# Finding shortest non-separating and non-disconnecting paths 

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#### Abstract

For a connected graph $G=(V, E)$ and $s, t \in V$, a non-separating $s$ - $t$ path is a path $P$ between $s$ and $t$ such that the set of vertices of $P$ does not separate $G$, that is, $G-V(P)$ is connected. An $s$ - $t$ path is non-disconnecting if $G-E(P)$ is connected. The problems of finding shortest non-separating and non-disconnecting paths are both known to be NP-hard. In this paper, we consider the problems from the viewpoint of parameterized complexity. We show that the problem of finding a non-separating $s-t$ path of length at most $k$ is $\mathrm{W}[1]$-hard parameterized by $k$, while the non-disconnecting counterpart is fixed-parameter tractable parameterized by $k$. We also consider the shortest non-separating path problem on several classes of graphs and show that this problem is NP-hard even on bipartite graphs, split graphs, and planar graphs. As for positive results, the shortest non-separating path problem is fixed-parameter tractable parameterized by $k$ on planar graphs and polynomial-time solvable on chordal graphs if $k$ is the shortest path distance between $s$ and $t$.


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## 1 Introduction

Lovász's path removal conjecture states the following claim: There is a function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that for every $f(k)$-connected graph $G$ and every pair of vertices $u$ and $v, G$ has a path $P$ between $u$ and $v$ such that $G-V(P)$ is $k$-connected. This claim still remains open, while some spacial cases have been resolved [4, 15, 16, 22]. Tutte [22] proved that $f(1)=3$, that is, every triconnected graph satisfies that for every pair of vertices, there is a path between them whose removal results a connected graph. Kawarabayashi et al. [15] proved a weaker version of this conjecture: There is a function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that for every $f(k)$-connected graph $G$ and every pair of vertices $u$ and $v, G$ has an induced path $P$ between $u$ and $v$ such that $G-E(P)$ is $k$-connected.

As a practical application, let us consider a network represented by an undirected graph $G$, and we would like to build a private channel between a specific pair of nodes $s$ and $t$. For some security reasons, the path used in this channel should be dedicated to the pair $s$ and $t$, and hence all other connections do not use all nodes and/or edges on this path while keeping their connections. In graph-theoretic terms, the vertices (resp. edges) of the path between $s$ and $t$ does not form a separator (resp. a cut) of $G$. Tutte's result [22] indicates that such a path always exists in triconnected graphs, but may not exist in biconnected graphs. In
addition to this connectivity constraint, the path between $s$ and $t$ is preferred to be short due to the cost of building a private channel. Motivated by such a natural application, the following two problems are studied.

- Definition 1. Given a connected graph $G, s, t \in V(G)$, and an integer $k$, Shortest Non-Separating Path asks whether there is a path $P$ between $s$ and $t$ in $G$ such that the length of $P$ is at most $k$ and $G-V(P)$ is connected.
- Definition 2. Given a connected graph $G, s, t \in V(G)$, and an integer $k$, ShORTEST Non-Disconnecting Path asks whether there is a path $P$ between $s$ and $t$ in $G$ such that the length of $P$ is at most $k$ and $G-E(P)$ is connected.

We say that a path $P$ is non-separating (in $G$ ) if $G-V(P)$ is connected and is nondisconnecting (in $G$ ) if $G-E(P)$ is connected.

Related work. The shortest path problem in graphs is one of the most fundamental combinatorial optimization problems, which is studied under various settings. Indeed, our problems Shortest Non-Separating Path and Shortest Non-Disconnecting Path can be seen as variants of this problem. From the computational complexity viewpoint, Shortest Non-Separating Path is known to be NP-hard and its optimization version cannot be approximated with factor $|V|^{1-\varepsilon}$ in polynomial time for $\varepsilon>0$ unless $\mathrm{P}=\mathrm{NP}$ [23]. Shortest Non-Disconnecting Path is shown to be NP-hard on general graphs and polynomial-time solvable on chordal graphs [18].

Our results. We investigate the parameterized complexity of both problems. We show that Shortest Non-Separating Path is W[1]-hard and Shortest Non-Disconnecting Path is fixed-parameter tractable parameterized by $k$. A crucial observation for the fixedparameter tractability of Shortest Non-Disconnecting Path is that the set of edges in a non-disconnecting path can be seen as an independent set of a cographic matroid. By applying the representative family of matroids [11], we show that Shortest Non-Disconnecting Path can be solved in $2^{\omega k}|V|^{O(1)}$ time, where $\omega$ is the exponent of the matrix multiplication. We also show that Shortest Non-Disconnecting Path is OR-compositional, that is, there is no polynomial kernelization unless coNP $\subseteq$ NP/poly. To cope with the intractability of Shortest Non-Separating Path, we consider the problem on planar graphs and show that it is fixed-parameter tractable parameterized by $k$. This result can be generalized to wider classes of graphs which have the diameter-treewidth property [9]. We also consider Shortest Non-Separating Path on several classes of graphs. We can observe that the complexity of Shortest Non-Separating Path is closely related to that of Hamiltonian Cycle (or Hamiltonian Path with specified end vertices). This observation readily proves the NP-completeness of Shortest Non-Separating Path on bipartite graphs, split graphs, and planar graphs. For chordal graphs, we devise a polynomial-time algorithm for Shortest Non-Separating Path for the case where $k$ is the shortest path distance between $s$ and $t$.

## 2 Preliminaries

We use standard terminologies and known results in matroid theory and parameterized complexity theory, which are briefly discussed in this section. See [6, 20] for details.

Graphs. Let $G$ be a graph. The vertex set and edge set of $G$ are denoted by $V(G)$ and $E(G)$, respectively. For $v \in V(G)$, the open neighborhood of $v$ in $G$ is denoted by
$N_{G}(v)$ (i.e., $N_{G}(v)=\{u \in V(G):\{u, v\} \in E(G)\}$ ) and the closed neighborhood of $v$ in $G$ is denoted by $N_{G}[v]$ (i.e., $N_{G}[v]=N_{G}(v) \cup\{v\}$ ). We extend this notation to sets: $N_{G}(X)=\bigcup_{v \in X} N_{G}(v) \backslash X$ and $N_{G}[X]=N_{G}(X) \cup X$ for $X \subseteq V(G)$. For $u, v \in V(G)$, we denote by $\operatorname{dist}_{G}(u, v)$ the length of a shortest path between $u$ and $v$ in $G$, where the length of a path is defined to the number of edges in it. We may omit the subscript of $G$ from these notations when no confusion arises. For $X \subseteq V(G)$, we write $G[X]$ to denote the subgraph of $G$ induced by $X$. For notational convenience, we may use $G-X$ instead of $G[V(G) \backslash X]$. For $F \subseteq E$, we also use $G-F$ to represent the subgraph of $G$ consisting all vertices of $G$ and all edges in $E \backslash F$. For vertices $u$ and $v$, a path between $u$ and $v$ is called a $u-v$ path. A vertex is called a pendant if its degree is exactly 1 .

Matroids and representative sets. Let $E$ be a finite set. If $\mathcal{I} \subseteq 2^{E}$ satisfies the following axioms, the pair $\mathcal{M}=(E, \mathcal{I})$ is called a matroid: (1) $\emptyset \in \mathcal{I} ;(2) Y \in \mathcal{I}$ implies $X \in \mathcal{I}$ for $X \subseteq Y \subseteq 2^{E}$; and (3) for $X, Y \in \mathcal{I}$ with $|X|<|Y|$, there is $e \in Y \backslash X$ such that $X \cup\{e\} \in \mathcal{I}$. Each set in $\mathcal{I}$ is called an independent set of $\mathcal{M}$. From the third axiom of matroids, it is easy to observe that every (inclusion-wise) maximal independent set of $\mathcal{M}$ have the same cardinality. The rank of $\mathcal{M}$ is the maximum cardinality of an independent set of $\mathcal{M}$. A matroid $\mathcal{M}=(E, \mathcal{I})$ of rank $n$ is linear (or representable) over a field $\mathbb{F}$ if there is a matrix $M \in \mathbb{F}^{n \times|E|}$ whose columns are indexed by $E$ such that $X \in \mathcal{I}$ if and only if the set of columns indexed by $X$ is linearly independent in $M$.

Let $G=(V, E)$ be a graph. A cographic matroid of $G$ is a matroid $\mathcal{M}(G)=(E, \mathcal{I})$ such that $F \subseteq E$ is an independent set of $\mathcal{M}(G)$ if and only if $G-F$ is connected. Every cographic matroid is linear and its representation can be computed in polynomial time [20].

Our algorithmic result for Shortest Non-Disconnected Path is based on representative families due to [11].

- Definition 3. Let $\mathcal{M}=(E, \mathcal{I})$ be a matroid and let $\mathcal{F} \subseteq \mathcal{I}$ be a family of independent sets of $\mathcal{M}$. For an integer $q \geq 0$, we say that $\widehat{\mathcal{F}} \subseteq \mathcal{F}$ is $q$-representative for $\mathcal{F}$ if the following condition holds: For every $Y \subseteq E$ of size at most $q$, if there is $X \in \mathcal{F}$ with $X \cap Y=\emptyset$ such that $X \cup Y \in \mathcal{I}$, then there is $\widehat{X} \in \widehat{\mathcal{F}}$ with $\widehat{X} \cap Y=\emptyset$ such that $\widehat{X} \cup Y \in \mathcal{I}$.
- Theorem $4([11,17])$. Given a linear matroid $\mathcal{M}=(E, \mathcal{I})$ of rank $n$ that is represented as a matrix $M \in \mathbb{F}^{n \times|E|}$ for some field $\mathbb{F}$, a family $\mathcal{F} \subseteq \mathcal{I}$ of independent sets of size $p$, and an integer $q$ with $p+q \leq n$, there is a deterministic algorithm computing a $q$-representative family $\widehat{\mathcal{F}} \subseteq \mathcal{F}$ of size $\operatorname{np}\binom{p+q}{p}$ with

$$
O\left(|\mathcal{F}| \cdot\left(\binom{p+q}{p} p^{3} n^{2}+\binom{p+q}{q}^{\omega-1} \cdot(p n)^{\omega-1}\right)\right)+(n+|E|)^{O(1)}
$$

field operations, where $\omega<2.373$ is the exponent of the matrix multiplication.

Parameterized complexity. A parameterized problem is a language $L \subseteq \Sigma^{*} \times \mathbb{N}$, where $\Sigma$ is a finite alphabet. We say that $L$ is fixed-parameter tractable (parameterized by $k$ ) if there is an algorithm deciding if $(I, k) \in L$ for given $(I, k) \in \Sigma^{*} \times \mathbb{N}$ in time $f(k)|I|^{O(1)}$, where $f$ is a computable function. A kernelization for $L$ is a polynomial-time algorithm that given an instance $(I, k) \in \Sigma^{*} \times \mathbb{N}$, computes an "equivalent" instance $\left(I^{\prime}, k^{\prime}\right) \in \Sigma^{*} \times \mathbb{N}$ such that (1) $(I, k) \in L$ if and only if $\left(I^{\prime}, k^{\prime}\right) \in L$ and (2) $\left|I^{\prime}\right|+k^{\prime} \leq g(k)$ for some computable function $g$. We call $\left(I^{\prime}, k^{\prime}\right)$ a kernel. If the function $g$ is a polynomial, then the kernelization algorithm is called a polynomial kernelization and its output $\left(I^{\prime}, k^{\prime}\right)$ is called a polynomial kernel. An

OR-composition is an algorithm that given $p$ instances $\left(I_{1}, k\right), \ldots\left(I_{p}, k\right) \in \Sigma^{*} \times \mathbb{N}$ of $L$, computes an instance $\left(I^{\prime}, k^{\prime}\right) \in \Sigma^{*} \times \mathbb{N}$ in time $\left(\sum_{1 \leq i \leq p}\left|I_{i}\right|+k\right)^{O(1)}$ such that (1) $\left(I^{\prime}, k^{\prime}\right) \in L$ if and only if $\left(I_{i}, k\right) \in L$ for some $1 \leq i \leq p$ and (2) $k^{\prime}=k^{O(1)}$. For a parameterized problem $L$, its unparameterized problem is a language $L^{\prime}=\left\{x \# 1^{k}:(x, k) \in L\right\}$, where $\# \notin \Sigma$ is a blank symbol and $1 \in \Sigma$ is an arbitrary symbol.

- Theorem 5 ([3]). If a parameterized problem $L$ admits an OR-composition and its unparameterized version is $N P$-complete, then $L$ does not have a polynomial kernelization unless coNP $\subseteq$ NP/poly.


## 3 Shortest Non-Separating Path

We discuss our complexity and algorithmic results for Shortest Non-Separating Path.

### 3.1 Hardness on graph classes

We obverse that, in most cases, Shortest Non-Separating Path is NP-hard on classes of graphs for which Hamiltonian Path (with distinguished end vertices) is NP-hard. Let $G=(V, E)$ be a graph and $s, t \in V$ be distinct vertices of $G$. We add a pendant vertex $p$ adjacent to $s$ and denote the resulting graph by $G^{\prime}$. Then, we have the following observation.

- Observation 6. For every non-separating path $P$ between s and $t$ in $G^{\prime}, V(G) \backslash V(P)=\{p\}$.

Suppose that for a class $\mathcal{C}$ of graphs,

- the problem of deciding whether given graph $G \in \mathcal{C}$ has a Hamiltonian path between specified vertices $s$ and $t$ in $G$ is NP-hard and
- $G \in \mathcal{C}$ implies $G^{\prime} \in \mathcal{C}$.

By Observation 6. $G^{\prime}$ has a non-separating $s$ - $t$ path if and only if $G$ has a Hamiltonian path between $s$ and $t$. This implies that the problem of finding a non-separating path between specified vertices is NP-hard on class $\mathcal{C}$.

- Theorem 7. The problem of deciding if an input graph has a non-separating s-t path is NP-complete even on planar graphs, bipartite graphs, and split graphs.

The classes of planar graphs and bipartite graphs are closed under the operation of adding a pendant. Recall that a graph $G$ is a split graph if the vertex set $V(G)$ can be partitioned into a clique $C$ and an independent set $I$. Thus, for the class of split graphs, we need the assumption that the pendant added is adjacent to a vertex in $C$.

As the problem trivially belongs to NP, it suffices to show that Hamiltonian Path (with distinguished end vertices) is NP-hard on these classes of graphs ${ }^{1}$ For split graphs, it is known that Hamiltonian Path is NP-hard even if the distinguished end vertices are contained in the clique $C$ [19]. Let $G$ be a graph and let $v \in V(G)$. We add a vertex $v^{\prime}$ that is adjacent to every vertex in $N_{G}(v)$, that is, $v$ and $v^{\prime}$ are twins. The resulting graph is denoted by $G^{\prime}$. It is easy to verify that $G$ has a Hamiltonian cycle if and only if $G^{\prime}$ has a Hamiltonian path between $v$ and $v^{\prime}$. The class of bipartite graphs is closed under this operation, that is, $G^{\prime}$ is bipartite. For planar graphs, $G^{\prime}$ may not be planar in general. However, Hamiltonian Cycle is NP-complete even if the input graph is planar and has

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Figure 1 The left figure depicts an instance $G$ of Multicolored Clique and the right figure depicts the graph $H$ constructed from $G$. Some vertices and edges in $H$ are not drawn in this figure for visibility. The edges of a clique $C$ and the corresponding non-separating $s-t$ path $P$ are drawn as thick lines.
a vertex of degree 2 [14]. We apply the above operation to this degree- 2 vertex, and the resulting graph $G^{\prime}$ is still planar. As the problem of finding a Hamiltonian cycle is NP-hard even on bipartite graphs [19] and planar graphs [14], Theorem 7 follows.

### 3.2 W[1]-hardness

Next, we show that Shortest Non-Separating Path is W[1]-hard parameterized by $k$. The proof is done by giving a reduction from Multicolored Clique, which is known to be $\mathrm{W}[1]$-complete [10]. In Multicolored Clique, we are given a graph $G$ with a partition $\left\{V_{1}, V_{2}, \ldots, V_{k}\right\}$ of $V(G)$ and asked to determine whether $G$ has a clique $C$ such that $\left|V_{i} \cap C\right|=1$ for each $1 \leq i \leq k$.

From an instance ( $G,\left\{V_{1}, \ldots, V_{k}\right\}$ ) of Multicolored Clique, we construct an instance of Shortest Non-Separating Path as follows. Without loss of generality, we assume that $G$ contains more than $k$ vertices. We add two vertices $s$ and $t$ and edges between $s$ and all $v \in V_{1}$ and between $t$ and all $v \in V_{k}$. For any pair of $u \in V_{i}$ and $v \in V_{j}$ with $|i-j| \geq 2$, we do the following. If $\{u, v\} \in E$, then we remove it. Otherwise, we add a path $P_{u, v}$ of length 2 and a pendant vertex that is adjacent to the internal vertex $w$ of $P_{u, v}$. Finally, we add a vertex $v^{*}$, add an edge between $v^{*}$ and each original vertex $v \in V(G)$, and add a pendant vertex $p$ adjacent to $v^{*}$. The constructed graph is denoted by $H$. See Figure 1 for an illustration of the graph $H$.

- Lemma 8. There is a clique $C$ in $G$ such that $\left|C \cap V_{i}\right|=1$ for $1 \leq i \leq k$ if and only if there is a non-separating s-t path of length at most $k+1$ in $H$.

Proof. Suppose first that $G$ has a clique $C$ with $C \cap V_{i}=\left\{v_{i}\right\}$ for $1 \leq i \leq k$. Then, $P=\left\langle s, v_{1}, v_{2}, \ldots, v_{k}, t\right\rangle$ is an $s$ - $t$ path of length $k+1$ in $H$. To see the connectivity of $H-V(P)$, it suffices to show that every vertex is reachable to $v^{*}$ in $H-V(P)$. By the construction of $H$, each vertex in $V(G) \backslash V(P)$ is adjacent to $v^{*}$ in $H-V(P)$. Each vertex $z$ in $V(H) \backslash\left(V(G) \cup\left\{v^{*}, p\right\}\right)$ is either the internal vertex $w$ of $P_{u, v}$ for some $u, v \in V(G)$ or the pendant vertex adjacent to $w$. In both cases, at least one of $u$ and $v$ is not contained in $P$ as $V(P) \backslash\{s, t\}$ is a clique in $G$, implying that $z$ is reachable to $v^{*}$.

Conversely, suppose that $H$ has a non-separating $s$ - $t$ path $P$ of length at most $k+1$ in $H$. By the assumption that $G$ has more than $k$ vertices, there is a vertex of $G$ that does not belong to $P$. Observe that $P$ does not contain any internal vertex $w$ of some $P_{u, v}$ as otherwise the pendant vertex adjacent to $w$ becomes an isolated vertex by deleting $V(P)$, which implies $H-V(P)$ has at least two connected components. Similarly, $P$ does not contain $v^{*}$. These facts imply that the internal vertices of $P$ belong to $V(G)$, and we have
$\left|V(P) \cap V_{i}\right|=1$ for $1 \leq i \leq k$. Let $C=V(P) \backslash\{s, t\}$. We claim that $C$ is a clique in $G$. Suppose otherwise. There is a pair of vertices $u, v \in C$ that are not adjacent in $G$. This implies that $H$ contains the path $P_{u, v}$. However, as $P$ contains both $u$ and $v$, the internal vertex of $P_{u, v}$ together with its pendant vertex forms a component in $H-V(P)$, yielding a contradiction that $P$ is a non-separating path in $H$.

Thus, we have the following theorem.

- Theorem 9. Shortest Non-Separating Path is W[1]-hard parameterized by $k$.


### 3.3 Graphs with the diameter-treewidth property

By Theorem 9 Shortest Non-Separating Path is unlikely to be fixed-parameter tractable on general graphs. To overcome this intractability, we focus on sparse graph classes. We first note that algorithmic meta-theorems for FO Model Checking [12, 13] does not seem to be applied to Shortest Non-Separating Path as we need to care about the connectivity of graphs, while it can be expressed by a formula in MSO logic, which is as follows. The property that vertex set $X$ forms a non-separating $s$ - $t$ path can be expressed as:

$$
\operatorname{conn}(V \backslash X) \wedge \operatorname{hampath}(X, s, t)
$$

where $\operatorname{conn}(Y)$ and hampath $(Y, s, t)$ are formulas in $\mathrm{MSO}_{2}$ that are true if and only if the subgraph induced by $Y$ is connected and has a Hamiltonian path between $s$ and $t$, respectively. We omit the details of these formulas, which can be found in [6] for exampl $\underbrace{2}$ By Courcelle's theorem [5] and its extension due to Arnborg et al. [1], we can compute a shortest nonseparating s-t path in $O(f(\operatorname{tw}(G)) n)$ time, where $n$ is the number of vertices and $\operatorname{tw}(G)$ is the treewidth ${ }^{3}$ of $G$. As there is an $O\left(\operatorname{tw}(G)^{\operatorname{tw}(G)^{3}} n\right)$-time algorithm for computing the treewidth of an input graph $G$ [2], we have the following theorem.

- Theorem 10. Shortest Non-Separating Path is fixed-parameter tractable parameterized by the treewidth of input graphs.

A class $\mathcal{C}$ of graphs is minor-closed if every minor of a graph $G \in \mathcal{C}$ also belongs to $\mathcal{C}$. We say that $\mathcal{C}$ has the diameter-treewidth property if there is a function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that for every $G \in \mathcal{C}$, the treewidth of $G$ is upper bounded by $f(\operatorname{diam}(G))$, where $\operatorname{diam}(G)$ is the diameter of $G$. It is well known that every planar graph $G$ has treewidth at most $3 \cdot \operatorname{diam}(G)+1[21]$ which implies that the class of planar graphs has the diameter-treewidth property. This can be generalized to more wider classes of graphs. A graph is called an apex graph if it has a vertex such that removing it makes the graph planar.

- Theorem 11 ([7] ). Let $\mathcal{C}$ be a minor-closed class of graphs. Then, $\mathcal{C}$ has the diametertreewidth property if and only if it excludes some apex graph.

For $C \subseteq V(G)$ that induces a connected subgraph $G[C]$, we denote by $G_{C}$ the graph obtained from $G$ by contracting all edges in $G[C]$ and by $v_{C}$ the vertex corresponding to $C$ in $G_{C}$. Since $G[C]$ is connected, vertex $v_{C}$ is well-defined.

[^1]- Lemma 12. Let $C \subseteq V(G)$ be a vertex subset such that $G[C]$ is connected. Let $P$ be an s-t path in $G$ with $V(P) \cap C=\emptyset$. Then, $P$ is non-separating in $G$ if and only if is non-separating in $G_{C}$.

Proof. Suppose first that $P$ is non-separating in $G$. Let $u, v \in V(G) \backslash V(P)$ be arbitrary. As $P$ is non-separating, there is a $u-v$ path $P^{\prime}$ in $G-V(P)$. Let $u^{\prime}$ be the vertex of $G_{C}$ such that $u^{\prime}=u$ if $u \notin C$ and $u^{\prime}=v_{C}$ if $u \in C$. Let $v^{\prime}$ be the vertex defined analogously. We show that there is a $u^{\prime}-v^{\prime}$ path in $G_{C}-V(P)$ as well. If $P^{\prime}$ does not contain any vertex in $C$, it is also a $u^{\prime}-v^{\prime}$ path in $G_{C}$, and hence we are done. Suppose otherwise. Let $x$ and $y$ be the vertices in $V\left(P^{\prime}\right) \cap C$ that are closest to $u$ and $v$, respectively. Note that $x$ and $y$ can be the end vertices of $P^{\prime}$, that is, $C$ may contain $u$ and $v$. Let $P_{u, x}$ and (resp. $P_{y, v}$ ) be the subpath of $P^{\prime}$ between $u$ and $x$ (resp. $y$ and $v$ ). Then, the sequence of vertices obtained by concatenating $P_{u, x}$ after $P_{y, v}-\{y\}$ and replacing exactly one occurrence of a vertex in $C$ with $v_{C}$ forms a path between $u^{\prime}$ and $v^{\prime}$. Since we choose $u, v$ arbitrarily, there is a path between any pair of vertices in $G_{C}-V(P)$ as well. Hence, $P$ is non-separating in $G_{C}$.

Conversely, suppose that $P$ is non-separating in $G_{C}$. For $u, v \in V\left(G_{C}\right) \backslash V(P)$, there is a path $P^{\prime}$ in $G_{C}-V(P)$. Suppose that neither $u=v_{C}$ nor $v=v_{C}$. Then, we can construct a $u-v$ path in $G-V(P)$ as follows. If $v_{C} \notin V\left(P^{\prime}\right), P^{\prime}$ is also a path in $G-V(P)$ and hence we are done. Otherwise, $v_{C} \in V\left(P^{\prime}\right)$. Let $P_{u}$ and $P_{v}$ be the subpaths in $P^{\prime}-\left\{v_{C}\right\}$ containing $u$ and $v$, respectively. From $P_{u}$ and $P_{v}$, we have a $u-v$ path in $G$ by connecting them with an arbitrary path in $G[C]$ between the end vertices other than $u$ and $v$. Note that such a bridging path in $G[C]$ always exists since $G[C]$ is connected. Moreover, as $V\left(P^{\prime}\right) \cap C=\emptyset$ and $V(P) \cap C=\emptyset$, this is also a $u-v$ path in $G-V(P)$. Suppose otherwise that either $u=v_{C}$ or $v=v_{C}$, say $u=v_{C}$. In this case, we can construct a path between every vertex $w$ in $C$ and $v$ by concatenating $P^{\prime}$ and an arbitrary path in $G[C]$ between $w$ and the end vertex of the subpath $P^{\prime}-\left\{v_{C}\right\}$ other than $v$. Therefore, $P$ is non-separating in $G$.

Now, we are ready to state the main result of this subsection.

- Theorem 13. Suppose that a minor-closed class $\mathcal{C}$ of graphs has the diameter-treewidth property. Then, Shortest Non-Separating Path on $\mathcal{C}$ is fixed-parameter tractable parameterized by $k$.

Proof. Let $G \in \mathcal{C}$. We first compute $B=\{v \in V(G): \operatorname{dist}(s, v) \leq k\}$. This can be done in linear time. If $t \notin B$, then the instance $(G, s, t, k)$ is trivially infeasible. Suppose otherwise that $t \in B$. Let $C$ be a component in $G-B$. By definition, every non-separating s-t path $P$ of length at most $k$ does not contain any vertex of $C$. Let $G^{\prime}$ be the graph obtained from $G$ by contracting all edges in $E(G-B)$. For each component $C$ in $G-B$, we denote by $v_{C}$ the vertex of $G^{\prime}$ corresponding to $C$ (i.e., $v_{C}$ is the vertex obtained by contracting all edges in $C$ ). Then, we have $\operatorname{diam}\left(G^{\prime}\right) \leq 2 k+2$ as $\operatorname{diam}(G[B]) \leq k$ and every vertex in $V\left(G^{\prime}\right) \backslash B$ is adjacent to a vertex in $B$. By Lemma 12, $G$ has a non-separating $s$ - $t$ path of length at most $k$ if and only if so does $G^{\prime}$. Since $\mathcal{C}$ is minor-closed, we have $G^{\prime} \in \mathcal{C}$ and hence the treewidth of $G^{\prime}$ is upper bounded by $f(2 k+2)$ for some function $f$. By Theorem 10 , we can check whether $G^{\prime}$ has a non-separating $s$ - $t$ path of length at most $k$ in $O\left(g(k)\left|V\left(G^{\prime}\right)\right|\right)$ time for some function $g$.

### 3.4 Chordal graphs with $k=\operatorname{dist}(s, t)$

In Section 3.1 we have seen that Shortest Non-Separating Path is NP-complete even on split graphs (and thus on more general chordal graphs as well). To overcome this intractability, we restrict ourselves to finding a non-separating $s$ - $t$ path of length dist $(s, t)$ on chordal graphs.

A graph $G$ is choral if it has no cycles of length at least 4 as an induced subgraph. In the following, fix a connected chordal graph $G$.

- Lemma 14. Let $S \subseteq V(G)$ be a vertex set such that $G[S]$ is connected. For $u, v \in S$, every induced u-v path $P$ in $G$ satisfies that $V(P) \subseteq N[S]$.

Proof. Suppose to the contrary that an induced $u$-v path $P$ contains a vertex $x \notin N[S]$. Since $P$ starts and ends in $S$, it contains a subpath $Q=\langle a, \ldots, x, \ldots, b\rangle$ such that $a, b \in N(S)$ and all other vertices in $Q$ belong to $V-N[S]$. As $a, b \in N(S)$ and $G[S]$ is connected, $G$ has an induced $a-b$ path $R$ with all internal vertices belonging to $S$. Since the internal vertices of $Q$ have no neighbors in $S, Q \cup R$ induces a cycle. Both $a-b$ paths $Q$ and $R$ have length at least 2 as $a$ and $b$ are not adjacent, and thus the cycle $G[Q \cup R]$ has length at least 4. This contradicts that $G$ is chordal.

For $u, v \in V(G)$, a set of vertices $S \subseteq V(G) \backslash\{u, v\}$ is called a $u-v$ separator of $G$ if there is no $u-v$ path in $G-S$. An inclusion-wise minimal $u-v$ separator of $G$ is called a minimal $u-v$ separator. A minimal separator of $G$ is a minimal $u-v$ separator for some $u, v \in G$. Dirac's well-know characterization [8] of chordal graphs states that a graph is chordal if and only of every minimal separator induces a clique.

- Lemma 15. Let $s, t \in V(G)$ be such that $\{s, t\} \notin E(G)$. If $v \in V(G) \backslash\{s, t\}$ is an internal vertex of a shortest s-t path $P$, then $N[v] \backslash\{s, t\}$ is an $s$-t separator of $G$.

Proof. Let $d=\operatorname{dist}(s, t)$. For $0 \leq i \leq d$, let

$$
D_{i}=\{v \in V(G): \operatorname{dist}(s, v)=i \wedge \operatorname{dist}(v, t)=d-i\}
$$

and $V(P) \cap D_{i}=\left\{u_{i}\right\}$. Let $j(0<j<d)$ be the index such that $v=u_{j}$.
Suppose to the contrary that there is an induced s-t path $Q$ such that $V(Q) \cap\left(N\left[u_{j}\right] \backslash\right.$ $\{s, t\})=\emptyset$. By Lemma 14, $V(Q) \subseteq N[V(P)]=\bigcup_{0 \leq i \leq d} N\left[u_{i}\right]$ holds. Since $Q$ starts in $N\left[u_{0}\right]$ and ends in $N\left[u_{d}\right]$, there are indices $i$ and $k$ with $0 \leq i<j<k \leq d$ such that $Q$ consecutively visits a vertex $v_{i} \in N\left[u_{i}\right]$ and then a vertex $v_{k} \in N\left[u_{k}\right]$ in this order. Since $\operatorname{dist}\left(u_{i}, u_{k}\right)=k-i \geq 2$ and $\left\{v_{i}, v_{k}\right\} \in E$, at least one of $v_{i} \neq u_{i}$ and $v_{k} \neq u_{k}$ holds. By symmetry, we assume that $v_{i} \neq u_{i}$.

If $v_{k}=u_{k}$, then $v_{i} \in N\left(u_{i}\right) \cap N\left(u_{k}\right)$. In this case, we have $i=j-1$ and $k=j+1$ since otherwise $P$ admits a shortcut using the subpath $\left\langle u_{i}, v_{i}, u_{k}\right\rangle$. This implies that $\operatorname{dist}\left(s, v_{i}\right) \leq$ $\operatorname{dist}\left(s, u_{i}\right)+1=i+1=j$ and $\operatorname{dist}\left(v_{i}, t\right) \leq 1+\operatorname{dist}\left(v_{k}, t\right)=1+\operatorname{dist}\left(u_{k}, t\right)=1+(d-k)=d-j$. Since $\operatorname{dist}\left(s, v_{i}\right)+\operatorname{dist}\left(v_{i}, t\right) \geq d$, we have $\operatorname{dist}\left(s, v_{i}\right)=j$ and $\operatorname{dist}\left(v_{i}, t\right)=d-j$. This implies that $v_{i} \in D_{j} \subseteq N\left[u_{j}\right] \backslash\{s, t\}$, a contradiction

Next we consider the case $v_{k} \neq u_{k}$. Recall that we also have $v_{i} \neq u_{i}$ as an assumption. In this case, we have $k-i \leq 3$ as $\left\langle u_{i}, v_{i}, v_{k}, u_{k}\right\rangle$ is not a shortcut for $P$. Assume first that $k-i=3$. By symmetry, we may assume that $i=j-1$ and $k=j+2$. Since $\operatorname{dist}\left(s, v_{i}\right) \leq \operatorname{dist}\left(s, u_{i}\right)+1=j$ and $\operatorname{dist}\left(v_{i}, t\right) \leq 2+\operatorname{dist}\left(u_{k}, t\right) \leq 2+(d-k)=d-j$, we have $v_{i} \in D_{j} \subseteq N\left[u_{j}\right] \backslash\{s, t\}$, a contradiction. Next assume that $k-i=2$. That is, $i=j-1$ and $k=i+1$. Since $v_{i}, v_{k} \notin N\left[u_{j}\right] \backslash\{s, t\}$ and $P$ is shortest, the vertices $v_{i}, u_{i}, u_{j}, u_{k}, v_{k}$ are distinct and form a cycle of length 5 . Observe that $v_{i} \notin\{s, t\}$ since otherwise $\left\langle v_{i}=s, v_{k}, u_{k}\right\rangle$ or $\left\langle u_{i}, v_{i}=t\right\rangle$ is a shortcut. Similarly, $v_{k} \notin\{s, t\}$. Hence, $v_{i}, v_{k} \notin N\left[u_{j}\right]$. Therefore, the possible chords for the cycle $\left\langle v_{i}, u_{i}, u_{j}, u_{k}, v_{k}\right\rangle$ are $\left\{u_{i}, v_{k}\right\}$ and $\left\{u_{k}, v_{i}\right\}$. In any combination of them, the graph has an induced cycle of length at least 4.

Let $d$ and $D_{i}$ be defined as in the proof of Lemma 15, and let $D=\bigcup_{0 \leq i \leq d} D_{i}$. Note that each $D_{i}$ is a clique: if $i \in\{0, d\}$, then it is a singleton; otherwise, it is a minimal $s-t$
separator of the chordal graph $G[D]$. Observe that if $\left|D_{i}\right|=1$ for all $0 \leq i \leq d$, then $G$ contains a unique shortest $s$ - $t$ path, and thus the problem is trivial. Otherwise, we define $\ell$ to be the minimum index such that $\left|D_{\ell}\right|>1$ and $r$ to be the maximum index such that $\left|D_{r}\right|>1$. Since $\left|D_{0}\right|=\left|D_{d}\right|=1$, we have $0<\ell \leq r<d$. Our algorithm works as follows.

1. If $G$ contains a unique shortest $s$ - $t$ path $P$, then test if $P$ is non-separating.
2. Otherwise, find a shortest $s$ - $t$ path $P$ satisfying the following conditions.
a. $V(P)$ does not contain a minimal $a-b$ separator for $a \in D_{\ell}$ and $b \in V \backslash D$.
b. $V(P)$ does not contain a minimal $a-b$ separator for $a \in D_{\ell}$ and $b \in D_{r}$.

- Lemma 16. The algorithm is correct.

Proof. The first case is trivial. In the following, we prove the correctness of the second case.
First we show that the condition 2 aa is necessary. Let $a \in D_{\ell}$ and $b \in V \backslash D$. Since $\left|D_{\ell}\right|>1$ and $\left|V(P) \cap D_{\ell}\right|=1$, there is a vertex $a^{\prime} \in D_{\ell} \backslash V(P)$, where $a^{\prime}$ may be $a$ itself. Since $V(P) \subseteq D$, it holds that $b \notin V(P)$. Hence, $a^{\prime}$ and $b$ belong to the same connected component of $G-V(P)$. Since $D_{\ell}$ is a clique, $a \in N\left[a^{\prime}\right]$. Thus, $a$ and $b$ belong to the same connected component of $G-(V(P) \backslash\{a, b\})$. Therefore, $V(P)$ does not contain any $a-b$ separator.

Next we show that the condition 2 b is necessary. Let $a \in D_{\ell}$ and $b \in D_{r}$. As before, it suffices to show that $a$ and $b$ belong to the same connected component of $G-(V(P) \backslash\{a, b\})$. By the same reasoning in the previous case, there are vertices $a^{\prime} \in D_{\ell} \backslash V(P)$ and $b^{\prime} \in D_{r} \backslash V(P)$ and they belong to the same connected component of $G-V(P)$. Now, since $a \in N\left[a^{\prime}\right]$ and $b \in N\left[b^{\prime}\right], a$ and $b$ belong to the same connected component of $G-(V(P) \backslash\{a, b\})$.

Finally we show that the conditions 2 a and 2 b together form a sufficient condition for $P$ to be non-separating. Assume that a shortest $s$ - $t$ path $P$ satisfies the conditions 2a and 2b Since $\left|D_{\ell}\right|>1$ and $\left|V(P) \cap D_{\ell}\right|=1$, there is a connected component $C$ of $G-V(P)$ that contains at least one vertex of $D_{\ell}$. Now the condition 2 a implies that $V \backslash D \subseteq V(C)$ (recall that $V(P) \subseteq D$ ), and the condition 2b implies that $\left(D_{\ell} \cup D_{r}\right) \backslash V(P) \subseteq V(C)$ holds. To complete the proof, it suffices to show that $D_{i} \backslash V(P) \subseteq V(C)$ for all $i$. If $i<\ell$ or $i>r$, then $D_{i} \backslash V(P)=\emptyset$. Let $v \in D_{i} \backslash V(P)$ for some $i$ with $\ell \leq i \leq r$. Observe that $v$ is an internal vertex of a shortest path from the unique vertex $u \in D_{\ell-1}$ to the unique vertex $w \in D_{r+1}$. By Lemma $15, N[v] \backslash\{u, w\}$ is a $u-w$ separator. Since $C$ is connected and $u, w$ have neighbors in $C, G[V(C) \cup\{u, w\}]$ contains a $u$-w path $Q$. Since $N[v] \backslash\{u, w\}$ is a $u$ - $w$ separator, $Q$ contains a vertex $q$ such that

$$
q \in V(Q) \cap(N[v] \backslash\{u, w\})=(V(Q) \backslash\{u, w\}) \cap N[v] \subseteq V(C) \cap N[v] .
$$

Therefore, $v$ has a neighbor (i.e., $q$ ) in $V(C)$, and thus $v$ itself belongs to $C$.

- Lemma 17. The algorithm has a polynomial-time implementation.

Proof. Since $G$ is chordal, each minimal separator of $G$ is a clique. Since $P$ is a shortest path, the size of a clique in $G[V(P)]$ is at most 2 . Therefore, every minimal separator of $G$ contained in $V(P)$ has size at most 2 . Furthermore, every size-2 minimal separator $\{u, v\}$ is an edge of $G$. This observation gives us the following implementation of the algorithm that clearly runs in polynomial time.

For $i \in\{1,2\}$, let $\mathcal{F}_{i}$ be the set of size- $i$ minimal $a-b$ separators of $G$ such that $a \in D_{\ell}$ and $b \in(V \backslash D) \cup D_{r}$. It suffices to find a shortest $s$ - $t$ path $P$ such that no element of $\mathcal{F}_{1} \cup \mathcal{F}_{2}$ is a subset of $V(P)$. To forbid the elements of $\mathcal{F}_{1}$, we just remove the vertices that form the size-1 separators in $\mathcal{F}_{1}$. Similarly, to forbid the elements of $\mathcal{F}_{2}$, we remove the
edges corresponding to the size- 2 separators in $\mathcal{F}_{2}$. Now we find a shortest $s$ - $t$ path $P$ in the resultant graph. If $P$ has length $d=\operatorname{dist}_{G}(s, t)$, then $P$ is a non-separating shortest $s-t$ path in $G$. Otherwise, $G$ does not have such a path.

- Theorem 18. There is a polynomial-time algorithm for Shortest Non-Separating Path on chordal graphs, provided that $k$ is equal to the shortest path distance between $s$ and $t$.


## 4 Shortest Non-Disconnecting Path

The goal of this section is to establish the fixed-parameter tractability and a conditional lower bound on polynomial kernelizations for Shortest Non-Disconnecting Path.

### 4.1 Fixed-parameter tractability

- Theorem 19. Shortest Non-Disconnecting Path can be solved in time $2^{\omega k} n^{O(1)}$, where $\omega$ is the matrix multiplication exponent and $n$ is the number of vertices of the input graph $G$.

To prove this theorem, we give a dynamic programming algorithm with the aid of representative families of cographic matroids. Let $(G, s, t, k)$ be an instance of Shortest Non-Disconnecting Path. For $0 \leq i \leq k$ and $v \in V(G)$, we define $\operatorname{dp}(i, v)$ as the family of all sets of edges $F$ satisfying the following two conditions: (1) $F$ is the set of edges of an $s$ - $v$ path of length $i$ and (2) $G-F$ is connected. An edge set $F$ is legitimate if $F$ forms a path and $G-F$ is connected. For a family of edge sets $\mathcal{F}$ and an edge $e$, we define $\mathcal{F} \bowtie e:=\{F \cup\{e\}: F \in \mathcal{F}\}$ and $\operatorname{leg}(\mathcal{F})$ as the subfamily of $\mathcal{F}$ consisting of all legitimate $F \in \mathcal{F}$. The following simple recurrence correctly computes $\mathrm{dp}(i, v)$.

$$
\operatorname{dp}(i, v)= \begin{cases}\{\emptyset\} & i=0 \text { and } s=v  \tag{1}\\ \emptyset & i=0 \text { and } s \neq v \\ \operatorname{leg}\left(\bigcup_{u \in N(v)}(\operatorname{dp}(i-1, u) \bowtie\{u, v\})\right) & i>0 .\end{cases}
$$

A straightforward induction proves that $\operatorname{dp}(i, t) \neq \emptyset$ if and only if $G$ has a nondisconnecting $s$ - $t$ path of length exactly $i$ and hence it suffices to check whether $\operatorname{dp}(i, t) \neq \emptyset$ for $0 \leq i \leq k$. However, the running time to evaluate this recurrence is $n^{O(k)}$. To reduce the running time of this algorithm, we apply Theorem 4 to each $\mathrm{dp}(i, v)$. Now, instead of (3), we define

$$
\begin{equation*}
\operatorname{dp}(i, v)=\operatorname{rep}_{k-i}\left(\operatorname{leg}\left(\bigcup_{u \in N(v)}(\operatorname{dp}(i-1, u) \bowtie\{u, v\})\right)\right) \tag{3'}
\end{equation*}
$$

where $\operatorname{rep}_{k-i}(\mathcal{F})$ is a $(k-i)$-representative family of $\mathcal{F}$ for the cographic matroid $\mathcal{M}=$ $(E(G), \mathcal{I})$ defined on $G$. In the following, we abuse the notation of dp to denote the families of legitimate sets that are computed by the recurrence composed of (11), (2), and (3).

- Lemma 20. The recurrence composed of (1), (2), and (3) is correct, that is, G has a non-disconnecting s-t path of length at most $k$ if and only if $\bigcup_{0 \leq i \leq k} \mathrm{dp}(i, t) \neq \emptyset$.

Proof. It suffices to show that $\operatorname{dp}\left(k^{\prime}, t\right) \neq \emptyset$ if $G$ has a non-disconnecting $s$ - $t$ path $P$ of length $k^{\prime} \leq k$. Let $P=\left(v_{0}=s, v_{1}, \ldots, v_{k^{\prime}}=t\right)$ be a non-disconnecting path in $G$. For $0 \leq i \leq k^{\prime}$, we let $P_{i}=\left(v_{i}, v_{i+1}, \ldots, v_{k}\right)$. In the following, we prove, by induction on $i$, a slightly stronger claim that there is a legitimate set $F \in \mathrm{dp}\left(i, v_{i}\right)$ such that $F \cup E\left(P_{i}\right)$ forms a non-disconnecting $s$ - $t$ path in $G$ for all $0 \leq i \leq k^{\prime}$. As $\operatorname{dp}(0, s)=\{\emptyset\}$ and $P_{0}=P$ itself is a non-disconnecting path, we are done for $i=0$. Suppose that $i>0$. By the induction hypothesis, there is a legitimate $F \in \operatorname{dp}\left(i-1, v_{i-1}\right)$ such that $F \cup E\left(P_{i-1}\right)$ forms a non-disconnecting $s$ - $t$ path in $G$. Let $\mathcal{F}=\operatorname{leg}\left(\bigcup_{u \in N\left(v_{i}\right)}\left(\operatorname{dp}(i-1) \bowtie\left\{u, v_{i}\right\}\right)\right)$. Since $F \cup E\left(P_{i-1}\right)$ is legitimate, $F \cup\left\{\left\{v_{i-1}, v_{i}\right\}\right\}$ is also legitimate, implying that $\mathcal{F}$ is nonempty. Let $\widehat{\mathcal{F}}=\operatorname{rep}_{k-i}(\mathcal{F})$ be $(k-i)$-representative for $\mathcal{F}$ and let $Y=\left\{\left\{v_{j}, v_{j+1}\right\}: i \leq j<k^{\prime}\right\}$. As $|Y| \leq k-i, X \cap Y=\emptyset$, and $X \cup Y \in \mathcal{I}, \widehat{\mathcal{F}}$ contains an edge set $\widehat{X}$ with $\widehat{X} \cap Y$ and $\widehat{X} \cup Y \in \mathcal{I}$, implying that there is $\widehat{X} \in \operatorname{dp}\left(i, v_{i}\right)$ such that $\widehat{X} \cup E\left(P_{i}\right)$ forms a non-disconnecting $s$ - $t$ path in $G$. Thus, the lemma follows.

- Lemma 21. The recurrence can be evaluated in time $2^{\omega k} n^{O(1)} \subset 5.18^{k} n^{O(1)}$, where $\omega<2.373$ is the exponent of the matrix multiplication.

Proof. By Theorem 4, $\mathrm{dp}(i, v)$ contains at most $2^{k} k n$ sets for $0 \leq i \leq k$ and $v \in V(G)$ and can be computed in time $2^{\omega k} n^{O(1)}$ by dynamic programming.

Thus, Theorem 19 follows.

### 4.2 Kernel lower bound

It is well known that a parameterized problem is fixed-parameter tractable if and only if it admits a kernelization (see [6, for example). By Theorem 19. Shortest Non-Disconnecting Path admits a kernelization. A natural step next to this is to explore the existence of polynomial kernelizations for Shortest Non-Disconnecting Path. However, the following theorem conditionally rules out the possibility of polynomial kernelization. To prove this, we first show the following lemma.

- Lemma 22. Let $H$ be a connected graph. Suppose that $H$ has a cut vertex $v$. Let $C$ be a component in $H-\{v\}$ and let $F \subseteq E(H[C \cup\{v\}])$. Then, $H-F$ is connected if and only if $H[C \cup\{v\}]-F$ is connected.

Proof. If $H-F$ is connected, then all the vertices in $C \cup\{v\}$ are reachable from $v$ in $H-F$ without passing through any vertex in $V(H) \backslash(\{C\} \cup\{v\})$. Thus, such vertices are reachable from $v$ in $H[C \cup\{v\}]-F$. Conversely, suppose $H[C \cup\{v\}]-F$ is connected. Then, every vertex in $C$ is reachable from $v$ in $H-F$. Moreover, as $F$ does not contain any edge outside $H[C \cup\{v\}]$, every other vertex is reachable from $v$ in $H-F$ as well.

- Theorem 23. Unless coNP $\subseteq \mathrm{NP} /$ poly, Shortest Non-Disconnecting Path does not admit a polynomial kernelization (with respect to parameter $k$ ).

Proof. We give an OR-composition for Shortest Non-Disconnecting Path. Let $\left(G_{1}, s_{1}, t_{1}, k\right), \ldots,\left(G_{p}, s_{p}, t_{p}, k\right)$ be $p$ instances of Shortest Non-Disconnecting Path. We assume that for $1 \leq i \leq p, t_{i}$ is not a cut vertex in $G_{i}$. To justify this assumption, suppose that $t_{i}$ is a cut vertex in $G_{i}$. Let $C$ be the component in $G_{i}-\left\{t_{i}\right\}$ that contains $s_{i}$. By Lemma 22, for any $s_{i}-t_{i}$ path, it is non-disconnecting in $G_{i}$ if and only if so is in $G_{i}\left[C \cup\left\{t_{i}\right\}\right]$. Thus, by replacing $G_{i}$ with $G_{i}[C]$, we can assume that $t_{i}$ is not a cut vertex in $G_{i}$.

From the disjoint union of $G_{1}, \ldots, G_{p}$, we construct a single instance ( $G, s, t, k^{\prime}$ ) as follows. We first add a vertex $s$ and an edge between $s$ and $s_{i}$ for each $1 \leq i \leq p$. Then, we identify


Figure 2 An illustration of the graph $G$ obtained from $q=4$ instances.
all $t_{i}$ 's into a single vertex $t$. See Figure 2 for an illustration. In the following, we may not distinguish $t$ from $t_{i}$. Now, we claim that $(G, s, t, k+1)$ is a yes-instance if and only if $\left(G_{i}, s_{i}, t_{i}, k\right)$ is a yes-instance for some $i$.

Consider an arbitrary $s-t$ path in $G$. Observe that all edges in the path except for that incident to $s$ are contained in a single subgraph $G_{i}$ for some $1 \leq i \leq p$ as $\{s, t\}$ separates $V\left(G_{i}\right) \backslash\left\{t_{i}\right\}$ from $V\left(G_{j}\right) \backslash\left\{t_{j}\right\}$ for $j \neq i$. Moreover, the path $P$ forms $P=\left(s, s_{i}, v_{1}, \ldots, v_{q}, t\right)$, meaning that the subpath $P^{\prime}=\left(s_{1}, v_{1}, \ldots, v_{q}, t_{i}\right)$ is an $s_{i}-t_{i}$ path in $G_{i}$. This conversion is reversible: for any $s_{i}-t_{i}$ path $P^{\prime}$ in $G_{i}$, the path obtained from $P^{\prime}$ by attaching $s$ adjacent to $s_{i}$ is an $s$ - $t$ path in $G$. Thus, it suffices to show that for $F \subseteq E\left(G_{i}\right), F \cup\left\{\left\{s, s_{i}\right\}\right\}$ is a cut of $G$ if and only if $F$ is a cut of $G_{i}$. Since $t$ is a cut vertex in $G-\left\{\left\{s, s_{i}\right\}\right\}$, by Lemma 22 the claim holds.

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[^0]:    1 These results for bipartite graphs and planar graphs seem to be folklore but we were not able to find particular references.

[^1]:    ${ }^{2}$ In [6], they give an $\mathrm{MSO}_{2}$ sentence hamiltonicity expressing the property of having a Hamiltonian cycle, which can be easily transformed into a formula expressing hampath $(X, s, t)$.
    ${ }^{3}$ We do not give the definition of treewidth and (the optimization version of) Courcelle's theorem. We refer to 6 for details.
    ${ }^{4}$ More precisely, the treewidth of a planar graph is upper bounded by $3 r+1$, where $r$ is the radius of the graph.

