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Finding Stabbing Lines in 3-Space*

M. Pellegrini[1] and P. W. Shor[2]

[1] Courant Institute, New York University, 251 Mercer Street, New York, NY 10012, USA

[2] AT&T Bell Laboratories, 2D-149, 600 Mountain Avenue, Murray Hill, NJ 07974, USA

Abstract. A line intersecting all polyhedra in a set $\mathscr{B}$ is called a "stabber" for the set $\mathscr{B}$. This paper addresses some combinatorial and algorithmic questions about the set $\mathscr{S}(\mathscr{B})$ of all lines stabbing $\mathscr{B}$. We prove that the combinatorial complexity of $\mathscr{S}(\mathscr{B})$ has an $O(n^3 2^{c\sqrt{\log n}})$ upper bound, where n is the total number of facets in $\mathscr{B}$, and c is a suitable constant. This bound is almost tight. Within the same time bound it is possible to determine if a stabbing line exists and to find one.

1. Introduction

The first algorithm for finding line stabbers for a set $\mathscr{B}$ of polyhedra in R^3 with total complexity n, due to Avis and Wenger [AW1], [AW2], has an $O(n^4 \log n)$-time bound. McKenna and O'Rourke [MO] improve the time complexity to $O(n^4\alpha(n))$, where $\alpha(n)$ is a functional inverse of the Ackerman function. The algorithm in [MO] finds *all the isotopy classes*[1] of lines generated by the polyhedra in $\mathscr{B}$ and it is within an $\alpha(n)$ factor from the optimal for that problem. The set $\mathscr{S}(\mathscr{B})$ of stabbing lines coincides with the union of some of the isotopy classes and it was conjectured that the complexity of $\mathscr{S}(\mathscr{B})$ could be less than the complexity of all the isotopy classes.

Jaromczyk and Kowaluk [JK] claimed an $O(n^3 2^{\alpha(n)})$ upper bound to the complexity of the set of stabbing lines, but unfortunately there are cases in which the analysis used in [JK] is incorrect (see [P2]).

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[1] Two lines are in the same isotopy class if it is possible to move one into the other without crossing edges of $\mathscr{B}$.

A line is characterized by four parameters, therefore a natural representation for a line in R^3 is a point in R^4. The locus of lines intersecting a given line is a quadric (hyperbolic) surface in R^4. Given a set of polyhedra $\mathscr{B}$, consider the lines spanning edges of the polyhedra, and for each such line the corresponding surface in R^4. We obtain an arrangement $\mathscr{A}(\mathscr{B})$ of surfaces in R^4 that divide the space into cells (McKenna and O'Rourke [MO] implicitly construct this arrangement). Each cell of $\mathscr{A}(\mathscr{B})$ contains points whose stabbed set is invariant within the cell. The set $\mathscr{S}(\mathscr{B})$ of all stabbing lines is therefore the union of some cells in the arrangement. Our aim is to find a worst-case upper bound on the combinatorial complexity of $\mathscr{S}(\mathscr{B})$ which is significantly lower than the complexity of the whole arrangement $\mathscr{A}(\mathscr{B})$. We consider the lines spanning edges in $\mathscr{B}$ to be in *general position* when no four lines are on the same ruled surface (planes, one sheet hyperboloids, and hyperbolic paraboloids [B]). For simplicity, we deal mostly with edges in general position and we give additional arguments to cope with degeneracies.

The complexity of $\mathscr{S}(\mathscr{B})$ is bounded by the number of zero-dimensional faces (vertices) of $\mathscr{S}(\mathscr{B})$. Each vertex represents an *extremal stabbing line.* An *extremal stabbing line* l is a stabbing line for $\mathscr{B}$ which falls into one of the following three categories:

1. l intersects four edges in four distinct polyhedra in $\mathscr{B}$ and is tangent to the same four polyhedra.
2. l intersects one vertex and two edges in three distinct polyhedra in $\mathscr{B}$ and is tangent to the same three polyhedra.
3. l meets two vertices in two distinct polyhedra in $\mathscr{B}$ and is tangent to those two polyhedra.

For subclasses 2 and 3 an $O(n^3)$ upper bound is trivially established. In this paper we concentrate our attention on subclass 1 of extremal stabbing lines.

An $\Omega(n^3)$ lower bound for the complexity of $\mathscr{S}(\mathscr{B})$ is shown in [P2] and [P1] and can also be derived by results in [CEGS]. We prove in this paper an almost matching $O(n^3 2^{c\sqrt{\log n}})$ upper bound on the complexity of $\mathscr{S}(\mathscr{B})$ and on the time to answer the question of whether a stabbing line exists (*stabbing problem*). Initially (Section 3) we consider only triangles to show the analysis in a simplified setting. The main tools used are the *Plücker coordinates of lines*, which are introduced in Section 2, and the *random sampling* technique of Clarkson [C]. All randomized algorithms in this paper can be turned into deterministic algorithms within the same time bounds using the methods of Matoušek [Ma2]. The results for triangles are extended to the general case of convex polyhedra in Section 4.

The *query problem* (given a line, is it a stabber?) is solved using $O(n^{2+\varepsilon})$ preprocessing and storage, and $O(\log n)$ query time (Section 5). The query algorithm is used as a subroutine of an algorithm to find a stabbing line which uses $O(n^3 2^{c\sqrt{\log n}})$ time (Section 5). The techniques used in [AW], [MO] and [JK] are not suitable for solving stabbing queries efficiently nor for dealing with special cases. Section 6 is an overview of results about special cases of the stabbing problem.

2. Plücker Coordinates of Lines

Triangles in three-dimensional space are plane figures bounded by edges. Therefore we need to be able to represent and manipulate segments and lines in R^3 efficiently.

We use a representation for lines called *Plücker coordinates of the line* [So], [CEGS]. Two points $x = (x_0, x_1, x_2, x_3)$ and $y = (y_0, y_1, y_2, y_3)$ in three-dimensional homogeneous coordinates define a line l in 3-space. The six quantities[2]

$$\xi_{ij} = x_i y_j - x_j y_i, \tag{1}$$

$$ij = 01, 02, 03, 12, 23, 31, \tag{2}$$

are called *Plücker coordinates* of the line l (oriented from x to y). They correspond to the two-by-two minors of the two-by-four matrix formed by the coordinates of the point x (on the first row) and y (on the second row):

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{pmatrix}. \tag{3}$$

The six parameters are not independent; they must satisfy the following equation (whose solution constitutes the Plücker hypersurface or Klein quadric or Grassman manifold $\mathscr{F}_4^2$ [St], [HP]):

$$\Pi\colon \xi_{01}\xi_{23} + \xi_{02}\xi_{31} + \xi_{03}\xi_{12} = 0. \tag{4}$$

The six Plücker coordinates represent uniquely an oriented line modulo multiplication by positive scalars, therefore the six Plücker coordinates are homogeneous coordinates of a projective oriented five-dimensional space $\mathscr{P}^5$.

The incidence relation between two lines l and l' can be expressed using the Plücker coordinates of l and l'. Let a_1, b_1 (resp. a_2, b_2) be two points on l (resp. l') oriented the same as l (resp. l'). The incidence between l and l' is expressed as the vanishing of the determinant of a four-by-four matrix whose rows are the coordinates of a_1, b_1, a_2, b_2 in this order from top to bottom.[3] We use the notation $D(a, b, c, d)$ for the determinant formed by the coordinates of the points a, b, c, d placed on the rows in this order from top to bottom:

$$D(a_1, b_1, a_2, b_2) \stackrel{\text{def}}{=} \begin{vmatrix} a_{10} & a_{11} & a_{12} & a_{13} \\ b_{10} & b_{11} & b_{12} & b_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ b_{20} & b_{21} & b_{22} & b_{23} \end{vmatrix} = 0. \tag{5}$$

[2] The indices of the six quantities are chosen so as to obtain only positive signs in the following formulas.

[3] The absolute value of this determinant represents the volume of the tetrahedron whose vertices are a_1, b_1, a_2, and b_2, after normalizing the fourth homogeneous coordinate to 1. The sign of the determinant is positive if the quadrupole of points has the same orientation as the reference frame chosen in R^3. When the volume vanishes the four points are coplanar, therefore the two lines intersect (or are parallel).

If we expand the determinant according to the two-by-two minors of the two submatrices formed by the first two rows and by the last two rows we obtain the following equation in which only Plücker coordinates are involved:

$$\xi_{01}\xi'_{23} + \xi_{02}\xi'_{31} + \xi_{03}\xi'_{12} + \xi'_{01}\xi_{23} + \xi'_{02}\xi_{31} + \xi'_{03}\xi_{12} = 0. \tag{6}$$

Let us introduce two mappings: π: $l \to \pi_l$ maps a line in R^3 to a hyperplane in projective oriented five-dimensional space $\mathscr{P}^5$, whose plane coordinates are the Plücker coordinates of l appropriately reordered. p: $l \to p_l$ maps a line in R^3 to a point in $\mathscr{P}^5$ whose coordinates are the Plücker coordinates of the line.

The incidence relation between the two lines l and l' (expressed by equation (6)) can be reformulated as an incidence relation between points and hyperplanes in $\mathscr{P}^5$. Equation (6) can be rewritten in the form $\pi_l(p_{l'}) = 0$, which is equivalent to requiring point $p_{l'}$ to belong to hyperplane π_l. Standard geometric computations can be performed in oriented projective spaces using techniques due to Stolfi [St].

2.1. *Characterization of Stabbing Lines Using Plücker Coordinates*

Definition 1. Given the point a and the triangle t in 3-space the cone $\mathscr{C}_{a,t}$ is the set of rays from a intersecting t (see Fig. 1).

A set of triangles T and a point a define a family of cones $\mathscr{C}_{a,T} = \{\mathscr{C}_{a,t} | t \in T\}$. We say that the family of cones $\mathscr{C}_{a,T}$ is *based* on T with *apex a*. In the following we restrict our attention to stabbing lines *intersecting a reference plane P* and we assume without loss of generality that the triangles in T are all above P. It is easy to prove the following lemma:

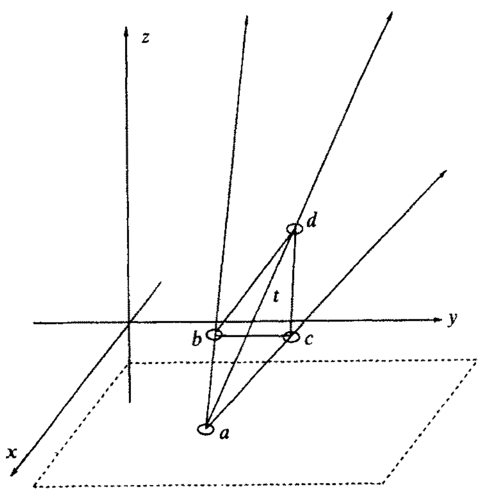

Fig. 1. Cone with apex a based on t.

Lemma 1. *There exists a stabbing line for T intersecting P if and only if there is a point q on P such that $\bigcap \mathscr{C}_{q,T} \neq \varnothing$.*

Let $\mathscr{C}_{q,t}$ be the cone based on triangle $t = (p_1, p_2, p_3)$ and with apex $q \in P$. Consider the three planes spanned by two vertices of t and of the apex q; the half-space containing the third vertex of t is called *positive*. When q belongs to the plane spanned by t the cone degenerates in a two-dimensional object, but for simplicity of exposition we ignore degenerate cases. The set of points in the rays belonging to the cone $\mathscr{C}_{q,t}$ is the intersection of the three positive half-spaces determined by q and t (see Fig. 1).

We denote by aff(t) the plane spanning t. If aff(t) is parallel to the reference plane P, then a variable point Q in the cone $\mathscr{C}_{q,t}$ satisfies the following system of linear inequalities:

$$\begin{aligned} D(q, p_1, p_2, Q) &\geq 0, \\ D(q, p_2, p_3, Q) &\geq 0, \\ D(q, p_3, p_1, Q) &\geq 0. \end{aligned} \tag{7}$$

If necessary we relabel the vertices of t to ensure the inequalities all have the same sign.

If aff(t) is not parallel to P, we consider the position of apex q with respect to the line $l_t = P \cap \text{aff}(t)$. The cone $\mathscr{C}_{q,t}$ is defined by the two systems of linear inequalities: system (7) and system (7) with the direction of the inequalities reversed, depending on the position of q relative to l_t.

We need two systems of inequalities because for apexes q on different sides of l_t the three positive half-spaces switch with the nonpositive ones (see Fig. 2).

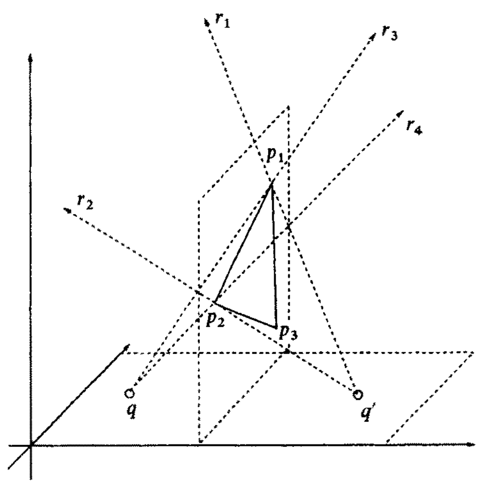

Fig. 2. Different cones based at q and q'.

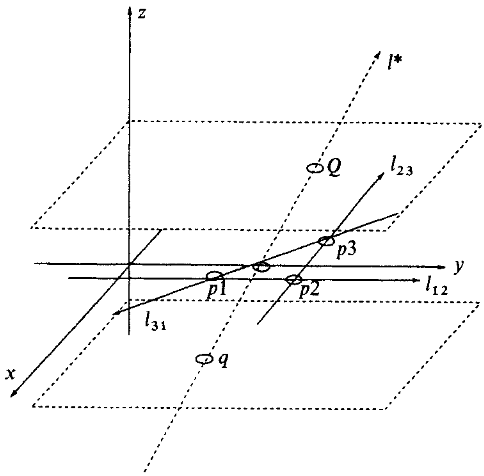

Fig. 3. Stabbing line and triangle.

Using the *row exchange rule* of determinants and changing the sign of the inequalities accordingly we can put all the determinants in system (7) in the form $D(p_1, p_2, q, Q)$, $D(p_2, p_3, q, Q)$, and $D(p_3, p_1, q, Q)$. Then we expand those determinants according to the two-by-two minors of the submatrices formed by the first two rows and by the last two rows. We obtain linear expressions in terms of Plücker coordinates (see (5) and (6)). Note that the inequalities involve the Plücker coordinates of lines supporting edges of triangles in T and the (variable) line passing through q and Q (see Fig. 3).

The discussion above shows how to characterize stabbing lines intersecting a reference plane P using Plücker coordinates. Let us consider three mutually orthogonal planes P_1, P_2, P_3; every line in R^3 must intersect at least one of them. A bound on the number of extremal stabbing lines intersecting a given plane extends immediately to a bound on the number of extremal stabbing lines with no restrictions.

We are now ready to state the main lemma of this section. Let $\mathscr{L}$ be the set of lines on P induced by the set aff(T) of planes spanning triangles in T, and denote by $\mathscr{M}$ the arrangement generated by $\mathscr{L}$ on P. Clearly, all points of a region σ of $\mathscr{M}$ have the same relative position with respect to the lines in $\mathscr{L}$, therefore the stabbing lines through σ satisfy *one* system of linear inequalities. The solution set to such a system is the possible empty polytope[4] $K_\sigma(T)$ in $\mathscr{P}^5$. The following lemma therefore holds:

Lemma 2. *Given a line l such that $l \cap P \in \sigma$, $l \in \mathscr{S}(T) \Rightarrow p_l \in K_\sigma(T)$.*

[4] More precisely, we have a polyhedron in 5-space. To avoid confusion between polyhedra in 3-space and polyhedra in 5-space we use the word *polytope* in a nonstandard way.

3. A Combinatorial Bound for Triangles

Relaxing the conditions of Lemma 2, we now consider a generic connected region σ on P. Let T_σ^1 be the subset of triangles in T whose spanning planes do not intersect σ. We have the following lemma:

Lemma 3. *For a connected region σ on P, let T_σ^1 be the set of triangles whose spanning planes do not intersect σ,*

$$l \in \mathscr{S}(T_\sigma^1) \quad \Rightarrow \quad p_l \in K_\sigma(T_\sigma^1).$$

Given the set of lines $\mathscr{L}$ on P we use Agarwal's technique [Ag] (or Matoušek's technique [Ma1], [Ma2]) to partition the plane P into a set $\mathscr{M}'(\mathscr{L})$ of $O(r^2)$ triangles so that no triangle meets more than $O(n/r)$ lines of $\mathscr{L}$. Let σ be one of these triangles on P. We partition T into two sets of triangles in 3-space: T_σ^1, whose supporting planes *do not cut* σ, and T_σ^2, whose supporting planes *cut* σ. Furthermore, by the properties of the partition, $|T_\sigma^1| < n$ and $|T_\sigma^2| < O(n/r)$.

Definition 2. Given a set T, a region σ, and sets T_σ^1 and T_σ^2 as above, $A_\sigma(i, j, B)$ is the set of lines that touch i edges in T_σ^1, j edges in T_σ^2, intersect every triangle in the set $B \subseteq T$, and intersect P in the region σ.

Let $N(n)$ be the maximum number of extremal stabbing lines[5] for a set of n triangles in *general position*. We will show a uniform upper bound on $N(n)$. Suppose without loss of generality that T, of size n, attains the maximum value $N(n)$. Let $N_\sigma(n)$ be the number of extremal stabbing lines for T passing through the region σ. Clearly, the following holds:

$$N(n) = \sum_{\sigma \in \mathscr{M}'(\mathscr{L})} N_\sigma(n) \tag{8}$$

and

$$\begin{aligned} N_\sigma(n) = {} & |A_\sigma(4, 0, T)| + |A_\sigma(3, 1, T)| + |A_\sigma(2, 2, T)| \\ & + |A_\sigma(1, 3, T)| + |A_\sigma(0, 4, T)|. \end{aligned} \tag{9}$$

The following lemma allows us to relax the stabbing conditions.

Lemma 4. *If $C \subset B \subseteq T$, then $|A_\sigma(i, j, C)| \geq |A_\sigma(i, j, B)|$.*

Proof. The set $A_\sigma(i, j, \varnothing)$ of lines touching i edges in T_σ^1 and j edges in T_σ^2 is a given finite set of lines because of the general position assumption. A line in $A_\sigma(i, j, \varnothing)$ that stabs B stabs every subset of B. $\square$

[5] By considering edges as relatively closed sets, we treat uniformly stabbing lines of type 1, 2, and 3 according to the classification of Section 1.

Note that Lemma 4 is true also for $C \subseteq B \not\subseteq T$, but in the proof of Theorem 1 we exploit the fact that $B \subset T_\sigma^1 \cup T_\sigma^2 = T$, therefore we state the lemma in this weaker form. Using Lemma 4 we can bound from above each term in (9) using a suitable subset of T:

$$N_\sigma \leq |A_\sigma(4, 0, T_\sigma^1)| + |A_\sigma(3, 1, T_\sigma^1)| + |A_\sigma(2, 2, T_\sigma^1)| + |A_\sigma(1, 3, T)| + |A_\sigma(0, 4, T_\sigma^2)|. \tag{10}$$

Now we bound separately every term in (10):

1. $|A_\sigma(4, 0, T_\sigma^1)|$ represents the number of lines touching four edges in T_σ^1 and stabbing T_σ^1. Since the Plücker mappings preserve incidence, if $l \in A_\sigma(4, 0, T_\sigma^1)$, then the Plücker point p_l must be on an edge of the boundary of $K_\sigma(T_\sigma^1)$. From the general position assumption, we know that every edge of $K_\sigma(T_\sigma^1)$ intersects the Plücker surface Π in no more than two points. Therefore each edge can contribute at most two lines to $A_\sigma(4, 0, T_\sigma^1)$. The number of edges is $O(n^2)$, by the upper bound theorem for polytopes (see Theorem 6.12 of Chapter 6 of [E]).
2. $|A_\sigma(3, 1, T_\sigma^1)|$ represents the number of lines touching one edge in T_σ^2 and three edges in T_σ^1 and stabbing T_σ^1. If $l \in A_\sigma(3, 1, T_\sigma^1)$, then the Plücker point p_l must be on the boundary[6] of $K_\sigma(T_\sigma^1)$ and on a Plücker hyperplane whose corresponding line spans an edge in T_σ^2. From the general position assumption, a 2-face, a hyperplane, and Π intersect in at most two points.

   The combinatorially different classes of extremal stabbing lines are determined by cutting the Plücker polytope $K_\sigma(T_\sigma^1)$ using the Plücker hyperplanes corresponding to lines spanning edges of T_σ^2. On each Plücker hyperplane derived from a line spanning an edge in T_σ^2, we have the intersections of n half-spaces in a four-dimensional space. Each intersection of half-spaces has complexity $O(n^2)$; the total complexity is $O(n^3/r)$.
3. $|A_\sigma(2, 2, T_\sigma^1)|$ represents the number of lines touching two edges in T_σ^2, two edges in T_σ^1 and stabbing T_σ^1. Consider all pairs of lines spanning edges of triangles in T_σ^2 and take their corresponding Plücker hyperplanes. Intersecting each pair of hyperplanes we obtain $O((n/r)^2)$ linear spaces of dimension 3. As in the previous case we must bound the complexity of their intersection with $K_\sigma(T_\sigma^1)$. We have $O((n/r)^2)$ subproblems; each one is the intersection of n half-spaces in a three-dimensional space. Each subproblem has complexity $O(n)$. The total bound is $O(n^3/r^2)$.
4. $|A_\sigma(1, 3, T)|$ represents the number of lines touching three edges in T_σ^2, one edge in T_σ^1, and stabbing T. We partition T_σ^1 into r disjoint sets $T_1, \ldots, T_r$ of size at most $\lceil n/r \rceil$. We form r sets $Q_i = T_i \cup T_\sigma^2$ for $i = 1, \ldots, r$. From the definition of $A_\sigma(1, 3, T)$ and the observation that a stabbing line for T is a stabbing line for any set Q_i, we have that *every line in* $A_\sigma(1, 3, T)$ *is an extremal stabbing line for exactly one of the sets* Q_i. The maximum

[6] To be precise, on a two-dimensional face.

number of extremal stabbing lines for any set of $2n/r$ triangles is $N(2n/r)$. Therefore $rN(2n/r)$ is an upper bound for $|A_\sigma(1, 3, T)|$.
5. $|A_\sigma(0, 4, T_\sigma^2)|$ counts the number of extremal stabbing lines for the set T_σ^2 of size n/r. Therefore $|A_\sigma(0, 4, T_\sigma^2)|$ is bounded from above by $N(n/r)$.

From the above discussion and (8) and (10), we obtain the following recursive inequality for $N(n)$:

$$N(n) \leq r^2(c_3 n^2 + c_4 n^3/r + c_5 n^3/r^2 + rN(c_1 n/r) + N(c_2 n/r)), \tag{11}$$

where c_1, c_2, c_3, c_4, c_5 are constants. Distributing r^2:

$$N(n) \leq r^3 N(c_1 n/r) + r^2 N(c_2 n/r) + c_3 r^2 n^2 + c_4 r n^3 + c_5 n^3. \tag{12}$$

Lemma 5. *$N(n) \leq c_1 n^3 2^{c_2\sqrt{\log n}}$ for some constants c_1 and c_2.*

Proof. First, we observe that

$$N(kn) \leq \binom{k}{4} N(4n)$$

for any integer $k \geq 4$. This holds because we can break the kn triangles into subsets of size n, and then group these subsets into subsets of size $4n$ in every possible way. Each extremal stabbing line of the original set of triangles will be a stabbing line of at least one of these groups. Using $r < n$ and the above observation, we obtain from (12) the simpler equation

$$N(n) \leq cr^3 N(n/r) + cn^3 r \tag{13}$$

for some constant c. Now, let $N(n) = n^3 f(n)$. Substituting in (13) and dividing by n^3 gives

$$f(n) \leq cf(n/r) + cr. \tag{14}$$

Now, let $r = 2^{\sqrt{\log n}}$. We will show that we can choose c_1 and c_2 so that $f(n) \leq c_1 2^{c_2\sqrt{\log n}}$. Assume by induction that $f(n/r)$ satisfies this equation. Then

$$\begin{aligned} cf(n/r) + cr &\leq cc_1 2^{c_2\sqrt{\log(n/r)}} + cr \\ &= cc_1 2^{c_2\sqrt{\log n - \sqrt{\log n}}} + c2^{\sqrt{\log n}} \\ &\leq cc_1 2^{c_2\sqrt{\log n} - c_2/2} + c2^{\sqrt{\log n}} \\ &= cc_1 2^{-c_2/2} 2^{c_2\sqrt{\log n}} + c2^{c_2\sqrt{\log n}}. \end{aligned}$$

This last expression will be less than $c_1 2^{c_2\sqrt{\log n}}$ if $c_2 > 1$ and $cc_1 2^{-c_2/2} + c < c_1$. This can be ensured with the appropriate choice of c_1 and c_2. Now, if we choose c_1 and c_2 so that the bounds hold for small n, then by induction they will hold for all n. □

In the case when the set of triangles is not in general position, a standard perturbation argument shows that the maximum of $|\mathscr{S}(T)|$ is attained by a set T in general position [CEG+, p. 5], [E].

We summarize the result of this section with the following theorem:

Theorem 1. *Given a set T of n triangles in R^3 the complexity of $\mathscr{S}(T)$ is bounded by $c_1 n^3 2^{c_2\sqrt{\log n}}$, where c_1 and c_2 are constants.*

4. A Combinatorial Bound for Polyhedra

Section 3 gives an almost cubic *upper bound* to the number of extremal stabbing lines of a set of triangles. The proof of a similar result for convex polygons in R^3 follows by simple modifications of the argument for triangles. The upper bound for polygons does not give immediately a similar result for polyhedra, because polygons are just a subclass of all polyhedra. Also, it is not convenient to reduce directly the problem of finding stabbing lines of polyhedra to the problem of finding stabbing lines for the faces of those polyhedra. A line intersecting a polyhedron must intersect one of its faces, therefore a stabbing line for a set of polyhedra is a stabbing line for a subset of faces, where each face is drawn from a distinct polyhedron. It is easy to check that there is a superpolynomial number of sets of faces to consider in the worst case. The approach we follow in this section is to reconsider the proof for triangles and adapt it for polyhedra.

Definition 3. Given a convex polyhedron B and a point q disjoint from B, the cone $\mathscr{C}_{q,B}$ is the set of rays from q intersecting B.

A family $\mathscr{B}$ of polyhedra and a point q define a family of cones $\mathscr{C}_{q,\mathscr{B}} = \{\mathscr{C}_{q,B} | B \in \mathscr{B}\}$. We assume that the reference plane P leaves all polyhedra in $\mathscr{B}$ on one side and that $q \in P$. It is easy to prove the following lemma:

Lemma 6. *There exists a stabbing line for $\mathscr{B}$ intersecting P if and only if there is a point q on P such that $\bigcap \mathscr{C}_{q,\mathscr{B}} \neq \varnothing$.*

From a point q external to the polyhedron B only a connected subset of the faces of B is visible.

Definition 4. Given a polyhedron B and a point q external to B, the *silhouette* of B from q is the set of edges of B adjacent to a facet visible from q and to a facet not visible from q. This set is denoted as sil(q, B).

The set aff($\mathscr{B}$) of planes spanning facets of $\mathscr{B}$ induce a planar arrangement $\mathscr{M}$ on P. We have an equivalent of Lemma 2 for polyhedra:

Lemma 7. *Given a region $\sigma \in \mathscr{M}$, a line l such that $l \cap P \in \sigma$, and a point $q \in \sigma$, the following holds: $l \in \mathscr{S}(\mathscr{B}) \Rightarrow p_l \in K_\sigma(\mathrm{sil}(q, \mathscr{B}))$.*

Proof. For any $B \in \mathscr{B}$, and each region $\sigma \in \mathscr{M}$ the silhouette sil(q, B) is the same for any $q \in \sigma$, We can express the stabbing condition for all lines through σ as the intersection of a set of cones based on the silhouettes as seen from σ and a variable point $q' \in \sigma$. We can express the intersection of this family of cones by a set of linear inequalities, which define a polytope $K_\sigma(\mathrm{sil}(q, \mathscr{B}))$ in $\mathscr{P}^5$. □

Given any connected region σ on P, let $E_\sigma = \bigcup_{B \in \mathscr{B}} \bigcap_{p \in \sigma} \mathrm{sil}(p, B)$ be the set of edges of silhouettes that are common for all points $q \in \sigma$. The equivalent of Lemma 3 is:

Lemma 8. *For a connected region σ on P and a line l such that $l \cap P \in \sigma$, the following holds: $l \in \mathscr{S}(\mathscr{B}) \Rightarrow p_l \in K_\sigma(E_\sigma)$.*

In order to use the recursive argument in the upper bound proof we need to be able to reconstruct a set of polyhedra from a set of edges. Given a polyhedron B, we consider B as the intersection of positive half-spaces based on the planes spanning the facets and we pair the half-spaces to form wedges.

Definition 5. Given a polyhedron B and an edge e of B, the wedge $w(e, B)$ is the intersection of two positive half-spaces defined by planes spanning facets adjacet to e.

Definition 6. Let B be a polyhedron, let $E(B)$ be the set of edges of B, and let E' be any subset of $E(B)$. Then

$$W(E', B) \stackrel{\text{def}}{=} \bigcap_{e \in E'} w(e, B).$$

For the properties of the intersection we have the following lemma:

Lemma 9.

1. $W(E(B), B) = B$.
2. *If $E' \subseteq E'' \subseteq E(B)$, then $W(E'', B) \subseteq W(E', B)$.*

In order to preserve separation properties the polyhedra of the form $W(E', B)$ are intersected with the region of R^3 above the plane P. For simplicity of exposition we omit this detail in the rest of the discussion. From Lemma 9 the following lemma is easily obtained.

Lemma 10. *If a line l stabs B, then, for every $E' \subseteq E(B)$, l stabs $W(E', B)$.*

All definitions and lemmas of this section extend to sets of polyhedra. Given a connected region σ of P and a set of polyhedra $\mathscr{B}$ of total complexity n we define the set E_σ of edges of silhouettes *common* to all points in σ:

$$E_\sigma = \bigcup_{B \in \mathscr{B}} \bigcap_{p \in \sigma} \operatorname{sil}(p, B)$$

and the set Q_σ of edges of silhouettes that are *not common* to all points in σ:

$$Q_\sigma = \bigcup_{B \in \mathscr{B}} \bigcup_{p \in \sigma} \operatorname{sil}(p, B) - E_\sigma.$$

Now we partition P in r^2 regions such that for each one the set Q_σ has small size. Consider the set $\mathscr{L}$ of lines induced by aff($\mathscr{B}$) on P; we associate to every line in $\mathscr{L}$ a weight equal to the number of edges incident to the corresponding facet of $\mathscr{B}$. Clearly, the sum of the weights is equal to the number of facet/edge incidences, that is $O(n)$.

The partitioning technique [Ag], [Ma1], [Ma2] allows us to subdivide P into a set $\mathscr{M}'(\mathscr{L})$ of r^2 triangular regions such that each region is cut by lines of $\mathscr{L}$ whose total weight is $O(n/r)$.

Consider two points q' and q on P with the same relative position with respect to all lines in $\mathscr{L}$ except for one line $l \in \mathscr{L}$. The silhouettes of a polyhedron from q and q' differ in the worst case on the edges incident to the facet corresponding to l. The number of these edges is the weight of l. Therefore the sum of the weights of lines cutting σ is an upper bound to the cardinality of Q_σ. To summarize, for every region σ of P,

$$|E_\sigma| \leq O(N),$$
$$|Q_\sigma| \leq O(n/r).$$

A stabber l through σ is extremal if it touches four edges of silhouettes in four different objects in $\mathscr{B}$, at least two of which are distinct, and is tangent to them. We can distribute the four contacts on the sets E_σ and Q_σ and define all extremal stabbing lines through σ.

Definition 7. Given a set $\mathscr{B}$ of polyhedra in general position, a region σ, sets E_σ and Q_σ as above, and a subset $E \subseteq E(\mathscr{B})$, $A_\sigma(i, j, E)$ is the set of lines that touch i edges in E_σ, j edges in Q_σ, intersect every polyhedron in the set $W(E,\mathscr{B})$, and intersect P in the region σ.

We denote by $N(n)$ the maximum number of extremal stabbing lines of type 1, 2, and 3 for a set of convex polyhedra with n edges in *general position*. We assume without loss of generality that our set $\mathscr{B}$ attains the maximum. By $N_\sigma(n)$ we denote the number of extremal stabbing lines for $\mathscr{B}$ through σ. Recalling that $\mathscr{B} = W(E(\mathscr{B}), \mathscr{B})$, the following holds:

$$N(n) = \sum_{\sigma \in \mathscr{M}'(\mathscr{L})} N_\sigma(n)$$

and

$$N_\sigma(n) = |A_\sigma(4, 0, E(\mathscr{B}))| + |A_\sigma(3, 1, E(\mathscr{B}))| + |A_\sigma(2, 2, E(\mathscr{B}))| + |A_\sigma(1, 3, E(\mathscr{B}))| + |A_\sigma(0, 4, E(\mathscr{B}))|. \tag{15}$$

The following lemma allows us to use subsets of the edges of $\mathscr{B}$ to relax the stabbing conditions:

Lemma 11. *If $E' \subset E'' \subseteq E(\mathscr{B})$, then $|A_\sigma(i, j, E')| \geq |A_\sigma(i, j, E'')|$.*

Proof. From Lemma 9, $W(E'', B) \subset W(E', B)$ for every $B \in \mathscr{B}$ and from Lemma 10 every line stabbing $W(E'', B)$ also stabs $W(E', B)$. □

Using Lemma 11 we bound from above every term in (15) using suitable subsets of $E(\mathscr{B})$:

$$N_\sigma \leq |A_\sigma(4, 0, E_\sigma)| + |A_\sigma(3, 1, E_\sigma)| + |A_\sigma(2, 2, E_\sigma)| + |A_\sigma(1, 3, E(\mathscr{B}))| + |A_\sigma(0, 4, Q_\sigma)|. \tag{16}$$

We bound separately every term in (16) using observations similar to those in the proof of Theorem 1:

1. $|A_\sigma(4, 0, E_\sigma)|$ is bounded by the complexity of the Plücker polytope $K_\sigma(E_\sigma)$, which is $O(n^2)$.
2. $|A_\sigma(0, 4, Q_\sigma)|$ is the number of external stabbing lines for Q_σ, bounded from above by $N(O(n/r))$.
3. $|A_\sigma(3, 1, E_\sigma)|$ counts the lines touching one edge in E_σ and three edges in Q_σ and stabbing $W(E_\sigma, \mathscr{B})$. These lines are determined by cutting the Plücker polytope $K_\sigma(E_\sigma)$ using the hyperplanes corresponding to edges of Q_σ. The total complexity is $O(n^3/r)$.
4. $|A_\sigma(2, 2, E_\sigma)|$ counts the lines touching two edges in E_σ and two edges in Q_σ, Pairing the hyperplanes in Q_σ we obtain $O((n/r)^2)$ linear spaces to intersect with $K_\sigma(E_\sigma)$. The bound for this quantity is $O(n^3/r^2)$.
5. $|A_\sigma(1, 3, E(\mathscr{B}))|$ counts the lines touching three lines in Q_σ and one edge in E_σ and stabbing $\mathscr{B}$. An argument similar to that used for Theorem 1 gives an upper bound $rN(O(n/r))$.

From the above discussion we obtain the following recursive equation for $N(n)$:

$$N(n) \leq r^2(n^2 + N(O(n/r)) + n^3/r + n^3/r^2 + rN(O(n/r))), \tag{17}$$

which is the same recursion as in Lemma 5. We summarize the main result of this section with the following theorem:

Theorem 2. *Given a set $\mathscr{B}$ of polyhedra with total complexity n, the complexity of $\mathscr{S}(\mathscr{B})$ is bounded by $c_1 n^3 2^{c_2\sqrt{\log n}}$, where c_1 and c_2 are constants.*

5. Finding a Stabbing Line

Lemma 12. *Given a set of n half-spaces in R^d, there exists a data structure which uses $O(n^{\lfloor d/2 \rfloor + \varepsilon})$ storage and can be built in $O(n^{\lfloor d/2 \rfloor + \varepsilon})$ expected time that in $O(\log n)$ time determines whether any point $p \in R^d$ belongs to the intersection of the half-spaces.*

Proof. Given n half-spaces, draw a random sample of size r. Compute the intersection of the sample and triangulate it. Carry on the construction recursively in each of the simplices over the $O(n/r \log r)$ half-spaces intersecting each simplex [C]. The time and storage needed to compute the set of simplices is proportional to the number of vertices of the intersection of the sampled half-spaces, which is $O(r^{\lfloor d/2 \rfloor})$ by the upper bound theorem. The function $\mathscr{T}(n)$ which is the maximum storage and preprocessing time of the recursive data structure satisfies the following inequality:

$$\mathscr{T}(n) \leq O(r^{\lfloor d/2 \rfloor})\mathscr{T}((n \log r)/r) + O(r^{\lfloor d/2 \rfloor} \log r). \tag{18}$$

The solution is $O(n^{\lfloor d/2 \rfloor + \varepsilon})$. For r constant, the query time is $O(\log^2 n)$. To improve the query time we set $r = n^{\nu}$, and we choose ν depending on ε in the range $(\lfloor d/2 \rfloor + \varepsilon)/(\lfloor d/2 \rfloor(d + \varepsilon)) > \nu > 0$. We set up fast point-location data structures for locating in $O(\log r)$ time the simplex containing the query Plücker point. The additional data structure is a standard point-location data structure in a set of hyperplanes which uses $O(r^{\lfloor d/2 \rfloor(d+\varepsilon)})$ storage. The additional $O(n^{\lfloor d/2 \rfloor + \varepsilon})$ term in (18) does not change the asymptotic bound on $\mathscr{T}(n)$. The new search tree has constant depth, therefore a total query time $O(\log n)$. Using the techniques in [St] this results hold also in oriented projective d-space $\mathscr{P}^d$. □

Theorem 3. *There exists an algorithm for answering line stabbing queries on a set $\mathscr{B}$ of polyhedra in three dimensions with total complexity n, that requires $O(n^{2+\varepsilon})$ expected randomized preprocessing time and space, with $O(\log n)$ worst-case query time.*

Proof. Consider facets of polyhedra in $\mathscr{B}$ and give to each of them a weight equal to the number of edges incident to the facet. We draw a random sample R of the set of weighted facets, obtaining an induced arrangement A_R on the plane P. Each simplex of the triangulated arrangement $\sigma \in \triangle(A_R)$ is intersected by planes spanning facets whose total weight is $O(n/r \log r)$, by results in [C].

For each edge e not incident to planes cutting σ, we can decide whether e is part of the silhouette for all points in σ, by comparing e with a point $q \in \sigma$. The overhead for silhouettes computation is $O(r^2 n)$. Let E_σ be the set of silhouette edges common to all points in σ. If l is a stabber for $\mathscr{B}$ and l intersects σ, then $p_l \in K_\sigma(E_\sigma)$. Therefore, we set up a point-location-in-a-polytope data structure of Lemma 12 for each region σ and each set E_σ. We continue recursively the construction within each simplex in $\triangle(A_R)$ for the lines defined by planes in aff($\mathscr{B}$) passing through the simplex. Denoting by $\mathscr{T}(n)$ the total storage (and preprocessing time) for the stabbing query data structure, we obtain the following recurrence equation:

$$\mathscr{T}(n) = r^2\mathscr{T}((n/r)\log r) + O(r^2 n^{2+\varepsilon}) + O(r^2 n), \tag{19}$$

whose solution is $\mathscr{T}(n) = O(r^2 n^{2+\varepsilon})$, for a different, slightly greater, ε.

The query algorithm is the following: given the line l, consider the point $q = l \cap P$ and locate it in a simplex $\sigma \in \triangle(A_R)$; then locate the Plücker point p_l in the associated Plücker polytope K_σ. If the point p_l is external to K_σ the line l is not a stabber; otherwise, we recurse the query on the data structure associated with σ. The depth of the recursion is at most $\log_r n$. We set $r = n^\nu$, and we choose ν depending on ε. We add fast planar point-location data structures to locate the simplex σ such that $\sigma \cap l \neq \emptyset$. The size of the planar point-location data structure is $O(r^2)$. The additional term in (19) does not change the asymptotic solution. The depth of the tree is constant and we obtain a total $O(\log n)$ query time. We can make sure that the depth of the search tree is constant in the *worst case* by requiring that, if a random sample does not have the required property (which holds with high probability), it is discarded and a new sample is drawn. With high probability we will not have to resample often and the asymptotic expected complexity is increased only by a multiplicative constant (see p. 216 of [C]). □

Theorem 4. *Given a set $\mathscr{B}$ of polyhedra in general position with total complexity n, the set of all extremal stabbing lines can be found in time $O(n^3 2^{c\sqrt{\log n}})$ for a suitable constant c.*

Proof. Suppose the set $\mathscr{B}$ is in general position. The counting argument used in Sections 3 and 4 to bound the number of extremal stabbing lines can be easily modified to give an algorithm whose output is a set of lines L that is a superset of the set of all extremal stabbing lines. All counting arguments in the proof are based either on recursive steps or on the worst-case complexity of the intersection of half-spaces. We find actual lines by intersecting the edges of the resulting polytopes with the Plücker hypersurface Π.

Rewriting (12) for the time needed to compute L it is easy to see that the critical term is $c_4 r n^3$, which corresponds to solving convex-hull problems in four-dimensional space. The optimal convex-hull algorithm of Seidel [E] for even dimension constructs the intersection without any extra factor over the worst-case combinatorial complexity of the output. The final bound, resulting from solving a recursive inequality similar to (12), is the same as in Lemma 5.

The superset L of lines computed in the first phase has size $O(n^3 2^{c_2\sqrt{\log n}})$ and can be found in time $O(n^3 2^{c_2\sqrt{\log n}})$. Using the stabbing query algorithm of Theorem 3 after $O(n^{2+\varepsilon})$ preprocessing we can determine in $O(\log n)$ time whether a line in L is a stabber for $\mathscr{B}$. The total complexity is $O(n^{2+\varepsilon} + c_1 n^3 2^{c_2\sqrt{\log n}} \log n)$ which is $O(n^3 2^{c\sqrt{\log n}})$ for a constant $c > c_2$. □

The combinatorial bound on $|\mathscr{S}(\mathscr{B})|$ for a set of triangles (polyhedra) in general position is extended to a set of triangles (polyhedra) not in general position by using a standard perturbation argument. In order to obtain an algorithm, however, we have to deal with degeneracies explicitly. The algorithm of Theorem 4 has three phases. In the first phase a set of Plücker polytopes is generated. In the second phase we intersect the Plücker polytopes with the Plücker surface Π. In the general case, the intersection of an edge e with Π is a finite set of Plücker points, which are tested, in the third phase, using the procedure of Theorem 3.

1. A degenerate situation arises when the edge e is fully contained in Π. The edge e is (a portion of) a one-dimensional set Γ of lines in 3-space. If all the lines in Γ are stabbing lines for $\mathscr{B}$, then we just include one point of e in the test set of lines. If not all the lines in Γ are stabbing lines, but there is at least one stabber in Γ, then there exists an extremal stabbing line in Γ defined by three of the lines whose Plücker hyperplanes contain e and a fourth line whose Plücker hyperplane *does not* contain e. This extremal stabbing line is either an endpoint of e or it is a test line for some of the Plücker polytopes produced in the first phase of the algorithm. To account for this case, we include the endpoints of e in the test set.
2. We have a second degenerate case when a two-dimensional face f of a Plücker polytope is completely contained in Π. Then f is (a portion of) a two-dimensional family of lines Δ. If all the lines in Δ are stabbing lines for $\mathscr{B}$, we include one line of Δ in our test set. If not all the lines of Δ are stabbing lines for $\mathscr{B}$, but there is at least one stabber in Δ, then two of the lines whose Plücker hyperplanes contain f and a line whose Plücker hyperplane *does not* contain f must define a one-dimensional family of lines Γ which contains a stabbing line. Γ contains some edge e of some Plücker polytope. This case is covered by the analysis at 1.
3. A three-dimensional face of a Plücker polytope cannot be completely contained in Π [So], therefore, there are no other degeneracies to consider.

The above discussion proves the following corollary:

Corollary 1. *In time $O(n^3 2^{c\sqrt{\log n}})$ we can determine whether a set of polyhedra with n facets has a stabbing line and find one.*

6. Other Results on Line Stabbing and Open Problems

• If we have a set of parallel triangles the number of extremal stabbing lines is $\Theta(n^2)$ and the set of all stabbing lines $\mathscr{S}(T)$ is connected. A stabber can be found in time $O(n)$ [P2]. For a set of polyhedra of total complexity n whose facets lay on c different plane directions there exists an algorithm to find a stabbing line in time $O(c^2 n^2 \log n)$ [P2], [P1].

• Hohmeyer and Teller [HT] give an $O(n \log n)$ algorithm for finding a stabbing line of a set of axis-oriented boxes. Recently, Megiddo[7] [Me] and Amenta

[7] Actually, Megiddo's algorithm finds a stabber in linear time for axis-oriented boxes in any fixed dimension.

[Am] have found linear-time algorithms, based on linear programming techniques, for this problem. The problem of finding an algorithm close to linear for c-oriented polyhedra is still open.

• Theorem 2 gives the best-known upper bound on the number of components of $\mathscr{S}(\mathscr{B})$. The best-known lower bound for the number of components is $\Omega(n^2)$ (see [P2]). The challenge here is to narrow the gap between the two bounds.

• There is an $\Omega(n \log n)$ lower bound on the time needed to find a stabbing line of a set of polyhedra in 3-space. This bound is obtained by extending a lower bound for finding a stabbing line in a set of segments in R^2 [ARW]. We are still far from a provably optimal algorithm for the general stabbing problem.

• For a set of *disjoint polyhedra*, it is easy to show an $\Omega(n^2)$ lower bound for $|\mathscr{S}(\mathscr{B})|$, by exploiting a planar construction in [ES]. It would be interesting to narrow the gap between the upper and the lower bound for disjoint polyhedra.

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