Finding Two Disjoint Paths Between Two Pairs of Vertices in a Graph

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ABSTRACT. Given a graph $G = (V, E)$ and four vertices s_1, t_1, s_2, and t_2, the problem of finding two disjoint paths, P_1 from s_1 to t_1 and P_2 from s_2 to t_2, is considered This problem may arise as a transportation network problem and in printed circuits routing The relations between several versions of the problem are discussed Efficient algorithms are given for the following special cases—acyclic directed graphs and 3-connected planar and chordal graphs.

KEY WORDS AND PHRASES: graph algorithms, disjoint paths, pairs of vertices, efficiency of algorithms, connectivity, planar graphs, chordal graphs, acyclic directed graphs, polynomial reductions, transportation networks, routing

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1. *Introduction*

Given a graph $G = (V, E)$ and four vertices $s_1, t_1, s_2, t_2 \in V$, we consider the problem of finding two disjoint paths, $P_1(s_1, t_1)$ from s_1 to t_1 and $P_2(s_2, t_2)$ from s_2 to t_2.

This problem has four versions corresponding to the following cases: G is a directed/undirected graph and the paths are vertex-disjoint/edge-disjoint. The problem in general is denoted by $P2(s_1, t_1; s_2, t_2; G)$ or by $P2$ when G and s_1, t_1, s_2, t_2 have already been specified. The letters D and U indicate whether we deal with directed or undirected graphs respectively, while V and E stand for vertex-disjoint and edge-disjoint paths, respectively. For example $DVP2$ denotes the problem $P2$ for directed graphs and vertex-disjoint directed paths. We also use the notation $P2(s_1, t_1; s_2, t_2; G)$ for the predicate "There exist two disjoint paths in G, $P_1(s_1, t_1)$ and $P_2(s_2, t_2)$." $\neg P2(s_1, t_1; s_2, t_2; G)$ is the negation of this predicate.

The more general problem of finding $k + 1$ pairwise edge (vertex) disjoint paths, k paths between s_1 and t_1 and one path between s_2 and t_2, is shown by Even, Itai, and Shamir [1] to be NP-complete. Actually another general problem of finding k pairwise disjoint paths between k pairs of vertices $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$, respectively, is also NP-complete. This can be shown by reduction from the previous problem to the last one where $k + 1$ pairwise disjoint paths are required between the $k + 1$ pairs of vertices $(s_1, t_1), (s_1, t_1), \ldots, (s_1, t_1), (s_2, t_2)$.

In Section 2 we present polynomial reductions between several pairs of these four problems, and their relation to connectivity is discussed.

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In Section 3 we solve $DVP2$ for acyclic directed graphs in $O(|V| \cdot |E|)$ operations.

$UVP2$ is solved for 3-connected planar graphs and 3-connected chordal graphs in $O(|E|)$ operations in Sections 4 and 5, respectively.

Itai [6] showed that the problem for a planar or a chordal graph G which is not 3-connected can also be solved in $O(|E|)$ operations by reducing the problem into several separate problems for the 3-connected components of G. He makes use of Hopcroft and Tarjan's algorithm [4] for the decomposition of a graph into 3-connected components in $O(|E|)$ operations.

2. *Reductions and Relation to Connectivity*

Let $P_1 \alpha P_2$ denote that P_1 is polynomially reducible to P_2. For the exact definition of polynomial reducibility, see [7].

THEOREM 2.1. *DVP2 α DEP2.*

PROOF Given a directed graph $G = (V, E)$, we define $G' = (V', E')$ as follows:

$$V' = \bigcup_{v \in V} \{v', v''\}; \quad E' = \{v' \to v'' | v \in V\} \cup \{u'' \to v' | u \to v \in E\}.$$

Let $P = (v_1, \ldots, v_m)$ be a directed path in G. Its corresponding directed path P' in G' is $(v_1', v_1'', v_2', v_2'', \ldots, v_m', v_m'')$. One can easily verify that two directed paths P_1 and P_2 in G are vertex-disjoint iff P_1' and P_2' are edge-disjoint in G'. Q.E.D.

THEOREM 2.2 *DEP2 α DVP2.*

PROOF. Given are $G = (V, E)$ and $s_1, s_2, t_1, t_2 \in V$. Add to G a vertex u and 4 edges, $u \to s_1$, $u \to s_2$, $t_1 \to u$, $t_2 \to u$, obtaining a graph G'. Let $G'' = (V'', E'')$ denote the directed line graph of G' [3], and let $a, b, c, d \in V''$ correspond to the additional edges of G', respectively. Obviously

$$DEP2(s_1, t_1, s_2, t_2; G) \leftrightarrow DVP2(a, c; b, d; G'').$$

Q.E.D.

THEOREM 2.3. *UVP2 α DVP2.*

PROOF. Each undirected edge $u - v$ is replaced by the pair $u \to v$ and $v \to u$. Q.E.D.

THEOREM 2.4. *UEP2 α UVP2.*

PROOF. If there are two vertex disjoint paths $P_1(s_1, t_1)$ and $P_2(s_2, t_2)$ they can be found by a $UVP2$ algorithm. Let G' be the graph obtained from G by adding one vertex v and connecting it by four edges to s_1, t_1, s_2, and t_2. If there are two edge-disjoint paths $P_1(s_1, t_1)$ and $P_2(s_2, t_2)$ which are not vertex-disjoint, they have a common vertex u

CLAIM. *Such P_1 and P_2 exist iff there exists a vertex u in G and four edge-disjoint paths connecting u and v in G'.*

The proof is trivial.

In order to find P_1 and P_2 we choose a vertex u in G and search for four such paths using flow techniques (see, for example, [2]). This process is applied at most $|V|$ times. Q.E.D.

Theorems 2.1–2.4 can be summarized by

$$UEP2 \; \alpha \; UVP2 \; \alpha \; DVP2 \equiv DEP2.$$

P2 AND CONNECTIVITY

THEOREM 2.5. *If G is a 3-edge-connected undirected graph then $UEP2(s_1, t_1; s_2, t_2; G)$ is true for any choice of s_1, t_1, s_2, and t_2.*

PROOF. There exist three edge-disjoint paths $P_1(s_1, t_1)$, $P_2(s_1, s_2)$, $P_3(s_1, t_2)$ in G. (This is a variant of Menger's theorem.) P_2 and P_3 form a path $P(s_2, t_2)$ which is edge-disjoint to P_1. Q.E.D.

The relation between vertex connectivity and $UVP2$ is discussed in several papers (see [8, 11]). It is shown in [11] that 5-vertex-connectivity does not assure $UVP2$. It is conjectured there that 6-connectivity and even 4-connectivity with nonplanarity imply *UVP2*.

There are similar problems concerning *DVP2* and *DEP2*. One of the most interesting is: What are the minimal values of K_1 and K_2 such that vertex strong K_1-connectivity implies *DVP2* and edge strong K_2-connectivity implies *DEP2*? It is shown in Figure 1 that $K_1 \geq 3$.

3. *Acyclic Directed Graphs*

In this section we present an $O(|V| \cdot |E|)$ algorithm solving *DVP2* for acyclic directed graphs.

An edge which emanates from t_1 or t_2 cannot participate in any solution to $P2$. Thus we may delete such edges from G and assume t_1 and t_2 are sinks in G. For each $v \in V$ define the level $l(v)$ as the length of a longest path emanating from v. $l(v)$ can be efficiently determined by the familiar process of a successive deletion of all the sinks of the graph at a time.

Let $\bar{G} = (\bar{V}, \bar{E})$ be defined by:

$$\bar{V} = \{\langle u, v\rangle | u, v \in V \text{ and } u \neq v\},$$

$$\bar{E} = \{\langle u, v\rangle \to \langle u, w\rangle | v \to w \in E \text{ and } l(v) \geq l(u)\} \cup \{\langle v, u\rangle \to \langle w, u\rangle | v \to w \in E \text{ and } l(v) \geq l(u)\}.$$

THEOREM 3.1. *$P2(s_1, t_1; s_2, t_2; G)$ if and only if there exists a directed path $P(\langle s_1, s_2\rangle, \langle t_1, t_2\rangle)$ in $\bar{G}$.*

PROOF. The "only if" direction: Let $P_1(s_1, t_1)$ and $P_2(s_2, t_2)$ be two disjoint paths in G. The proof is by induction on $L(P_1) + L(P_2)$, i.e. the sum of the lengths of P_1 and P_2. If $L(P_1) + L(P_2) = 2$ then $P_1 = (s_1, t_1)$ and $P_2 = (s_2, t_2)$. If $l(s_1) \geq l(s_2)$, set $P = (\langle s_1, s_2\rangle, \langle t_1, s_2\rangle, \langle t_1, t_2\rangle)$ in $\bar{G}$. If $l(s_1) < l(s_2)$ then $P = (\langle s_1, s_2\rangle, \langle s_1, t_2\rangle, \langle t_1, t_2\rangle)$ is the desired path in $\bar{G}$.

Assume that $L(P_1) + L(P_2) > 2$. Let $P_1 = (s_1 = v_1, \ldots, v_k = t_1)$ and $P_2 = (s_1 = w_1, \ldots, w_m = t_2)$. If $l(s_1) \geq l(s_2)$ then $\langle s_1, s_2\rangle \to \langle v_2, s_2\rangle$ is the first edge of P. The rest of P is provided by the inductive hypothesis on the paths $P_1' = (v_2, \ldots, v_k = t_1)$ and P_2. If $l(s_1) < l(s_2)$, the first edge of P is $\langle s_1, s_2\rangle \to \langle s_1, w_2\rangle$ while the rest of it is given again by the inductive hypothesis on P_1 and $P_2' = (w_2, \ldots, w_m = t_2)$. This completes the proof of the "only if" direction.

The "if" direction: The proof is by induction on $L(P)$. If $L(P) = 2$ then $s_1 \to t_1 \in E$ and $s_2 \to t_2 \in E$ and $P2(s_1, t_1; s_2, t_2; G)$ holds.

If $P = (\langle s_1, s_2\rangle = \langle v_1, w_1\rangle, \ldots, \langle v_k, w_k\rangle = \langle t_1, t_2\rangle)$ then $P_1 = (s_1 = v_1, \ldots, v_k = t_1)$ and $P_2 = (s_2 = w_1, \ldots, w_k = t_2)$ are directed paths from s_1 to t_1 and from s_2 to t_2, respectively. Note that the definition of $\bar{G}$ implies that for each $1 \leq i < k$, either $v_i = v_{i+1}$ and $w_i \neq w_{i+1}$ or $w_i = w_{i+1}$ and $v_i \neq v_{i+1}$. Thus $L(P_1) + L(P_2) = L(P)$. If the first edge of P is $\langle s_1, s_2\rangle \to \langle v_1, s_2\rangle$, then $l(s_1) \geq l(s_2)$. By the inductive hypothesis, $P_1' = (v_2, \ldots, v_k = t_1)$ and P_2 are disjoint. s_2 is the first vertex of P_2 and therefore $l(w_j) < l(s_2)$ for $j = 2, \ldots, k$. Since $l(s_1) \geq l(s_2)$ and $s_1 \neq s_2$, $s_1 \notin P_2$ and therefore P_1 and P_2 are disjoint. A symmetric argument applies to the case in which $\langle s_1, s_2\rangle \to \langle s_1, w_2\rangle$ is the first edge of P. Q.E.D.

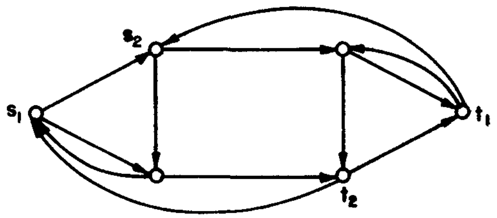

FIG. 1

ALGORITHMIC ASPECTS Generating the graph $\bar{G}$ and finding a path $P(\langle s_1, s_2\rangle, \langle t_1, t_2\rangle)$ in $\bar{G}$ takes $O(|\bar{E}|)$ operations. $|\bar{E}| \le 2\,(|V| - 2)|E|$ since each edge $u \to v \in E$ yields at most $2(|V| - 2)$ edges in $\bar{G}$, namely the edges of the form $\langle w, u\rangle \to \langle w, v\rangle$ and $\langle u, w\rangle \to \langle v, w\rangle$ for $w \in V - \{u, v\}$. Thus $P2$ can be solved by an $O(|V|\cdot|E|)$ algorithm in terms of the original graph $-G$.

4. *Planar 3-Connected Graphs*

In this section $P2$ means $UVP2$.

THEOREM 4.1. *Let G be a planar graph. If G has a planar representation such that four vertices s_1, s_2, t_1, t_2 are on one face F in this cyclic order, then $\neg P2(s_1, t_1; s_2, t_2; G)$.*

PROOF Assume $P2(s_1, t_1; s_2, t_2; G)$. Construct a graph G' by adding to G a vertex v and 4 edges (v, s_1), (v, s_2), (v, t_1), (v, t_2). The graph G' is planar too, since we may place v inside F

The subgraph of G' containing any two disjoint paths $P_1(s_1, t_1)$, $P_2(s_2, t_2)$, F, v and its incident edges is contractible to the complete graph K_5. (See Figure 2.)

Thus by Kuratowski's theorem (see for example [3]) G' is not planar, a contradiction. Q.E.D.

THEOREM 4 2. *Let G be a planar graph and let s_1, t_1, s_2, t_2 be four vertices of G. If*

(a) the vertices s_1, s_2, t_1, t_2 are not on one face in this cyclic order in any planar representation of G,

(b) there exist three disjoint paths P_1, P_2, P_3 between s_1 and t_1 and three disjoint paths Q_1, Q_2, Q_3 between s_2 and t_2,

then $P2[s_1, t_1; s_2, t_2; G]$

Note that three disjoint paths for each pair are necessary since otherwise s_1 and t_1 (s_2 and t_2) can disconnect all the Q's (P's).

Theorems 4.1 and 4 2 yield the following theorem.

THEOREM 4 3. *Let G be a 3-connected planar graph. Then $P2(s_1, t_1, s_2, t_2; G)$ iff there exists no planar representation of G in which the vertices s_1, s_2, t_1, t_2 are on one face in this cyclic order*

PROOF OF THEOREM 4.2. We assume to the contrary $\neg P2$ and conditions (a) and (b) through the following lemmas which establish some order on the paths, which enables us to prove the theorem

Let $P_1(u, v)$ and $P_2(v, w)$ be two paths. Denote by $P_1(u, v) * P_2(v, w)$ the path $P(u, w)$ obtained by concatenation of $P_1(u, v)$ and $P_2(v, w)$

Let u, v belong to a path P; then $P[u; v]$ denotes the subpath of P between u and v.

LEMMA 4.1. *We may assume that if for some $1 \le i, j \le 3$, (1) $u, v \in Q_j \cap P_i$ and (2) $Q_j[u; v] \cap P_k = \emptyset$ for $k \ne i$, then $P_i[u; v] = Q_j[u; v]$.*

PROOF. Whenever 1 and 2 hold for some $1 \le i, j \le 3$ but $P_i[u; v] \ne Q_j[u; v]$ we change P_i by replacing $P_i[u; v]$ by $Q_j[u; v]$.

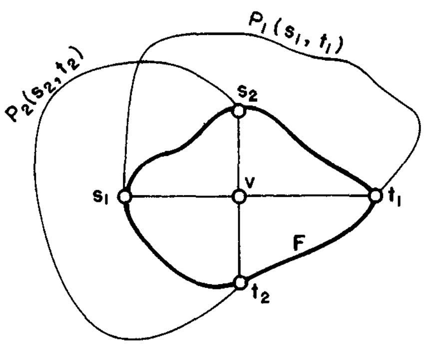

FIG 2

Condition 2 implies that the new P_i is disjoint to P_k for $k \neq i$. We still have to prove that this process is finite since, as shown in Figure 6, dealing with such a pair (u, v) may create a new pair (w, z) violating the conditions of the lemma.

A common subpath of P_i and Q_j is maximal if it is not a proper subpath of any other common subpath of P_i and Q_j. (A maximal common subpath may be a single vertex.) The total number of maximal common subpaths of the P_i and the Q_j is reduced, at least by one, each time such a replacement occurs. Thus, after a finite number of replacements, the lemma holds. Q.E.D.

Henceforth we consider the order of the vertices of each $P_i(Q_j)$ with respect to its walk from $s_1(s_2)$ to $t_1(t_2)$. $\neg P2$ implies that every Q_j intersects every P_i

At least one of the three paths Q_j does not contain either s_1 or t_1; assume it is Q_1. Assume that Q_1 intersects the P_i for the first time in the order P_1, P_2, P_3; otherwise rename the P_i. Let v_{ji} and w_{ji} denote the first and last vertices on Q_j which belong to P_i.

LEMMA 4.2. *The last P-path which intersects* Q_1 *is* P_3.

PROOF. Assume the last P-path which intersects Q_1 is P_i, $i \neq 3$. Then the path

$$Q_1[s_2; v_{1,i}] * P_i[v_{1,i}; w_{1,i}] * Q_1[w_{1,i}, t_2]$$

is disjoint to P_3 – a contradiction to $\neg P2$. Q.E.D.

LEMMA 4.3. *Let* $v_1 \in Q_1 \cap P_1$, $v_3 \in Q_1 \cap P_3$, *then* $Q_1[v_1, v_3]$ *intersects* P_2.

PROOF. Assume that $Q_1[v_1, v_3]$ is disjoint to P_2. By Lemma 4.2 $Q_1[w_{1,3}; t_2]$ is disjoint to P_2. Thus the path

$$Q_1[s_2; v_{1,1}] * P_1[v_{1,1}; v_1] * Q_1[v_1, v_3] * P_3[v_3; w_{1,3}] * Q_1[w_{1,3}; t_2]$$

is disjoint to P_2 – a contradiction to $\neg P2$. Q.E.D.

LEMMA 4.4. *All the* Q_j *intersect the* P_i *for the first time in the order* P_1, P_2, P_3.

Remark. The vertices s_1 and t_1 belong to all the P_i. In the case where a Q-path passes through s_1 or t_1, it is regarded as intersecting the P_i in the order P_1, P_2, P_3.

PROOF. The claim is true for Q_1 by definition We prove it for Q_2. The proof for Q_3 is similar.

Assume that P_2 is the first P_i which intersects Q_2. We first show that $Q_1[w_{1,2}; t_2]$ is disjoint to P_1. Assume there exists a vertex x such that $x \in Q_1[w_{1,2}; t_2] \cap P_1$. Lemma 4.2 implies that there exists a vertex y such that $y \in Q_1[x; t_2] \cap P_3$. Lemma 4.3 implies that $Q_1[x; y] \cap P_2 \neq \varnothing$, contradicting the definition of $w_{1,2}$. Thus $Q_1[w_{1,2}; t_2] \cap P_1 = \varnothing$.

Thus the path,

$$Q_2[s_2; v_{2,2}] * P_2[v_{2,2}; w_{1,2}] * Q_1[w_{1,2}; t_2]$$

is disjoint to P_1 – a contradiction to $\neg P2$.

Similarly, it can be shown that P_3 is not the first P_i which intersects Q_2. Thus the first P_i which intersects Q_2 is P_1. In a similar way we can show that P_2 is the second P-path which intersects Q_2, completing the proof. Q.E.D.

LEMMA 4.5. *Lemmas* 4.2 *and* 4.3 *are valid for every* Q_j.

PROOF. Implied immediately by Lemma 4.4.

We choose now a planar representation of G such that P_2 is inside the region R which is bounded by P_1 and P_3 (see Figure 3). We assume that the order of $v_{j,1}$ on P_1 is $v_{1,1}$, $v_{2,1}$, $v_{3,1}$ (otherwise renumber the Q_j).

LEMMA 4.6. *The vertices* s_2 *and* t_2 *are either outside* R *or on its boundary.*

PROOF. Assume that s_2 is inside R. (The proof for t_2 is symmetric.) s_2 is inside the region bounded by P_1 and P_2 since all the Q_j intersect P_1 first (Lemma 4.4).

Consider the cycle

$$C = Q_1[s_2; v_{1,1}] * P_1[v_{1,1}; s_1] * P_3 * P_1[t_1; v_{3,1}] * Q_3[v_{3,1}; s_2]$$

(heavy lines in Figure 3). We show that $C \cap Q_2[v_{2,1}; v_{2,2}] = \varnothing$.

Lemma 4.4 implies that $P_3 \cap Q_2[v_{2,1}; v_{2,2}] = \varnothing$.

Assume that there exists a vertex $u \in P_1[v_{1,1}; s_1] \cap Q_2[v_{2,1}; v_{2,2}]$. $v_{2,2}$ is the first

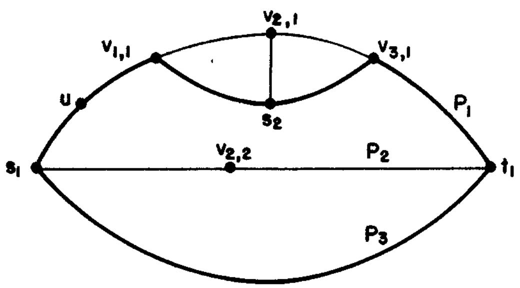

FIG. 3

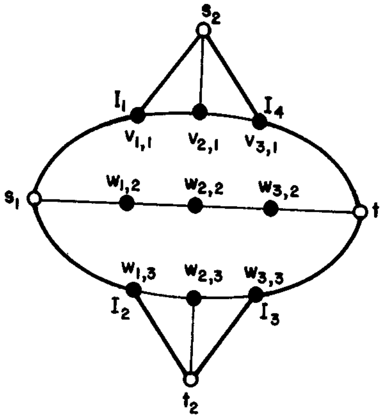

FIG 4

intersection of Q_2 with $P_2 \cup P_3$. Since u precedes $v_{2,2}$ on Q_2, Lemma 4.1 implies that $Q_2[v_{2,1}; u] = P_1^3\text{v2,1}_+ u]$. Thus $v_{1,1} \in Q_1 \cap Q_2$, a contradiction. Therefore $P_1[v_{1,1}; s_1] \cap Q_2[v_{2,1}; v_{2,2}] = \varnothing$. Similarly $P_1[t_1; v_{3,1}] \cap Q_2[v_{2,1}; v_{2,2}] = \varnothing$. Hence $C \cap Q_2[v_{2,1}; v_{2,2}] = \varnothing$. But $v_{2,1}$ and $v_{2,2}$ are outside and inside the region bounded by C, respectively. Thus $C \cap Q_2[v_{2,1}; v_{2,2}] \neq \varnothing$—a contradiction. Q.E.D.

LEMMA 4.7. *The vertices $w_{1,i}$; $w_{2,i}$; $w_{3,i}$ are in this order on P_i, for $i = 2, 3$.* (See Figure 4.)

PROOF. We first prove the lemma for $i = 3$. The vertices $v_{1,1}$, $v_{2,1}$, $v_{3,1}$ are in this order on P_1. Let us prove that $Q_j[v_{j,1}; w_{j,3}]$ is inside R (the region bounded by P_1 and P_3). If $Q_j[v_{j,1}; w_{j,3}]$ leaves R through a vertex of P_1 (P_3) then it cannot enter R again either through P_1 (P_3) or through P_3 (P_1), by Lemmas 4.1 and 4.5, respectively.

Let G' denote the planar subgraph of G contained in R (including the boundary). The boundary of R is the external face of G'. Since the paths $Q_j[v_{j,1}; w_{j,3}]$ and $Q_k[v_{k,1}; w_{k,3}]$, $1 \leq j < k \leq 3$, are disjoint, the case $i = 3$ is implied by Theorem 4.1.

A similar argument shows that if $w_{1,3}$, $w_{2,3}$, and $w_{3,3}$ are in this order on P_3 then $w_{1,2}$, $w_{2,2}$, and $w_{3,2}$ are in this order on P_2. This proves the case $i = 2$. Q.E.D.

The lemmas above enable us to complete the proof of Theorem 4.2. By Lemma 4.6, s_2 and t_2 are either outside R or on its boundary and by Lemma 4.7 the vertices $v_{1,1}$; $v_{2,1}$; $v_{3,1}$ and $w_{1,3}$; $w_{2,3}$; $w_{3,3}$ are in this order on P_1 and P_3, respectively.

Consider the four following paths (see Figure 4):

$$I_1 = Q_1[s_2; v_{1,1}] * P_1[v_{1,1}; s_1], \quad I_2 = P_3[s_1; w_{1,3}] * Q_1[w_{1,3}; t_2],$$
$$I_3 = Q_3[t_2; w_{3,3}] * P_3[w_{3,3}; t_1], \quad I_4 = P_1[t_1; v_{3,1}] * Q_3[v_{3,1}; s_2].$$

Note that if s_2 is on P_1, then $Q_1[s_2; v_{1,1}]$ and $Q_3[v_{3,1}; s_2]$ are empty.

The cycle $I_1 * I_2 * I_3 * I_4$ encloses a region denoted by F. Let

$$J_1 = I_1 * I_4 - \{s_1, t_1\}, \quad J_2 = I_2 * I_3 - \{s_1, t_1\},$$
$$J_3 = I_1 * I_2 - \{s_2, t_2\}, \quad J_4 = I_3 * I_4 - \{s_2, t_2\}.$$

(see Figures 4 and 5).

The vertices s_1, s_2, t_1, t_2 are on the boundary of F in this cyclic order. Thus by assumption *a* of the theorem, the exterior of F is not a face of G. Hence, at least one of the following cases occurs.

Case 1 (2). There exist $u \in J_1(J_3)$, $v \in J_2(J_4)$ and a path $R(u, v)$; all of its vertices except u and v are outside of F. In the first case the path $Q(s_2, t_2) = J_1[s_2; u] * R(u, v) * J_2[v; t_2]$ is disjoint to P_2, which is inside F (see Figure 5(a)).

In case 2 let $P(s_1, t_1) = J_3[s_1; u] * R(u, v) * J_4[v, t_1]$ and let $Q(s_2, t_2) = Q_2[s_2; v_{2,2}] * P_2[v_{2,2}; w_{2,2}] * Q_2[w_{2,2}; t_2]$ (see Figure 5(b)).

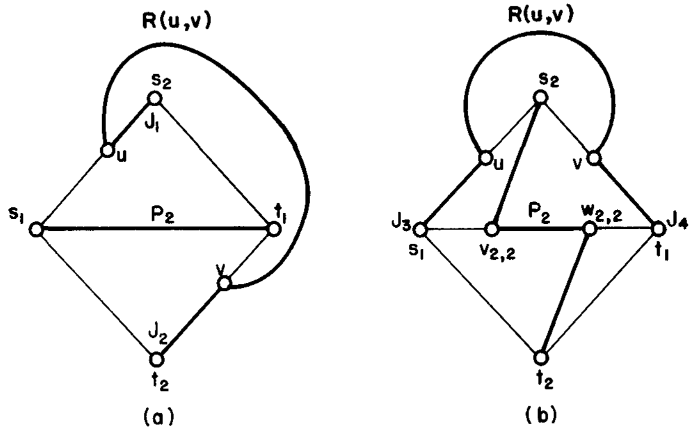

FIG. 5

Similar arguments to those which were used to show that $P_1[v_{1,1}; s_1] \cap Q_2[v_{2,1}; v_{2,2}] = \emptyset$ in the proof of Lemma 4.6 can be used here to show that $P \cap Q_2[s_2; v_{2,2}] = \emptyset$. Lemma 4.7 (for $i = 2$) implies that $w_{2,2} \neq s_1, t_1$ and therefore $P \cap P_2[v_{2,2}; w_{2,2}] = \emptyset$. It follows directly from Lemma 4.5 and the definition of $w_{2,2}$ that $P \cap Q_2[w_{2,2}; t_2] = \emptyset$. Thus $P \cap \bar{Q} = \emptyset$. Hence, in both cases $P2$ is true—a contradiction. Q.E.D.

THE ALGORITHM

The algorithm actually follows the lines of Theorem 4 1 and the proof of Theorem 4 2

1 If s_1, s_2, t_1, t_2 are on the same face F of G in this cyclic order, then $\neg P2$, stop
2 Find three disjoint paths $P_i(s_1, t_1)$ and three disjoint paths $Q_j(s_2, t_2)$
3. Change the P_i such that if $u, v \in Q_j \cap P_i$ and $Q_j[u, v] - \{u, v\} \cap P_k = \emptyset$ for $k = 1, 2, 3$, then $P_i[u; v] = Q_j[u, v]$ (see Lemma 4 1) A linear implementation of this step is given later
4. Scan each of the Q_j checking the following conditions
   a Q_j intersects every P_i
   b. All the Q_j intersect the P_i, for the first time, in the same order (say, P_1, P_2, P_3)
   c P_3 is the last P-path intersected by Q_j
   d If $u \in P_1 \cap Q_j$, $v \in P_3 \cap Q_j$, then $Q_j[u, v]$ intersects P_2
   In case any of these conditions is violated, construct two disjoint paths $P(s_1, t_1)$ and $Q(s_2, t_2)$ according to the proof of the appropriate lemma (one of 4 2, 4 3, 4 4, 4 5) While scanning the Q_j, determine the vertices $v_{j,i}$, $w_{j,i}$
5 If there exists a path $Q(s_2, t_2)$ in $G - P_2$, stop. The desired paths are Q and P_2 ($G - P_2$ is obtained by removing the vertices of P_2 from G)
6 Let $Q(s_2, t_2) = Q_2[s_2, v_{2,2}] * P_2[v_{2,2}, w_{2,2}] * Q_2[w_{2,2}, t_2]$ Find a path $P(s_1, t_1)$ in $G - Q$ The desired paths are Q and P

The validity of steps 5 and 6 can be easily derived from the end of the proof of Theorem 4.2.

THE COMPLEXITY OF THE ALGORITHM. We prove now that the algorithm is linear. We assume that G is planar and its faces are given. Anyway, this is the output of the linear planarity testing algorithm of Hopcroft and Tarjan [5, 10]. Steps 1, 4, 5, and 6 are obviously linear. Step 2 is performed by applying flow techniques [2]. It is linear since only three augmenting paths are required for finding three disjoint paths.

A *straightforward implementation of step 3 requires* $O(n^2)$ operations. In the following we give a linear implementation of it which yields the linearity of the whole algorithm.

This implementation and its linearity proof are technically complicated and are not given in a full detailed form. However, an exact formal implementation and linearity proof can be derived from this description quite easily.

A LINEAR IMPLEMENTATION OF STEP 3

Definition. A vertex $v \in V$ is a *cross vertex* if it is either an end vertex of a maximal common subpath (defined in the proof of Lemma 4.1) or s_2 or t_2. Henceforth we consider each $Q_j(s_2, t_2)$ as going from left to right (s_2 is leftmost).

With every cross vertex on Q_j, we associate a *left pointer* and a *right pointer* to the closest cross vertices on Q_j to its left and right, respectively.

There are two types of pointers. A pointer from u to v is a *continuous pointer* if u and v are end vertices of the same maximal common subpath. Otherwise it is a *jumping pointer* (see Figure 6).

If a couple of cross vertices $u, v \in Q_j \cap P_i$ are pointing one to the other by jumping pointers, then $Q_j[u; v] \cap P_k = \emptyset$ for $k \neq i$. Thus $P_i[u; v]$ should be replaced by $Q_j[u; v]$. Such a couple u, v is a *candidate couple* and it is stored in a *replacement list*.

A general scheme of the implementation of step 3 is as follows:

1. (Initialization.) While scanning the Q_j, set the pointers of the cross vertices and put candidate couples into the replacement list.
2. Take a candidate couple (u, v) out of the replacement list. If there is no such couple—stop
3. If u and v do not belong to the same P_i, return to 2.
4. Scan $P_i[u; v]$ from u to v, excluding u and v Let w denote the current vertex in this scanning
5. Delete w from P_i.
6. If w is a cross vertex, then connect left (w) and right (w) through the appropriate pointers. (w is no longer a cross vertex) In the case where left (w) and right (w) belong to the same P_i, (left (w), right (w)) is added to the replacement list In the case where left (w) (right (w)) is a continuous pointer, then the next current vertex in the scanning of $P_i[u; v]$ is left (w) (right (w))
7. Change right (u) and left (v) into continuous pointers If left (u) or right (v) are continuous pointers, update the pointers appropriately.

Step 3 is required since a couple which was put into the replacement list may cease being a candidate couple because of changes of the P_i. The couples (u, v) and (x, y) in Figure 6 are put into the replacement list at step 1. If we treat (u, v) first, then (x, y) is not a candidate couple anymore.

We give now the main arguments for a linearity proof of this implementation. It is easy to see that the initialization is linear. A vertex of P_i which is not in any of the Q_j is treated at most once (step 5), since after its deletion from P_i it would never enter any of the P_k (only vertices of the Q_j may enter the P_k).

Vertices of the Q_j which are not cross vertices are not treated after the initialization. A cross vertex w is treated only if it belongs to $P_i[u; v]$ for some couple (u, v) in the replacement list. If $w \neq u, v$ it ceases from being a cross vertex and the number of cross

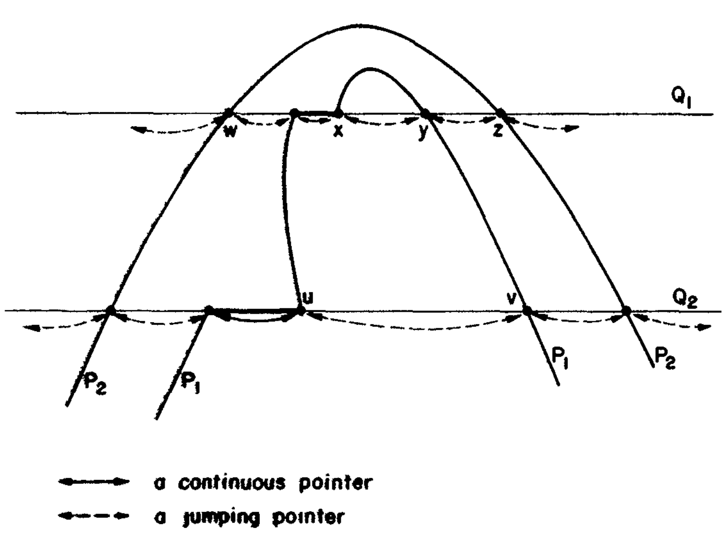

FIG. 6

vertices is reduced by one. Although treating u and v may not reduce the number of cross vertices it always reduces the number of maximal common subpaths, which does not exceed n. The treatment of a single cross vertex requires a constant number of operations. Thus, the whole implementation is linear.

5. *Chordal Graphs*

An undirected graph $G = (V, E)$ is *chordal* if every cycle of length $n \geq 4$ has a chord, i.e. an edge between two of its nonadjacent vertices. For applications of chordal graphs see [9].

A set $S \subset V$ is a *separating set* if the deletion of the vertices of S from G yields an unconnected graph. A separating set S is *minimal* if there is no separating proper subset of S. A path is *minimal* if there is no edge between nonadjacent vertices of it.

THEOREM 5.1. *Every minimal separating set in a chordal graph is a clique.* (See [9].)

THEOREM 5.2. *If G is a 3-connected chordal graph, $UVP2(s_1, t_1; s_2, t_2; G)$ holds for any four vertices s_1, t_1, s_2, and t_2.*

PROOF. Assume $\neg UVP2(s_1, t_1; s_2, t_2; G)$. Let $P(s_1, t_1)$ be a minimal path which neither passes through s_2 nor through t_2. (There exists such a path since G is 3-connected.) The vertices of P form a separating set S between s_2 and t_2. Let $S' \subseteq S$ be a minimal separating set between s_2 and t_2. $|S'| \geq 3$ since G is 3-connected. By Theorem 5.1 S' is a clique – a contradiction to the minimality of P. Q.E.D.

Theorem 5.2 suggests a simple algorithm to find disjoint paths $P_1(s_1, t_1)$ and $P_2(s_2, t_2)$ in a 3-connected chordal graph G.

THE ALGORITHM

1. Find a shortest path $P_1(s_1, t_1)$ in $G - \{s_2, t_2\}$
2. Find a path $P_2(s_2, t_2)$ in $G - P_1$.

The validity of the algorithm is easily derived from the proof of Theorem 5.2. The algorithm requires $O(|E|)$ operations.

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