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Finding Convex Sets Among Points in the Plane

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Abstract. Let $g(n)$ denote the least value such that any $g(n)$ points in the plane in general position contain the vertices of a convex n-gon. In 1935, Erdős and Szekeres showed that $g(n)$ exists, and they obtained the bounds

$$2^{n-2}+1 \le g(n) \le \binom{2n-4}{n-2}+1.$$

Chung and Graham have recently improved the upper bound by 1; the first improvement since the original Erdős–Szekeres paper. We show that

$$g(n) \le \binom{2n-4}{n-2}+7-2n.$$

1. Introduction

Esther Klein, early in the 1930s, considered the following problem: Is it true that for every n, there is a least value $g(n)$, such that any set of $g(n)$ points in the plane in general position always contain the vertices of a convex n-gon?

Her question was partially resolved in a famous paper by Erdős and Szekeres [2] in which they showed that

$$2^{n-2}+1 \le g(n) \le \binom{2n-4}{n-2}+1. \tag{1}$$

The upper bound was not improved for more than 50 years, which led Chung and Graham to raise the question: Can it be improved at all? They have recently shown [1]

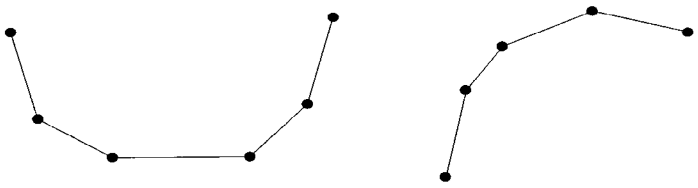

Fig. 1. A 6-cup and a 5-cap.

that it can be reduced by 1. This modest improvement is like a drop of water that appears downstream from a dam. This drop can be the harbinger of a trickle, then perhaps a stream, and finally the dam may collapse with a rush of water.

It is the purpose of this paper to replace the Chung–Graham drop by a trickle.[1]

The original Erdős–Szekeres argument is based upon the notions of n-caps and n-cups:

Definition 1. An n-cap is a set of n points which, when ordered from left to right, have the property that slopes of lines joining successive points are decreasing. Formally, an n-cap consists of n points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ satisfying $x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_{n-1} \leq x_n$, and $(y_i - y_{i-1})/(x_i - x_{i-1}) > (y_{i+1} - y_i)/(x_{i+1} - x_i)$ when $1 < i < n$. An n-cup is defined in the same way except the slopes are required to be increasing.

Both n-caps and n-cups are special cases of convex polygons. They also have the property that if a single point is both the left endpoint of an $(n-1)$-cap and also the right endpoint of an $(m-1)$-cup, then either the cap or the cup can be extended by one point.

Erdős and Szekeres used this idea together with an induction argument to find the least number of points in the plane that always contain an n-cap or an m-cup:

Theorem 1 (Erdős–Szekeres). *Let $f(n, m)$ be the least integer such that any $f(n, m)$ points in the plane in general position contain either an n-cap or an m-cup*

$$f(n, m) = \binom{n+m-4}{n-2} + 1. \tag{2}$$

We exploit the fact that the definition of caps and cups is dependent upon the orientation of the coordinate system used to describe the points in the plane. We show that if there are $\binom{n+m-4}{n-2} + 7 - m - n$ points there is always an orientation of coordinates so that there is an n-cap or an m-cup. In particular, if we choose two consecutive vertices a and b from the convex hull of the points, and orient our coordinates so that the line segment ab is vertical and forms the left end of the convex hull, then there is an n-cap or an m-cup in the configuration.

[1] Géza Tóth and Pavel Valtr have recently replaced our trickle with a stream by further improving the upper bound (see [5]).

2. Main Result

We begin by introducing some terminology which we will find useful.

Definition 2. A configuration of points in general position is said to be vertical if the two leftmost points on the convex hull have the same horizontal coordinate.

Definition 3. For given n, m, a point in a configuration is said to be a-defective if it is neither the left endpoint of an $(n-1)$-cap nor the right endpoint of an $(m-1)$-cup. Similarly, it is said to be b-defective if it is neither the right endpoint of an $(n-1)$-cap nor the left endpoint of an $(m-1)$-cup.

Theorem 2. *Let $f_V(n, m)$ be the least integer such that any $f_V(n, m)$ points in a vertical configuration contain an n-cap or an m-cup.*

$$f_V(n, m) = \binom{n+m-4}{n-2} + 7 - m - n. \tag{3}$$

Proof of the Upper Bound. We argue by induction. First, notice that if m or n are 3 we have $f_V(n, 3) = f_V(3, m) = 3$. So assume that $m, n \geq 4$.

Consider a vertical configuration, X, of $f_V(n, m) - 1$ points with no n-caps or m-cups. Let a and b be the two leftmost endpoints in X, chosen so that a is above b (see Fig. 2).

Our argument is based upon two observations: First, note that since a and b are on the left of our configuration, they cannot be right endpoints of $(n-1)$-caps or $(m-1)$-cups. We can also say that a is not the left endpoint of an $(n-1)$-cap because if it were we could use b to extend it to an n-cap. Similarly, b is not the left endpoint of an $(m-1)$-cup. It follows that a is a-defective and b is b-defective (the Chung–Graham theorem follows immediately from this observation).

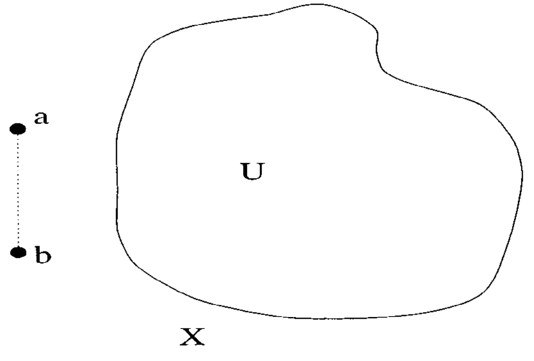

Fig. 2. A diagram of the configuration X.

Notes. 1. a and b form a vertical line.

2. All other points are in U to the right of ab.

3. Any cap with left endpoint a extends to a larger one ending at b.

4. Any cup with left endpoint b extends to a larger one ending at a.

Second, we can use the original Erdős–Szekeres argument for our specific orientation of points. We build a set R of points consisting of right endpoints of $(m-1)$-cups. We do this by finding an $(m-1)$-cup, adding the right endpoint to R, removing the point from X and repeating the procedure. Notice that as we do this X remains a vertical configuration. Since there are no n-caps in X, R contains $f_V(n, m) - f_V(n, m-1)$ points. We can also add the point a to R since it is a-defective. We therefore find $f_V(n, m) - f_V(n, m-1) + 1$ points which cannot contain an $(n-1)$-cap or an m-cup. We have a contradiction if

$$f_V(n, m) - f_V(n, m-1) + 1 \geq f(n-1, m), \tag{4}$$

in which case the set R would have an $(n-1)$-cap or m-cup.

We can also build a set consisting of $f_V(n, m) - f_V(n, m-1)$ right endpoints of $(n-1)$-caps together with the point b. We therefore have the recursions:

$$f_V(n, m) \leq f(n, m-1) + f_V(n-1, m) - 2, \tag{5}$$

$$f_V(n, m) \leq f(n-1, m) + f_V(n, m-1) - 2. \tag{6}$$

These recursions, together with the known values of $f(n, m)$ and the boundary conditions when the smaller argument is 3, have the solution:

$$f_V(n, m) \leq \binom{n+m-4}{n-2} + 7 - m - n. \tag{7}$$

Given a configuration of points, we can rotate it so that it is vertical. Setting $m = n$ in (7), we have

Corollary 1.

$$g(n) \leq \binom{2n-4}{n-2} + 7 - 2n. \tag{8}$$

Proof of the Lower Bound. We will show that the two recursions (5) and (6) are in fact equalities. Assume that $n, m \geq 4$. Let S_1 be a maximal vertical configuration with no $(n-1)$-cap or m-cup, and let S_2 be a maximal configuration (not necessarily vertical) with no n-cap or $(m-1)$-cup. The cardinality of S_1 will be $f_V(n-1, m) - 1$, and the cardinality of S_2 will be $f(n, m-1) - 1$. Choose S_2 so that the leftmost point is also the top point (this can always be arranged, see [4]).

Construct a vertical configuration as follows:

1. Transform S_1 and S_2 by affine transformations so that S_1 lies along the x-axis and S_2 along the line $y = -x$.
2. The leftmost point of S_1 (with smaller y-coordinate) coincides with the leftmost point of S_2. Call this point p.
3. The x-coordinates of points of S_1 are smaller than those of $S_2 - p$.
4. $S_1 - p$ lies above each line determined by points of S_2.
5. $S_2 - p$ lies above each line determined by points of S_1.

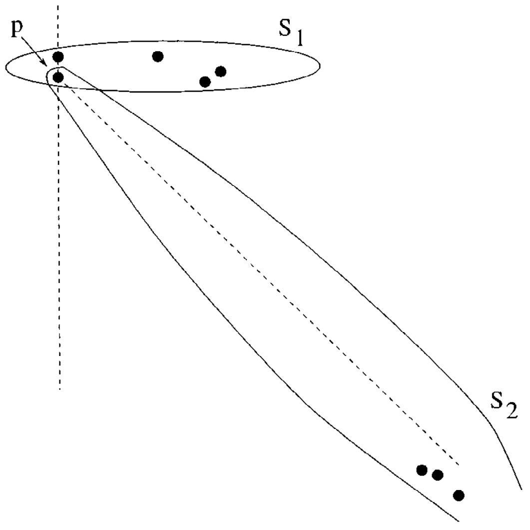

Fig. 3. Construction of a vertical configuration.

No $(n-2)$-cap in S_1 can be extended by more than 1 using a point in S_2. Similarly, no $(n-2)$-cup in S_2 can be extended by more than 1 using a point in S_1. Thus we see that our point set contains no n-caps or m-cups. Furthermore, the size of our vertical configuration implies that

$$f_V(n,m) \geq f(n,m-1) + f_V(n-1,m) - 2. \tag{9}$$

The above construction can be modified to place S_2 above S_1 along the line $y = x$, yielding

$$f_V(n,m) \geq f(n-1,m) + f_V(n,m-1) - 2. \tag{10}$$

3. Comments

What propels the argument of the last section is the observation that a and b must be defective under the given orientation. Furthermore, a and b remain defective even after the removal of right endpoints of $(n-1)$-caps and $(m-1)$-cups.

A point that is the left or right endpoint of an $(n-1)$-cap or $(m-1)$-cup, can, upon rotating the coordinate system, become an a-defective or b-defective point. This suggests that there may be angles at which there are many defective points. A careful analysis of the conditions under which a point is defective, coupled with the above observation, may allow an improvement in the upper bound for $g(n)$.

Acknowledgments

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