Finding shortest non-separating and non-disconnecting paths

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— Abstract

For a connected graph G = (V, E) and $s, t \in V$, a non-separating s-t path is a path P between s and t such that the set of vertices of P does not separate G, that is, G - V(P) is connected. An s-t path is non-disconnecting if G - E(P) is connected. The problems of finding shortest non-separating and non-disconnecting paths are both known to be NP-hard. In this paper, we consider the problems from the viewpoint of parameterized complexity. We show that the problem of finding a non-separating s-t path of length at most k is W[1]-hard parameterized by k, while the non-disconnecting counterpart is fixed-parameter tractable parameterized by k. We also consider the shortest non-separating path problem on several classes of graphs and show that this problem is NP-hard even on bipartite graphs, split graphs, and planar graphs. As for positive results, the shortest non-separating path problem is fixed-parameter tractable parameterized by k on planar graphs and polynomial-time solvable on chordal graphs if k is the shortest path distance between s and t.

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1 Introduction

Lovász's path removal conjecture states the following claim: There is a function $f: \mathbb{N} \to \mathbb{N}$ such that for every f(k)-connected graph G and every pair of vertices u and v, G has a path P between u and v such that G - V(P) is k-connected. This claim still remains open, while some spacial cases have been resolved [4, 15, 16, 22]. Tutte [22] proved that f(1) = 3, that is, every triconnected graph satisfies that for every pair of vertices, there is a path between them whose removal results a connected graph. Kawarabayashi et al. [15] proved a weaker version of this conjecture: There is a function $f: \mathbb{N} \to \mathbb{N}$ such that for every f(k)-connected graph G and every pair of vertices u and v, G has an induced path P between u and v such that G - E(P) is k-connected.

As a practical application, let us consider a network represented by an undirected graph G, and we would like to build a private channel between a specific pair of nodes s and t. For some security reasons, the path used in this channel should be dedicated to the pair s and t, and hence all other connections do not use all nodes and/or edges on this path while keeping their connections. In graph-theoretic terms, the vertices (resp. edges) of the path between s and t does not form a separator (resp. a cut) of G. Tutte's result [22] indicates that such a path always exists in triconnected graphs, but may not exist in biconnected graphs. In

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addition to this connectivity constraint, the path between s and t is preferred to be short due to the cost of building a private channel. Motivated by such a natural application, the following two problems are studied.

▶ **Definition 1.** Given a connected graph G, $s, t \in V(G)$, and an integer k, SHORTEST NON-SEPARATING PATH asks whether there is a path P between s and t in G such that the length of P is at most k and G - V(P) is connected.

▶ Definition 2. Given a connected graph G, $s, t \in V(G)$, and an integer k, SHORTEST NON-DISCONNECTING PATH asks whether there is a path P between s and t in G such that the length of P is at most k and G - E(P) is connected.

We say that a path P is non-separating (in G) if G - V(P) is connected and is nondisconnecting (in G) if G - E(P) is connected.

Related work. The shortest path problem in graphs is one of the most fundamental combinatorial optimization problems, which is studied under various settings. Indeed, our problems SHORTEST NON-SEPARATING PATH and SHORTEST NON-DISCONNECTING PATH can be seen as variants of this problem. From the computational complexity viewpoint, SHORTEST NON-SEPARATING PATH is known to be NP-hard and its optimization version cannot be approximated with factor $|V|^{1-\varepsilon}$ in polynomial time for $\varepsilon > 0$ unless P = NP [23]. SHORTEST NON-DISCONNECTING PATH is shown to be NP-hard on general graphs and polynomial-time solvable on chordal graphs [18].

Our results. We investigate the parameterized complexity of both problems. We show that SHORTEST NON-SEPARATING PATH is W[1]-hard and SHORTEST NON-DISCONNECTING PATH is fixed-parameter tractable parameterized by k. A crucial observation for the fixedparameter tractability of SHORTEST NON-DISCONNECTING PATH is that the set of edges in a non-disconnecting path can be seen as an independent set of a cographic matroid. By applying the representative family of matroids [11], we show that SHORTEST NON-DISCONNECTING PATH can be solved in $2^{\omega k} |V|^{O(1)}$ time, where ω is the exponent of the matrix multiplication. We also show that SHORTEST NON-DISCONNECTING PATH is OR-compositional, that is, there is no polynomial kernelization unless $coNP \subseteq NP/poly$. To cope with the intractability of SHORTEST NON-SEPARATING PATH, we consider the problem on planar graphs and show that it is fixed-parameter tractable parameterized by k. This result can be generalized to wider classes of graphs which have the *diameter-treewidth property* [9]. We also consider SHORTEST NON-SEPARATING PATH on several classes of graphs. We can observe that the complexity of SHORTEST NON-SEPARATING PATH is closely related to that of HAMILTONIAN CYCLE (or HAMILTONIAN PATH with specified end vertices). This observation readily proves the NP-completeness of SHORTEST NON-SEPARATING PATH on bipartite graphs, split graphs, and planar graphs. For chordal graphs, we devise a polynomial-time algorithm for SHORTEST NON-SEPARATING PATH for the case where k is the shortest path distance between s and t.

2 Preliminaries

We use standard terminologies and known results in matroid theory and parameterized complexity theory, which are briefly discussed in this section. See [6, 20] for details.

Graphs. Let G be a graph. The vertex set and edge set of G are denoted by V(G) and E(G), respectively. For $v \in V(G)$, the open neighborhood of v in G is denoted by

 $N_G(v)$ (i.e., $N_G(v) = \{u \in V(G) : \{u, v\} \in E(G)\}$) and the closed neighborhood of vin G is denoted by $N_G[v]$ (i.e., $N_G[v] = N_G(v) \cup \{v\}$). We extend this notation to sets: $N_G(X) = \bigcup_{v \in X} N_G(v) \setminus X$ and $N_G[X] = N_G(X) \cup X$ for $X \subseteq V(G)$. For $u, v \in V(G)$, we denote by dist_G(u, v) the length of a shortest path between u and v in G, where the length of a path is defined to the number of edges in it. We may omit the subscript of G from these notations when no confusion arises. For $X \subseteq V(G)$, we write G[X] to denote the subgraph of G induced by X. For notational convenience, we may use G - X instead of $G[V(G) \setminus X]$. For $F \subseteq E$, we also use G - F to represent the subgraph of G consisting all vertices of Gand all edges in $E \setminus F$. For vertices u and v, a path between u and v is called a u-v path. A vertex is called a *pendant* if its degree is exactly 1.

Matroids and representative sets. Let E be a finite set. If $\mathcal{I} \subseteq 2^E$ satisfies the following axioms, the pair $\mathcal{M} = (E, \mathcal{I})$ is called a *matroid*: (1) $\emptyset \in \mathcal{I}$; (2) $Y \in \mathcal{I}$ implies $X \in \mathcal{I}$ for $X \subseteq Y \subseteq 2^E$; and (3) for $X, Y \in \mathcal{I}$ with |X| < |Y|, there is $e \in Y \setminus X$ such that $X \cup \{e\} \in \mathcal{I}$. Each set in \mathcal{I} is called an *independent set* of \mathcal{M} . From the third axiom of matroids, it is easy to observe that every (inclusion-wise) maximal independent set of \mathcal{M} have the same cardinality. The *rank* of \mathcal{M} is the maximum cardinality of an independent set of \mathcal{M} . A matroid $\mathcal{M} = (E, \mathcal{I})$ of rank n is *linear* (or *representable*) over a field \mathbb{F} if there is a matrix $M \in \mathbb{F}^{n \times |E|}$ whose columns are indexed by E such that $X \in \mathcal{I}$ if and only if the set of columns indexed by X is linearly independent in M.

Let G = (V, E) be a graph. A cographic matroid of G is a matroid $\mathcal{M}(G) = (E, \mathcal{I})$ such that $F \subseteq E$ is an independent set of $\mathcal{M}(G)$ if and only if G - F is connected. Every cographic matroid is linear and its representation can be computed in polynomial time [20].

Our algorithmic result for SHORTEST NON-DISCONNECTED PATH is based on *representa*tive families due to [11].

▶ **Definition 3.** Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid and let $\mathcal{F} \subseteq \mathcal{I}$ be a family of independent sets of \mathcal{M} . For an integer $q \ge 0$, we say that $\widehat{\mathcal{F}} \subseteq \mathcal{F}$ is q-representative for \mathcal{F} if the following condition holds: For every $Y \subseteq E$ of size at most q, if there is $X \in \mathcal{F}$ with $X \cap Y = \emptyset$ such that $X \cup Y \in \mathcal{I}$, then there is $\widehat{X} \in \widehat{\mathcal{F}}$ with $\widehat{X} \cap Y = \emptyset$ such that $\widehat{X} \cup Y \in \mathcal{I}$.

▶ **Theorem 4** ([11, 17]). Given a linear matroid $\mathcal{M} = (E, \mathcal{I})$ of rank *n* that is represented as a matrix $M \in \mathbb{F}^{n \times |E|}$ for some field \mathbb{F} , a family $\mathcal{F} \subseteq \mathcal{I}$ of independent sets of size *p*, and an integer *q* with $p + q \leq n$, there is a deterministic algorithm computing a *q*-representative family $\widehat{\mathcal{F}} \subseteq \mathcal{F}$ of size $np {p \choose p}$ with

$$O\left(|\mathcal{F}| \cdot \left(\binom{p+q}{p}p^3n^2 + \binom{p+q}{q}^{\omega-1} \cdot (pn)^{\omega-1}\right)\right) + (n+|E|)^{O(1)}$$

field operations, where $\omega < 2.373$ is the exponent of the matrix multiplication.

Parameterized complexity. A parameterized problem is a language $L \subseteq \Sigma^* \times \mathbb{N}$, where Σ is a finite alphabet. We say that L is fixed-parameter tractable (parameterized by k) if there is an algorithm deciding if $(I, k) \in L$ for given $(I, k) \in \Sigma^* \times \mathbb{N}$ in time $f(k)|I|^{O(1)}$, where f is a computable function. A kernelization for L is a polynomial-time algorithm that given an instance $(I, k) \in \Sigma^* \times \mathbb{N}$, computes an "equivalent" instance $(I', k') \in \Sigma^* \times \mathbb{N}$ such that (1) $(I, k) \in L$ if and only if $(I', k') \in L$ and (2) $|I'| + k' \leq g(k)$ for some computable function g. We call (I', k') a kernel. If the function g is a polynomial, then the kernelization algorithm is called a polynomial kernel. An

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OR-composition is an algorithm that given p instances $(I_1, k), \ldots, (I_p, k) \in \Sigma^* \times \mathbb{N}$ of L, computes an instance $(I', k') \in \Sigma^* \times \mathbb{N}$ in time $(\sum_{1 \leq i \leq p} |I_i| + k)^{O(1)}$ such that (1) $(I', k') \in L$ if and only if $(I_i, k) \in L$ for some $1 \leq i \leq p$ and (2) $k' = k^{O(1)}$. For a parameterized problem L, its unparameterized problem is a language $L' = \{x \# 1^k : (x, k) \in L\}$, where $\# \notin \Sigma$ is a blank symbol and $1 \in \Sigma$ is an arbitrary symbol.

▶ Theorem 5 ([3]). If a parameterized problem L admits an OR-composition and its unparameterized version is NP-complete, then L does not have a polynomial kernelization unless $coNP \subseteq NP/poly$.

3 Shortest Non-Separating Path

We discuss our complexity and algorithmic results for SHORTEST NON-SEPARATING PATH.

3.1 Hardness on graph classes

We obverse that, in most cases, SHORTEST NON-SEPARATING PATH is NP-hard on classes of graphs for which HAMILTONIAN PATH (with distinguished end vertices) is NP-hard. Let G = (V, E) be a graph and $s, t \in V$ be distinct vertices of G. We add a pendant vertex padjacent to s and denote the resulting graph by G'. Then, we have the following observation.

▶ **Observation 6.** For every non-separating path P between s and t in G', $V(G) \setminus V(P) = \{p\}$.

Suppose that for a class \mathcal{C} of graphs,

- the problem of deciding whether given graph $G \in C$ has a Hamiltonian path between specified vertices s and t in G is NP-hard and

By Observation 6, G' has a non-separating *s*-*t* path if and only if G has a Hamiltonian path between *s* and *t*. This implies that the problem of finding a non-separating path between specified vertices is NP-hard on class C.

▶ **Theorem 7.** The problem of deciding if an input graph has a non-separating s-t path is NP-complete even on planar graphs, bipartite graphs, and split graphs.

The classes of planar graphs and bipartite graphs are closed under the operation of adding a pendant. Recall that a graph G is a *split graph* if the vertex set V(G) can be partitioned into a clique C and an independent set I. Thus, for the class of split graphs, we need the assumption that the pendant added is adjacent to a vertex in C.

As the problem trivially belongs to NP, it suffices to show that HAMILTONIAN PATH (with distinguished end vertices) is NP-hard on these classes of graphs¹. For split graphs, it is known that HAMILTONIAN PATH is NP-hard even if the distinguished end vertices are contained in the clique C [19]. Let G be a graph and let $v \in V(G)$. We add a vertex v'that is adjacent to every vertex in $N_G(v)$, that is, v and v' are twins. The resulting graph is denoted by G'. It is easy to verify that G has a Hamiltonian cycle if and only if G' has a Hamiltonian path between v and v'. The class of bipartite graphs is closed under this operation, that is, G' is bipartite. For planar graphs, G' may not be planar in general. However, HAMILTONIAN CYCLE is NP-complete even if the input graph is planar and has

¹ These results for bipartite graphs and planar graphs seem to be folklore but we were not able to find particular references.



Figure 1 The left figure depicts an instance G of MULTICOLORED CLIQUE and the right figure depicts the graph H constructed from G. Some vertices and edges in H are not drawn in this figure for visibility. The edges of a clique C and the corresponding non-separating *s*-*t* path P are drawn as thick lines.

a vertex of degree 2 [14]. We apply the above operation to this degree-2 vertex, and the resulting graph G' is still planar. As the problem of finding a Hamiltonian cycle is NP-hard even on bipartite graphs [19] and planar graphs [14], Theorem 7 follows.

3.2 W[1]-hardness

Next, we show that SHORTEST NON-SEPARATING PATH is W[1]-hard parameterized by k. The proof is done by giving a reduction from MULTICOLORED CLIQUE, which is known to be W[1]-complete [10]. In MULTICOLORED CLIQUE, we are given a graph G with a partition $\{V_1, V_2, \ldots, V_k\}$ of V(G) and asked to determine whether G has a clique C such that $|V_i \cap C| = 1$ for each $1 \le i \le k$.

From an instance $(G, \{V_1, \ldots, V_k\})$ of MULTICOLORED CLIQUE, we construct an instance of SHORTEST NON-SEPARATING PATH as follows. Without loss of generality, we assume that G contains more than k vertices. We add two vertices s and t and edges between s and all $v \in V_1$ and between t and all $v \in V_k$. For any pair of $u \in V_i$ and $v \in V_j$ with $|i - j| \ge 2$, we do the following. If $\{u, v\} \in E$, then we remove it. Otherwise, we add a path $P_{u,v}$ of length 2 and a pendant vertex that is adjacent to the internal vertex w of $P_{u,v}$. Finally, we add a vertex v^* , add an edge between v^* and each original vertex $v \in V(G)$, and add a pendant vertex p adjacent to v^* . The constructed graph is denoted by H. See Figure 1 for an illustration of the graph H.

▶ Lemma 8. There is a clique C in G such that $|C \cap V_i| = 1$ for $1 \le i \le k$ if and only if there is a non-separating s-t path of length at most k + 1 in H.

Proof. Suppose first that G has a clique C with $C \cap V_i = \{v_i\}$ for $1 \le i \le k$. Then, $P = \langle s, v_1, v_2, \ldots, v_k, t \rangle$ is an s-t path of length k + 1 in H. To see the connectivity of H - V(P), it suffices to show that every vertex is reachable to v^* in H - V(P). By the construction of H, each vertex in $V(G) \setminus V(P)$ is adjacent to v^* in H - V(P). Each vertex z in $V(H) \setminus (V(G) \cup \{v^*, p\})$ is either the internal vertex w of $P_{u,v}$ for some $u, v \in V(G)$ or the pendant vertex adjacent to w. In both cases, at least one of u and v is not contained in P as $V(P) \setminus \{s, t\}$ is a clique in G, implying that z is reachable to v^* .

Conversely, suppose that H has a non-separating s-t path P of length at most k + 1 in H. By the assumption that G has more than k vertices, there is a vertex of G that does not belong to P. Observe that P does not contain any internal vertex w of some $P_{u,v}$ as otherwise the pendant vertex adjacent to w becomes an isolated vertex by deleting V(P), which implies H - V(P) has at least two connected components. Similarly, P does not contain v^* . These facts imply that the internal vertices of P belong to V(G), and we have

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 $|V(P) \cap V_i| = 1$ for $1 \le i \le k$. Let $C = V(P) \setminus \{s, t\}$. We claim that C is a clique in G. Suppose otherwise. There is a pair of vertices $u, v \in C$ that are not adjacent in G. This implies that H contains the path $P_{u,v}$. However, as P contains both u and v, the internal vertex of $P_{u,v}$ together with its pendant vertex forms a component in H - V(P), yielding a contradiction that P is a non-separating path in H.

Thus, we have the following theorem.

► Theorem 9. SHORTEST NON-SEPARATING PATH is W[1]-hard parameterized by k.

3.3 Graphs with the diameter-treewidth property

By Theorem 9, SHORTEST NON-SEPARATING PATH is unlikely to be fixed-parameter tractable on general graphs. To overcome this intractability, we focus on sparse graph classes. We first note that algorithmic meta-theorems for FO MODEL CHECKING [12, 13] does not seem to be applied to SHORTEST NON-SEPARATING PATH as we need to care about the connectivity of graphs, while it can be expressed by a formula in MSO logic, which is as follows. The property that vertex set X forms a non-separating s-t path can be expressed as:

 $\texttt{conn}(V \setminus X) \land \texttt{hampath}(X, s, t),$

where $\operatorname{conn}(Y)$ and $\operatorname{hampath}(Y, s, t)$ are formulas in MSO₂ that are true if and only if the subgraph induced by Y is connected and has a Hamiltonian path between s and t, respectively. We omit the details of these formulas, which can be found in [6] for example². By Courcelle's theorem [5] and its extension due to Arnborg et al. [1], we can compute a shortest nonseparating s-t path in $O(f(\operatorname{tw}(G))n)$ time, where n is the number of vertices and $\operatorname{tw}(G)$ is the treewidth³ of G. As there is an $O(\operatorname{tw}(G)^{\operatorname{tw}(G)^3}n)$ -time algorithm for computing the treewidth of an input graph G [2], we have the following theorem.

▶ **Theorem 10.** SHORTEST NON-SEPARATING PATH is fixed-parameter tractable parameterized by the treewidth of input graphs.

A class C of graphs is *minor-closed* if every minor of a graph $G \in C$ also belongs to C. We say that C has the *diameter-treewidth property* if there is a function $f \colon \mathbb{N} \to \mathbb{N}$ such that for every $G \in C$, the treewidth of G is upper bounded by $f(\operatorname{diam}(G))$, where $\operatorname{diam}(G)$ is the diameter of G. It is well known that every planar graph G has treewidth at most $3 \cdot \operatorname{diam}(G) + 1$ [21]⁴, which implies that the class of planar graphs has the diameter-treewidth property. This can be generalized to more wider classes of graphs. A graph is called an *apex graph* if it has a vertex such that removing it makes the graph planar.

▶ **Theorem 11** ([7, 9]). Let C be a minor-closed class of graphs. Then, C has the diametertreewidth property if and only if it excludes some apex graph.

For $C \subseteq V(G)$ that induces a connected subgraph G[C], we denote by G_C the graph obtained from G by contracting all edges in G[C] and by v_C the vertex corresponding to C in G_C . Since G[C] is connected, vertex v_C is well-defined.

² In [6], they give an MSO₂ sentence hamiltonicity expressing the property of having a Hamiltonian cycle, which can be easily transformed into a formula expressing hampath(X, s, t).

 $^{^{3}}$ We do not give the definition of treewidth and (the optimization version of) Courcelle's theorem. We refer to [6] for details.

⁴ More precisely, the treewidth of a planar graph is upper bounded by 3r + 1, where r is the radius of the graph.

▶ Lemma 12. Let $C \subseteq V(G)$ be a vertex subset such that G[C] is connected. Let P be an s-t path in G with $V(P) \cap C = \emptyset$. Then, P is non-separating in G if and only if it is non-separating in G_C .

Proof. Suppose first that P is non-separating in G. Let $u, v \in V(G) \setminus V(P)$ be arbitrary. As P is non-separating, there is a u-v path P' in G - V(P). Let u' be the vertex of G_C such that u' = u if $u \notin C$ and $u' = v_C$ if $u \in C$. Let v' be the vertex defined analogously. We show that there is a u'-v' path in $G_C - V(P)$ as well. If P' does not contain any vertex in C, it is also a u'-v' path in G_C , and hence we are done. Suppose otherwise. Let x and y be the vertices in $V(P') \cap C$ that are closest to u and v, respectively. Note that x and y can be the end vertices of P', that is, C may contain u and v. Let $P_{u,x}$ and (resp. $P_{y,v}$) be the subpath of P' between u and x (resp. y and v). Then, the sequence of vertices obtained by concatenating $P_{u,x}$ after $P_{y,v} - \{y\}$ and replacing exactly one occurrence of a vertex in C with v_C forms a path between u' and v'. Since we choose u, v arbitrarily, there is a path between any pair of vertices in $G_C - V(P)$ as well. Hence, P is non-separating in G_C .

Conversely, suppose that P is non-separating in G_C . For $u, v \in V(G_C) \setminus V(P)$, there is a path P' in $G_C - V(P)$. Suppose that neither $u = v_C$ nor $v = v_C$. Then, we can construct a u-v path in G - V(P) as follows. If $v_C \notin V(P')$, P' is also a path in G - V(P) and hence we are done. Otherwise, $v_C \in V(P')$. Let P_u and P_v be the subpaths in $P' - \{v_C\}$ containing u and v, respectively. From P_u and P_v , we have a u-v path in G by connecting them with an arbitrary path in G[C] between the end vertices other than u and v. Note that such a bridging path in G[C] always exists since G[C] is connected. Moreover, as $V(P') \cap C = \emptyset$ and $V(P) \cap C = \emptyset$, this is also a u-v path in G - V(P). Suppose otherwise that either $u = v_C$ or $v = v_C$, say $u = v_C$. In this case, we can construct a path between every vertex w in C and v by concatenating P' and an arbitrary path in G[C] between w and the end vertex of the subpath $P' - \{v_C\}$ other than v. Therefore, P is non-separating in G.

Now, we are ready to state the main result of this subsection.

▶ **Theorem 13.** Suppose that a minor-closed class C of graphs has the diameter-treewidth property. Then, SHORTEST NON-SEPARATING PATH on C is fixed-parameter tractable parameterized by k.

Proof. Let $G \in \mathcal{C}$. We first compute $B = \{v \in V(G) : \operatorname{dist}(s, v) \leq k\}$. This can be done in linear time. If $t \notin B$, then the instance (G, s, t, k) is trivially infeasible. Suppose otherwise that $t \in B$. Let C be a component in G - B. By definition, every non-separating *s*-*t* path P of length at most k does not contain any vertex of C. Let G' be the graph obtained from G by contracting all edges in E(G - B). For each component C in G - B, we denote by v_C the vertex of G' corresponding to C (i.e., v_C is the vertex obtained by contracting all edges in C). Then, we have diam $(G') \leq 2k + 2$ as diam $(G[B]) \leq k$ and every vertex in $V(G') \setminus B$ is adjacent to a vertex in B. By Lemma 12, G has a non-separating *s*-*t* path of length at most k if and only if so does G'. Since C is minor-closed, we have $G' \in C$ and hence the treewidth of G' is upper bounded by f(2k + 2) for some function f. By Theorem 10, we can check whether G' has a non-separating *s*-*t* path of length at most k in O(g(k)|V(G')|) time for some function g.

3.4 Chordal graphs with k = dist(s, t)

In Section 3.1, we have seen that SHORTEST NON-SEPARATING PATH is NP-complete even on split graphs (and thus on more general chordal graphs as well). To overcome this intractability, we restrict ourselves to finding a non-separating s-t path of length dist(s, t) on chordal graphs.

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A graph G is *choral* if it has no cycles of length at least 4 as an induced subgraph. In the following, fix a connected chordal graph G.

▶ Lemma 14. Let $S \subseteq V(G)$ be a vertex set such that G[S] is connected. For $u, v \in S$, every induced u-v path P in G satisfies that $V(P) \subseteq N[S]$.

Proof. Suppose to the contrary that an induced u-v path P contains a vertex $x \notin N[S]$. Since P starts and ends in S, it contains a subpath $Q = \langle a, \ldots, x, \ldots, b \rangle$ such that $a, b \in N(S)$ and all other vertices in Q belong to V - N[S]. As $a, b \in N(S)$ and G[S] is connected, G has an induced a-b path R with all internal vertices belonging to S. Since the internal vertices of Q have no neighbors in $S, Q \cup R$ induces a cycle. Both a-b paths Q and R have length at least 2 as a and b are not adjacent, and thus the cycle $G[Q \cup R]$ has length at least 4. This contradicts that G is chordal.

For $u, v \in V(G)$, a set of vertices $S \subseteq V(G) \setminus \{u, v\}$ is called a *u-v* separator of G if there is no *u-v* path in G-S. An inclusion-wise minimal *u-v* separator of G is called a *minimal u-v* separator. A *minimal separator* of G is a minimal *u-v* separator for some $u, v \in G$. Dirac's well-know characterization [8] of chordal graphs states that a graph is chordal if and only of every minimal separator induces a clique.

▶ Lemma 15. Let $s, t \in V(G)$ be such that $\{s, t\} \notin E(G)$. If $v \in V(G) \setminus \{s, t\}$ is an internal vertex of a shortest s-t path P, then $N[v] \setminus \{s, t\}$ is an s-t separator of G.

Proof. Let d = dist(s, t). For $0 \le i \le d$, let

 $D_i = \{ v \in V(G) : \operatorname{dist}(s, v) = i \wedge \operatorname{dist}(v, t) = d - i \}$

and $V(P) \cap D_i = \{u_i\}$. Let $j \ (0 < j < d)$ be the index such that $v = u_j$.

Suppose to the contrary that there is an induced s-t path Q such that $V(Q) \cap (N[u_j] \setminus \{s,t\}) = \emptyset$. By Lemma 14, $V(Q) \subseteq N[V(P)] = \bigcup_{0 \leq i \leq d} N[u_i]$ holds. Since Q starts in $N[u_0]$ and ends in $N[u_d]$, there are indices i and k with $0 \leq i < j < k \leq d$ such that Q consecutively visits a vertex $v_i \in N[u_i]$ and then a vertex $v_k \in N[u_k]$ in this order. Since dist $(u_i, u_k) = k - i \geq 2$ and $\{v_i, v_k\} \in E$, at least one of $v_i \neq u_i$ and $v_k \neq u_k$ holds. By symmetry, we assume that $v_i \neq u_i$.

If $v_k = u_k$, then $v_i \in N(u_i) \cap N(u_k)$. In this case, we have i = j - 1 and k = j + 1 since otherwise P admits a shortcut using the subpath $\langle u_i, v_i, u_k \rangle$. This implies that $\operatorname{dist}(s, v_i) \leq \operatorname{dist}(s, u_i) + 1 = i + 1 = j$ and $\operatorname{dist}(v_i, t) \leq 1 + \operatorname{dist}(v_k, t) = 1 + \operatorname{dist}(u_k, t) = 1 + (d - k) = d - j$. Since $\operatorname{dist}(s, v_i) + \operatorname{dist}(v_i, t) \geq d$, we have $\operatorname{dist}(s, v_i) = j$ and $\operatorname{dist}(v_i, t) = d - j$. This implies that $v_i \in D_j \subseteq N[u_j] \setminus \{s, t\}$, a contradiction

Next we consider the case $v_k \neq u_k$. Recall that we also have $v_i \neq u_i$ as an assumption. In this case, we have $k - i \leq 3$ as $\langle u_i, v_i, v_k, u_k \rangle$ is not a shortcut for P. Assume first that k - i = 3. By symmetry, we may assume that i = j - 1 and k = j + 2. Since $\operatorname{dist}(s, v_i) \leq \operatorname{dist}(s, u_i) + 1 = j$ and $\operatorname{dist}(v_i, t) \leq 2 + \operatorname{dist}(u_k, t) \leq 2 + (d - k) = d - j$, we have $v_i \in D_j \subseteq N[u_j] \setminus \{s, t\}$, a contradiction. Next assume that k - i = 2. That is, i = j - 1 and k = i + 1. Since $v_i, v_k \notin N[u_j] \setminus \{s, t\}$ and P is shortest, the vertices v_i, u_i, u_j, u_k, v_k are distinct and form a cycle of length 5. Observe that $v_i \notin \{s, t\}$ since otherwise $\langle v_i = s, v_k, u_k \rangle$ or $\langle u_i, v_i = t \rangle$ is a shortcut. Similarly, $v_k \notin \{s, t\}$. Hence, $v_i, v_k \notin N[u_j]$. Therefore, the possible chords for the cycle $\langle v_i, u_i, u_j, u_k, v_k \rangle$ are $\{u_i, v_k\}$ and $\{u_k, v_i\}$. In any combination of them, the graph has an induced cycle of length at least 4.

Let d and D_i be defined as in the proof of Lemma 15, and let $D = \bigcup_{0 \le i \le d} D_i$. Note that each D_i is a clique: if $i \in \{0, d\}$, then it is a singleton; otherwise, it is a minimal s-t

separator of the chordal graph G[D]. Observe that if $|D_i| = 1$ for all $0 \le i \le d$, then G contains a unique shortest *s*-*t* path, and thus the problem is trivial. Otherwise, we define ℓ to be the minimum index such that $|D_{\ell}| > 1$ and r to be the maximum index such that $|D_r| > 1$. Since $|D_0| = |D_d| = 1$, we have $0 < \ell \le r < d$.

Our algorithm works as follows.

- 1. If G contains a unique shortest s-t path P, then test if P is non-separating.
- 2. Otherwise, find a shortest s-t path P satisfying the following conditions.
 - **a.** V(P) does not contain a minimal *a-b* separator for $a \in D_{\ell}$ and $b \in V \setminus D$.
 - **b.** V(P) does not contain a minimal *a-b* separator for $a \in D_{\ell}$ and $b \in D_r$.

▶ Lemma 16. The algorithm is correct.

Proof. The first case is trivial. In the following, we prove the correctness of the second case.

First we show that the condition 2a is necessary. Let $a \in D_{\ell}$ and $b \in V \setminus D$. Since $|D_{\ell}| > 1$ and $|V(P) \cap D_{\ell}| = 1$, there is a vertex $a' \in D_{\ell} \setminus V(P)$, where a' may be a itself. Since $V(P) \subseteq D$, it holds that $b \notin V(P)$. Hence, a' and b belong to the same connected component of G - V(P). Since D_{ℓ} is a clique, $a \in N[a']$. Thus, a and b belong to the same connected component of $G - (V(P) \setminus \{a, b\})$. Therefore, V(P) does not contain any a-b separator.

Next we show that the condition 2b is necessary. Let $a \in D_{\ell}$ and $b \in D_r$. As before, it suffices to show that a and b belong to the same connected component of $G - (V(P) \setminus \{a, b\})$. By the same reasoning in the previous case, there are vertices $a' \in D_{\ell} \setminus V(P)$ and $b' \in D_r \setminus V(P)$ and they belong to the same connected component of G - V(P). Now, since $a \in N[a']$ and $b \in N[b']$, a and b belong to the same connected component of $G - (V(P) \setminus \{a, b\})$.

Finally we show that the conditions 2a and 2b together form a sufficient condition for P to be non-separating. Assume that a shortest *s*-*t* path P satisfies the conditions 2a and 2b. Since $|D_{\ell}| > 1$ and $|V(P) \cap D_{\ell}| = 1$, there is a connected component C of G - V(P) that contains at least one vertex of D_{ℓ} . Now the condition 2a implies that $V \setminus D \subseteq V(C)$ (recall that $V(P) \subseteq D$), and the condition 2b implies that $(D_{\ell} \cup D_r) \setminus V(P) \subseteq V(C)$ holds. To complete the proof, it suffices to show that $D_i \setminus V(P) \subseteq V(C)$ for all *i*. If $i < \ell$ or i > r, then $D_i \setminus V(P) = \emptyset$. Let $v \in D_i \setminus V(P)$ for some *i* with $\ell \leq i \leq r$. Observe that *v* is an internal vertex of a shortest path from the unique vertex $u \in D_{\ell-1}$ to the unique vertex $w \in D_{r+1}$. By Lemma 15, $N[v] \setminus \{u, w\}$ is a *u*-*w* path *Q*. Since $N[v] \setminus \{u, w\}$ is a *u*-*w* separator, *Q* contains a vertex *q* such that

$$q \in V(Q) \cap (N[v] \setminus \{u, w\}) = (V(Q) \setminus \{u, w\}) \cap N[v] \subseteq V(C) \cap N[v].$$

Therefore, v has a neighbor (i.e., q) in V(C), and thus v itself belongs to C.

▶ Lemma 17. The algorithm has a polynomial-time implementation.

Proof. Since G is chordal, each minimal separator of G is a clique. Since P is a shortest path, the size of a clique in G[V(P)] is at most 2. Therefore, every minimal separator of G contained in V(P) has size at most 2. Furthermore, every size-2 minimal separator $\{u, v\}$ is an edge of G. This observation gives us the following implementation of the algorithm that clearly runs in polynomial time.

For $i \in \{1, 2\}$, let \mathcal{F}_i be the set of size-*i* minimal *a*-*b* separators of *G* such that $a \in D_\ell$ and $b \in (V \setminus D) \cup D_r$. It suffices to find a shortest *s*-*t* path *P* such that no element of $\mathcal{F}_1 \cup \mathcal{F}_2$ is a subset of V(P). To forbid the elements of \mathcal{F}_1 , we just remove the vertices that form the size-1 separators in \mathcal{F}_1 . Similarly, to forbid the elements of \mathcal{F}_2 , we remove the

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edges corresponding to the size-2 separators in \mathcal{F}_2 . Now we find a shortest s-t path P in the resultant graph. If P has length $d = \text{dist}_G(s, t)$, then P is a non-separating shortest s-t path in G. Otherwise, G does not have such a path.

▶ Theorem 18. There is a polynomial-time algorithm for SHORTEST NON-SEPARATING PATH on chordal graphs, provided that k is equal to the shortest path distance between s and t.

4 Shortest Non-Disconnecting Path

The goal of this section is to establish the fixed-parameter tractability and a conditional lower bound on polynomial kernelizations for SHORTEST NON-DISCONNECTING PATH.

4.1 Fixed-parameter tractability

▶ Theorem 19. SHORTEST NON-DISCONNECTING PATH can be solved in time $2^{\omega k} n^{O(1)}$, where ω is the matrix multiplication exponent and n is the number of vertices of the input graph G.

To prove this theorem, we give a dynamic programming algorithm with the aid of representative families of cographic matroids. Let (G, s, t, k) be an instance of SHORTEST NON-DISCONNECTING PATH. For $0 \le i \le k$ and $v \in V(G)$, we define dp(i, v) as the family of all sets of edges F satisfying the following two conditions: (1) F is the set of edges of an s-v path of length i and (2) G - F is connected. An edge set F is legitimate if F forms a path and G - F is connected. For a family of edge sets \mathcal{F} and an edge e, we define $\mathcal{F} \bowtie e := \{F \cup \{e\} : F \in \mathcal{F}\}$ and $\operatorname{leg}(\mathcal{F})$ as the subfamily of \mathcal{F} consisting of all legitimate $F \in \mathcal{F}$. The following simple recurrence correctly computes dp(i, v).

$$\{\emptyset\} \qquad \qquad i = 0 \text{ and } s = v \tag{1}$$

$$i = 0 \text{ and } s \neq v$$
 (2)

$$dp(i,v) = \begin{cases} \{\emptyset\} & i = 0 \text{ and } s = v & (1) \\ \emptyset & i = 0 \text{ and } s \neq v & (2) \\ leg\left(\bigcup_{u \in N(v)} (dp(i-1,u) \bowtie \{u,v\})\right) & i > 0 \ . & (3) \end{cases}$$

A straightforward induction proves that $dp(i,t) \neq \emptyset$ if and only if G has a nondisconnecting s-t path of length exactly i and hence it suffices to check whether $dp(i, t) \neq \emptyset$ for $0 \le i \le k$. However, the running time to evaluate this recurrence is $n^{O(k)}$. To reduce the running time of this algorithm, we apply Theorem 4 to each dp(i, v). Now, instead of (3), we define

$$d\mathbf{p}(i,v) = \operatorname{rep}_{k-i}\left(\operatorname{leg}\left(\bigcup_{u \in N(v)} (d\mathbf{p}(i-1,u) \bowtie \{u,v\})\right)\right),\tag{3'}$$

where $\operatorname{rep}_{k-i}(\mathcal{F})$ is a (k-i)-representative family of \mathcal{F} for the cographic matroid \mathcal{M} = $(E(G),\mathcal{I})$ defined on G. In the following, we abuse the notation of dp to denote the families of legitimate sets that are computed by the recurrence composed of (1), (2), and (3').

Lemma 20. The recurrence composed of (1), (2), and (3') is correct, that is, G has a non-disconnecting s-t path of length at most k if and only if $\bigcup_{0 \le i \le k} dp(i, t) \neq \emptyset$.

Proof. It suffices to show that $dp(k', t) \neq \emptyset$ if G has a non-disconnecting s-t path P of length $k' \leq k$. Let $P = (v_0 = s, v_1, \ldots, v_{k'} = t)$ be a non-disconnecting path in G. For $0 \leq i \leq k'$, we let $P_i = (v_i, v_{i+1}, \ldots, v_k)$. In the following, we prove, by induction on i, a slightly stronger claim that there is a legitimate set $F \in dp(i, v_i)$ such that $F \cup E(P_i)$ forms a non-disconnecting s-t path in G for all $0 \leq i \leq k'$. As $dp(0, s) = \{\emptyset\}$ and $P_0 = P$ itself is a non-disconnecting path, we are done for i = 0. Suppose that i > 0. By the induction hypothesis, there is a legitimate $F \in dp(i-1, v_{i-1})$ such that $F \cup E(P_{i-1})$ forms a non-disconnecting s-t path in G. Let $\mathcal{F} = leg(\bigcup_{u \in N(v_i)} (dp(i-1) \bowtie \{u, v_i\}))$. Since $F \cup E(P_{i-1})$ is legitimate, $F \cup \{v_{i-1}, v_i\}\}$ is also legitimate, implying that \mathcal{F} is nonempty. Let $\widehat{\mathcal{F}} = rep_{k-i}(\mathcal{F})$ be (k-i)-representative for \mathcal{F} and let $Y = \{\{v_j, v_{j+1}\} : i \leq j < k'\}$. As $|Y| \leq k - i, X \cap Y = \emptyset$, and $X \cup Y \in \mathcal{I}, \widehat{\mathcal{F}}$ contains an edge set \widehat{X} with $\widehat{X} \cap Y$ and $\widehat{X} \cup Y \in \mathcal{I}$, implying that there is $\widehat{X} \in dp(i, v_i)$ such that $\widehat{X} \cup E(P_i)$ forms a non-disconnecting s-t path in G. Thus, the lemma follows.

▶ Lemma 21. The recurrence can be evaluated in time $2^{\omega k}n^{O(1)} \subset 5.18^k n^{O(1)}$, where $\omega < 2.373$ is the exponent of the matrix multiplication.

Proof. By Theorem 4, dp(i, v) contains at most $2^k kn$ sets for $0 \le i \le k$ and $v \in V(G)$ and can be computed in time $2^{\omega k} n^{O(1)}$ by dynamic programming.

Thus, Theorem 19 follows.

4.2 Kernel lower bound

It is well known that a parameterized problem is fixed-parameter tractable if and only if it admits a kernelization (see [6], for example). By Theorem 19, SHORTEST NON-DISCONNECTING PATH admits a kernelization. A natural step next to this is to explore the existence of polynomial kernelizations for SHORTEST NON-DISCONNECTING PATH. However, the following theorem conditionally rules out the possibility of polynomial kernelization. To prove this, we first show the following lemma.

▶ Lemma 22. Let *H* be a connected graph. Suppose that *H* has a cut vertex *v*. Let *C* be a component in $H - \{v\}$ and let $F \subseteq E(H[C \cup \{v\}])$. Then, H - F is connected if and only if $H[C \cup \{v\}] - F$ is connected.

Proof. If H - F is connected, then all the vertices in $C \cup \{v\}$ are reachable from v in H - F without passing through any vertex in $V(H) \setminus (\{C\} \cup \{v\})$. Thus, such vertices are reachable from v in $H[C \cup \{v\}] - F$. Conversely, suppose $H[C \cup \{v\}] - F$ is connected. Then, every vertex in C is reachable from v in H - F. Moreover, as F does not contain any edge outside $H[C \cup \{v\}]$, every other vertex is reachable from v in H - F as well.

▶ **Theorem 23.** Unless coNP \subseteq NP/poly, SHORTEST NON-DISCONNECTING PATH does not admit a polynomial kernelization (with respect to parameter k).

Proof. We give an OR-composition for SHORTEST NON-DISCONNECTING PATH. Let $(G_1, s_1, t_1, k), \ldots, (G_p, s_p, t_p, k)$ be p instances of SHORTEST NON-DISCONNECTING PATH. We assume that for $1 \leq i \leq p$, t_i is not a cut vertex in G_i . To justify this assumption, suppose that t_i is a cut vertex in G_i . Let C be the component in $G_i - \{t_i\}$ that contains s_i . By Lemma 22, for any s_i - t_i path, it is non-disconnecting in G_i if and only if so is in $G_i[C \cup \{t_i\}]$. Thus, by replacing G_i with $G_i[C]$, we can assume that t_i is not a cut vertex in G_i .

From the disjoint union of G_1, \ldots, G_p , we construct a single instance (G, s, t, k') as follows. We first add a vertex s and an edge between s and s_i for each $1 \le i \le p$. Then, we identify

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Figure 2 An illustration of the graph G obtained from q = 4 instances.

all t_i 's into a single vertex t. See Figure 2 for an illustration. In the following, we may not distinguish t from t_i . Now, we claim that (G, s, t, k + 1) is a yes-instance if and only if (G_i, s_i, t_i, k) is a yes-instance for some i.

Consider an arbitrary *s*-*t* path in *G*. Observe that all edges in the path except for that incident to *s* are contained in a single subgraph G_i for some $1 \le i \le p$ as $\{s, t\}$ separates $V(G_i) \setminus \{t_i\}$ from $V(G_j) \setminus \{t_j\}$ for $j \ne i$. Moreover, the path *P* forms $P = (s, s_i, v_1, \ldots, v_q, t)$, meaning that the subpath $P' = (s_1, v_1, \ldots, v_q, t_i)$ is an s_i - t_i path in G_i . This conversion is reversible: for any s_i - t_i path P' in G_i , the path obtained from P' by attaching *s* adjacent to s_i is an *s*-*t* path in *G*. Thus, it suffices to show that for $F \subseteq E(G_i), F \cup \{\{s, s_i\}\}$ is a cut of *G* if and only if *F* is a cut of G_i . Since *t* is a cut vertex in $G - \{\{s, s_i\}\}$, by Lemma 22, the claim holds.

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