tfrac{1}{18}\tfrac{18-23x+4x^2-20x^3+12x^4}{(x-1)^4x^3}$$

From generalized exponent, we can obtain the following:

- $S_{re\sigma} = \emptyset$, so no known zeroes.
- the polar part of f is $\frac{\pm 2i}{\sqrt{t_0}}$ at x=0, and $\frac{\pm 1}{\sqrt{2}t_1}$ at x=1.
- the polar part of g is $\frac{-4}{t_0}$ at x=0, and $\frac{1}{2t^2}+\frac{?}{t_1}$ at x=1
- $B = x(x-1)^2$, $d_A = 3$.
- $\nu \in \{\frac{1}{2}\}$

How to compute A?



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Introduction

Linear Equations

Assume
$$L_B \xrightarrow{f} C M \sim_{EG} L$$
 and $g = f^2 = \frac{A}{B}$ and $A = \sum_{i=0}^{u_A} a_i x^i$.

Roots

$$p \in S_{reg} \implies p \text{ is a root of } A$$

 \implies one linear equation of a_i 's.

Poles

If
$$p \in S_{irr} \implies p$$
 is a pole of g (assume m_p is the pole order) $\Longrightarrow \lceil \frac{m_p}{2} \rceil$ linear equations of a_i 's.

We get at least $\#S_{reg} + \frac{1}{2}d_A$ linear equations in total.



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Continuation of the Example

In our example we can assume

$$g = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3}{x(x-1)^2}$$

Roots

 $S_{reg} = \emptyset \Longrightarrow$ no linear equations from regular singularities.

Poles

- polar part of g at x = 0 is $\frac{a_0}{t_0} + O(t_0^0) \Longrightarrow a_0 = -4$.
- polar part of g at x = 1 is $\frac{a_0+a_1+a_2+a_3}{t^2}+O(t_1^{-1})\Longrightarrow a_0+a_1+a_2+a_3=\frac{1}{2}.$

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The First Difficulty

Difficulties

Assume
$$L_B \xrightarrow{f} C M \sim_{EG} L$$
, $g = f^2 = \frac{A}{B}$.

Not enough equations to compute A

- Only know about half of polar parts of g



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The First Difficulty

Introduction

Difficulties

Assume
$$L_B \xrightarrow{f} C M \sim_{EG} L$$
, $g = f^2 = \frac{A}{B}$.

Not enough equations to compute A

- Only know about half of polar parts of g
- Only have about $\frac{1}{2}d_A$ linear equations from irregular singularities to get A.
- With disappearing singularities, we do not have enough equations to get A.



Conclusions

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Assume
$$L_B \xrightarrow{f}_C M \sim_{EG} L$$
, $g = f^2 = \frac{A}{B}$.

Not enough equations to compute A

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- With disappearing singularities, we do not have enough equations to get A.



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Difficulties

Assume $L_B \xrightarrow{f} C M \sim_{EG} L$, where $g = f^2 = \frac{A}{B}$ and $\nu \in \mathbb{Q} \setminus \mathbb{Z}$.

- $S_{irr} = \{ \text{Poles of } f \}.$
- $S_{reg} \subseteq \{ \text{Roots of } f \}$



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Assume $L_B \xrightarrow{f} C M \sim_{EG} L$, where $g = f^2 = \frac{A}{B}$ and $\nu \in \mathbb{Q} \setminus \mathbb{Z}$.

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Problem: \subseteq is not =
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the Reason for the First difficulty

Assume $L_B \xrightarrow{f} C M \sim_{EG} L$, where $g = f^2 = \frac{A}{B}$ and $\nu \in \mathbb{Q} \setminus \mathbb{Z}$.

- $S_{irr} = \{ \text{Poles of } f \}.$
- $S_{reg} \subseteq \{ \text{Roots of } f \}$

Problem: \subseteq is not =

Reason: Regular singularities may become removable under $\stackrel{t}{\longrightarrow}_{C}$. thus may disappear under \sim_{FG}

Note: If $f \in K$, this is not a problem, because we do not need as many equations in that case.



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Difficulties

the Solution for the First Difficulty

Assume $L_B \xrightarrow{f} C M \sim_{EG} L$, where $g = f^2 = \frac{A}{B}$. Let d be the denominator of ν and m_p be the multiplicity of f at p.

Solution:

- Singularity p disappears only if $\nu \in \mathbb{Q} \setminus \mathbb{Z}$ and $d \mid 2m_p$.
- We can write $A = C \cdot A_1 \cdot A_2^d$. Here A_1 contains all known roots, A_2 is the disappeared part.
- Now we need to compute A_2 .
- Since $d \ge 3$, so we only need roughly $\frac{1}{3}d_A$ equations to get A_2



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the Solution for the First Difficulty

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Conclusions

Assume $L_B \xrightarrow{f} C M \sim_{EG} L$, where $g = f^2 = \frac{A}{B}$. Let d be the denominator of ν and m_p be the multiplicity of f at p.

Solution:

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Difficulties

Continuation of the Example

In our example: assume $A = C \cdot A_1 \cdot A_2^3$

- $S_{re\sigma} = \emptyset \Longrightarrow A_1 = 1$;
- Fix C = -4. (We will discuss how to find C later.)
- Assume $A_2 = a_0 + a_1 x$.

Now we get

$$g = \frac{-4(a_0 + a_1 x)^3}{x(x-1)^2}$$

- polar part of g at x = 0 is $\frac{-4a_0^3}{t_0} + O(t_0^0) \Longrightarrow -4a_0^3 = -4$.

$$\frac{-4(a_0+a_1)^3}{t^2} + O(t_1^{-1}) \Longrightarrow -4(a_0+a_1)^3 = \frac{1}{2}$$

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Continuation of the Example

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Difficulties

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- polar part of g at x=0 is $\frac{-4a_0^3}{t_0}+O(t_0^0)\Longrightarrow -4a_0^3=-4$.
- polar part of g at x = 1 is $\frac{-4(a_0 + a_1)^3}{t^2} + O(t_1^{-1}) \Longrightarrow -4(a_0 + a_1)^3 = \frac{1}{2}.$
- The equations are not linear. (In this case, the equations are easy to solve because there is only one term in each power series. But in general, it is hard.)

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Continuation of the Example

In our example: assume $A = C \cdot A_1 \cdot A_2^3$

- $S_{reg} = \emptyset \Longrightarrow A_1 = 1$;
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The Second Difficulty

Non-linear equations

- To get enough equations, we write $A = C \cdot A_1 \cdot A_2^d$.
- But the approach on the previous slide provides non-linear equations, that can be solved with Gröbner basis. (Problem: doubly-exponential complexity).



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The Second Difficulty

Non-linear equations

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The Second Difficulty

Non-linear equations

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- But the approach on the previous slide provides non-linear equations, that can be solved with Gröbner basis. (Problem: doubly-exponential complexity).

the Solution:

From power series of A_2^d , try to get a power series of A_2 , then we will have linear equations.

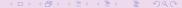


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Continuation of the Example

Assume
$$A = -4(a_0 + a_1 x)^3$$
, $\mu_3 = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$.

• the power series of $g=\frac{CA_2^3}{B^2}$ at 0 is $\frac{-4}{t_0}+O(t_0^0)$.

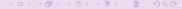


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Difficulties

Assume
$$A = -4(a_0 + a_1 x)^3$$
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- the power series of $g=\frac{CA_2^3}{B}$ at 0 is $\frac{-4}{t_0}+O(t_0^0)$.
- The series of A_2^3 is $1 + O(t_0)$.



Continuation of the Example

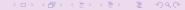
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- The series of A_2^3 is $1 + O(t_0)$.
- $(\mu_3 + O(t_0), \text{ or } \mu_3^2 + O(t_0)).$ • The series of A_2 is $1 + O(t_0)$.



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- We get $a_0 = 1$. (uniqueness theorem)



Difficulties

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- We get $a_0 = 1$. (uniqueness theorem)
- the power series of $g=\frac{CA_2^3}{B}$ at 1 is $\frac{1}{2t_1^2}+O(t_1^{-1})$.
- the series of A_2^3 is $-\frac{1}{8} + O(t_1)$.
- The series of A_2 is $S = -\frac{1}{2} + O(t_1)$. $(\mu_3 S \text{ or } \mu_3^2 S)$.
- We get $a_0 + a_1 = -\frac{1}{2}$.



Difficulties

Continuation of the Example

Assume
$$A = -4(a_0 + a_1 x)^3$$
, $\mu_3 = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$.

- the power series of $g = \frac{CA_2^3}{R}$ at 0 is $\frac{-4}{t_0} + O(t_0^0)$.
- The series of A_2^3 is $1 + O(t_0)$.
- The series of A_2 is $1 + O(t_0)$. $(\mu_3 + O(t_0), \text{ or } \mu_3^2 + O(t_0))$.
- We get $a_0 = 1$. (uniqueness theorem)
- the power series of $g = \frac{CA_2^3}{B}$ at 1 is $\frac{1}{2t^2} + O(t_1^{-1})$.
- the series of A_2^3 is $-\frac{1}{9} + O(t_1)$.
- The series of A_2 is $S = -\frac{1}{2} + O(t_1)$. $(\mu_3 S \text{ or } \mu_3^2 S).$
- We get $a_0 + a_1 = -\frac{1}{2}$.
- solve both equations we get $A_2 = 1 \frac{3}{2}x$.



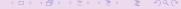
Solution

By computing the relation under \sim_{FG} , we find two independent solutions:

$$\sqrt{x(3x-2)}(x-1)I_{\frac{1}{3}}(\sqrt{\frac{(3x-2)^3}{2x(x-1)^2}})$$

and

$$\sqrt{x(3x-2)}(x-1)K_{\frac{1}{3}}(\sqrt{\frac{(3x-2)^3}{2x(x-1)^2}})$$



Fix A_1

Introduction

$$\nu \in \mathbb{Q}$$
, $A = C \cdot A_1 \cdot A_2^d$.
We can fix A_1 this way:

- If we don't have regular singularities, then $A_1 = 1$
- Each $p \in S_{reg}$ corresponds to each root of A_1
- Exponent differences and d will give a set of candidates for the multiplicities. (Diophantine equations)
- Try all candidates.



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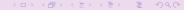
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Introduction

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We can fix A_1 this way:

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- Each $p \in S_{reg}$ corresponds to each root of A_1 .
- Exponent differences and d will give a set of candidates for the multiplicities. (Diophantine equations)
- Try all candidates.

For our example, $S_{reg} = \emptyset$, so $A_1 = 1$.



About C

- We know that no algebraic extension of C_K is needed for g.
- However without the right value for C in $g = \frac{CA_1A_2^d}{R}$, an algebraic extension of C_K will be needed in A_2 .
- Define $C_1 \sim C_2$ if $C_1 = c^d \cdot C_2$, where $c \in C_K$.
- C is unique (up to \sim) if there exist $p \in S_{irr}$ such that $p \in C_K \cup \{\infty\}.$
- If $p \in \overline{C_K} \setminus C_K$ then finding all C's up to \sim involves a number theoretical problem.



Fix C

Introduction

Pick $p \in S_{irr}$ such that $p \in C_K \cup \{\infty\}$. If no such p exists, pick any $p \in S_{irr}$ and consider everything over $C_K(p)$



Fix C

Introduction

Pick $p \in S_{irr}$ such that $p \in C_K \cup \{\infty\}$. If no such p exists, pick any $p \in S_{irr}$ and consider everything over $C_K(p)$ We know the power series of $g = \frac{CA_1A_2^d}{R}$ at p. $(\Delta(L,p))$



Fix C

Introduction

Pick $p \in S_{irr}$ such that $p \in C_K \cup \{\infty\}$. If no such p exists, pick any $p \in S_{irr}$ and consider everything over $C_K(p)$. We know the power series of $g = \frac{CA_1A_2^d}{B}$ at p. $(\Delta(L,p))$ \Rightarrow the series of $CA_2^d = \frac{gB}{A_1}$.



Fix C

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Pick $p \in S_{irr}$ such that $p \in C_K \cup \{\infty\}$. If no such p exists, pick any $p \in S_{irr}$ and consider everything over $C_K(p)$

We know the power series of $g=\frac{CA_1A_2^d}{B}$ at p. $(\Delta(L,p))$

- \Rightarrow the series of $CA_2^d = \frac{gB}{A_1}$.
- \Rightarrow Let C equal the coefficient of the first term of this series.



Fix C

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We know the power series of $g=\frac{CA_1A_2^d}{B}$ at p. $(\Delta(L,p))$

- \Rightarrow the series of $CA_2^d = \frac{gB}{A_1}$.
- \Rightarrow Let C equal the coefficient of the first term of this series.

For our examples, we can fix C=-4 (if we start with p=0) or $\frac{1}{2}$ (if we start with p=1). There are equivalent, since $-4=\frac{1}{2}\cdot (-2)^3$.



Uniqueness

Introduction

Theorem 1

If *L* has a solution $\exp(\int r)(r_0B_{\nu}(f_1) + r_1(B_{\nu}(f_1))')$ and $\exp(\int \hat{r})(\hat{r}_0B_{\nu}(f_2) + \hat{r}_1(B_{\nu}(f_2))')$ where $r, r_0, r_1, \hat{r}, \hat{r}_0, \hat{r}_1, f_1, f_2 \in \overline{\mathbb{Q}(x)}$, then $f_1 = \pm f_2$.



Uniqueness

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Why Need Uniqueness

- Theoretically, it to prove the completeness of our algorithm.
- Practically, if we get a candidate of f and $f^2 \notin K$, we can discard f without further computation, which increases the speed of algorithm significantly.

(Note: In our example, it reduced the number of combinations from 9 to 1.)



To prove the theorem, we need to use

- Classification of differential operators mod p (p-curvature).
- Number theory (Chebotarev's density theorem).
- Differential Galois theory.



the Sketch of the proof

- If $\nu \in \frac{1}{2} + \mathbb{Z}$ (non-interesting case in algorithm), then L_B has exponential solutions.
- Use Chebotarev's density theorem, there are infinitely many p, for which ν reduces to an element in \mathbb{F}_p .
- Thus $\nu \equiv \frac{1}{2} \mod p$.
- So we know the solutions mod such p in these cases.
- by classification theory (p-curvature), we get $\pm f' \equiv 1 \mod p$.
- Since there exist infinity many such p, we get $\pm f$ is unique up to a constant.
- The rest of the proof is based on the differential Galois theory.



Conclusions

Introduction

Conclusions

Our contribution in the thesis:

- Developed a complete Bessel solver for second order differential equations.
- Combine Bessel Solver with Whittaker/Kummer solver to get a solver for ${}_{0}F_{1}$, ${}_{1}F_{1}$ functions.
- Proved the completeness of our algorithm.
- As an application, found relations between Heun functions and Bessel functions.



Acknowledgement

Introduction

Thanks

- Thanks to my advisor Mark van Hoeij for his support, patience, and friendship.
- Thanks to the members of my committee for their time and efforts.
- Thanks to my family and friends for their support.

