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FINDING FINITE B_2-SEQUENCES FASTER

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Abstract. A B_2-sequence is a sequence $a_1 < a_2 < \cdots < a_r$ of positive integers such that the sums $a_i + a_j$, $1 \le i \le j \le r$, are different. When q is a power of a prime and θ is a primitive element in $GF(q^2)$ then there are B_2-sequences $A(q, \theta)$ of size q with $a_q < q^2$, which were discovered by R. C. Bose and S. Chowla.

In Theorem 2.1 I will give a faster alternative to the definition. In Theorem 2.2 I will prove that multiplying a sequence $A(q, \theta)$ by integers relatively prime to the modulus is equivalent to varying θ. Theorem 3.1 is my main result. It contains a fast method to find primitive quadratic polynomials over $GF(p)$ when p is an odd prime. For fields of characteristic 2 there is a similar, but different, criterion, which I will consider in "Primitive quadratics reflected in B_2-sequences", to appear in *Portugaliae Mathematica* (1999).

1. Introduction

A sequence of positive integers $a_1 < a_2 < \cdots < a_r$ is called a B_2-*sequence* (or Sidon sequence) if the sums $a_i + a_j$, $1 \le i \le j \le r$, are different. Erdös and Turán proved in [4] that $a_r \le n$ implies that $r < n^{1/2} + O(n^{1/4})$. This was improved by the author in [5] to $r < n^{1/2} + n^{1/4} + 1$. Erdös asked in [3] if $r < n^{1/2} + C$ is true for a constant C.

B_2-sequences with $r > n^{1/2}$ are known to exist by a theorem of Bose and Chowla [1]. Let q be a power of a prime and θ primitive in $GF(q^2)$; then

$$A(q, \theta) = \{a : 1 \le a < q^2, \theta^a - \theta \in GF(q)\} \tag{1.1}$$

will give a B_2-sequence of size q. These Bose-Chowla B_2-sequences have the stronger property that the sums $a_i + a_j$, $1 \le i \le j \le q$, are different modulo $q^2 - 1$. This has important consequences for the problem of Erdös, which Zhang noticed and used in [7].

By Lemma 3.3 in [7], if $\{a_i\}_1^r$ is a B_2-sequence $(\bmod\, m)$, then $\{a_i + b\}_1^r$ will also be a B_2-sequence $(\bmod\, m)$ for any integer b. Assume that $a_1 < a_2 < \cdots < a_r$ and define $a_{r+1} = a_1 + m$. Determine the largest interval (a_i, a_{i+1}) for $1 \le i \le r$. Let $b = m + 1 - a_{i+1}$. Then the largest number in the new sequence is, in general, smaller.

Another idea of Zhang was to generate a large number of B_2-sequences for each q by varying the primitive element $\theta \in GF(q^2)$. There are $\varphi(q^2 - 1)$ primitive elements θ, where φ is Euler's function. This number can be reduced to

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$\varphi(q^2-1)/4$ due to symmetries of the B_2-sequences. Then he determines one with largest possible interval giving a smallest possible upper bound by the previous idea. It is laborious to check each time that θ is primitive. But it is only necessary to do this for one $A(q,\theta)$. The other sequences can be found if we multiply the sequence by integers which are relatively prime to q^2-1 and reduce modulo q^2-1. This is contained in Theorem 2.2. In Theorem 2.1 I prove that $A(q,\theta)$ can be determined q times faster than suggested by (1.1).

Zhang considered only the case when $q=p$ is an odd prime. To check that θ is primitive in $GF(p^2)$ he used the following necessary and sufficient conditions: (i) θ^{p+1} is primitive in $GF(p)$; (ii) $\theta, \theta^2, \dots, \theta^p \notin GF(p)$ (Lemma 4.3 in [7]).

In Theorem 3.1 I give a new criterion for θ to be primitive in $GF(p^2)$. If θ satisfies the quadratic equation $\theta^2 = u\theta - v$ with u, $v \in GF(p)$ my criterion poses conditions on u^2/v and v.

2. Finding $A(q,\theta)$ faster

In this section I will assume that q is a power of a prime. The following Lemma 2.2 generalizes Lemma 4.3 in [7].

Lemma 2.1. *Let θ be a root of an irreducible quadratic $X^2 - uX + v$ with u, $v \in GF(q)$. Then we have*

$$\theta^q + \theta = u, \qquad \theta^{q+1} = v. \tag{2.1}$$

Proof. There are two roots θ and θ^q. The relations (2.1) follow since u is the sum and v is the product of the roots of the quadratic. □

Lemma 2.2. *Let $\theta \in GF(q^2)$ and write $\theta^{q+1} = v$. Then θ is a primitive element if and only if*

(i) *$\theta^i \notin GF(q)$ for $1 \le i \le q$; and*

(ii) order$(v) = q-1$.

Proof. Assume that θ is primitive in $GF(q^2)$. Then order$(\theta) = q^2-1$. If $\theta^i \in GF(q)$ for some i, $1 \le i \le q$, then $\theta^{i(q-1)} = 1$ gives a contradiction. Therefore (i) holds. If order$(v) = n < q-1$, then $\theta^{(q+1)n} = 1$ gives another contradiction since $(q+1)n < q^2-1$. Therefore (ii) holds.

Conversely, assume that (i) and (ii) are satisfied. Note that $v \in GF(q)$ since $v^{q-1} = \theta^{q^2-1} = 1$. Let order$(\theta) = n = (q+1)k + r$, $0 \le r \le q$. Then $\theta^n = 1$ implies that $\theta^r = v^{-k} \in GF(q)$ and $r = 0$ follows by (i). Then $v^k = 1$ and $k = q-1$ follows by (ii). Hence $n = q^2-1$. □

Let θ be primitive in $GF(q^2)$. Define u_i and $v_i \in GF(q)$ by

$$\theta^i = u_i\theta - v_i. \tag{2.2}$$

We have $u_i \neq 0$ for $1 \le i \le q$ by Lemma 2.2(i). Since v is primitive in $GF(q)$ by (ii), there are integers t_i such that

$$u_i = v^{t_i} = \theta^{(q+1)t_i}, \qquad 1 \le i \le q. \tag{2.3}$$

If we divide (2.2) by u_i, then we find

$$\theta^{i-(q+1)t_i} - \theta = -v_i u_i^{-1} \in GF(q) \tag{2.4}$$

and since, by definition

$$A(q,\theta) = \{a\colon 1 \le a < q^2, \theta^a - \theta \in GF(q)\}, \tag{2.5}$$

it follows that

$$i-(q+1)t_i \in A(q,\theta), \qquad 1 \le i \le q. \tag{2.6}$$

We have

Theorem 2.1. *Let θ be a primitive element in $GF(q^2)$ and define the integers t_i for $1 \le i \le q$ by (2.3) and $A(q,\theta)$ by (2.5). Then we have*

$$A(q,\theta) = \{i-(q+1)t_i \pmod{q^2-1} \colon 1 \le i \le q\}. \tag{2.7}$$

Proof. With regard to (2.6) it remains to prove that the elements are distinct modulo q^2-1. If $i-(q+1)t_i \equiv j-(q+1)t_j \pmod{q^2-1}$, then $i \equiv j \pmod{q+1}$ and we have $i=j$ since $1 \le i,\, j \le q$. □

Example 2.1. Let $q=7$ and $\theta^2 = \theta - 3$ (cf. Example 3.1 in [7]). We find $u_1 = u_2 = 1$, $u_3 = 5$, $u_4 = 2$, $u_5 = 1$, $u_6 = 2$, $u_7 = 3$ and, since $v=3$, $t_1 = t_2 = 0$, $t_3 = 5$, $t_4 = 2$, $t_5 = 0$, $t_6 = 2$, $t_7 = 3$, which gives $A(7,\theta) = \{1,2,5,11,31,36,38\}$ after sorting. □

If c is relatively prime to q^2-1, then $M_c(x) = cx$ defines a one-one mapping of the integers modulo q^2-1. For any integer t we define another one-one mapping $\pmod{q^2-1}$ by $T_t(x) = x-(q+1)t$.

Theorem 2.2. *Let θ and θ_1 be primitive elements in $GF(q^2)$ and $\theta = \theta_1^c = u_c\theta_1 - v_c (u_c, v_c \in GF(q))$, $u_c = \theta_1^{(q+1)t}$. Then $A(q,\theta_1) = T_t M_c A(q,\theta)$.*

Proof. Let $a \in A(q,\theta)$. Then we have $\theta^a - \theta \in GF(q)$ and $\theta_1^{ca} - u_c\theta_1 \in GF(q)$. If we divide this by u_c $(\neq 0)$, we find that $ca-(q+1)t \in A(q,\theta_1)$ and $T_t M_c A(q,\theta) = A(q,\theta_1)$ follows since both sets have q elements. □

3. A criterion for primitive quadratics

I will prove a new criterion for a quadratic $X^2 - uX + v$ over $GF(p)$, p an odd prime, to be primitive, i.e., with a root θ, which is a primitive element in $GF(p^2)$. I am looking for a criterion which is suitable for computations and faster than the one in Lemma 2.2. There is a criterion by Bose, Chowla and Rao, Theorem 3A in [2], which depends on cyclotomic polynomials. I do not think it is what I am looking for, but I have use of the *integral order* of $\alpha \in GF(p^2)$. It is the least positive number n for which $\alpha^n \in GF(p)$. I found this notion in [2].

I will need polynomials $Q_m(X)$ of degree $m \ge 0$ defined recursively by

$$Q_0(X) = 1, \qquad Q_1(X) = X, \tag{3.1}$$

$$Q_{m+1}(X) = XQ_m(X) - Q_{m-1}(X) \quad \text{when } m \ge 1. \tag{3.2}$$

Lemma 3.1. *Let α be a root of the irreducible quadratic $X^2 - uX + v$ over $GF(p)$ with $u,\, v \neq 0$. Write $u^2/v = w$ and let $n = 2(m+1)$. Then $(\alpha^2/v)^n = 1$ if and only if $Q_m(w-2) = 0$.*

Proof. We have $(\alpha^2+v)^2 = u^2\alpha^2$. Hence $\alpha^4 + v^2 = (u^2-2v)\alpha^2$ and

$$(\alpha^2/v) + (v/\alpha^2) = w-2. \tag{3.3}$$

Write $\alpha^2/v = \beta$ for brevity. Observe that $\beta \neq \pm 1$. Hence $\beta^2 - 1 \neq 0$.

Assume that $\beta^n = 1$, $n = 2(m+1)$. If we divide $\beta^n - 1 = 0$ by $\beta^2 - 1 \neq 0$ we find $\beta^{2m} + \beta^{2m-2} + \cdots + 1 = 0$. Divide this by β^m. Now

$$\beta^m + \beta^{m-2} + \cdots + \beta^{-m} = 0. \tag{3.4}$$

The left-hand side of (3.4) can be written as a polynomial in $\beta+\beta^{-1}$. In fact, it is $Q_m(\beta+\beta^{-1})$. For obviously $Q_1(X)=X$, $Q_2(X)=X^2-2$ and (3.2) follows since $(\beta+\beta^{-1})Q_m(\beta+\beta^{-1})=(Q_{m+1}+Q_{m-1})(\beta+\beta^{-1})$. Since $\beta+\beta^{-1}=w-2$ by (3.3), we have $Q_m(w-2)=0$.

Conversely, assume that $Q_m(w-2)=0$. Then, working backward, we find that $\beta^n=1$. □

Lemma 3.2. *If $\alpha^m\in GF(p)$ and n is the integral order of α, then $n|m$.*

Proof. Write $m=kn+r$, $0\le r<n$. Then $\alpha^r=\alpha^m(\alpha^n)^{-k}\in GF(p)$ and $r=0$ follows by the definition of n. □

Theorem 3.1. *Consider a quadratic X^2-uX+v with u, $v\in GF(p)$, $v\neq 0$ and p an odd prime. Write $u^2/v=w$. The quadratic is primitive if and only if the following conditions are satisfied* ((iv) *or* (iv′))

(i) *v is primitive* $(\operatorname{mod} p)$,
(ii) *$w\not\equiv 0$ is a quadratic nonresidue* $(\operatorname{mod} p)$,
(iii) *$w-4$ is a quadratic residue* $(\operatorname{mod} p)$,
(iv) $Q_m(w-2)\not\equiv 0\ (\operatorname{mod} p)$ *when* $m\le[(p+1)/6]-1$,
(iv′) *for all odd primes q dividing $p+1$* $Q_{m(q)}(w-2)\not\equiv 0\ (\operatorname{mod} p)$, *where* $m(q)=((p+1)/2q)-1$.

Proof. When we prove the necessity of one condition we may assume that the preceding ones are satisfied.

Condition (i) is necessary by Lemma 2.2(ii). Assume that (i) holds. Then v is nonsquare in $GF(p)$. It follows that w is nonsquare in $GF(p)$ ($u=0$ is impossible). This gives (ii). A primitive quadratic is irreducible. Then the discriminant u^2-4v must be nonsquare in $GF(p)$. If we divide by nonsquare v we will get a square by the rules. This is (iii).

Assume that the conditions (i)–(iii) are satisfied. The quadratic is then irreducible and we have $v=\theta^{p+1}$ by Lemma 2.1, where θ is a root.

Assume that $Q_m(w-2)\equiv 0\ (\operatorname{mod} p)$ for some $m\le[(p+1)/6]-1$. By Lemma 3.1 we have $1=(v/\theta^2)^n=\theta^{(p-1)n}$ with $n\le(p+1)/3$. This is impossible when θ is a primitive element in $GF(p^2)$. This gives (iv) and (iv′).

Assume that (i)–(iii) and (iv′) are satisfied. Let n be the integral order of θ. Since $\theta^{p+1}=v\in GF(p)$, $p+1=kn$ follows by Lemma 3.2.

Note that v is nonsquare in $GF(p)$ and $v=\theta^{p+1}=(\theta^n)^k$, $\theta^n\in GF(p)$. It follows that k is an odd integer. We claim that $k=1$.

Assume that $k>1$. Let q be an odd prime divisor of k. Then $\bar{n}=(p+1)/q$ will be a multiple of $n=(p+1)/k$. Observe that $(v/\theta^2)^n=\theta^{n(p-1)}=1$ since $\theta^n\in GF(p)$. Then we have $(\theta^2/v)^{\bar{n}}=1$. By Lemma 3.1 it follows that $Q_{m(q)}(w-2)\equiv 0\ (\operatorname{mod} p)$, a contradiction to (iv′). Therefore $k=1$ and $n=p+1$.

We have proved that the integral order of θ is $p+1$. I will prove that this implies that θ is primitive. If $N=\operatorname{order}(\theta)$, then $\theta^N=1$ and we have $n\mid N$ by Lemma 3.2, i.e., $p+1\mid N$. Write $N=(p+1)a$ and we find that $1=\theta^N=v^a$. Since v is primitive in $GF(p)$, it follows that $p-1\mid a$. Hence $N=p^2-1$, which was to be proved. □

In calculations using a computer one could use (iv) and (3.1), (3.2). If the calculations are done by hand, then (iv′) is better. In both cases start with a list L1 of all quadratic nonresidues $(\operatorname{mod} p)$. The length of this list is $(p-1)/2$. Delete from this list all integers w for which $w-4 \pmod p$ belongs to the list. Then we obtain a list L2, which is about half as long (the length of L2 is $(p+1)/4$ when -1 is a quadratic nonresidue $\pmod p$ and $(p-1)/4$ when -1 is a quadratic residue $\pmod p$)). Then go to (iv) or (iv′) and check the numbers in L2. Suppose we have found a number w, which satisfies all four conditions. Then find a primitive element $\pmod p$ from a table and determine u such that $u^2 \equiv vw \pmod p$. Then we have the coefficients u and v of a primitive polynomial. If we apply (iv) or (iv′) to all numbers on the list L2 we may determine all primitive quadratic polynomials.

It is easy to prove by induction over $m \geq 1$ that

$$Q_m(X) = \sum_{i=1}^{[m/2]} (-1)^i \binom{m-i}{i} X^{m-2i}.$$

Example 3.1. Let $p = 29$. The odd primes dividing $p+1$ are 3 and 5. We find that $m(3) = 4$ and $m(5) = 2$. We have $Q_2(X) = X^2-1$, $Q_4(X) = X^4-3X^2+1$. The list of quadratic nonresidues is $\mathrm{L1} = \{2, 3, 8, 10, 11, 12, 14, 15, 17, 18, 19, 21, 26, 27\}$. We delete all w for which $w-4$ belongs to the list and find $\mathrm{L2} = \{3, 8, 10, 11, 17, 26, 27\}$. From L2 we delete "3" since $3-2 = 1$ is a root of Q_2 and we delete "8" and "26" because 6 and 24 are roots of $Q_4 \pmod{29}$. There remains: 10, 11, 17, 27, which satisfy conditions (ii), (iii) and (iv′). There are $\varphi(28) = 12$ primitive elements v in $GF(29)$. Hence there are $4 \cdot 12 \cdot 2 = 96$ primitive polynomials (4 numbers w, 12 numbers v, and 2 numbers u for each combination of v and w). This gives 192 primitive elements in $GF(29^2)$ in agreement with $\varphi(29^2-1) = 192$. If we choose $w = 10$ and $v = 2$, we find $u = 7$ (or -7) and $X^2 - 7X + 2$ is a primitive polynomial $\pmod{29}$. □

Corollary. *If $p = 2^k - 1$ is a (Mersenne) prime or if $p = 2q - 1$ for an odd prime q, then the conditions* (i)–(iii) *are necessary and sufficient for the quadratic $X^2 - uX + v$ to be primitive.*

Proof. In the first case (iv′) is vacuously satisfied. In the second case $m(q) = 0$ and $Q_0 = 1$. □

4. A very fast construction

There is a new construction of B_2-sequences by I. Z. Ruzsa in [6], Theorem 4.4, which gives B_2-sequences of the size $p-1$ for each odd prime p. The computations are straightforward and therefore very fast. I have extended the construction by the introduction of a factor f, an integer in $1 \leq f < p-1$, which is relatively prime to $p-1$. Let g be a primitive element $\bmod p$ and define

$$R(p, f) = \{pfi + (p-1)g^i \bmod p(p-1) \colon 1 \leq i \leq p-1\}. \tag{4.1}$$

The integers of $R(p, f)$ are smaller than $p(p-1)$.

Theorem 4.1. *$R(p, f)$ is a B_2-sequence modulo $p(p-1)$.*

Proof. Let $pf(i+j)+(p-1)(g^i+g^j) \equiv a \pmod{p(p-1)}$ be the sum of two elements. Then we find

$$g^i + g^j \equiv -a \pmod p \tag{4.2}$$

and $f(i+j) \equiv a \pmod{p-1}$. Since f is relatively prime to $p-1$, there is an integer h such that $fh \equiv 1 \pmod{p-1}$. It follows that $i+j \equiv ah \pmod{p-1}$ and we have by Fermat's little theorem

$$g^i g^j \equiv g^{ah} \pmod{p}. \tag{4.3}$$

By (4.2) and (4.3) g^i and g^j are the roots of $X^2 + aX + g^{ah} = 0$ in $GF(p)$. Hence, g^i and g^j are unique and determine $\{i, j\}$ uniquely. □

If we replace the primitive element g by another primitive g^b we will get $R(p, fd)$, where $bd \equiv 1 \pmod{p-1}$. If we multiply $R(p, f)$ by an integer c relatively prime to $p(p-1)$ we get a translate of $R(p, fc)$. Thus we have essentially only $\varphi(p-1)$ B_2-sequences for each prime p. This "count" is much smaller than the count of the Bose-Chowla sequences $A(p, \theta)$. The estimates for C using $R(p, f)$ are worse than those of $A(p, \theta)$.

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