Finding sets of points without empty convex 6-gons

Mark H. Overmars

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Mark Overmars

Department of Information and Computing Sciences, Utrecht University,
P.O. Box 80.089, 3508 TB Utrecht, the Netherlands, Email: markov@cs.uu.nl

Abstract

Erdös asked whether every large enough set of points in general position in the plane contains six points that form a convex 6-gon without any points from the set in its interior. In this note we show how a set of 29 points was found that contains no empty convex 6-gon. To this end a fast incremental algorithm for finding such 6-gons was designed and implemented and a heuristic search approach was used to find promising sets. Also some observations are made that might be useful in proving that large sets always contain an empty convex 6-gon.

1 The problem

Given a set V of n points in the plane, no three of which are collinear, we consider subsets $\Phi_k(V)$ of V of cardinality k that lie in convex position, i.e., they form the vertices of some convex k-gon, and there is no point of V in the interior of this polygon (the polygon is empty). Let F_k denote the smallest number such that any set V of cardinality at least F_k contains some subset $\Phi_k(V)$. Erdös [3] proposed the study of finding bounds on F_k. See [6] for an overview of the state of the art for this problem and related problems. It is trivial to prove that $F_1 = 1$, $F_2 = 2$, $F_3 = 3$ and $F_4 = 5$. The following results are known:

Theorem 1.1 *(Harborth[4])* $F_5 = 10$.

Theorem 1.2 *(Horton[5])* $F_7 = \infty$. *(In other words, for any size n there exists a set of n points without an empty convex 7-gon.)*

Clearly this last result also holds for $k > 7$. For $k = 6$ no exact bounds on F_k are known. In [1] Avis and Rappaport give a method to determine whether a given set of points does contain an empty convex 6-gon. Using this method they succeeded in finding a set of 20 points that does not contain an empty convex 6-gon, showing that $F_6 \geq 21$.

About ten years ago we devised a new incremental algorithm for checking whether a set contains an empty convex 6-gon. Using an implementation of that algorithm we managed to find a set of 26 points without an empty 6-gon, showing that $F_6 \geq 27$. The results were never properly published ([7]). Recently we reimplemented the algorithm and combined it with heuristic search techniques, resulting in a set of 29 points without an empty convex 6-gon. This note discusses this result and the algorithm and implementation used. Also some observations are made that might be helpful in proving that large enough sets always contain an empty convex 6-gon.

2 The algorithm

Because our implementation creates sets in an incremental way we want to solve the following problem:

Problem: Given a set V of n points without an empty convex 6-gon, and a point p, test whether $V \cup \{p\}$ contains an empty convex 6-gon.

To be able to test many different points an efficient algorithm (and, in particular, efficiently implementable) for this problem was required. To this end an algorithm by Dobkin, Edelsbrunner and Overmars [2] for finding empty polygons was adapted to our incremental approach. Let me give a brief sketch of the algorithm. For more details consult [7]. An empty convex 6-gon in $V \cup \{p\}$ must have p as a vertex. Consider the rays r^+ that runs from p upwards to infinity and r^- that runs downwards to infinity. An empty convex 6-gon including p cannot intersect both r^+ and r^-. We treat both cases separately.

Let us consider the case in which the 6-gon does not intersect r^+. The other case can be treated similar. We sort all points of V counter-clockwise by angle around p, starting with the direction of r^+, leading to a sequence $p_1, ..., p_n$. $p_1, ..., p_n$ together with p forms a starshaped polygon P with p in the kernel. We want to determine whether this polygon contains an empty convex 6-gon involving p. To this end we compute the visibility graph inside P, ignoring all edges involving p. For this we can use a slight adaptation of the approach by Dobkin et al. [2], which takes $O(n^2)$ time.

Finding an empty convex 6-gon involving p is equivalent to finding a convex chain of 4 edges in the visibility graph, whose total angle around p is less than π. To this end we walk counter-clockwise along the vertices of the polygon. For each edge we will determine the "best" convex chain of length 3 and 2 that ends on this edge. With "best" we mean: that starts the latest. This means that the chain makes the smallest angle with p. For an edge e we use C_e^3 to denote the starting point of the best chain of length 3 and C_e^2 to denote the starting point of the best chain of length 2. For consistency we use C_e^1 to denote the starting point of edge e but we do not actually have to store it. When we reach a point p_i we assume that for all incoming edges the C^3 and C^2 values are known (if they exist).

Note that both the incoming edges and outgoing edges of p_i are sorted by angle (that is the way the visibility graph algorithm produces them). We now determine which outgoing edges form a convex angle with what incoming edges. We start with the first outgoing edge e, which can be connected to the smallest number of incoming edges, and determine all incoming edges with which it forms a convex angle. Of these we determine the best C^3, C^2 and C^1 values. If C^3 exists and together with e and the point p forms an empty convex 6-gon we are done. Otherwise, we set C_e^3 to the best C^2 value and C_e^2 to the best C^1 value. We then continue with the next outgoing edge. We do not have to recheck all the incoming edges because we already know the best values of C^3, C^2 and C^1 among them. We only have to check untreated incoming edges that form a convex angle with the outgoing edge. In this way we proceed until we either find a convex 6-gon or reach the last vertex. It is easy to see that the algorithm takes $O(n^2)$ time.

In the implementation of the method, the construction of the visibility graph and the checking of convex chains are interleaved. So for every edge of the visibility graph that is created, immediately the chain information is computed. As a result, the program can stop as soon as it finds a chain of length 4, saving a lot of work because most of the time an empty convex 6-gon does indeed exist.

The algorithm was implemented in Delphi. A careful look at the algorithm reveals that the only geometric operation required is to determine whether a sequence of three points makes a left turn, a right turn, or is straight. This can be implemented by comparing two numbers created using two additions and a multiplication. As point coordinates we use 30 bits positive integers. As a result 62 bits of precision is required to do the calculation exact. Using 64 bit integers for the operation guarantees that the algorithm is robust. The algorithm is also very fast. For example, on a Pentium III 500 MHz the program can check about 15000 points per second in a set of 29 points.

3 The result

Just testing random sets of points never resulted in a set of more that 18 points. (A similar observation was made in [1].) Using an incremental approach, adding points in a slowly growing region increased this number to 22. To obtain larger sets we used a combination of random backtracking techniques and Brownian motion to slowly change point sets. After some tuning we just let the program run. After about 4 days it reported a set of 29 points without an empty convex 6-gon, leading to the following result:

Theorem 3.1 $F_6 \geq 30$.

Proof: See Figure 1 for a set of 29 points that does not contain an empty convex 6-gon. □

Since this set was found the program has run for months. It reports sets of 29 points every few days but never found a larger set. Anybody who is interested to run the program on their machine can download it from the website

`http://www.cs.uu.nl/people/markov/sixgon/`

If you want to check or adapt the source, please contact the author.

There are two interesting observations one can make about the resulting sets. First of all, even though we use 30-bit integers when creating sets, we managed to reduce the coordinate values such that only integers between 0 and about 1300 were required (the program can do this automatically). All the additional precision does not seem to help.

Secondly, all the different sets of 29 points that we looked at have a very special structure. If we look at the convex layers they have the following sizes: The outer layer is a triangle. The

```
(   1,1260)
(  16, 743)
(  22, 531)
(  37,   0)
( 306, 592)
( 310, 531)
( 366, 552)
( 371, 487)
( 374, 525)
( 392, 575)
( 396, 613)
( 410, 539)
( 416, 550)
( 426, 526)
( 434, 552)
( 436, 535)
( 446, 565)
( 449, 518)
( 450, 498)
( 453, 542)
( 458, 526)
( 489, 537)
( 492, 502)
( 496, 579)
( 516, 467)
( 552, 502)
( 754, 697)
( 777, 194)
(1259, 320)
```

Figure 1: A set of 29 points without an empty convex 6-gons.

next layer is a convex quadralateral. Then there are three convex 7-gons and finally a point in the center. From lots of tests we reach the conjecture that a set of points without an empty convex 6-gon can have at most 7 points on the convex hull. This would immediately imply that all layers have at most 7 points because peeling off a layer cannot introduce empty convex 6-gons. This in turn would mean that the number of layers is linear in the number of points, which might be useful for proving a bound on F_6. (B.t.w., it is trivial to proof that there are $\Omega(\log n)$ layers.)

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