Finding Eigenvalue Problems for Solving Nonlinear Evolution Equations*⁾

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The problem of determining what nonlinear evolution equations are exactly solvable by inverse scattering techniques is simplified by considering a linear limit. By linearizing a given eigenvalue problem and the associated time evolution operator, it is possible to determine the class of linearized dispersion relation(s) of the exactly solvable nonlinear evolution equations. Examples are given to illustrate the method.

§ 1. Introduction

In recent years, we have seen a large number of nonlinear evolution equations,[1]~[11] which can be exactly solved by what is called the method of "inverse scattering transforms",[7] hereafter called IST. The difficulty of finding what evolution equations are solvable by this method can be well appreciated by investigators in this field, and as of now, no systematic method exists for determining when a given nonlinear evolution equation can or cannot be solved exactly by this method. Some progress in this direction has recently been made by Ablowitz, Kaup, Newell and Segur.[7] They showed for the Schrödinger and the Zakharov-Shabat[2] eigenvalue problems, that if given a meromorphic dispersion relation, $\omega(k)$, with only a finite number of poles, there exists a nonlinear evolution equation exactly solvable by the Zakharov-Shabat IST. And if $\omega(k)$ is also odd, then there existed another and different nonlinear evolution equation which is exactly solvable by the Schrödinger IST. Furthermore, in the linear limit of these nonlinear evolution equations, the frequency-wave number relation is exactly the initially assumed $\omega(k)$. This clearly indicates that those nonlinear evolution equations which are exactly solvable by an IST, can be characterized by two items: i) the dispersion relation of the linearized evolution equations and ii) an eigenvalue problem. Thus, the number of different nonlinear evolution equations solvable by IST (all having the same linearized dispersion relation) is limited by our ingenuity for finding eigenvalue problems with solvable inverse scattering problems.

Although we are still not able to determine whether or not a given equation is, in general, solvable by an IST, we can extend (or at least show how) the results of Ref. 7). We do this by noting that one of the important distinguishing

characteristics of these equations is their *linearized* dispersion relation (s).[7] Since these nonlinear evolution equations arise from an integrability condition, one of the more difficult tasks in finding them is the difficulty of going through the necessary algebra for obtaining the integrability condition, which is fully nonlinear. But if we linearize this procedure, we can obtain the final result in a very general form. And since we only need the linearized dispersion relation (s), this is certainly quite sufficient.

In brief, what we do in § 2 is the following: We take a general eigenvalue equation with an arbitrary set of potentials. We then proceed to find the most general set of linearized dispersion relations for these potentials if the nonlinear evolution equations for these potentials is to be solvable by an IST. Thus, when we have a nonlinear evolution equation which we suspect may be solvable by an IST, we have a relative quick means of determining if a particular eigenvalue problem contains this equation. Finally, in § 3, we will illustrate this technique with specific examples, and in the process, find one new equation which can be solved by an IST.

§ 2. Derivation

Consider the general eigenvalue problem

$$[D(-i\partial_x) + Q(-i\partial_x, \zeta, x, t)]\Psi(x, t) = \lambda\Psi(x, t) \tag{1}$$

on the interval $-\infty < x < \infty$ and where Ψ is a general column vector, $D(l)$ is assumed to be a matrix polynomial in l, of some finite order, Q is a general matrix potential which vanishes as $x \to \pm\infty$ and λ is the eigenvalue which will be related to ζ by

$$\lambda = f(\zeta), \tag{2}$$

with f to be specified. The inclusion of ζ in (1) is to allow the potential matrix to contain eigenvalue-dependent terms as in the eigenvalue problem for the sine-Gordon equation in laboratory coordinates.[3] If this is to be a problem solvable by an IST, the time evolution of Ψ must be of the form

$$i\partial_t\Psi(x, t) = H(\zeta, \lambda, x, t)\Psi(x, t), \tag{3}$$

where H is a *local* operator involving, at worst, derivatives. (By local, we mean that H does not contain any integrals over Ψ.)

Let us now look at (1) and (3) as $Q \to 0$ and take Q to be the order of ε. In the linear limit, we may use Fourier analysis, and without loss of generality, we can consider only one Fourier component of Q at a time. Let

$$Q(-i\partial_x, \zeta, x, t) = \varepsilon e^{i(kx-\omega t)}\bar{q}(-i\partial_x, \zeta), \tag{4}$$

where $\omega(k)$ is the linear dispersion relation for Q (i.e., in the linear limit, $\omega(k)$ is frequency-wave number relation for the fields contained in Q, and may be

multivalued). Also take

$$\Psi(x, t) = \Psi_0(l, t) e^{ilx} + \varepsilon \Psi_1(\zeta, l, k, t) e^{i(l+k)x} + O(\varepsilon^2), \tag{5}$$

$$H(\zeta, \lambda, x, t) = H_0(-i\partial_x, \zeta, \lambda) + \varepsilon e^{i(kx - \omega t)} H_1(-i\partial_x, k, \zeta, \lambda) + O(\varepsilon^2). \tag{6}$$

Then Eqs. (1) and (3) yield, to first order in ε,

$$[D(l) - \lambda] \Psi_0 = 0, \tag{7}$$

$$[D(l) - \lambda] H_0(l, \zeta, \lambda) \Psi_0 = 0, \tag{8}$$

$$H_1(l, k, \zeta, \lambda) \Psi_0(l, t) = \left\{ \begin{matrix} H_0(k+l, \zeta, \lambda) E(k+l, \lambda) \bar{q}(l, \zeta) \\ -E(k+l, \lambda) \bar{q}(l, \lambda) [H_0(l, \zeta, \lambda) + I\omega(k)] \end{matrix} \right\} \Psi_0(l, t), \tag{9}$$

where E is defined by

$$E(l, \lambda) [D(l) - \lambda] = I, \tag{10}$$

and I is the identity operator (matrix). If the operator H is to be local, at least to first order in ε, then $H_1(l, k, \zeta, \lambda)$ [modulo equations (2) and (7)] must be a *finite polynomial* in l. It is this requirement which will determine the linearized dispersion relation(s) associated with a particular eigenvalue problem.

§ 3. Examples

To illustrate this technique, we shall consider three examples of eigenvalue problems. Two for which the complete set of exactly solvable nonlinear evolution equations is essentially known[7] (the Schrödinger equation and the Zakharov-Shabat equation), and one new eigenvalue problem to illustrate the ingenuity which may be required to find exact solutions for other nonlinear evolution equations.

A. *The standard Schrödinger equation*

First consider the Schrödinger equation where

$$D(l) = l^2, \tag{11}$$

and for the moment, we shall retain the ζ dependence in (4). Equation (8) is then trivially satisfied [modulo (7)] and without loss of generality, we may take

$$H_0(l, \zeta, \lambda) = h_0(\zeta, \lambda) + l h_1(\zeta, \lambda), \tag{12}$$

$$\bar{q}(l, \zeta) = \bar{q}(\zeta). \tag{13}$$

In (12), this choice can be made because modulo (7, 11), $l^2 = \lambda$ and any higher powers of l can be collapsed into the λ dependence of h_0 and h_1. In (13), the same is still true, except that also the inclusion of any $U(x)\partial_x \Psi$ term in the Schrödinger equation can always be transformed away.

Then evaluating (9) gives

$$H_1(l, k, \zeta, \lambda) = \frac{\bar{q}(\zeta)}{2l+k} \left[h_1(\zeta, \lambda) - \frac{\omega(k)}{k} \right]. \tag{14}$$

Clearly, for H_1 to be a finite polynomial in l, we must have the residue of the pole at $l = -\frac{1}{2}k$ vanish, or

$$\bar{q}(\zeta)[h_1(\zeta, \lambda) - \omega(k)/k] = 0, \tag{15}$$

modulo the condition

$$f(\zeta) = \lambda = k^2/4. \tag{16}$$

For the standard Schrödinger equation, there is no eigenvalue-dependent potential terms, so we can ignore the ζ-dependence of $\bar{q}$ and h_1. Then (15) gives

$$\omega(k) = kh_1(k^2/4), \tag{17}$$

as the class of linearized dispersion relations associated with the standard equation. General means for finding these solvable nonlinear evolution equations have recently been given by Ablowitz, Kaup, Newell and Segur.[7]

B. *The Schrödinger equation with ζ-dependent potentials*

Let us now slightly modify the Schrödinger equation and see if anything different happens. We can insert an eigenvalue-dependent term into the potential of the Schrödinger equation and change the preceding eigenvalue problem to be

$$[-\partial_x{}^2 + Q(x, t) + \zeta R(x,t)]\Psi = (\zeta^2 + \mu)\Psi, \tag{18}$$

where μ is some constant. This is still of the form given by (11) if we define $f(\zeta)$ to be

$$\lambda = \zeta^2 + \mu = f(\zeta). \tag{19}$$

The Fourier transform of the potentials is of the form

$$\bar{q}(\zeta) = q + \zeta r, \tag{20}$$

and we may take

$$h_1(\zeta, \lambda) = \eta_0(\lambda) + \zeta\eta_1(\lambda). \tag{21}$$

Since we are still using the Schrödinger equation, (15) is still true, as well as (16) if $f(\zeta)$ is given by (19). Combining these results gives

$$q[\eta_0(k^2/4) - \omega(k)/k + \zeta\eta_1(k^2/4)]$$
$$+ r[\eta_1(k^2/4)(k^2/4 - \mu) + \zeta\eta_0(k^2/4) - \zeta\omega(k)/k] = 0. \tag{22}$$

By (16) and (19), ζ has the *two* possible solutions

$$\zeta = \pm\sqrt{k^2/4 - \mu} \tag{23}$$

and (22) must be valid for either. Therefore, we have two linear homogeneous equations for q and r and a solution exists iff

$$\omega(k) = k[\eta_0(k^2/4) \pm \eta_1(k^2/4)\sqrt{k^2/4 - \mu}]. \tag{24}$$

Equation (24) is clearly different from (17) if $\eta_1(\lambda) \neq 0$ and is the same if $\eta_1(\lambda)$

$=0$. Thus, with the very simple modification of the Schrödinger equation made in (18), we suddenly have an entirely new class of nonlinear evolution equations which can be solved by another IST. The inverse scattering problem for (18) has been solved and will be published shortly.[13] When $\eta_0(\lambda)$ is zero, we clearly have a set of nonlinear evolution equations which are second order in time (ω^2 =polynomial in k^2) and which has waves traveling in both directions. To see what one of these nonlinear evolution equations might be, let's pick a case which might be interesting. If we choose $\eta_0(\lambda)=0$, $\eta_1(\lambda)=2i$ and $\mu=1/4$, (24) becomes

$$\omega = \pm k\sqrt{1-k^2}\,, \tag{25}$$

which is the linearized dispersion relation of the Boussinesq equation.[12] To see what the nonlinear evolution equation is which corresponds to this eigenvalue problem and (25), we simply work backwards. At zeroth order in ε, from (6), (12) and (21), we have

$$H = h_0 - i h_1 \partial_x$$
$$= h_0 - i(2i\zeta)\partial_x\,, \tag{26}$$

which then tells us what the form of H must be when $q=r=0$ at $x=\pm\infty$. So we can take

$$-\Psi_{xx} + (q+\zeta r)\Psi = (\zeta^2 + 1/4)\Psi\,, \tag{27}$$

as our eigenvalue problem, and

$$i\Psi_t = [A + B\partial_x]\Psi\,, \tag{28}$$

as our time evolution operator, where $B \to 2\zeta$ as $x \to \pm\infty$. One may readily verify that the desired solution is

$$\Psi_{xx} + [\zeta^2 + \tfrac{1}{4} - \tfrac{1}{2}\phi_t - \tfrac{3}{4}(\phi_x)^2 + i\zeta\phi_x]\Psi = 0\,, \tag{29}$$

$$i\Psi_t = (2\zeta - i\phi_x)\Psi_x + \tfrac{1}{2}i\phi_{xx}\Psi\,, \tag{30}$$

with the integrability condition for (29) and (30) being

$$\pi_t = \phi_{xxxx} + \phi_{xx} - 2(\phi_x\pi)_x\,, \tag{31a}$$

$$\pi \equiv \phi_t + \phi_x^2\,. \tag{31b}$$

Of course, (31) is not the Boussinesq equation due to the presence of a cubic nonlinearity. However, it is interesting to note that (31) is derivable from the water wave equations, as is the Boussinesq equation, if one goes to one order higher in the nonlinearity. Furthermore, the second-order eigenvalue problem (29) *can* be readily inverted,[13] whereas the eigenvalue problem for the Boussinesq equation is a third-order problem,[11] which has not been inverted.

C. *The Zakharov-Shavat equation*

As a final example, we will apply this procedure to another second-order equation which will illustrate further use of the conditions (7) and (8). This

is the generalized Zakharov-Shabat equation[2],[4]

$$V_{1x} + i\zeta V_1 = qV_2, \tag{32a}$$

$$V_{2x} - i\zeta V_2 = rV_1, \tag{32b}$$

for which we have

$$D(l) = l\sigma_3, \tag{33}$$

where

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{34}$$

and is the third Pauli spin matrix.   In this case, we again ignore all ζ dependence and we have

$$E(l, \lambda) = [I\lambda + l\sigma_3]/(l^2 - \lambda^2). \tag{35}$$

From (8), one finds that H_0 must be restricted to be of the form

$$H_0(l, \lambda) = h_0(\lambda) I + h_3(\lambda) \sigma_3, \tag{36}$$

and as before, one finds

$$H_1(l, k, \lambda) \Psi_0 = \left[\frac{Il\omega/k + h_3\sigma_3}{k + 2l} \overline{Q} - \sigma_3 \overline{q}\omega/k \right] \Psi_0, \tag{37}$$

where

$$\overline{Q} \equiv \sigma_3 \overline{q} - \overline{q}\sigma_3 \tag{38}$$

and we have used (7), which for (33) is

$$\lambda \Psi_0 = l\sigma_3 \Psi_0. \tag{39}$$

Note that at the pole $(l = -k/2)$ in (37), due to (39), λ does not have a definite value, although λ^2 does.   Thus, we must break $h_3(\lambda)$ into even and odd parts by

$$h_3(\lambda) = h_{3e}(\lambda^2) + \lambda h_{3o}(\lambda^2). \tag{40}$$

Then requiring the residue at the pole to vanish gives

$$\tfrac{1}{2}k[\omega/k - h_{3o}(k^2/4)]\overline{Q} - h_{3e}(k^2/4)\sigma_3\overline{Q} = 0. \tag{41}$$

Since $\overline{Q}$ and $\sigma_3\overline{Q}$ are linear independent, (41) has a nontrivial solution iff

$$\omega(k) = kh_{3o}(\tfrac{1}{4}k^2) \pm 2h_{3e}(\tfrac{1}{4}k^2). \tag{42}$$

The general method for finding these nonlinear evolution equations has also been given by Ablowitz, Kaup, Newell and Segur.[7]

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