

**Appendix B: Finding Four Dimensional Symplectic Maps with Reduced
Chaos: Preliminary Results**

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Finding Four Dimensional Symplectic Maps with Reduced Chaos: Preliminary Results

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Abstract. A method for finding integrable four-dimensional symplectic maps is outlined. The method relies on solving for parameter values at which the linear stability factors of the fixed points of the map have the values corresponding to integrability. This method is applied to accelerator lattices in order to increase dynamic aperture. Results show a increase of the dynamic aperture after correction, which implies the validity of the method.

INTRODUCTION

Much progress has been made in the problem of determining whether a dynamical system is chaotic. Poincaré (1) showed that for a general perturbation of an integrable multidimensional oscillator, there is no invariant analytic in the perturbation parameter. The problem of small denominators in normal form theory (2) prevents one from finding a convergent invariant in the neighborhood of a linearly stable fixed point. The understanding of what happens when an integrable is perturbed was greatly increased by the KAM theorem (3-6) which showed that some invariant surfaces remain (provided that the linear frequencies in normal form theory are not linearly related with integers smaller than 4). Complimentarily, Melnikov (7) and subsequent work (8) on the intersection of stable and unstable manifold showed that chaotic motion is very easy to find. These are the approaches one must take to determine the stability of a given Hamiltonian system, such as the motion of the asteroids or the stability of the solar system; one would like to know whether the motion for given initial conditions is stable, and so one would like methods for determining whether a trajectory found in given region of phase space for a given Hamiltonian is, or is likely, to be integrable or chaotic.

whereas before the parameters were the Fourier harmonics of the winding law of the coils.

The method proved to be successful here too, insofar as only one transverse degree of freedom was concerned, i.e., for trajectories with initial conditions (momentum and position) entirely in the horizontal plane (an invariant plane for the lattices we considered). However, an examination of the dynamics of the four dimensional system (by considering trajectories with initial conditions out of the horizontal plane) found that the dynamics was worse. That is, the four dimensional volume of confined initial conditions was found to be smaller even though the uncoupled horizontal dynamics was improved.

These results meant that to be able to reduce the chaotic dynamics in Hamiltonian systems of two and a half degrees of freedom, one would have to consider the full dynamics. In this paper we show how this could be done. We begin, in the following section, by reviewing Hamiltonian dynamics and the one-turn map of an accelerator. Then we discuss the linear stability of fixed points in four dimensional maps and its connection to the integrability of the dynamical systems, which in turn presents a method of finding integrable Hamiltonian systems of two and a half degrees of freedom. This is followed by an example demonstrating the application of the method to an accelerator lattice which shows an increase of the dynamic aperture. Implications of this method are discussed in the last section.

HAMILTONIAN DYNAMICS AND THE ONE-TERM MAP OF AN ACCELERATOR

The motion of particles in a circular accelerator is governed locally by a Hamiltonian. However, for understanding the long-term dynamics it is sufficient to analyze the return map. In this section we review the relationship between the Hamiltonian and the return map.

We consider the motion defined by a Hamiltonian having N degrees of freedom plus temporal variation, $H(q_1, \dots, q_N, P_1, \dots, P_N, s)$. In keeping with the practice in accelerator physics tracking, the temporal variable s in an accelerator is a variable that parameterizes the path around the accelerator ring, while the energy and time form the third canonical pair (Ref. 16, Sec. 3.3, Ref. 17 and 18). The Hamiltonian is then the function that gives the rate of change of the other variables with respect to s as the accelerator is circumnavigated:

$$\frac{dq_i}{ds} = \frac{\partial H}{\partial p_i} \quad (1)$$

and

$$\frac{dp_i}{ds} = -\frac{\partial H}{\partial q_i} \quad (2)$$

In fact, for single-particle dynamics in accelerators, N is at most three, and the coordinates are the transverse coordinates and momenta, the energy, and the time (which is needed to calculate the temporally varying accelerating kick).

This Hamiltonian theory neglects the phase-space contracting effect of synchrotron radiation and the diffusion caused by quantum fluctuations. These effects are usually neglected in proton machines. In addition they can be neglected during the first several hundred turns in electron machines, when it is nevertheless important to know the dynamic aperture (the region of confined trajectories in phase space), as that controls the success of the injection process.

For most analyses of the single-particle dynamics in accelerators, the energy-time canonical pair is considered separately. The characteristic (*synchrotron*) frequency of the energy-time oscillations is much smaller than the characteristic (*betatron*) frequencies of the transverse oscillations, and so can be treated later by adiabatic theory. When this decoupling applies, the picture of the motion in phase space can be pieced together by understanding the constant-energy dynamics for the range of energies implied by the synchrotron oscillations.

For the systems we consider, the temporal variable is periodic. By this we mean not only that the Hamiltonian is periodic in this variable,

$$H(s + S) = H(s) \quad (3)$$

with some period S , but also that phase space is toroidal in the variable s so that $(q_1, \dots, q_N, p_1, \dots, p_N, s)$ and $(q_1, \dots, q_N, p_1, \dots, p_N, s + S)$ are the same point. This obviously holds for the case of the accelerator as described above. This requirement implies that single-valuedness and periodicity as in Eq. (3) are the same condition.

The integration of the trajectories through one period of the Hamiltonian gives a map,

$$\bar{z} = M(z) \quad (4)$$

of the phase space of initial conditions at $s = 0$ onto the phase space at $s = S$. Here, \bar{z} is the points obtained by starting at z and integrating the equations of motion for one period S of the temporal variable. This map is symplectic (Ref. 19, Sec. 7-1) because the equations of motion come from a Hamiltonian. Furthermore, this map is the same for each period, as the Hamiltonian is periodic. This implies that one can find the trajectory after going through n periods of the Hamiltonian by applying the map n times.

A Hamiltonian system of N degrees of freedom is integrable if it has N single-valued invariants $\mathbf{J} = (J_1, \dots, J_N)$ in *involution* (See Ref. 20, sec. 147), known as *actions*. *In involution* means that the Poisson brackets of the actions

with themselves vanish. Assuming, as we will, that the surfaces of constant action are compact, the surfaces must be tori in phase-space-time. The variables canonical to the actions are the *angles* $(\theta_1, \dots, \theta_N)$, which, with time, specify a point on a given torus. Because the actions are constants of the motion, the Hamiltonian does not depend on the angles. Further, it can be shown that the time-dependence of the Hamiltonian can be eliminated by a subsequent transformation. Hence, the Hamiltonian for an integrable system has the form, $H(\mathbf{J})$. Because the Hamiltonian has this form, the angle variables increase linearly in time,

$$\frac{d\theta_i}{ds} = \frac{\partial H}{\partial J_i} \equiv \omega_i(\mathbf{J}) \quad (5)$$

Any observable O has time dependence only through the evolution of the angles,

$$O = f(\theta_1, \dots, \theta_N, s) \quad (6)$$

Furthermore, the dependence on these angles is periodic. This, together with the fact that the angle variables increase linearly in time, implies that the time-dependence of any observable is *quasiperiodic*,

$$O = f(\theta_1 = \theta_{10} + \omega_1 s, \dots, \theta_N = \theta_{N0} + \omega_N s, s) \quad (7)$$

which means, in essence, that the evolution has the form of Eq. (7) - that of a function periodic in $N + 1$ variables with the evolution of each of these variables being linear. Such evolution has $N + 1$ fundamental frequencies, one (ω_i) for each of the degrees of freedom and one,

$$\Omega \equiv 2\pi/S \quad (8)$$

associated with the periodicity of the temporal variable.

In fact, the fundamental temporal frequency (8) provides a scale for the other frequencies. The scaled frequencies,

$$\nu_i \equiv \omega_i/\Omega \quad (9)$$

are known as the winding numbers in most of the nonlinear dynamics literature, as these give the number of times a trajectory circulates around the torus "the i th way" (i.e., increasing θ_i by 2π) per each circulation around the torus "the s th way", that is, per increase of the temporal variable by one fundamental period S . In accelerator physics the quantities (9) are known as the tunes, as they give the number oscillations of the various degrees of freedom per circulation around the accelerator.

To design a system which is stable under small perturbations, the tunes have to be linearly unrelated to integers smaller than or equal to 4, as the KAM theory requires. To be specific, the tunes have to satisfy

$$\sum_i l_i \nu_i - m \neq 0, \quad (10)$$

discontinuous expression such as this can be awkward. But taking advantage of the natural periodicity to write the vector potential as a discrete Fourier series makes the expression for A_ϕ tractable. One can use a general time dependence to express the vector potential as a sum of traveling waves and then note that for nearly-harmonic time dependence, only one of the waves is resonant with the particle motion. By neglecting the non-resonant terms we can keep only one term in the Fourier expansion, making the expression for A_ϕ a single wave.

To develop the single-wave model, we first need to Fourier expand the cavity field. The periodicity of this problem makes such an expansion a natural step, unlike the beam-plasma case⁶ where there is no inherent periodicity and one relies on the differing growth rates to make one wave dominant. Because A_ϕ is constant inside the cavity (see Fig. 2), the Fourier expansion has the form:

$$A_\phi = \sum_{n=-\infty}^{\infty} \frac{A_0 h \sin(nh / 2R)}{2\pi R \quad nh / 2R} \cos(n\phi) X(t) \quad (12)$$

where R is the radius of the accelerator (see Fig. 1). We assume a general time-dependence for $X(t)$ that allows for varying frequency and amplitude:

$$X(t) = X_{0c}(t) \cos(\int \omega_r dt) + X_{0s}(t) \sin(\int \omega_r dt). \quad (13)$$

where ω_r is the real part of the wave frequency. Combining Eq. (13) and Eq. (12) yields the following expression for the vector potential:

$$A_\phi = \sum_{n=-\infty}^{\infty} \frac{A_0 h \sin(nh / 2R)}{2\pi R \quad nh / 2R} \frac{1}{2} [X_{0c}(\cos(n\phi + \int \omega_r dt) + \cos(n\phi - \int \omega_r dt)) + X_{0s}(\sin(n\phi + \int \omega_r dt) - \sin(n\phi - \int \omega_r dt))] \quad (14)$$

At this point, the vector potential for the problem is written as a sum of many traveling waves. Here we make the approximation we need only keep the one term in this sum which is the wave traveling in resonance with the particles. That resonance condition is

$$\frac{\omega_r}{n} \approx \omega_s. \quad (15)$$

If ω_r changes quickly, then as the system evolves different values of n will be resonant. For a single value of n to remain resonant over the time range considered, ω must not change too much. From Eq. (15) one can see that for $\Delta n \ll 1$, we need $\Delta\omega \ll \omega_s$. In this paper, we limit ourselves to such slowly-varying systems. Using Eq. (15) to select out only the term in Eq. (14) that is most nearly resonant (recall that there are contributions from both $\pm n$) and using the fact that $nh / 2R \ll 1$, we arrive at

$$A_\phi = \frac{A_0 h}{\pi R} \frac{1}{2} [X_{0c} \cos(n\phi - \int \omega_r dt) - X_{0s} \sin(n\phi - \int \omega_r dt)]. \quad (16)$$

This is the general, non-linear single-wave model of the vector potential.

The single-wave form of the vector potential implies that the distribution function will also be in terms of the wave and its harmonics. The effective force in the Vlasov equation that governs the kick the particles receive is

$$F = \ddot{\phi} = \frac{\eta \omega_s e}{p_s} \frac{A_0 h}{2\pi R} [(\dot{X}_{0c} + \omega_r X_{0s}) \cos(n\phi - \int \omega_r dt) - (\dot{X}_{0s} - \omega_r X_{0c}) \sin(n\phi - \int \omega_r dt)] \quad (17)$$

which implies that even nonlinearly, the distribution function will contain only terms like this and harmonics:

$$f = \sum_{k=0}^{\infty} \alpha_k \cos(k(n\phi - \int \omega_r dt)) + \beta_k \sin(k(n\phi - \int \omega_r dt)). \quad (18)$$

where α_k and β_k are found in the usual way,

$$\{\alpha_k, \beta_k\} = \frac{1}{\pi} \int d\phi \{ \cos(k(n\phi - \int \omega_r dt)), \sin(k(n\phi - \int \omega_r dt)) \}. \quad (19)$$

Only the fundamental mode ($k=1$) is resonant with the cavity, however, and so it is excited much more highly than the harmonics ($k > 1$) and we drop all but the $k=1$ term. This now goes into the current equation (10), which drives the cavity oscillations. The cavity equation (5) now becomes

$$\ddot{X} + \frac{\omega_c}{Q} \dot{X} + \omega_c^2 X = eR \iint \phi d\phi \sum_m \frac{A_0 h}{2\pi R} \frac{\sin(mh/2R)}{mh/2R} \cos(m\phi) [\alpha_1 \cos(n\phi - \int \omega_r dt) + \beta_1 \sin(n\phi - \int \omega_r dt)] \quad (20)$$

The orthogonality of cosines will select out only the $\pm m = n$ terms and we are left with the equation

$$\ddot{X} + \frac{\omega_c}{Q} \dot{X} + \omega_c^2 X = eA_0 h \int \phi d\phi [\alpha_1 \cos(\int \omega_r dt) - \beta_1 \sin(\int \omega_r dt)]. \quad (21)$$

Plugging in Eq. (13) for X and comparing coefficients on the two sides yields the following equations for the cavity oscillation amplitudes:

$$\ddot{Y} + \frac{\omega_c}{Q} \dot{Y} + (\omega_c^2 - \omega_r^2) Y - i \left(\frac{\omega_r \omega_c}{Q} Y + \frac{1}{Y} \frac{d}{dt} (\omega_r Y^2) \right) = eA_0 h \int \phi d\phi (\alpha_1 - i\beta_1). \quad (22)$$

Here we have introduced $Y = X_{0c} + iX_{0s}$ for simplicity. Along with the Vlasov equation,

$$\frac{\partial f}{\partial t} + \dot{\phi} \frac{\partial f}{\partial \phi} + F \frac{\partial f}{\partial \dot{\phi}} = 0, \quad (23)$$

these form a complete description of the non-linear, single-mode beam-cavity system. Because Eq. (22) is an equation for the amplitude of the cavity oscillations only, it does not change on the fast time scale of the cavity frequency as does an equation such as Eq. (5). This fact has tremendous advantages for numerical modeling of the nonlinear system and

will be discussed at length in a future paper.⁷

IV. Linear theory and the dispersion relation

In this section we develop the linear theory of our system in order to determine whether the system exhibits instability. We linearize the electric field and reduce the cavity equation to simpler form. Then we solve Eq. (19) for the coefficients of the perturbed distribution in the linearized regime. In the usual way, the amplitude of the field then drops out of Eq. (22), and we arrive at a typical dispersion relation, relating the frequency of the oscillation to the unperturbed distribution function. We obtain the dispersion relation in a form familiar to plasma physics and then we also recast it in terms of the impedance of a cavity as one most often sees in accelerator physics. Knowing the dispersion relation allows one to calculate the imaginary part of the frequency to determine whether instabilities exist.

The first step in obtaining a dispersion relation is modeling the linearized electric field and simplifying the cavity equation. We assume that the frequency is constant in time and amplitude of the electric field grows exponentially. By choosing the initial phase of the field properly we can write without any loss of generality,

$$X_{0c}(t) = X_0 e^{\omega t} \quad (24a)$$

$$X_{0s}(t) = 0, \quad (24b)$$

where X_0 and $\omega = \omega_r + i\omega_i$ are now constant. Plugging these into Eq. (22) gives the relation

$$(-\omega^2 - i \frac{\omega \omega_c}{Q} + \omega_c^2) X_{0c} = e A_0 h \int [\alpha_1 - i\beta_1] \dot{\phi} d\phi. \quad (25)$$

This is the cavity equation for the linearized electric field.

What remains is to solve for the coefficients of the perturbed distribution, α_1 and β_1 , in terms of the linearized electric field. For this we need the linearized Vlasov equation:

$$\frac{\partial f_1}{\partial t} + \dot{\phi} \frac{\partial f_1}{\partial \phi} + \ddot{\phi} \frac{\partial f_0}{\partial \dot{\phi}} = 0. \quad (26)$$

where F depends on the electric field via Eq. (17). In the usual way of ordering³, we have assumed the electric field is of the same order as f_1 and so only the unperturbed distribution function appears in the third term. Plugging in for f_1 the expression from Eq. (18) and for F from Eq. (17), we compare coefficients to find

$$\alpha_1 - i\beta_1 = \frac{\omega \omega_s \eta e A_0 h X_{0c}}{2\pi R p_s (n\dot{\phi} - \omega)} \frac{\partial f_0}{\partial \dot{\phi}}, \quad (27)$$

which is the same result one would obtain using a complex distribution function. To then generate a dispersion relation, we plug this into Eq. (25) from above and cancel the field amplitude, X_{0c} , to obtain

$$-\omega^2 - i \frac{\omega \omega_c}{Q} + \omega_c^2 = \text{sign}(\eta) \omega \omega_b^2 \int \frac{\dot{\phi}}{(n\dot{\phi} - \omega)} \frac{\partial g_0}{\partial \dot{\phi}} d\dot{\phi}, \quad (28)$$

where $f_0 = \rho g_0$ and ρ is the beam density. We introduce the constant ω_b , closely related to the plasma frequency³ familiar in plasma physics:

$$\omega_b^2 = \left(\frac{|\eta| \rho e^2 \omega_s}{p_s} \right) \left(\frac{h}{d^2 (2\pi R)} \frac{1}{\pi \epsilon_0 J_1^2(v_{01})} \right). \quad (29)$$

The first set of parenthesis contains the usual components of the plasma frequency; the second parenthesis contain only geometric factors of the accelerator. We use this parameter

in the same way as the plasma frequency in plasma physics and the classical particle radius in accelerator physics.² Equation (28) determines the frequency of oscillation in terms of the unperturbed distribution function. It is the general linear dispersion relation for a damped, beam-driven oscillation.

While Eq. (28) may be more familiar to plasma physicists, accelerator physicists typically recast the dispersion relation in terms of the impedance of the resonant cavity:²

$$Z = \frac{i\omega\omega_c R_s}{Q} \frac{1}{\omega^2 + i\frac{\omega\omega_c}{Q} - \omega_c^2}, \quad (30)$$

where $R_s = Q(A_0 h)^2 / \omega_c$ is the shunt resistance of the cavity. Substituting this in allows the dispersion relation to be rewritten as

$$1 = \text{sign}(\eta) \frac{iQ}{\omega_c R_s} \omega_b^2 Z \int \frac{\phi}{n\phi - \omega} \frac{\partial g_0}{\partial \phi} d\phi, \quad (31)$$

which is the form usually seen in accelerator physics. For the rest of this paper, however, we will work with Eq. (28).

V. Analysis of dispersion relation in various limits

Next we try to solve Eq. (28) analytically by using a Lorentzian distribution function to evaluate the integral. We will then follow the traditional approach of plasma physics⁸ and solve the dispersion relation in various regions of the $\omega - \omega_s$ plane, looking at the usual cavity and beam mode solutions as well as the solutions in the crossing regime (see Fig. 3). For the cavity mode we make a special note that this mode, which is usually unimportant in accelerator physics, could begin to play a more significant role in accelerator dynamics as higher-Q cavities are introduced. We also note that one familiar limit

encompassed in the beam mode is that of the negative mass instability² of accelerator physics. For each region we look at the role of temperature and its stabilizing effects on the different modes, leading to the Keil-Schnell criterion of accelerator physics. Finally we discuss the fact that the name Landau damping is used in both accelerator physics and plasma physics, but to refer to very different phenomena.

To make further analytic progress with Eq. (28), we must assume a form for the unperturbed distribution function. For this we use the Lorentzian distribution,

$$g_0 = \frac{\omega_c \Delta}{\pi} \frac{1}{[(\phi - \omega_s)^2 + \omega_c^2 \Delta^2]}, \quad (32)$$

where Δ is the beam temperature, normalized to ω_c . Accelerator physicists² and plasma physicists⁸ often use the Lorentzian distribution to evaluate dispersion relations because it allows them to analytically evaluate the integral involved. O'Neil has shown⁸ that the results using a Lorentzian do not differ significantly from those using a more realistic distribution like a Gaussian. When we integrate Eq. (28) by parts and plug in the Lorentzian distribution, we reduce the dispersion relation to the following:

$$-\omega^2 - i \frac{\omega \omega_c}{Q} + \omega_c^2 = \frac{\text{sign}(\eta) \omega^2 \omega_b^2 \omega_c \Delta}{\pi} \int \frac{1}{(n\phi - \omega)^2 [(\phi - \omega_s)^2 + \omega_c^2 \Delta^2]} d\phi, \quad (33)$$

which we can integrate using contour integration in the complex- ϕ plane. This leads to the expression

$$\omega^2 + i \frac{\omega \omega_c}{Q} - \omega_c^2 = \frac{-\text{sign}(\eta) \omega^2 \omega_b^2}{(\omega - n\omega_s + i n \omega_c \Delta)^2}. \quad (34)$$

For small ω_b (discussed fully below), the cold beam limit gives the same dispersion relation

as for the traveling wave tube⁹ used to study the damped beam-plasma problem¹⁰.

In plasma physics, one typically considers the weak-beam limit⁶ where ω_b is the smallest frequency in the system; most accelerators operate in the weak-beam limit as well. For the Booster¹ and Main Ring¹¹ at Fermilab, ω_b is typically on the order of a few Hertz compared to MHz for ω_c or $n\omega_s$. The weak-beam limit suggests that one look for solutions in three separate regimes (see Fig. 3). To illustrate this, we rewrite Eq. (34) in dimensionless terms by introducing the variable $\bar{\omega} = \omega / \omega_c$:

$$\left(\bar{\omega}^2 + \frac{i}{Q} \bar{\omega} - 1 \right) \left(\bar{\omega} - \frac{n\omega_s}{\omega_c} + i\eta \right)^2 = -\text{sign}(\eta) \bar{\omega}_b^2 \bar{\omega}^2. \quad (35)$$

Given that $\bar{\omega}_b^2 = \omega_b^2 / \omega_c^2$ on the right-hand side of Eq. (35) is small ($\approx 10^{-10}$ for the Booster at Fermilab), solutions will only occur in three cases: the region where the expression in the first set of paranthesis on the left-hand side is near zero but the expression in the second set is not (the cavity mode), the region where the first expression is far from zero but the second is near zero (the beam mode), and finally the region where both are near zero (the crossing region). This is the approach O'Neil used for the beam-plasma case⁸ where one neglects damping. The presence of damping will modify our approach just slightly. We will also examine the cavity mode and the beam mode, however in the crossing region, we will only be able to examine the exact crossing point, where $n\omega_s = \omega_c$. This analysis is similar to that of the Robinson instability² of accelerator physics but fundamentally different because there is no synchrotron bounce frequency here.

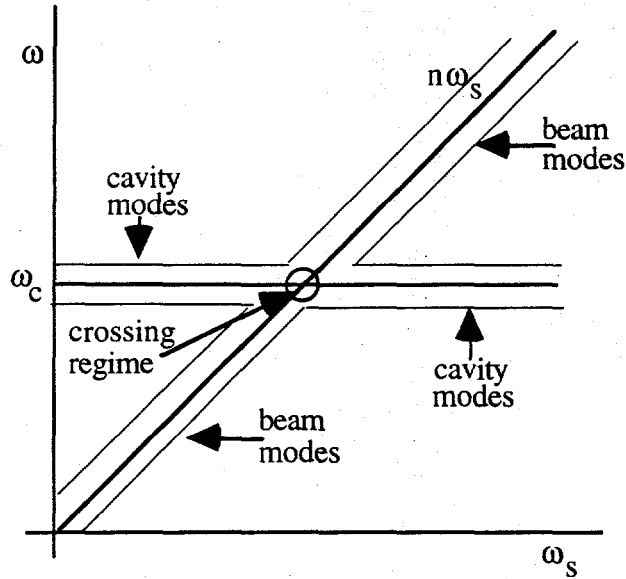


Fig. 3. Plot of ω versus ω_s showing the regions in which we find solutions to the dispersion relation, Eq. (35).

A. Cavity mode

The first solution we look at is the solution where the first term on the left-hand side of Eq. (35) is near zero while the second is not. The solutions for ω obtained in this case are known as the cavity modes of the system. In this limit we find that the slope of the distribution function determines whether the system is unstable and so there is no stabilizing temperature for the cavity mode. We also find that cavity modes exist for weakly-damped systems, but not strongly-damped ones. Consequently, cavity modes are generally neglected in accelerator physics where damping is usually prevalent. We point out that for systems with higher-Q cavities, cavity modes could become important.

To solve for the cavity mode, we assume the terms in the first set of paranthesis on the left-hand side of Eq. (35) are near zero. Consequently, we first find the two complex roots of $\bar{\omega}^2 + i\bar{\omega}/Q - 1 = 0$. These roots are the shifted resonance frequencies:

$$\bar{\omega}_c^\pm = \pm(1 - \frac{1}{4Q^2})^{1/2} - \frac{i}{2Q}. \quad (36)$$

Because we are looking for solutions near these frequencies and we have chosen n to be positive, we can drop the $\bar{\omega}_c^-$ solution as it will not be resonant with the beam. We substitute $\bar{\omega} = \bar{\omega}_c^+ + \delta$ into Eq. (35) and expand to leading order on both sides, assuming $|\delta| \ll 1$, since $|\bar{\omega}_c^+| = 1$. This yields

$$\delta = \frac{-\text{sign}(\eta)(\bar{\omega}_c^+)^2 \bar{\omega}_b^2}{(\bar{\omega}_c^+ - \bar{\omega}_c^-)[\Omega + in\Delta]^2}. \quad (37)$$

where we have introduced

$$\Omega = \bar{\omega}_c^+ - \frac{n\omega_s}{\omega_c}. \quad (38)$$

For this mode to be unstable, the imaginary part of Eq. (37) would have to be positive and larger than $1/2Q$.

The limit of small damping is the limit plasma physicists most often consider.³ In a plasma, damping is caused by Coulomb collisions between the particles and for high-frequency (electron) plasma oscillations, the time between collisions is much larger than the time to complete an oscillation. Thus, damping is typically neglected in the beam-plasma problem.

In general, small damping (high Q) means that the second term in the first set of parenthesis on the left-hand side of Eq. (35) is negligible compared to the difference of the other terms:

$$\frac{1}{Q} \ll |1 - \bar{\omega}^2|. \quad (39)$$

For $Q \gg 1$, where $\bar{\omega}_c^+ \approx 1$, Eq. (39) can be rewritten:

$$\frac{1}{2Q} \ll |\delta|. \quad (40)$$

Because we have already assumed the shift, δ , is small compared to unity, the condition $Q \gg 1$ is automatically satisfied. We now see that the physical interpretation of the condition for small damping is that the frequency shift be large enough to shift the mode outside the cavity resonance width. Using $Q \gg 1$ in Eq. (37) yields the shift for the weakly-damped cavity mode:

$$\delta = \frac{-\text{sign}(\eta)\bar{\omega}_b^2}{2(\Omega^2 + (n\Delta)^2)^2} \left[(\Omega^2 - (n\Delta)^2) - i2n\Delta\Omega \right], \quad (41)$$

where we have separated out the imaginary part of frequency in the last term. From this equation we can see how the energetics of the cavity mode work. Because the cavity mode is at frequencies near the cavity frequency, there will be instability only if the cavity frequency is on the low-energy side of the beam. Below transition ($\eta < 0$) the low energy side of the beam is at frequencies lower than $n\omega_s$ which means $\omega_c < n\omega_s$ and $\Omega < 0$, and we see this leads to a positive imaginary part in Eq. (41). Above transition, the low energy side of the beam is at frequencies higher than $n\omega_s$ which means we need positive Ω for an instability to be energetically favorable. This is the opposite condition of the more-familiar negative mass instability in accelerator physics, which we discuss below.

Here we also note that the imaginary part is related to the derivative of the distribution function:

$$\delta = \frac{-\text{sign}(\eta)\bar{\omega}_b^2}{2} \left[\frac{\Omega^2 - (n\Delta)^2}{(\Omega^2 + (n\Delta)^2)^2} + i \frac{\omega_c^2 \pi}{n^2} \frac{\partial g_0}{\partial \phi} \bigg|_{\frac{\omega_c}{n}} \right] \quad (42)$$

This shows the familiar result of Landau damping from plasma physics³: the growth rate is proportional to the slope of the distribution function. For $\eta < 0$, we see that a positive slope in ϕ means growth and a negative slope means damping. This makes sense energetically; if there are more particles, with slightly higher energy then this excess energy can feed the instability. For $\eta > 0$, a negative slope means positive instability, but the energetics are still correct so that, as we expect intuitively, $\partial g / \partial E > 0$ implies an instability. While this result was obtained using a Lorentzian, it is true for any distribution function that the rate of damping in the cavity mode depends on the slope of the distribution function.³

Another interesting feature brought out by writing the growth rate this way is that an arbitrarily warm beam is unstable. We will find this is not the case with the beam mode and some of the crossing modes. But as is evident from Eq. (42), the stability of the cavity mode is determined by the slope of the distribution function, which does not change sign as temperature grows. Consequently, there is no temperature above which the instability stabilizes as is familiar to accelerator physicists in the Keil-Schnell criterion.

The assumptions we have made to arrive at the weakly-damped cavity mode reduce to the following:

$$\frac{1}{Q} \ll \frac{\bar{\omega}_b^2}{\Omega^2 + n^2 \Delta^2} \ll 1, \quad (43)$$

where the inequality between the first and second terms insures we are in the small damping regime while the inequality between the second and third terms makes sure the

shift is small compared to the cavity frequency, keeping us in the cavity mode regime.

To compare with the weakly-damped limit, one might think of looking for a cavity solution to Eq. (37) in the highly-damped limit. As the damping grows, however, the solution for the shifted resonance frequency tends to $\bar{\omega}_c^+ = -iQ$, which is a purely damped mode incapable of supporting any kind of growth.

The absence of a highly-damped limit of the cavity mode is of special significance to accelerator physics. A rough analysis shows that most accelerators operate in the highly-damped regime and so have no cavity mode instability. If we replace $|\Omega|$ by ω_s / ω_c (the largest value it can have for constant ω_s), we see that the cold-beam condition for a weakly-damped cavity instability to develop is

$$\frac{1}{Q} \ll \frac{\omega_b^2}{\omega_s^2}. \quad (44)$$

For typical Booster values¹, Q would have to be greater than approximately 10^8 for this to hold. This suggests that cavity modes have not been a significant problem as quality factors as high as this are not typical. But for superconducting cavities² Q can be as high as 10^9 which means that the cavity mode could become unstable in accelerators with high- Q cavities. This analysis gives only a qualitative estimate of the Q value needed; a proper analysis is difficult since in reality $|\Omega|$ changes over a wide range of values as particles are accelerated and then one must answer the more complicated question of whether an instability can develop in the time $|\Omega|$ is in the proper range for that instability.

B. Beam mode

The second regime in which we can solve Eq. (35) is where the terms in the second set of parenthesis on the left-hand side of Eq. (35) are near zero while the terms in the first

are far from zero. This mode of oscillation is known as the beam mode. We apply the same perturbative technique as with the cavity mode, letting $\delta = (\omega - n\omega_s) / \omega_c$ and expanding both sides of Eq. (35) to leading order in δ by assuming $|\delta| \ll n\omega_s / \omega_c$. The solution for δ in this case is

$$\delta = \pm \left[\frac{\text{sign}(\eta) \bar{\omega}_b^2}{\left(\left(\frac{\Omega \omega_c}{n\omega_s} \right)^2 + 2\Omega \frac{\omega_c}{n\omega_s} \right) (1 - i\alpha)} \right]^{1/2} - i n \Delta, \quad (45)$$

where $\alpha = (1/2Q) / [\Omega(1 + \Omega\omega_c / 2n\omega_s)]$ is a measure of how the damping compares to Ω , the detuning between ω_c and $n\omega_s$. In accelerator physics, this is equivalent to a measure of the resistive part of the impedance compared to the reactive part. One interesting feature of the beam mode solution is that while the first term may produce a positive imaginary part, a large enough temperature will stabilize the system. This fact is familiar in accelerator physics², and we will discuss it at length later.

As with the cavity mode, we first look at the limit of small damping:

$$\delta = \pm \left[\frac{(n\omega_s)^2 \bar{\omega}_b^2}{2|\Omega^2 + 2n\omega_s\Omega|} \right]^{1/2} [(1 + \text{sign}(\eta)\text{sign}(\Omega))^{1/2} + \text{sign}(\eta)i(1 - \text{sign}(\eta)\text{sign}(\Omega))^{1/2}] - i n \Delta \quad (46)$$

Neglecting the damping term is equivalent to having a purely imaginary impedance. Note that either the first or second term will be zero; for a cold beam this means the shift is either a purely real or purely imaginary. The sign of Ω determines whether the impedance is capacitive ($\Omega < 0$) or inductive ($\Omega > 0$). Accelerator physicists often consider the case of

purely capacitive impedance² (such as the impedance due to space charge). For our system, this is the limit of $n\omega_s \gg \omega_c$, where $\Omega \approx -n\omega_s / \omega_c$. For this type of impedance, Eq. (46) reduces to

$$\delta = \pm \frac{\bar{\omega}_b}{\sqrt{2}} [(1 - \text{sign}(\eta))^{1/2} + i(1 + \text{sign}(\eta))^{1/2}] - i n \Delta. \quad (47)$$

This equation shows the familiar result for a cold beam of stability below transition and instability above, known as the negative mass instability in accelerator physics. If the beam is not cold, the instability stabilizes for

$$\Delta > \frac{\bar{\omega}_b}{n\sqrt{2}}. \quad (48)$$

This criterion is known in accelerator physics as a Keil-Schnell criterion for temperature stabilization of the system. We see the well-known feature that the stabilization temperature decreases for higher harmonics² (large n). If the beam is warmer than this temperature, accelerator physicists say the instability is Landau damped.

The first condition for which Eq. (47) holds is simply the condition for small damping, namely that the cavity resonance width is small compared to the mismatch of beam harmonic and background frequencies:

$$\frac{1}{2Q} \ll \left| \Omega \left(1 + \Omega \frac{\omega_c}{2n\omega_s} \right) \right|. \quad (49)$$

This is the small α condition. The second assumption we made was that the shift be small compared to the harmonic of the beam frequency:

$$\left\{ \bar{\omega}_b^2 \frac{\omega_c^2}{(n\omega_s)^2} (1 - \text{sign}(\eta)) + \left[\bar{\omega}_b \frac{\omega_c}{n\omega_s} (1 + \text{sign}(\eta))^{1/2} - n\Delta \frac{\omega_c}{n\omega_s} \right]^2 \right\}^{1/2} \ll 1. \quad (50)$$

Unlike the cavity mode, the beam mode also has a highly-damped limit. This highly-damped regime is equivalent to using a broad-band, purely resistive impedance. For this case, Eq. (46) becomes

$$\delta = \pm \left(\frac{n\omega_s \bar{\omega}_b^2 Q}{2\omega_c} \right)^{1/2} (\text{sign}(\eta) + i) - n\Delta, \quad (51)$$

which is the familiar accelerator physics result for a broad impedance.² For this case, the temperature stabilizes the system at

$$\Delta > \left(\frac{\omega_s \bar{\omega}_b^2 Q}{2n\omega_c} \right)^{1/2}. \quad (52)$$

The highly-damped limit requires that α be large compared to one,

$$\frac{1}{2Q} \gg \left| \Omega \left(1 + \Omega \frac{\omega_c}{2n\omega_s} \right) \right|, \quad (53)$$

and that the size of the shift compared to the beam harmonic be much smaller than one,

$$\left\{ \frac{Q\omega_b^2}{\omega_c n\omega_s} + \left[\left(\frac{Q\omega_b^2}{\omega_c n\omega_s} \right)^{1/2} - \Delta \frac{\omega_c}{\omega_s} \right]^2 \right\}^{1/2} \ll 1. \quad (54)$$

The condition in Eq. (53) applies only for $\Omega > \delta$; if $\Omega < \delta$ then the system is in the crossing mode regime treated below. Eq. (51) gives the shift away from the beam

harmonic frequency and Eqs. (53) and (54) give the regime of validity for this highly-damped beam mode.

C. Crossing mode

The third regime where we solve Eq. (35) is where $n\omega_s = \omega_c$. This is a regime important to plasma physics as it is the regime where transitions⁸ between the cavity mode and the beam mode occur and is the point of largest⁴ instability growth rate. Again we will look at the limits of low and high damping. For small damping, there is substantial modification as neither the beam mode or cavity mode strictly applies but for large damping we find that the beam mode solution from above applies unchanged.

For small damping we look all the way back to the dispersion relation, Eq. (35), in the regime where Eq. (39) holds:

$$\bar{\omega}^2 - 1 = \frac{-\text{sign}(\eta)\bar{\omega}^2\bar{\omega}_b^2}{[\bar{\omega} - 1 + i\text{in}\Delta]^2}. \quad (55)$$

where we have neglected the damping term and have made the replacement $n\omega_s = \omega_c$. We will again solve this equation with perturbation technique, substituting $\delta = \bar{\omega} - 1$ into the equation and assuming $\delta \ll 1$ so we can expand and keep only first order terms. However, because both the left-hand side and the right-hand denominator can be small, this now yields the following equation for δ

$$\delta(\delta + i\text{in}\Delta)^2 = \frac{-\text{sign}(\eta)}{2}\bar{\omega}_b^2. \quad (56)$$

We can solve this cubic polynomial directly, but for our case it is sufficient simply to look at the two limits of low and high temperature.

The first case of the weakly-damped limit we consider is the cold-beam case,

$\delta \gg n\Delta$. This is the well-known cold beam instability of plasma physics.⁶ Making the approximation $\delta \gg n\Delta$, Eq. (56) becomes

$$\delta^3 = \frac{-\text{sign}(\eta)\bar{\omega}_b^2}{2}. \quad (57)$$

The two solutions to this equation which have imaginary parts are

$$\delta = \left(\frac{\bar{\omega}_b^2}{2}\right)^{1/3} \left(\frac{\text{sign}(\eta)}{2} \pm i\frac{\sqrt{3}}{2}\right), \quad (58)$$

one of which is an instability. This result is the familiar plasma physics result of O'Neil.⁶ Note that the sign of the real part of the frequency shift is the same as the sign of η . From energy arguments we understand this fact: the wave will always shift frequency to the lower energy side of the beam. In an accelerator, lower energy particles have lower frequency if the machine is below transition ($\eta < 0$) but higher frequency is above transition ($\eta > 0$). Plasmas are always in the $\eta < 0$ regime. The conditions for which this cold-beam plasma equation is valid are

$$\frac{1}{Q} \ll (\bar{\omega}_b)^{2/3} \ll 1, \quad (59)$$

where the first inequality insures that we are in the small damping, low temperature regime and the second that the shift is small compared to the frequency itself.

The second limit for which we wish to look at the weakly-damped equation is the warm-beam limit, $n\Delta \gg \delta$. We would like to see if the weakly-damped instability shown above will shut off at large temperatures. The first order solution does not have an imaginary part, so we keep higher orders simply to see the sign of the imaginary term:

$$\delta = \frac{\text{sign}(\eta)\bar{\omega}_b^2}{2n^2\Delta^2} + i\frac{\bar{\omega}_b^4}{4n^5\Delta^5}. \quad (60)$$

We see from this expression that the imaginary part is always positive. This shows that in the case of small damping, there is always an instability in the crossing region regardless of beam temperature. Also, as we saw previously, the sign of the real part of the frequency shift is the same as the sign of η , as is required for the instability to be energetically favorable. The conditions under which this equation holds are

$$\frac{1}{Q} \ll \frac{\bar{\omega}_b^2}{(n\Delta)^2} \ll \Delta \frac{\omega_c}{\omega_s} \quad (61)$$

where we've made the assumption that Δ can never be much larger than ω_s / ω_c .

To see that the highly-damped solution is unchanged, note that in deriving Eq. (51), we neglected all but the damping term in the first set of parenthesis on the left-hand side of Eq. (35). The same solution therefore will apply in the crossing regime where we again neglect all but the damping term and solve for δ , this time with $n\omega_s = \omega_c$:

$$\delta = \pm \left(\frac{\bar{\omega}_b^2 Q}{2} \right)^{1/2} (\text{sign}(\eta) + i) - i n \Delta, \quad (62)$$

with the simplified Keil-Schnell criterion for stabilization

$$\Delta > \frac{\bar{\omega}_b}{n} \left(\frac{Q}{2} \right)^{1/2}. \quad (63)$$

The difference in the highly-damped mode in the crossing regime is in the conditions for validity. For $\Omega < \delta$, the leading term in expanding $\bar{\omega}^2 - 1$ is 2δ , so the large damping condition of Eq. (53) becomes:

$$\frac{1}{Q} \gg (2\bar{\omega}_b^2 Q)^{1/2}, \quad (64)$$

which, along with the small shift criterion, reduces to the condition on Q that

$$(\bar{\omega}_b)^{2/3} \ll \frac{1}{Q} \ll 1. \quad (65)$$

We summarize our results for the three different modes with a table (see Table 1) showing the frequency shifts, requirements to be in the given regime, and thresholds for stability for the different regimes discussed here.

The last point we address is the difference between Landau damping in plasma physics and accelerator physics. As mentioned, plasma physicists call Landau damping the damping that results from a negative slope of the distribution function with respect to energy (see Eq. (42)), while accelerator physicists call the stabilization due to a warm beam Landau damping (see Eq. (45)). Although the two mechanisms share the same name, they are clearly very distinct phenomena: Landau damping in plasma physics applies to high- Q cavity and crossing modes, where Landau damping in accelerator physics applies to the low- Q beam mode; Landau damping in plasma physics depends on the slope of the distribution function and therefore can produce either growth or damping, while Landau damping in accelerator physics depends only on the temperature and only produces damping, never growth. Robinson damping of accelerator physics does have the property that it produces growth or damping, depending on the slope of the real part of the impedance at the beam revolution harmonic. However, Robinson damping differs from plasma Landau damping in that it is still damping of the low- Q beam mode and is damping of a bunched beam instability, not a coasting beam instability.

VI. Summary and discussion

We have given a comprehensive treatment of the beam-plasma and beam-cavity problems under the single theoretical framework of damped, beam-driven oscillations of a single wave. There are four main results of this work: we generalize the beam-plasma problem by introducing damping, we bring a new model to the beam-cavity problem by introducing a single-wave vector potential, we point out that cavity modes, absent in the highly-damped limit of a broad-band impedance, should not necessarily be ignored for high-Q cavities, and finally we note the differences between what is called Landau damping in accelerator physics and plasma physics.

Our single-wave model of the beam-cavity problem takes advantage of the natural periodicity in the system. Fourier analysis of the vector potential combined with a nearly harmonic time dependence allows one to cast the vector potential as traveling waves. By ignoring all but the term most resonant with the beam, one can treat this problem as a beam interacting with a single wave.

One of the features of the linearized solutions to the beam-cavity equations is the existence of a cavity mode when damping is weak, but the absence of such a mode in the presence of high damping. Most accelerators operate with highly-damped cavities, so cavity mode instabilities are not a problem. However, as cavities with higher quality factors (such as super-conducting cavities) become more widely-used, cavity modes could become a new source of instability in accelerators.

The damping of the beam mode and highly-damped crossing mode by a warm beam is known in accelerator physics as Landau damping. This is in contrast to the damping of the cavity mode and weakly-damped beam mode, which plasma physicists call Landau damping. There are many differences between the two effects: Landau damping in accelerator physics depends solely on the temperature of the beam and always results in

damping, while Landau damping in plasma physics depends on the slope of the distribution function, not the temperature, and so can result in damping or growth.

VII. Acknowledgments

The authors thank Scott Hendrickson for his suggestions regarding this work. This research was supported by U. S. Department of Energy Grant 95-FG03-95ER40926.

Appendix: Magnitude of Space Charge Field

In this appendix, we show that due to magnetic cancellation, the relative size of the cavity and ring, and cavity resonance effects, the cavity field is much larger than space charge field.

From Faraday's Law, we know that a continuous, straight beam of particles traveling inside a perfectly-conducting pipe generates an on-axis longitudinal space charge field of:

$$E_{sc} = -\frac{1}{\gamma^2} \frac{\ln(b/a)}{2\pi\epsilon_0} \lambda', \quad (\text{A1})$$

where b is the pipe radius, a is the beam radius, λ is the linear charge density and the prime denotes a derivative with respect to the linear coordinate. For the linearized regime of Sec. IV, one can write the derivative of the charge density in terms of the normalized distribution function as follows:

$$E_{sc} = -\frac{\ln(b/a)}{\gamma^2} \frac{e\rho}{2\pi\epsilon_0 R^2} \int_{-\infty}^{\infty} g_1 d\phi, \quad (\text{A2})$$

where we have assumed the radius of the machine is large enough that the beam looks basically straight. One can cast the cavity field in a similar way, using the unperturbed

particle velocity in Eq. (10) for the linear regime and plugging in to Eq. (5) to get

$$E_{\text{cav}} = -\frac{i}{n} \frac{(A_0)^2 h e \rho}{(-\bar{\omega}^2 - i\bar{\omega}/Q + 1)} \int_{-\infty}^{\infty} g_1 d\phi. \quad (\text{A3})$$

comparing the two gives

$$\left| \frac{E_{\text{sc}}}{E_{\text{cav}}} \right| = \frac{\pi}{\gamma^2} \frac{d}{R} n^2 \left| \left(\bar{\omega}^2 + \frac{i\bar{\omega}}{Q} - 1 \right) \left[\frac{d}{h} \ln \left(\frac{b}{a} \right) J_1^2(v_{01}) \right] \right|, \quad (\text{A4})$$

where we have accounted for fact that the cavity field acts on the particles for only a fraction of their transit time by including a factor of $2\pi R/h$.

Magnetic cancellation and the relative sizes of the cavity and ring, via the factor $\gamma^{-2}(d/R)$, keep Eq. (A4) small for most accelerators ($\gamma^{-2}(d/R)$ is on the order of 10^{-5} for typical Fermilab parameters). Furthermore, near resonance, the term $\left| \left(\bar{\omega}^2 + i\bar{\omega}/Q - 1 \right) \right|$ can be extremely small, depending on the cavity Q , as shown below.

We first evaluate Eq. (A4) for the strongly-damped mode, applicable to Fermilab and other typical accelerators. We will consider a cold beam with zero detuning. The shift for that case is given in Sec. V by Eq. (62). This reduces Eq. (A4) to

$$\left| \frac{E_{\text{sc}}}{E_{\text{cav}}} \right| = \frac{\pi}{Q\gamma^2} \frac{d}{R} n^2 \left[\frac{d}{h} \ln \left(\frac{b}{a} \right) J_1^2(v_{01}) \right]. \quad (\text{A5})$$

For typical Fermilab values, $E_{\text{sc}}/E_{\text{cav}}$ will be of order 10^{-2} , thus neglecting space charge is justified.

For the weakly-damped crossing mode, familiar in plasma physics, the appropriate shift is given by Eq. (58). This in turn reduces Eq. (A4) to

$$\left| \frac{E_{sc}}{E_{cav}} \right| = \frac{\pi d}{\gamma^2 R} n^2 (2\bar{\omega}_b)^{2/3} \left[\frac{d}{h} \ln\left(\frac{b}{a}\right) J_1^2(v_{01}) \right]. \quad (A6)$$

Using the same Fermilab values as for the strongly-damped case, E_{sc} / E_{cav} will be of order 10^{-3} . This is simply for comparison, as Fermilab does not operate in the weakly-damped regime. Furthermore, as this expression depends on $\bar{\omega}_b^{2/3}$, it will decrease with decreasing beam intensities and so one can always find some regime where the cavity field dominates simply by considering a weak enough beam.

Tab. 1. The complex frequency shift, requirements for that shift to be valid, and temperature threshold for stability for four regimes of the beam-cavity system. The regimes with no threshold have different shifts (given in the text) which apply for high temperature.

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